Career Concerns with Cost Uncertainty*

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Abstract

I consider a continuous-time career concerns model. As in Holmström (1999), there is symmetric uncertainty about a worker's ability. Also, the worker has private information about her conscientiousness (cost of effort). The sequence of observed outputs allows learning about both attributes of the worker. As in the career concerns literature, incentives to exert effort are driven in part by a signal-jamming motive, i.e., the desire to manipulate the market's beliefs about the worker's ability. In line with prior results, this motive is present throughout the worker's lifetime, but its impact on the worker's effort gradually decreases over time as the market learns the worker's ability. In contrast, the motive to signal conscientiousness is more nuanced and changes sign as time progresses. I find that early in her career, cost uncertainty pushes the agent to work harder to signal that she is conscientious. During her middle and late career, the agent has an incentive to signal that she is lazy. In the second phase of her career, the agent lowers her effort to seem lazy. During her late career, the agent, surprisingly, increases her effort in order to convince the market of her laziness. This result, in which the impact of cost uncertainty on effort choice changes sign twice is used to explain the patterns of residuals in the relation between earnings and work experience specified in Mincer (1974) and noted in Murphy & Welch (1990).

JEL codes: D82, D83, J22, J24, L14

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1 Introduction

Motivation for the model: Ability and hard work are some of the most important, not-so-secret ingredients to success. The applied psychology literature abounds with studies on the importance of both characteristics in job performance. Hunter & Hunter (1984) finds that for job performance at entry-level jobs, cognitive ability has a mean validity (coefficient of correlation) of 0.53, which exceeds all other indicators that they tested. In fact, Gottfredson (1997) notes that while personnel psychologists still argue about the extent to which general intelligence, or ability, predicts job performance, there is no longer any dispute regarding the conclusion that intelligence helps predict performance in most, if not all, jobs.

The importance of conscientiousness is highlighted in (Barrick & Mount, 1991) which studies the relationship between the big five personality traits and job performance. They find that conscientiousness shows a positive correlation with job performance for all job types. For other personality traits, the results vary. Sometimes conscientiousness may matter more than intelligence or natural ability. Ericsson et al. (1993) finds that, with few exceptions, expert performance reflects years of deliberate effort rather than innate ability.[1]

In more recent work in labor economics, J. J. Heckman et al. (2006) argues that both, cognitive, and non-cognitive skills, must be considered in explaining labor market outcomes. J. Heckman et al. (2013) notes that most of the gains from the influential Perry pre-school program are attributable to positive personality changes rather than gains in cognitive ability. Chetty et al. (2011) studies the long-term impacts of having a small class size and finds suggestive evidence that the gains may be attributable to improvements in non-cognitive skills (including effort). Crucially, the paper also notes that "*noncognitive measures are highly correlated with earnings even conditional on test scores*"[2].

The importance of the two dimensions of ability and conscientiousness in predicting job performance implies that the expected output, and therefore the wages offered to workers must depend upon the perceived strength of these attributes for the workers.[3] Workers, therefore, have an interest in manipulating market perception of their ability as well as conscientiousness in order to gain higher wages. Students entering the job market

[1] Other key papers in this literature include Behling (1998), Gottfredson (1997), Wagerman & Funder (2007), Bakker et al. (2012) Barrick et al. (1993), and Ng & Feldman (2010)

[2] Where test scores are presumably generated from some production function that includes cognitive ability as well as non-cognitive skills

[3] This, of course, relies on imperfect contractibility of output.

signal their ability through scholarships, fellowships, grades, and other achievements. Insofar as output is observed, while the effort is imperfectly observable, one of the crucial means by which workers communicate their type is output. Scientists may use patents to signal ability, managers use profits, and academic researchers use publications. Signals of conscientiousness are also employed by workers. In 2017, US workers cumulatively left more than 705 million paid vacation days unused.[4] In a 2016 survey of more than 5000 US workers, 39 percent of the workers claimed that they "*think it is a good thing to be seen as a work martyr by the boss*".[5]. This proportion is higher for younger workers and declines as workers age. The key point to note is that workers consider the signaling implications of their effort choices.

Model: I consider a career concerns model in continuous time. As in Holmström (1999), there is symmetric uncertainty about a worker's ability. Also, the worker has private information about her conscientiousness (cost of effort). The sequence of observed outputs allows learning about both attributes of the worker. As in the career concerns literature, incentives to exert effort are driven in part by a signal-jamming motive, i.e., the desire to manipulate the market's beliefs about the worker's ability. In line with prior results, this motive is present throughout the worker's lifetime, but its impact on the worker's effort tends to gradually decrease over time as the market learns the worker's ability. In contrast, the motive to signal conscientiousness is more nuanced and changes sign as time progresses. I find that early in her career, cost uncertainty pushes the agent to work harder to signal that she is conscientious. During her middle and late career, the agent has an incentive to signal that she is lazy. In their mid-career, the agent lowers her effort to seem lazy. In the last phase of her career, the agent, surprisingly, increases her effort in order to convince the market of her laziness. I find that this result, in which the impact of cost uncertainty on effort choice changes sign twice fits well with, and provides an economic explanation for a long-standing puzzle in the empirical labor economics literature.

Empirical puzzle: Forty-four years after it was published, Mincer (1974)'s specification remains a workhorse of the labor economics literature. The usual Mincerian specifi-

[4] As reported by a coalition of US travel companies called Project Time Off. Link: https://projecttimeoff.com/wp-content/uploads/2018/05/StateofAmericanVacation2018.pdf

[5] A work martyr is an employee who is conscientious to a fault (literally). The survey is titled *The Work Martyr's Cautionary Tale* and was conducted by Project Time Off

cation treats log wages as a linear function of years of schooling, and a quadratic function of years of experience.

$$log(w) = \alpha_0 + \rho_s s + \beta_0 x + \beta_1 x^2 + \nu \tag{1}$$

where w denotes wages, s stands for years of schooling, and x represents the years of experience an individual has had. Mincer (1974) uses a model of human capital investment over a life cycle to motivate log earnings as a quadratic function of years of experience. Details of the micro-foundations for the specification may be found in J. J. Heckman et al. (2003) or Chiswick (2003).

Empirically, this model provides a succinct specification that does an exceptional job at fitting the data. As Lemieux (2006) notes, "*most studies still tend to estimate earnings regression that are very closely related to equation [1]. Though a list of other regressors are typically added to the basic Mincer equation, the three key variables in equation [1] still appear in most empirical estimates of earnings regressions.*"

However, it has also been the subject of criticism, particularly because of the magnitude and robustness of the pattern of residuals generated from this specification. As Murphy & Welch (1990) note and Lemieux (2006) confirms, the consistent pattern in the residuals generated using the Mincer equation is indicative of a specification issue. The predictions from the Mincerian specification overestimate earnings for the first five to seven years of experience, then underestimates earnings it till around year 17. For workers with 18-35 years of experience, the quadratic specification overestimates earnings, while underestimating earnings for workers with more than 35 years of experience. Murphy & Welch (1990) recommends adding cubic and quartic terms to the specification since it largely gets rid of the pattern in residuals. The "*fine-tuning*" of the Mincer equation is, of course, required, but these papers offer no theoretical justification for the patterns present in the residuals.

While the Mincerian specification has a solid theoretical foundation, the "*fine-tuning*" suggested in Murphy & Welch (1990) has none. In this paper, I focus on workers with between 7 and 40 years of experience.[6] I explain the pattern in residuals as a function of the desire of workers to signal their conscientiousness in a continuous-time career concerns model with the presence of uncertainty over effort costs.

[6] There are good reasons for lower than expected earnings during the first seven years, and this paper will not be focusing on explaining that phase. I discuss the reasons for expecting lower wages in that phase in section 4.

Preview of results: In the main result, I show that a career may be split into three distinct phases. In sharp contrast with conventional wisdom, I find that it is beneficial to be thought of as hard-working only for the first phase of a worker's career. In the second and third phases, the worker's continuation value decreases with the likelihood the market puts on the worker being hard-working.[7] The driving force of this result comes from the fact that an agent's wage is conditioned on her output history, and that conditioning on the agent being lazy would make the market believe the agent has a higher ability. During the agent's early career, effort is a higher component of output and the agent benefits from seeming hard-working. Over time, effort diminishes, and ability becomes the key contributor to output. During the agent's middle and late career, this creates an incentive for the agent to try and convince the market that she is lazy, and thereby raise the market's posterior on her expected ability.

In the early and middle two phases of an agent's career, the market's posteriors on the agent's ability conditioning on her cost of effort are not very different. At the same time, the agent's level of effort is still a relatively sizeable contributor to the output. In those phases, the market believes that a hard-working agent will produce more than a lazy agent with the same output history. However, during the agent's late-career, effort diminishes as a big contributor to output, while ability becomes relatively more important. During this phase, as the market conditions on an output history, the market believes that a lazy agent with that history would produce a higher output going forward. This is driven by the fact that the market's posterior on the ability of an agent conditioning on high effort costs exceeds the market's posterior on the ability of a conscientious agent with the same history. This phase where agents work harder in order to convince the market of their lack of conscientiousness is both surprising and novel. This phase will also be crucial in explaining the pattern of residuals in the Mincerian regression.

Technical contribution: The intractable nature of the problem implies that a closed-form solution cannot be found in this environment. All the main results come from studying the properties of the first-order condition for optimality. Two issues arise because of this – *sufficiency* of the first-order approach, and *existence* of the equilibrium. Using the concavity of the agent's Hamiltonian, as is done in Prat & Jovanovic (2014), is not a feasi-

[7] This result is also found by Kőszegi & Li (2008) in a three-period model with restrictions on effort choices available to workers.

ble way of showing sufficiency in this environment. Luckily, it is not needed. I show that any deviation (or sequence of deviations) would be followed by on-equilibrium path play by the agent, and the persistent private information generated by the off-equilibrium play will not impact post-deviation choices. To show the existence of an equilibrium I require sufficiently convex effort costs.[8] First, I consider a model with finite and discrete-time and employ a fixed point theorem attributed to Debreu, Fan, and Glicksberg[9] to show the existence of equilibria at a realized state of the world, and then use a measurable selection theorem to select a measurable unique equilibrium for each realized state. Then, I must decrease step size while showing the convergence of the equilibrium in discrete-finite time to an equilibrium in continuous time. It is unusual to employ such general mathematical theorems to show the existence of equilibrium in this environment. Therefore, this approach may help guide future research in the career concerns literature in showing the existence of equilibria for intractable environments.

Related Literature. The seminal paper of Holmström (1999), along with much of the literature that follows it, considers the worker's choice of effort in an environment where the only unknown is ability, and it is symmetrically unknown. The paper closest to my analysis is Kőszegi & Li (2008)[10], which considers asymmetric information about worker drive[11] within a career concerns framework. However, Kőszegi & Li (2008) restricts the worker's effort choices to be linear in drive, and deterministic in time. This implies that workers are allowed to create Bayesian posteriors, but not use them while choosing effort (this is particularly true for posteriors regarding ability). Profitable deviations in which workers and firms use the information regarding the posterior distribution of ability and *drive* exist in this model. Unlike Kőszegi & Li (2008), I impose no restrictions on the strategy set available to agents. That, in addition to the more general time horizon in my paper, allows me to get much richer results in this environment.

The asymmetric uncertainty over the cost of effort ties this model to the literature

[8] The *sufficiency* is satisfied for reasonable values and will be required to make the agent's continuation value concave in her effort choice. See appendix A.2 for details.

[9] This fixed point theorem comes from Debreu (1952), Fan (1952), and Glicksberg (1952)

[10] Penczynski (2007) also follows this approach and solves a model of infinite two-period games and assumes the same restriction on strategy choices as Kőszegi & Li (2008). The impact of cost uncertainty found by Penczynski (2007) is either to monotonically decrease effort or is similar to Kőszegi & Li (2008).

[11] Kőszegi & Li (2008) consider a similar setup in three periods with symmetric uncertainty over ability and asymmetric uncertainty over drive. Drive denotes the marginal utility of money in their model. In my model, if I impose quadratic costs, of effort (where $cost = \frac{(effort)^2}{\lambda}$), drive would be equivalent to $\frac{1}{\lambda}$.

on signaling that builds on Spence (1973). The equilibrium in this paper is separating in the sense that the two types of workers choose different levels of effort. However, the presence of symmetric uncertainty and noise implies that the market learns about the type of the agent slowly over time. In sharp contrast with Spence (1973), the agent doesn't always want to be perceived as the type with low cost of effort. In the later phases of an agent's career, she benefits from seeming lazy.

This paper is also related to others that have sought to tie together signaling and signal-jamming. Farber & Gibbons (1996) develop a dynamic model in which there is an initial period of education acquisition followed by a working career. The education acquired works as a signal for ability. Another model in which there is an initial phase of education followed by employment is MacLeod & Urquiola (2015). The distinguishing factor in their paper is that they give importance to the institute attended in addition to the number of years of education.[12]

This paper is similar in spirit to the study of two-dimensional asymmetric uncertainty in Kartik & Frankel (2017). They consider a model of signaling asymmetric uncertainty over the natural action (or ability), as well as over the cost of action (or effort). The difference between their environment and the one I study is that ability is symmetrically unknown in my setup, while they assume that the worker knows her ability and the market does not.

The paper also contributes to and builds on the many recent papers in the career concerns literature have employed continuous time techniques. Chief among these are Prat & Jovanovic (2014), Cisternas (2018), and Bonatti & Hörner (2017).[13]

Outline.

Section 2 develops the basic intuition of the model using a simple two-period model. In section 3, the full model is described, the existence of the equilibrium is shown, and its properties are characterized. The pattern in Mincerian residuals is explained using the results of this model in section 4. Section 6 concludes. Extensions of the model are considered in section 5. All proofs are in appendix A.

[12] Other important papers at the intersection of these two literatures of signaling and signal-jamming are Belenzon et al. (2017), Heinsalu (2015), and Hvide (2003).

[13] Other recent and important papers include Cisternas (2017), Jovanovic & Prat (2016)

2 Basic Model

In this section, I introduce the basic notation and solve a simple two-period model to illustrate the main tension in this environment. In the second period, the market values only ability. The market's posterior on ability is higher when conditioned on high effort costs. However, in the first period, a higher output makes the market believe that the agent has low effort costs. Therefore, effort cost uncertainty pushes agents (of all types) to reduce first-period output so that they can benefit from a higher posterior on ability in the second period. Note that this first-period output is lower in comparison to a benchmark without cost uncertainty – which is equivalent to the environment in Holmström (1999).

2.1 Setup

A manager with a life-span of two periods is hired by a firm in a competitive wage market. The worker is born with an unknown ability, η, which is generated from the distribution $N(m, \frac{1}{h_1})$. The output generated by the worker in each period (y_t) is given by

$$y_t = \eta + a_t + \varepsilon_t \tag{2}$$

where t denotes the period, a_t is the costly effort made by the worker, and ε_t is a stochastic noise term generated from $\varepsilon_t \sim N(0, \frac{1}{h_\varepsilon})$. The worker maximizes her expected discounted utility

$$U(w, a, \lambda_i) = \mathbb{E}\left[\sum_1^2 \beta^{t-1}\left[w_t - \frac{a^2}{\lambda_i}\right]\right] \tag{3}$$

where w_t is the wage that the worker earns in period t. $\beta \in (0,1)$ denotes the discount factor. The heterogeneity in cost of effort is denoted by λ_i, where $\lambda_L < \lambda_H$. Here, type L corresponds to the *Lazy* worker who has a high cost of effort, while H corresponds to a *Hard-working* worker who has a low cost of effort. Since the market is competitive, wages equal the expected output

$$w_1 = \mathbb{E}[\eta] + \mathbb{E}[a_1] \tag{4}$$

$$w_2 = \mathbb{E}[\eta|y_1] + \mathbb{E}[a_2|y_1] \tag{5}$$

I next solve the model to calculate the optimal effort made by the worker, and the wages paid by the market

2.2 Solution

Since it is a two period model, no effort is made in the second period. Also, in the first period, $\mathbb{E}_{t=1}[\eta] = m$. Therefore, wages in the first period equal

$$w_1 = m + \mathbb{E}[a_1] \tag{6}$$

while wages in the second period equal

$$w_2 = \mathbb{E}[\eta|y_1] \tag{7}$$

Proposition 1. *In a situation without cost uncertainty - as in the standard Holmström (1999) model, optimal effort in the first period follows the first order condition*

$$\frac{a_1}{\lambda} = \frac{\beta}{2}\left[\frac{h_\varepsilon}{h_1 + h_\varepsilon}\right]$$

With cost uncertainty, optimal effort in the first period follows the first order condition

$$\frac{a_1^i}{\lambda_i} = \frac{\beta}{2}\left[\frac{h_\varepsilon}{h_1 + h_\varepsilon}\right] - \frac{\beta}{2}\mathbb{E}\left[\frac{h_1 h_\varepsilon^2 \left(a_H^* - a_L^*\right)^2 e^{\frac{h_1 h_\varepsilon \left(a_H^* - a_L^*\right)\left(2y_1 - 2m - a_H^* - a_L^*\right)}{2(h_1 + h_\varepsilon)}}}{(h_1 + h_\varepsilon)^2 \left(1 + e^{\frac{h_1 h_\varepsilon \left(a_H^* - a_L^*\right)\left(2y_1 - 2m - a_H^* - a_L^*\right)}{2(h_1 + h_\varepsilon)}}\right)^2}\right]$$

It is therefore, clear that optimal effort with cost uncertainty is strictly lower than optimal effort without cost uncertainty.

Proof. The proof of this proposition is given in Appendix A.1. □

From the above proposition, it is clear that in the first period, a hard-working worker will expend greater effort than a lazy worker. In the second period, faced with a given history of output, the market will attempt to gauge the level of ability of the worker. The posterior on ability (given a history of output), will be higher if the market conditions on the worker being lazy. We know that in the second period, no effort is made, therefore wages equal expected ability. Since the expected ability of a lazy worker exceeds that of a hard-working worker, there is an incentive for either type worker to convince the market of her laziness. Adding to this tension is the fact that a hard-working worker is expected to produce more in the first period. This creates a disincentive for working hard and creating a higher output.[14] Hence, each type of worker will underproduce in the first period in order to manipulate the market perception of her cost of effort.

It is also clear from proposition 1 that despite the simplification offered by quadratic costs, the effort choices of hard-working and lazy agents are not in a ratio equalling the ratio of their cost parameters (λ_H and λ_L). This is because agents differ in their expectations on future output. This offers a quick and easy insight into why the imposition of linear strategies is not without loss of generality in this environment.

Relation with *Ratchet Effect*. The tension above is similar to the ratchet effect described in Laffont & Tirole (1988) as a situation in which a worker has an incentive to mimic the lower type because revealing a higher type is costly for the worker. However, Laffont & Tirole (1988) states that *"an agent with a high performance today will tomorrow face a demanding incentive scheme. He should thus be reluctant to convey favorable information early in the relationship"*. In this case, the incentive scheme is a competitive wage market that is the same for either type of worker. The cost incurred by a worker revealing her propensity to be hard-working is in the form of a lower posterior expectation of her ability. The nature of this cost is fundamentally different from a more demanding incentive scheme.

3 Full Model

The most commonly observed perception manipulation regarding effort cost type is that of early-career agents working extra hard to convince their bosses that they are hard-working. While the two-period model succinctly illustrates the tension between signaling and signal-jamming, it fails to capture this phenomenon. In the following section, I will show that during the early career of the agent, effort is still a large contributor to output, and agents benefit by manipulating the market to put a higher weight on the likelihood that they have low effort costs.[15] I also show a surprising and novel result that during the late-career phase for the agent, the agents exert **more effort** than they would have in the absence of cost uncertainty in order to convince the market that they have **high effort**

[14] Kőszegi & Li (2008) refers to this as *"backward attribution"*.

[15] This result validates the finding in the three-period model of Kőszegi & Li (2008) within a more general environment.

costs.

3.1 Setup

I consider the incentives of a worker to exert effort in an environment with symmetric uncertainty over ability and asymmetric uncertainty regarding the cost of effort. I work in a continuous-time, finite horizon framework.[16] Much of the notation I use follows from Prat & Jovanovic (2014).

A worker is born with an ability η generated from the commonly known normal distribution with mean m_0 and variance $\frac{1}{h_0}$. The agent works till she dies at age $\bar{T}$. At any instant of time, say t, the worker expends costly effort denoted by a_t. Worker's effort can take values within $[0, \bar{A}]$, where $\bar{A}$ is chosen to be sufficiently high. Let $\{B_t\}_{t\geq 0}$ be a standard Brownian motion. Y_t is the cumulative output up till time t.

$$Y_t = \int_{s=0}^{t} (\eta + a_s)ds + \int_{s=0}^{t} \frac{1}{\sqrt{h_\varepsilon}} dB_s \tag{8}$$

where h_ε captures the precision of the error term. Over time, the worker and the market, both, observe the output generated and (given their expectations about the worker's cost of effort) form posteriors about the worker's ability. The sole source of the market's information comes from the realizations of $(Y_t)_{t\geq 0}$. Define Y^t as the output history: $Y^t \equiv (Y_s)_{0\leq s\leq t}$. Define $\mathcal{F}_t$ as the public filtration generated by the realization of Y^t.

Ability. Following prior literature I interpret ability as the *time invariant* component of output. It may also be thought of as the *natural action* of the agent. For a worker, this corresponds to the output she would produce in the absence of any extrinsic incentives. As may be observed from tenured faculty at universities, intrinsic motivation is a non-negligible driver for output.

Effort. This is the component of output that is costly for the agent to generate. It is, therefore, generated solely because of the presence of extrinsic incentives. For a worker, this corresponds to the additional output she produces in order to earn a bonus or to influence her employers' beliefs about her ability and her diligence.

[16] All the main results also work with an infinite horizon. See subsection 5.2.

Effort cost. I assume that there are two different effort types - *Lazy* and *Hard-working*. Each type has an associated cost of effort. Both costs of effort are sufficiently convex.[17] The cost of effort of the *Lazy* type is greater, and increases at a higher rate than the cost of effort of the *Hard-working* type of worker.

When updating the posterior on ability, the worker will know what her cost type is, and the amount of effort she made at each instant. In equilibrium, the market will form two different posteriors on ability, one for each cost type. For each cost type, the market will posit the amount of effort that the worker must have put it at each instant, and form a posterior accordingly. In equilibrium, the amount of effort actually expended by a worker of type i must equal the market's expectation of the amount of effort made by type i at each instant. Suppose the market assumes that an individual of type i chooses an effort stream $(\hat{a}_t^i(Y^t))_{t\geq 0}$, where Y^t denotes the history of outputs up to time t. Given a history Y^t, $\hat{A}_t^i(Y^t)$ denotes the cumulative amount of effort that the market believes that an agent of type i must have exerted. Note that $\hat{A}_t^i(Y^t) = \int_0^t a_s^i(Y^s)ds$. This allows the market to form a posterior on ability using the following:

$$E_t[\eta|Y_t, \hat{A}_t^i, i] \equiv m_t^i = \frac{h_0 m_0 + h_\varepsilon \left(Y_t - \hat{A}_t^i\right)}{h_t} \tag{9}$$

In equilibrium, the market's expectations on effort must also be optimal for the worker, and therefore must be correct. The precision of the worker's posterior on ability and the market's conditional posterior on the worker's ability will be identical at all instants, and for all histories of output. This precision is denoted by h_t and evolves according to

$$h_t = h_0 + h_\varepsilon t \tag{10}$$

It is immediately clear from equation 10 above that precision on ability (after conditioning on effort cost type) evolves deterministically over time. It might be useful to note that the overall precision on ability is lower because of the presence of cost uncertainty. However, conditioning on a particular cost type implies that, in equilibrium, the market and the worker will learn at the same speed.

Since the labor market is assumed to be competitive, the wage equals the worker's expected output. In order to calculate the expected output of a worker and her wage,

[17] The *sufficiency* will be required to make the agent's continuation value concave in her effort choice. See appendix A.2 for details.

the market must (i) form posteriors on her ability conditioning on her type, (ii) posit the amount of effort the worker will make conditioning on her type. Then the market must add up the posterior on the ability to the posited effort for each type, and calculate the average weighted by the likelihood of the worker being of either type given the observed history

$$\hat{w}_t = \mathbb{P}_t^H \left(m_t^H + \hat{a}_t^H \right) + \mathbb{P}_t^L \left(m_t^L + \hat{a}_t^L \right) \tag{11}$$

where $\hat{a}_t^i$ is the market's conjecture on the amount of effort a worker of type i will exert. m_t^i is the posterior on ability conditioning on the agent being of type i. The posterior likelihood that an agent is of type i is given by $\mathbb{P}_t^i$. The wage offered under this conjectured effort choice is $\hat{w}_t$. The worker's utility function is simply the present discounted value of the difference between her wage and cost of effort at every instant. Her utility function is given by

$$U_i = \mathbb{E}^{\hat{a}_i} \left[\int_0^{\bar{T}} e^{-\rho t} \left(\hat{w}_t - g_i(\hat{a}_t) \right) dt \right] \tag{12}$$

where $\mathbb{E}_i^{\hat{a}}[\cdot]$ refers to expectations formed under the assumption of effort stream $(\hat{a}_i)_{t \geq 0}$. Wages are denoted by $\hat{w}_t$ and are equal to the expected output of an agent, given her history. $\rho \in (0, 1)$ captures discounting. The worker's continuation value (at time t) is given by:

$$v_t^i = \mathbb{E}_i^{\hat{a}} \left[\int_t^{\bar{T}} e^{-\rho(s-t)} \left(\hat{w}_s - g_i(\hat{a}_s) \right) ds \right] \tag{13}$$

I define K_t to be the log-likelihood that an agent is hard-working (versus lazy). $K_t \equiv log\left(\frac{\mathbb{P}_t^H}{\mathbb{P}_t^L}\right)$. To evaluate the likelihood of the worker being hard-working versus lazy, I calculate the ratio of the probabilities of the errors required to generate the output stream observed.

$$K_t = \int_0^t \frac{\left(dY_s - (m_s^L + \hat{a}_s^L)ds\right)^2}{2ds\left(\frac{1}{h_s} + \frac{1}{h_\varepsilon}\right)} - \int_0^t \frac{\left(dY_s - (m_s^H + \hat{a}_s^H)ds\right)^2}{2ds\left(\frac{1}{h_s} + \frac{1}{h_\varepsilon}\right)} \tag{14}$$

With the basic notation and structure out of the way, I can proceed to solve this model. In the next section, I first define the equilibrium, then show its existence. After that, I use the properties of the first-order condition to show the main results.

3.2 Solution

In this subsection, I shall use the structure described earlier to find the properties of an equilibrium. I begin by defining an equilibrium in this environment.

Definition 1. ***Perfect Bayesian Equilibrium***
A Perfect Bayesian Equilibrium is a set of progressively measurable strategies $\{(a_t^{L\star})_{t\geq 0}, (a_t^{H\star})_{t\geq 0}\}$ *such that:*

- $(a_t^{i\star})_{t\geq 0}$ *maximizes 12 given* $(a_t^{-i\star})_{t\geq 0}$ *and subject to 8, and 11*
- $(m_t^i)_{t\geq 0}$ *and* $(\mathbb{P}_t^i)_{t\geq 0}$ *are constructed using Bayes' rule.*

Note that the market's beliefs over the symmetric uncertainty (ability) is identified by the posterior means $\{m_t^H, m_t^L\}$ and the deterministic conditional posterior precision on ability h_t. The belief over the asymmetric uncertainty is captured by $\mathbb{P}_t^H$. In a Perfect Bayesian equilibrium, the agent must find it optimal to choose actions that are equal to the market's conjectures at all times.

Before characterizing the properties of an equilibrium, I show that a Pure Strategy Nash Equilibrium (PSNE) exists in this environment.[18]

Proposition 2. *There exists a Nash equilibrium with pure strategies in this environment.*

Proof. I discretize time. In the final period, I employ a fixed point theorem attributed to Debreu, Fan, and Glicksberg to show the existence of a pure-strategy Nash equilibrium. Using a measurable selection theorem, I select a unique, measurable equilibrium outcome for each state. I use the selected equilibria as payoffs associated with each possible continuation of the game and use the Dereu, Fan, and Glicksberg FPT to show the existence of PSNE in all earlier periods and realized states. Again, the use of a measurable selection theorem allows me to select an equilibrium for the complete game. In order to show the convergence of equilibrium outcomes to the equilibrium in continuous time, I decrease the step size for the discrete game. For details, see appendix A.2. □

The effort choices by agents in equilibrium must necessarily satisfy the first-order condition. I use results from that first-order condition to characterize the properties of the equilibrium.

[18] Note that this Nash equilibrium will be Perfect Bayesian since all learning takes place in a Bayesian manner.

$$g'(a_t) = \mathbb{E}^{a^{i\star}}\left[\int_t^{\bar{T}} e^{-\rho(s-t)} \frac{\partial(w_s^\star - g_i(a_s))}{\partial y_t} ds\right] \tag{15}$$

Which is to say that the marginal cost of effort must be equal to expected marginal benefit from exerting that effort. While satisfying the first-order condition does not guarantee that the strategy is optimal, it is a necessary condition for an equilibrium to satisfy. Since the production function is additive, a change in the effort at any instant changes the expected output by the same amount. Implicit in the first-order condition above is the fact that $\mathbb{E}\left[\frac{\partial y_t}{\partial a_t}\right] = 1$. The change in expected future wages due to a change in output at a given instant can be split into two separate effects.

$$\frac{\partial w_s^\star}{\partial y_t} = \frac{h_\varepsilon}{h_s} + \frac{1}{h_s}\frac{\partial}{\partial y_t}\left[h_0\left(\mathbb{P}_s^H a_s^{H\star} + \mathbb{P}_s^L a_s^{L\star}\right) - h_\varepsilon\left(\mathbb{P}_s^H\left(A_s^{H\star} - a_s^{H\star}s\right) + \mathbb{P}_s^L\left(A_s^{L\star} - a_s^{L\star}s\right)\right)\right] \tag{16}$$

where $t < s$. The first effect is summarized in the first term on the right-hand side of the above equation. It is the effect that a change in output has on the posterior on the ability of the worker. This is exactly the reputational effect studied in Holmström. The second effect is more complicated and is captured by the second part of the above expression. This part denotes how, through a manipulation of market beliefs over the worker's cost type, the worker may influence her current and future expected effort, ability, and therefore, wages. This effect disappears when a change in output doesn't affect market perception regarding the worker's cost type.

Lemma 1. *If at some instant, there is no learning about cost type, workers will only care about the impact their output has on their perceived ability. Then the effort made by workers will equal the Holmström prediction.*

Proof. See appendix A.3 for details. □

One of the useful implications of lemma 1 is that at any point where no learning about cost type takes place, a hard-working agent must necessarily exert more effort than a lazy agent.

In this environment, allowing for belief divergence is critical to ensuring that the equilibrium is optimal. In equilibrium, firms in the competitive market correctly anticipate the actions by agents but off-path actions create persistent private beliefs that may lead to complicated strategies. In this particular case, however, any deviation changes private belief only about the level of ability. In equilibrium, the level of ability plays no role in determining the optimal action of the agents. Therefore, after any period of sustained divergence, the optimal action for the agent is to play the equilibrium action despite the agent and the market having different beliefs about the level of ability.

Lemma 2. *Off-path play by the agent is sub-optimal is any equilibrium in this environment.*

Proof. See appendix A.4 for details. □

From lemma 2, it is clear that remaining on-path is optimal for the agent in equilibrium, and that the first-order approach is sufficient. I will use the properties of the first-order condition in deriving proposition 3.

As time progresses, reputation solidifies. Moreover, as the agent approaches the truncation time, the remaining time over which the potential benefits from building reputation may be realized diminish to zero. It is useful to note that while optimal effort tends to zero eventually regardless of the agent's type, the hard-working agent starts with a higher level of effort than the lazy agent. Therefore the stream of expected outputs from a hard-working agent slopes downwards more steeply than the output stream expected from a lazy agent. Using prior expectation and observed outputs, firms in the competitive market must try and fit these two curves to the data. It is clear that with two curves trying to fit the data, with one curve more steep than the other, these curves must intersect.

In other words, the observation of high output in the earlier phase of an agent's career makes the market believe that the agent must be hard-working. However, the expected output stream from a hard-working agent is more steeply downward sloping. Therefore an observation of high output late in the agent's career is better explained by the agent being lazy, but with high ability.

Proposition 3. *Based on the primitives (i, m_0, h_0, ρ, h_ε, $g_L(\cdot)$, $g_H(\cdot)$), there exists a unique ex-ante exptected cutoff time T, such that $0 < T < \bar{T}$. Before that time, a higher output makes the agent seem hardworking. After time T a higher output makes the agent seem lazy. Ex-ante, the worker expects that:*

$$\begin{aligned} \mathbb{E}_{t=0}[\tfrac{\partial k_t}{\partial Y_t}] > 0 \quad &\Leftrightarrow \quad \mathbb{E}_{t=0}[a_t^{H\star} + m_t^H] > \mathbb{E}_{t=0}[a_t^{L\star} + m_t^L] \quad \forall \quad t < T \\ \mathbb{E}_{t=0}\left[\frac{\partial k_t}{\partial Y_t}\right] < 0 \quad &\Leftrightarrow \quad \mathbb{E}_{t=0}\left[a_t^{H\star} + m_t^H\right] < \mathbb{E}_{t=0}\left[a_t^{L\star} + m_t^L\right] \quad \forall \quad t \in (T, \bar{T}] \end{aligned}$$

Proof. See appendix A.5 for the proof. □

The result in proposition 3 is surprising because conventional wisdom dictates that being perceived as having a low cost of effort, and thereby being hard-working should always benefit a worker. In this case, being thought of as hard-working lowers the output that the market expects the worker to produce. This is because conditioning on a history of outputs, the expectation of future output depends upon the posterior on ability, as well as on expected effort. While the expected effort of a hard-working employee will be higher, her expected ability must be lower. As time progresses, the effort component of total output tends to diminish, eventually approaching zero. Therefore, there is a cut-off time T, before which the market expects a hard-working employee to be more productive, and after which, the market expects a lazy worker to be more productive.

This cut-off point is unique ex-ante. However, noise in output realizations may create multiple such points. It is also clear that as time gets sufficiently large, we must be at a point after the cut-off. That is, for sufficiently large t, it must be that the lazy worker is expected to produce a greater output than the hard-working employee.

Note that proposition 3 doesn't imply that a hard-working employee is worse-off than a lazy employee. Given the same level of ability, the lifetime utility for a hard-working employee always exceeds that of a lazy employee.[19] The claim in proposition 3 fixes a given history of output, and then states that a lazy worker with that output history would create a higher output in the future than a hard-working employee with the same history.

It is clear that at time T and at all times after the cut-off, the worker benefits from being perceived as lazy. The more weight the market puts on the worker being lazy, the higher the expected ability the worker seems to have, and the greater is her wage. I use this result from proposition 3 and apply the Envelope theorem to show that at time T, the gain from being perceived as lazy is strictly positive for the worker. In proposition 4, I show that the time at which the worker is just indifferent in being perceived hard-working or lazy is τ.

Proposition 4. *Based on the primitives (i, m_0, h_0, ρ, h_ε, $g_L(\cdot)$, $g_H(\cdot)$), there exists a unique ex-ante expected cutoff time $\tau \in (0, T)$ such that before that time, the continuation value of the agent is expected to increase with her being perceived as hardworking. After time τ, her continuation value (in expectation) increases when the market places a higher probability on her being lazy.*

Proof. See appendix A.6 for details. □

The two cut-off times divide a career into three phases. In the first phase, the worker

[19] This is simply a consequence of applying Envelope theorem to equation 12

would like to seem hard-working, and a higher output makes the agent seem hard-working. In the second phase, the worker would like to seem lazy, and a higher output makes the agent seem hard-working. In the third phase, the worker would like to seem lazy, and a higher output makes the agent seem lazy. In figure 1, I show how the continuation value of the agent changes with respect to the market's posterior on the likelihood that the agent is hard-working. In the first phase of the agent's career, the continuation value changes positively with the market's posterior on the agent's conscientiousness. In this phase, the agent benefits from looking hard-working. In the second and third phases, the agent's continuation value is negative in the market's posterior on the agent's conscientiousness. The more likely the market thinks the agent is hard-working, the lower is the agent's continuation value.

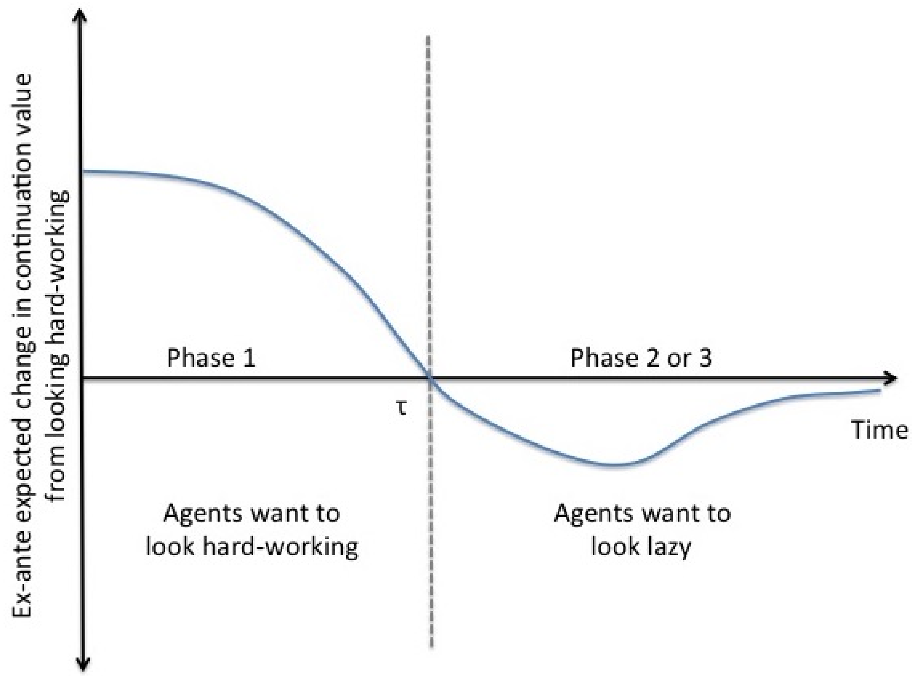
The difference in the market's posteriors on ability conditioning on the agent's effort-cost type is given by $m_t^L - m_t^H = \frac{h_\varepsilon}{h_t}\left(A_t^{H\star} - A_t^{L\star}\right)$. Over time, as optimal effort tends to zero for all types, the difference between their cumulative efforts ($A_t^{H\star} - A_t^{L\star}$) converges to a constant. If the truncation time $\bar{T}$ is large, the precision on conditional ability posteriors (h_t) also becomes large eventually. This implies that $m_t^L - m_t^H$ would tend to zero with a high enough truncation time. The loss from seeming hard-working, therefore, may decrease with time if the truncation time is large enough. This fact can also be seen in figure 1.

As can be seen in figure 2, in comparison with an environment with cost certainty, the presence of cost uncertainty results in (i) an increase in effort in the first phase as agents work harder to look hard-working, (ii) a decrease in effort in the second phase as agents work less to look lazy, and (iii) an increase in effort in the third phase as agents work harder to look lazy. The resultant change in effort may be seen as a difference between the two curves and is depicted in figure 3.

Corollary 1. *There are two cutoffs: τ and T. They divide a career into three phases:*

- *First phase:* $0 \leq t \leq \tau$
  - *Higher output makes the agent look hardworking* $\frac{\partial k_t}{\partial y_t} > 0$
  - *Agent wants to look hardworking* $\frac{dv_t^i}{dK_t} > 0$
  - *Agent works harder because of cost uncertainty*
- *Second phase:* $\tau \leq t \leq T$

Figure 1: Change in Continuation Value due to Market's Posterior on Conscientiousness

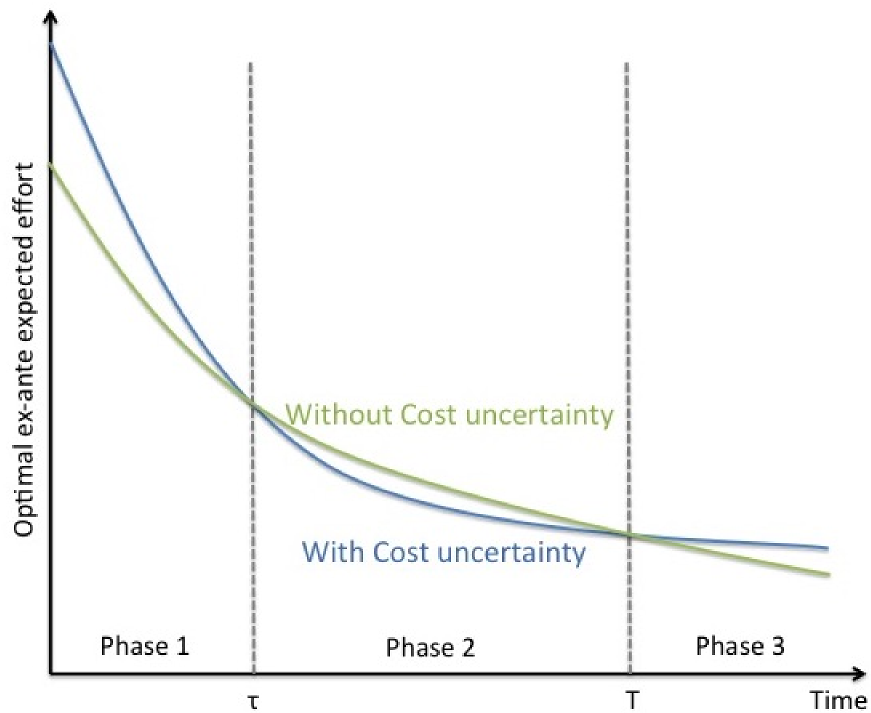


- *Higher output makes the agent look hardworking* $\frac{\partial k_t}{\partial y_t} > 0$
- *Agent wants to look lazy (with high ability)* $\frac{dv_t^i}{dK_t} < 0$
- *Agent works less because of cost uncertainty*

- *Third phase:* $T \leq t \leq \bar{T}$
  - *Higher output makes the agent look lazy* $\frac{\partial k_t}{\partial y_t} < 0$
  - *Agent wants to look lazy* $\frac{dv_t^i}{dK_t} < 0$
  - *Agent **works harder to look lazy***

It is important to note that in corollary 1, the statements about whether the agent works a greater or lesser amount are in comparison with a benchmark without cost uncertainty. As mentioned before, the focus of this paper is on how the presence of cost uncertainty impacts effort choices. The timeline of these effects can be seen in figure 4.

Figure 2: Effort Paths With and Without Cost Uncertainty



4 Explaining the Mincerian residuals

In the last forty-four years, Mincer (1974) has had a deep and lasting impact on empirical labor economics. The standard specification for the Mincerian regression is given in equation 1 and reproduced below.

$$log(w) = \alpha_0 + \rho_s s + \beta_0 x + \beta_1 x^2 + \nu$$

where w denotes wages, years of schooling is denoted by s, and x represents the years of experience for an individual. While this specification has enjoyed great success in fitting the data and predicting accurate results for most contexts, it has also faced criticism for the consistent pattern in the residuals generated by it. Murphy & Welch (1990) studies these patterns using a specification that relaxes the assumption of a linear relation between log wages and years of schooling. The specification separates individuals into school groups, estimating the following regression for each school group i for each year t.

$$y_{ixt} = b_0 + b_1 x + b_2 x^2 + \nu \tag{17}$$

Figure 3: Change in Effort due to the Presence of Cost Uncertainty

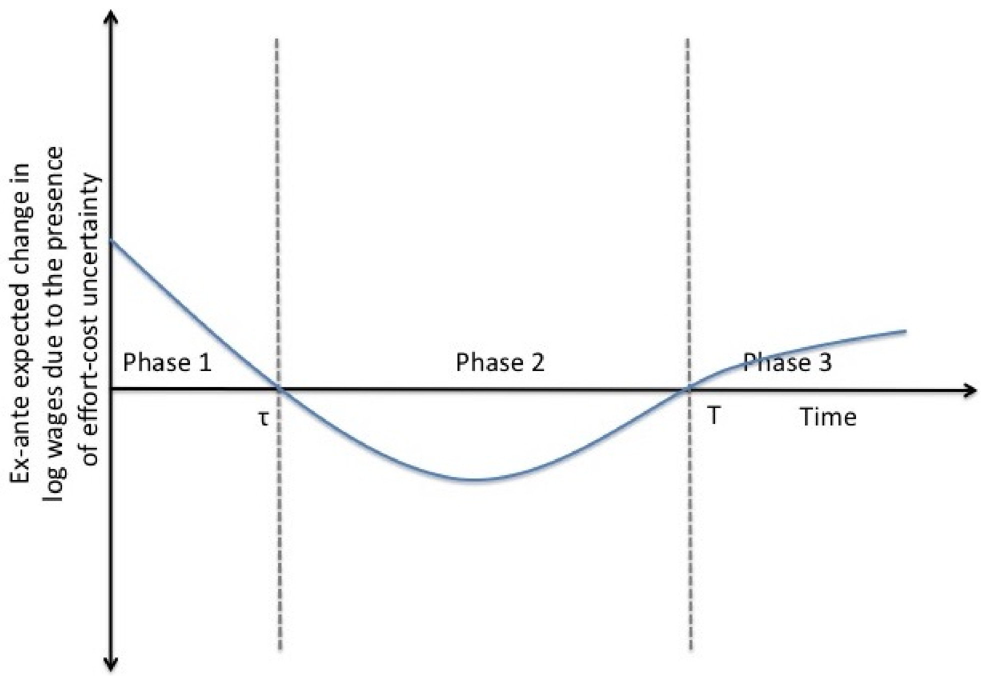
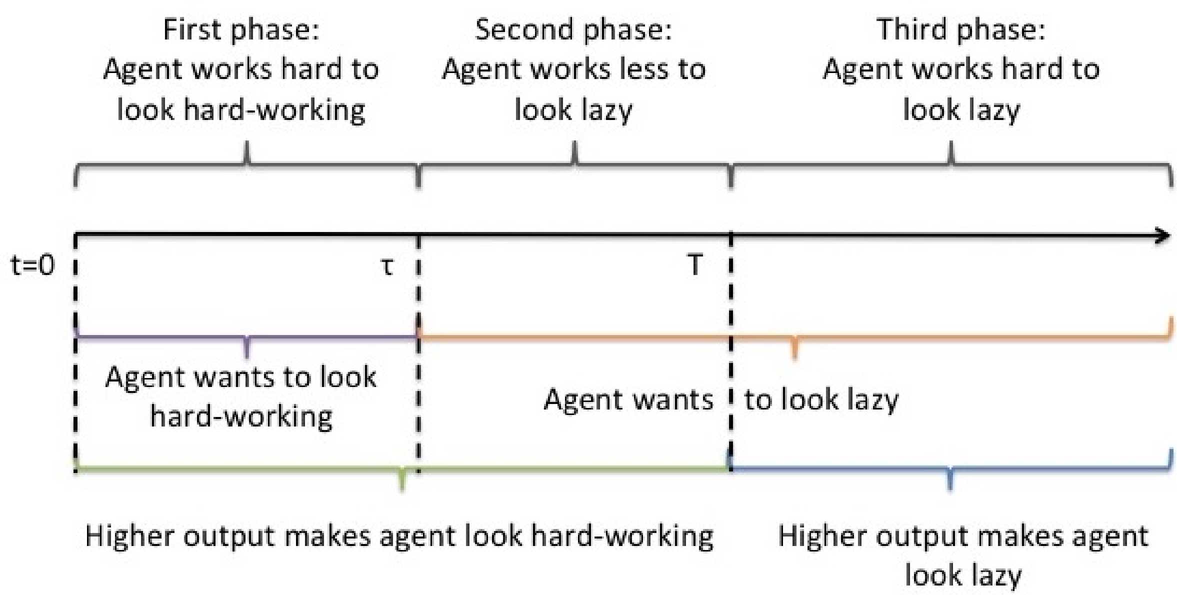


Figure 4: Impact of Cost Uncertaity by Phase



Note: All the results are in comparison with a benchmark of cost certainty.

where y_{ixt} is the log weekly wage earned by an average agent in school group i who has x years of experience in the year t. Murphy & Welch (1990) uses a weighted least squares regression to find the parameters (b_0, b_1, and b_2) that best fit the data. They find that the shape of the residuals from this regression follows a particular pattern that is exhibited in figure 5.

Figure 5: Murphy & Welch (1990) Residuals

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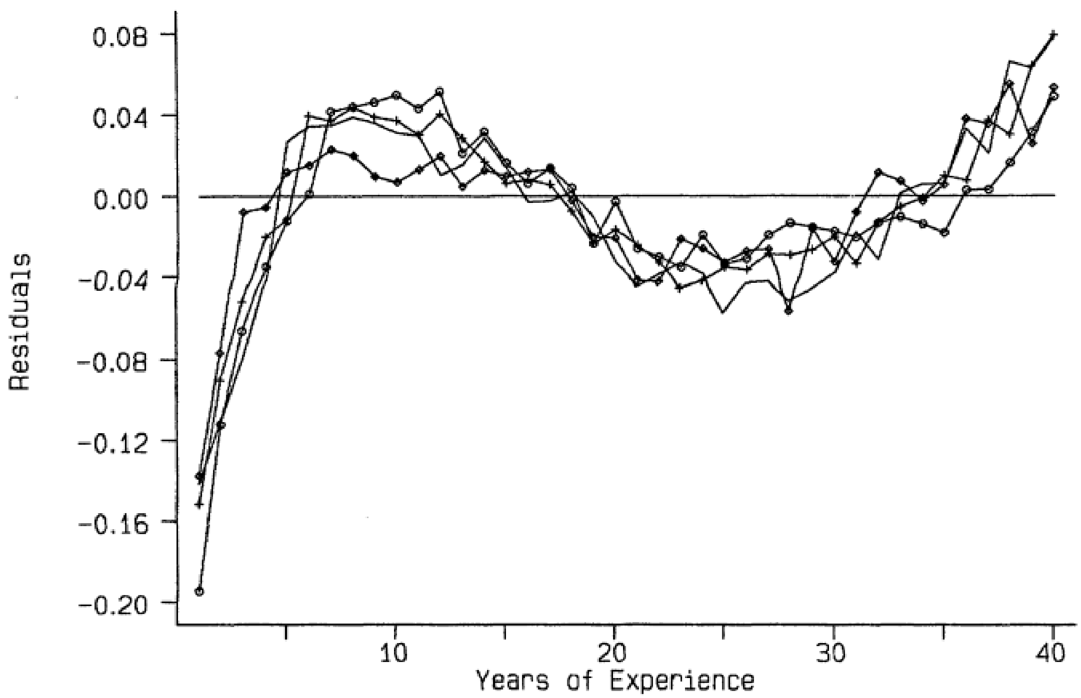
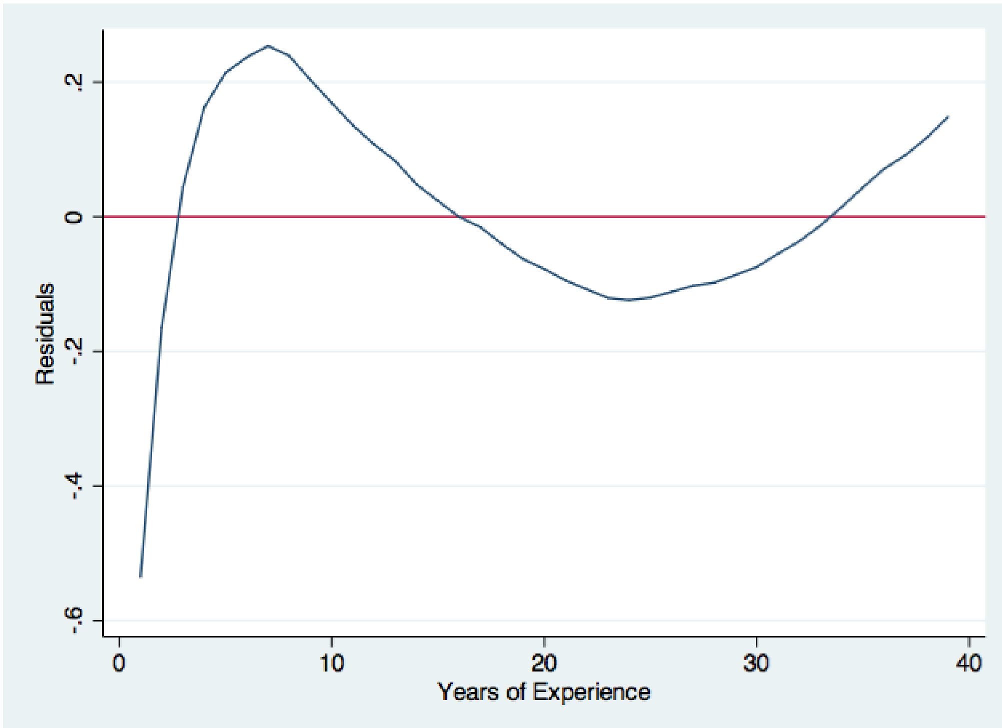


FIG. 4.—Residuals from quadratic specification: ○ = 8–11 years of schooling; + = high school graduates; □ = 13–15 years of schooling; plain line = college graduates.

Note: This figure is copied from Murphy & Welch (1990) and reproduced in its exact form.

In figure 5, the residuals are plotted against years of experience for four schooling groups. The similarity and magnitude of the residuals are striking. Note that the quadratic specification underestimates earnings for workers with 7-17 years of experience (peaking at about 6% during year 10). For workers with 18-35 years of experience, the residuals are negative - indicating that the quadratic specification overestimates wages during that phase. In the final phase of an agent's career (more than 35 years of experience), the quadratic specification underestimates earnings for the workers.[20] Lemieux (2006) confirms the presence of this pattern in residuals using CPS data from 1991-2001. In figure 6, I confirm that these residuals remain consistent in their pattern when calculated using 2012-16 ACS data from IPUMS.

Figure 6: Residuals using 2012-16 ACS data



Murphy & Welch (1990) note that the persistent pattern in the residuals is indicative of a specification bias present in the quadratic regression and suggest increasing the number of degrees in the polynomial expression used to explain the variance in log wages. They find that most of the residuals can be explained away by using a quartic expression. While structurally simple and easy to estimate, this solution lacks a micro-theoretic foundation. Further, according to the Weierstrass Approximation Theorem, given a sufficiently high degree polynomial, any continuous real-valued function over a bounded domain can be approximated to an arbitrarily high precision. Therefore, it is not surprising that adding polynomials to the expression would explain away residuals.

This paper posits that the cubic and quartic terms added by Murphy & Welch (1990)

[20] The quadratic specification also overestimates earnings for workers with 0-7 years of experience. Lower realized earnings in that phase may be explained by workers paying for learning on the job, experimentation by workers, apprenticeships, etc. It is not the focus of my analysis.

may be capturing the economically interesting behavior described in the earlier sections of this paper. Specifically, in the presence of effort-cost uncertainty, agents increase effort early in their careers, reduce effort during their middle career, and increase effort again in the last phase of their careers. In an environment with firms competing to hire workers, this increased or decreased effort translates to higher or lower wages directly. As I show in figure 3, the impact that cost uncertainty has on effort choice, and therefore wages, is positive in the first phase of an agent's career, then negative, and finally positive again. The pattern in residuals, after the first five to seven years, matches exactly with the main result of my model.

The lower than predicted wages during the first five to seven years of an agent's career are not explained by this approach and present a potential issue. Two arguments provide some resolution. First, several professions offer internships and apprenticeships. Workers may accept low wages to gain human capital essential to building a career in that field. Paralegals and Legal Assistants, for instance, gain experience and training, even though they earn relatively little (considering their education and work hours). Second, workers experiment and change industries at a higher rate early in their career - settling on a field after a few years of experience. As is noted in Kambourov & Manovskii (2008), "*occupational and industry mobility rates decline with worker's age*". Given that human capital accumulation is industry-specific[21], it makes sense that agents would earn lower than predicted wages early on in their career. While truncating data is usually not ideal, in this case, truncating the data from very early years of experience would allow one to consider the data in the absence of the above-mentioned issues.

At the very least, the fact that the change in effort and wages due to cost uncertainty is related to the years of experience. This implies that there is likely an omitted variable bias, with the effect of cost uncertainty being the variable omitted. In future research, it may be possible that the omitted variable bias inherent the quadratic structure in Mincer (1974) can be estimated structurally.

A more involved empirical exercise could use data from the American Community Survey (ACS) to conduct a calibration exercise that structurally estimates the parameters of the model that allow the change in log wages due to cost uncertainty to best fit the residuals in the data. The ACS, which collects workers years of schooling, wages, and age, allows one to conduct this exercise. Additionally, using the dataset from the National Longitudinal Survey of the Youth (NLSY) would allow one to go a step further.

[21] See, for instance, Parent (2000).

In addition to years of schooling, wages, and age, NLSY data include measures of cognitive ability and conscientiousness that would allow one to calibrate the model to find parameters corresponding to the underlying ability and effort-cost type of the worker.

Beyond the many empirical applications this model suggests, a key contribution of this paper is that it allows labor economists to retain the micro-foundations of the Mincerian regression while explaining the pattern in the residuals using an economically intuitive result.

5 Other Applications and Extensions

5.1 Distribution of outputs

In this section, I consider the implications of effort-cost uncertainty on the predicted distribution of outputs in this environment. I assume that a continuum of agents is randomly generated from the priors with the same amount of hard-working and lazy workers.

The distribution of outputs at any time depends upon the average actions of agents of each cost type, the dispersion due to ability, and the dispersion from noise. Irrespective of cost uncertainty, the distribution starts as the sum of two normal distributions with different means. All agents of the same cost type produce the same effort in the zeroth period. Also eventually, the distribution of outputs converges to a single normal distribution, centered around the prior mean of ability, as the effort of both hard-working and lazy type agents tends to zero.

Corollary 1 implies that, in comparison to a situation without cost uncertainty, with cost uncertainty, agents increase effort in the first phase of their career, decrease effort in the second phase, and increase effort in the third phase. It is crucial to note here that the effect of cost uncertainty depends on the amount of uncertainty present. The higher the cost uncertainty, the greater its effect on effort choice. For some types of agents, specifically hard-working, high ability agents or lazy, low-ability agents, effort cost uncertainty is resolved relatively quickly. For agents who have high ability and are lazy, or agents who have low ability, but are hard-working, it takes time and a long stream of outputs to resolve effort cost uncertainty – the impact of cost uncertainty is felt most acutely by those agents.

For each agent, and for each realization of noise, the cutoff times τ and T may differ. However, there must exist cutoffs where the average of the population generated switches to a different phase of their career. In this analysis, it is those cut-offs which are relevant. Before time τ (calculated for the average), agents work harder due to cost uncertainty. Hard-working-low-ability agents and lazy-high-ability agents increase their effort by an amount that exceeds other types of agents. This causes bunching of outputs at the higher end of the output distribution. This skewness is not present in any output stream generated for any parameters of the standard Holmström (1999) model. In the second phase, while most agents will have reduced their outputs due because of cost uncertainty, agents who are highly able and lazy, and agents who are less able and hard-working will reduce their outputs by more. This will result in a bunching towards the lower end of the outputs observed. In the final phase, since cost uncertainty tends to increase effort, there will be a skewing of the distribution of outputs towards the higher end of the outputs observed.

5.2 Infinite time model

In this section, I consider the same environment described in section 3, but with an infinite time horizon. A worker with an infinite lifespan has a symmetrically unknown ability, and privately known cost of effort. Both, the worker, and the market learn about the worker's ability over time. The market must also learn about the worker's effort cost, while the worker already knows her effort cost. The only observable signal is output, and I follow the career concerns literature in assuming that wages cannot be conditioned on output.

As is shown in appendix B.1, all of the results from section 3 can be reproduced in the infinite horizon extension of the model.

6 Conclusion

While the Mincer (1974) equation remains a workhorse of the empirical labor economics literature, its standard specification generates residuals that follow a consistent pattern. Murphy & Welch (1990) suggest solving this issue by adding cubic and quartic terms to the quadratic specification. While this solution allows economists to generate better predictions that better fit the data, it lacks any theoretical foundation. This paper contributes to the labor economics literature by providing an economic rationale for the pattern in the residuals.

I study a finite horizon, continuous-time environment in which agents differ in their effort costs, and ability. They are hired by a competitive market to perform a job in which their output is the sum of their ability, effort, and noise. Wages cannot be conditioned on current output and must be fixed before production takes place. However, conditioning wages on past outputs is possible.

In this environment, the market must use the observed output to update posteriors on the worker's ability, as well as effort costs. For the worker, this means that the choice of costly effort must take into consideration, the market's updating process on both, her ability and her effort cost. Since output informs the market along two dimensions of uncertainty, the key tension in this environment involves signaling effort cost and signal-jamming about ability optimally.

In the main results, I compare the optimal effort chosen in this environment to that chosen in an environment with cost certainty. I find that the presence of cost uncertainty may increase or decrease worker effort depending upon the phase of the worker's career. Early in her career, a worker increases effort to seem hard-working. During her middle and late career, a worker benefits from seeming lazy. Therefore, during her middle career, a worker would reduce effort below the benchmark with no cost uncertainty in order to look lazy, while increasing her effort in order to seem lazy during her late career.

The presence of private information regarding the agent's propensity to work hard, as well as public uncertainty regarding the agent's ability make this a highly intractable environment. Since the standard techniques of proving the existence of an equilibrium cannot be used here, I develop a method to prove the same. In addition to the theoretical contributions, this paper provides several testable implications that help explain the residuals in the Mincerian regression, and also predict patterns in the distribution of outputs.

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Appendix A Proofs

A.1 Proof of proposition 1

Proof. Given a risk-neutral competitive market, I have the following condition for the wage in period 2, conditioned on y_1

$$\begin{aligned}\mathbb{E}[\eta|y_1] &= \mathbb{P}\left(\lambda_i = \lambda_L|y_1\right)\left(\frac{h_1}{h_1+h_\varepsilon}m + \frac{h_\varepsilon}{h_1+h_\varepsilon}\left(y_1 - a_L^*\right)\right) \\ &\quad + \mathbb{P}\left(\lambda_i = \lambda_H|y_1\right)\left(\frac{h_1}{h_1+h_\varepsilon}m + \frac{h_\varepsilon}{h_1+h_\varepsilon}\left(y_1 - a_H^*\right)\right)\end{aligned}$$

where a_i^* is the optimal equilibrium effort of a worker of type i. I assume that in equilibrium, this will be known by the market

The above equation simplifies to:

$$\mathbb{E}[\eta|y_1] = (\frac{h_1}{h_1+h_\varepsilon})m + (\frac{h_\varepsilon}{h_1+h_\varepsilon})y_1 - (\frac{h_\varepsilon}{h_1+h_\varepsilon})a_H^* + \mathbb{P}(\lambda_i = \lambda_L|y_1)(\frac{h_\varepsilon}{h_1+h_\varepsilon})(a_H^* - a_L^*) \tag{18}$$

If it were the case that $\mathbb{P}(\lambda_i = \lambda_L|y_1)$ was independent of y_1, then our first-order condition for effort would have been identical to the situation without cost uncertainty. However, $\mathbb{P}(\lambda_i = \lambda_L|y_1)$ will depend upon y_1, we must incorporate that into our analysis. In plain English, because the market conditions the probability of the worker being of low cost (with high optimal effort) on the observed output, there is an incentive for workers to move towards a lower effort and thereby increase their likelihood of seeming to be a high cost (low optimal effort) type.

I now calculate value of $\mathbb{P}(\lambda_i = \lambda_L|y_1)$

$$\mathbb{P}(\lambda_i = \lambda_L|y_1) = \frac{\mathbb{P}(\varepsilon_1+\eta=y_1-a_L^*)}{\mathbb{P}(\varepsilon_1+\eta=y_1-a_L^*)+\mathbb{P}(\varepsilon_1+\eta=y_1-a_H^*)}$$

We know that: $\varepsilon_1 + \eta \sim N(m, \frac{h_1+h_\varepsilon}{h_1 h_\varepsilon})$

$$\mathbb{P}(\lambda_i = \lambda_L|y_1) = \frac{1}{1+\frac{\mathbb{P}(\varepsilon_1+\eta=y_1-a_H^*)}{\mathbb{P}(\varepsilon_1+\eta=y_1-a_L^*)}}$$

Where:

$$\frac{\mathbb{P}(\varepsilon_1+\eta=y_1-a_H^*)}{\mathbb{P}(\varepsilon_1+\eta=y_1-a_L^*)} = exp(-(\frac{(y_1-a_H^*-m)^2}{2(\frac{h_1+h_\varepsilon}{h_1h_\varepsilon})}) + (\frac{(y_1-a_L^*-m)^2}{2(\frac{h_1+h_\varepsilon}{h_1h_\varepsilon})}))$$

This simplifies to:

$$\frac{\mathbb{P}(\varepsilon_1+\eta=y_1-a_H^*)}{\mathbb{P}(\varepsilon_1+\eta=y_1-a_L^*)} = exp(\frac{h_1h_\varepsilon(a_H^*-a_L^*)(2y_1-2m-a_H^*-a_L^*)}{2(h_1+h_\varepsilon)}) \equiv e^K$$

This brings me back to the optimal wage in the second period

$$\mathbb{E}[\eta|y_1] = \mathbb{E}[w_2|y_1] = (\frac{h_1}{h_1+h_\varepsilon})m + (\frac{h_\varepsilon}{h_1+h_\varepsilon})y_1 - (\frac{h_\varepsilon}{h_1+h_\varepsilon})a_H^* + (\frac{h_\varepsilon}{h_1+h_\varepsilon})(a_H^*-a_L^*)(\frac{1}{1+e^K}) \tag{19}$$

Where: $K = \frac{h_1h_\varepsilon(a_H^*-a_L^*)(2y_1-2m-a_H^*-a_L^*)}{2(h_1+h_\varepsilon)}$

I now calculate the first order condition on effort

$$\frac{\partial g(a)}{\partial a} = \beta\frac{\partial \mathbb{E}[w_2|y_1]}{\partial a} \tag{20}$$

$$\Rightarrow \frac{a}{\lambda} = \frac{\beta}{2}\left[\frac{h_\varepsilon}{h_1+h_\varepsilon} - \frac{h_1h_\varepsilon^2(a_H^*-a_L^*)^2e^K}{(h_1+h_\varepsilon)^2(1+e^K)^2}\right] \tag{21}$$

The above forms two equations with two variables, and we should be able to solve for optimal a_H^* and a_L^*. The second part of the expression on the right-hand side comes from the change to the market's expectation on the worker's type. $\lambda_L < \lambda_H$ implies that $a_H^* > a_L^*$.

□

A.2 Proof of proposition 2

Proof. Structure of proof:

1. In discrete, finite time, show that an equilibrium exists at every point in time

2. Decrease step size. Show convergence to continuous time equilibrium.

In order to help with the proof, I assume that ability is initially drawn from a truncated normal distribution $\eta \sim N(m_0, \frac{1}{h_0})$, and lies within the interval $\eta \in [\underline{m}, \overline{m}]$. Assume that the $\underline{m}$ and $\overline{m}$ are sufficiently far away from the mean, such that the truncated normal distribution is arbitrarily close to a normal distribution.

For each period in the discrete-time game, I assume the following structure

$$y_t = l\left(\eta + a_t^i\right) + \sqrt{\frac{l}{h_\varepsilon}}\varepsilon_t$$

where, y_t is the output in period t, and the random error is weighted by the step-size (l). For now, assume that $l = 1$, I will be reducing this step size later in the proof to show convergence to the continuous-time equilibrium. Note, also that the error term ε_t is i.i.d. generated from $\sim N(0,1)$. The agents' rate of interest in the discrete-time game is $r \in (0,1)$.

Making sure that the workers' problem is always concave in her effort choice:

For either type of worker, the continuation value changes linearly with respect to her perceived ability. Costs are convex. In Holmström (1999), these two forces are the only ones present, and therefore the agent's problem becomes concave.

In my model, the change in perception regarding the agent's propensity to work hard ($\mathbb{P}_t^H$) could make the benefit from increasing effort convex for some cases. In order to ensure that the continuation value is concave in her effort cost, I find the maximum possible convexity in the benefit that could occur in this model, and then make the costs more convex than that.

Maximizing convexity in benefit: A change in effort results in an equal change in the expected output. Convexity in benefits is maximized when there is a discrete jump in benefits upon crossing a threshold (y_c)[22] in observed output. If there were no noise present in the output function, any discrete jump in benefits due to observed output would have resulted in infinite convexity in the benefits as a function of effort at the point of the jump. Since noise is present, any change in effort changes the probability of crossing the threshold in output continuously. Therefore, it is possible to put a finite upper bound on the convexity in benefit arising from a change in effort.

I maximize the discrete jump in benefit from the observed output by supposing that the best case continuation value for the agent is realized if she produces more than the cutoff, and the worst-case continuation value is realized if she produces less than the cutoff. In any theoretically possible equilibrium, the best-case scenario for an agent is a continuation path in which she produces the optimal effort ($a_{opt}^i = g_i^{-1}(1)$) in each instant and is rewarded for it through a competitive wage. The worst-case scenario is a

[22] I had called this $\bar{c}$ in the earlier version. I changed it because $\bar{c}$ seemed like an upper limit on cost, rather than a cutoff on output.

continuation path in which she produces no effort, and is therefore paid only the posterior on her ability. The maximum difference in continuation value is $\frac{a^H_{opt} - g_i(a^H_{opt})}{\rho}$ where $a^H_{opt} = g_H^{-1}(1)$. This corresponds to the largest possible discrete jump in continuation value with respect to output.

The maxmimum possible convexity in continuation value with respect to effort is given by:

$$max \left\{ \frac{\partial^2 \mathbb{E}[v_i]}{(\partial a_t)^2} \right\} = \text{Max convexity of} \left\{ \mathbb{P}(y_t > y_c | a_t) \times \frac{a^H_{opt} - g_i(a^H_{opt})}{\rho} \right\}$$

Since the random noise is brownian, at any instant, error is normally distributed. The maximum convexity of $\mathbb{P}(y_t > y_c|a_t)$ with respect to a_t is $\frac{e^{0.5} h_\varepsilon}{\sqrt{2\pi}}$. The higher the precision of the error term h_ε, the more convex the probability of crossing the threshold is - at complete precision, this convexity would be infinite. For any value of precision less than infinity, this value is finite. Therefore, the upper limit on the possible convexity in benefits arising from a change in effort is:

$$max \left\{ \frac{\partial^2 \mathbb{E}[v_i]}{(\partial a_t)^2} \right\} = \left(\frac{e^{0.5} h_\varepsilon}{\sqrt{2\pi}} \right) \left(\frac{a^H_{opt} - g_H(a^H_{opt})}{\rho} \right)$$

Ensuring costs are sufficiently convex: In order to ensure that the continuation value for the agent is concave in her choice of effort, I will impose that the cost of the effort at any instant is more convex than the upper bound calculated above.

That is:

$$g_i''(a) > \left(\frac{e^{-0.5} h_\varepsilon}{\sqrt{2\pi}} \right) \left(\frac{a^H_{opt} - g_H(a^H_{opt})}{\rho} \right)$$

$$\forall i \in \{L, H\}, \forall a \in [0, \bar{A}]$$

In the case of quadratic cost of effort, $g_i(a) = \frac{a^2}{\lambda_i}$, this simplifies to a sufficient condition:

$$\lambda_H < 2\sqrt{\frac{\rho\sqrt{2\pi e}}{h_\varepsilon}}$$

I assume for the discrete case that:

$$g_i''(a) > \left(\frac{e^{-0.5}h_\varepsilon}{\sqrt{2\pi}}\right)\left(\frac{a_{opt}^H - g_H(a_{opt}^H)}{r}\right)$$

$$\forall i \in \{L, H\}, \forall a \in [0, \bar{A}]$$

In the case of quadratic cost of effort, $g_i(a) = \frac{a^2}{\lambda_i}$, this simplifies to a sufficient condition:

$$\lambda_H < 2\sqrt{\frac{r\sqrt{2\pi e}}{h_\varepsilon}}$$

A.2.1 Part 1: Existence in discrete, finite time

I use **Debreu, Glicksberg, Fan theorem** to show existence of an equilibrium at every period and then work backwards from the truncation point.

Theorem. *Consider a strategic form game $\langle I, (S_i)_{i\in I}, (u_i)_{i\in I}\rangle$ such that for each $i \in I$*

- *S_i is compact and convex*
- *$u_i(s_i, s_{-i})$ is continuous and quasi-concave in s_i*

Then, a pure strategy Nash equilibrium exists.

I will work in a discrete-time framework and will start in the last period, $\bar{T}$. There are $N+1$ players in this game. $N > 1$ firms that together constitute the market, and an employee who is either hard-working or lazy. The information set available to the firms is the history of outputs ($y^{\bar{T}} \equiv (y_s)_{0<s<\bar{T}}$). The information set available to the agent is the history of outputs ($y^{\bar{T}}$), the history of effort choices ($a^{\bar{T}} \equiv (a_s)_{0<s<\bar{T}}$), and the knowledge of the agent's true type ($i \in \{L, H\}$). The information available to each agent defines the state of the world.

The actions that firm k ($k \in \{1, 2, ...N\}$) chooses are ($m^H_{\bar{T},k}$, $m^L_{\bar{T},k}$, $\mathbb{P}^H_{\bar{T},k}$, $\hat{a}^H_{\bar{T},k}$, $\hat{a}^L_{\bar{T},k}$)$\in$ $[\underline{m}, \overline{m}] \times [\underline{m}, \overline{m}] \times [0, 1] \times [0, \bar{A}] \times [0, \bar{A}]$. That is, each of the $N(> 1)$ firms must choose a conjecture on a posterior mean for each type of agent, a probability that the agent is hard-working, and the action that is expected from each type of agent. Firms must also choose a wage. Since the wage is determined from their other choices, I have not included it as a separate action. The firms will use the output history $(y^{\bar{T}})$ to make these predictions, and their payoffs are generated as a difference between the amount produced by the worker, and the wage offered by the firm if the firm offers the highest wage to that worker. If the firm's wage offer is not highest, then the firm earns a payoff of zero in that period.

The action chosen by the agent is an optimal effort $a_{\bar{T}} \in [0, \bar{A}]$. The agent uses the information available to her $(y^{\bar{T}}, a^{\bar{T}}, i)$ in order to make this decision.

Claim 1: *The set of strategies S_i is compact and convex at time $\bar{T}$*

Proof. At time $\bar{T}$, the state of the world $(y^{\bar{T}}, a^{\bar{T}}, i)$ is realised. The strategy set is the product set of the actions available to each player.

$$S_i = [\underline{m}, \overline{m}]^N \times [\underline{m}, \overline{m}]^N \times [0,1]^N \times [0, \bar{A}]^N \times [0, \bar{A}]^N \times [0, \bar{A}]$$

This set is compact and convex □

Claim 2: *The utility function for each player is continuous and quasiconcave in her action choices at time $\bar{T}$.*

Proof. For firms:

The firms exist in a competitive wage market. The actual expected utility for each firm is the probability that the firm will be able to hire the agent times the expected difference between the agent's output and the wage offered. In such a market, the optimal strategy for each firm is to offer a wage that equals the agent's expected output given her history.

I will endow the firms with an observationally equivalent utility function that is both, continuous and quasiconcave. Suppose that firm k's utility function is:

$$U_k = -\left((\mathbb{P}^H_{\bar{T},k} - \mathbb{I}_H)^2 + \mathbb{I}_H \left((m^H_{\bar{T},k} - m^A_{\bar{T}})^2 + (a^H_{\bar{T},k} - a_{\bar{T}})^2 \right) + (1 - \mathbb{I}_H) \left((m^L_{\bar{T},k} - m^A_{\bar{T}})^2 + (a^L_{\bar{T},k} - a_{\bar{T}})^2 \right) \right)$$

where $\mathbb{I}_H = \begin{cases} 1, i = H \\ 0, i = L \end{cases}$

Clearly, the utility function is such that each firm would use the information available optimally to create correct posteriors. This incentivizes firms to take actions that observationally equivalent to the behavior of firms in a competitive wage market.

For the agent:

Wages are non-contractible. In the last period, there are no reputational concerns. Therefore the utility function of the agent is:

$$U_A(a_{\bar{T}}) = \hat{w} - g_i(a_{\bar{T}})$$

The wage offered ($\hat{w}$) is irrespective of the output observed, or effort undertaken. The costs ($g_i(a_{\bar{T}})$) are continuous and convex. Therefore, the agent's utility function is continuous and quasiconcave in her action.

□

Using the Debreu, Fan, Glicksberg Fixed Point Theorem (DFG FPT), I can state that a pure strategy Nash equilibrium (PSNE) must exist in this period. For every realization of the state of the world at time $\bar{T}$ there exists at least one PSNE. The Axiom of Choice guarantees that a selection of PSNEs from each realization must exist. However, the axiom does not guarantee that the selection is measurable. To make sure that the selection of PSNEs satisfies the usual conditions of probability theory, I must make use of a measurable selection theorem.

Claim 3: *For each realization of the state of the world $(y^{\bar{T}-1}, a^{\bar{T}-1}, i)$ at time $\bar{T}$, it is possible to generate a measurable selection of pure strategy Nash equilibria.*

Proof. Theorem 6.9.3 from (Bogachev, 2007) states that:

Theorem. *Let X be a complete separable metric space and let Ψ be a mapping on $(\Omega, \mathcal{B})$ with values in the set of nonempty closed subsets of X. Suppose that for every open set $U \subset X$, we have*

$$\Psi(U) := \{\omega : \Psi(\omega) \cap U \neq \varnothing\} \in \mathcal{B}.$$

Then Ψ has a selection ζ that is measurable with respect to the pair of σ-algebras $\mathcal{B}$ and $\mathcal{B}(X)$.

Here, think of X as the strategy set (S). $\mathcal{E}(\omega)$ is the set of equilibria that occur when a state of the world $\omega \in \Omega$ is realized. S is a complete, separable metric space. For any open set $U \subset X$, $\Psi(U) := \{\omega : \Psi(\omega) \cap U \neq \varnothing\}$ is the set of all states of the world that give rise to the set of equilibrium strategies in U. The state of the world is defined by all the information that exists in the world. $\omega = (y^{\bar{T}-1}, a^{\bar{T}-1}, i)$, $\Omega = \mathbb{R}^{\bar{T}-1} \times [0, \bar{A}]^{\bar{T}-1} \times 2$. The Borel σ-algebra of Ω contains all open sets. Therefore, no matter what set of states (or the empty set, if no such states exist) of the world give rise to equilibria in U, that set is contained in the Borel σ-algebra of Ω.

Therefore, there exists a measurable selection that maps one equilibrium to each state of the world.

□

Therefore, an equilibrium exists in the last period of the discrete, finite time game. This implies that in the last period, there is a continuation value associated with each realization of the state of the world realized at the beginning of time $\bar{T}$. Denote the continuation value by $v(\omega_{\bar{T}})$. Note that I have allowed for this continuation value to be discontinuous in ω. However, at time $\bar{T}-1$, I can still have the expected continuation value be continuous in effort because of the presence of noise in the observed output.

Claim 4. There exists a pure strategy Nash equilibrium at time $\bar{T}-1$.

Proof. At the beginning of time $\bar{T}-1$, the state of the world is $(y^{\bar{T}-2}, a^{\bar{T}-2}, i) \in \mathbb{R}^{\bar{T}-2} \times [0, \bar{A}]^{\bar{T}-2} \times 2$. Since an equilibrium is associated with each realization at the beginning of time $\bar{T}$, there is a continuation value associated with each realization. These continuation values need not be continuous. However, the presence of random noise in the output function ensures that the expected continuation value is a continuous function of the amount of effort exerted by the worker.

The convexity in costs is sufficiently high. In fact, it is high enough to ensure that even at the highest discontinuous jump in continuation value, the costs are convex enough to ensure that the agent's expected continuation value is a concave function of the effort exerted by her.

The strategy set S at this stage is

$$S = [\underline{m}, \overline{m}]^N \times [\underline{m}, \overline{m}]^N \times [0,1]^N \times [0, \bar{A}]^N \times [0, \bar{A}]^N \times [0, \bar{A}]$$

This set is compact and convex.

The firm's utility is continuous and quasiconcave in the firm's actions. The agent's utility function is continuous and concave in her action. Therefore, by the DFG FPT, there exists a PSNE at this stage.

Using the same measurable selection theorem in the same way, we can ensure that there is a unique measurable PSNE associated with each realization at the beginning of time $\bar{T}-1$.

$\square$

Claim 5: There exists a PSNE at each period of the game.

Proof. I use the same method as the one I used for the previous claim. All arguments are the same. $\square$

The measurable unique PSNE associated with each realization for every period of the game together constitutes the PSNE for the complete game.

A.2.2 Part 2: Decrease step size

For each period in the discrete-time game, I assume the following structure

$$y_t = l\left(\eta + a_t^i\right) + \sqrt{\frac{l}{h_\varepsilon}}\varepsilon_t$$

where, y_t is the ouput in period t, and the random error is weighted by the step-size (l). For now, assume that $l = 1$, I will be reducing this step size later in the proof to show convergence to the continuous time equilibrium. Note, also that the error term ε_t is i.i.d. generated from $\sim N(0,1)$. The agents' discount factor in the discrete time game is $r \in (0,1)$, where $r = \rho l$ (the smaller the step size, the smaller is r.

The continuous time limit, i.e., the limit as $l \to 0$ of the output process above is

$$y_t = (\eta + a_t^i)d_t + \frac{1}{\sqrt{h_\varepsilon}}dB_t$$

where B_t is a standard Brownian motion, and y_t is instantaneous output. This generates the following expression for cumulative output

$$Y_t = \int_{s=0}^{t}(\eta + a_s)ds + \int_{s=0}^{t}\frac{1}{\sqrt{h_\varepsilon}}dB_s$$

Which is identical to the expression in equation 8 the main model.

In discrete-time, the update on ability follows

$$E_t[\eta|Y_t, \hat{A}_t^i, i] \equiv m_t^i = \frac{h_0 m_0 + h_\varepsilon\left(Y_t - \hat{A}_t^i\right)}{h_t}$$

This is exactly the expression from equation 9. The only difference here is that cumulative output (Y_t) and cumulative effort (A_t) must now be defined as the sum, rather than the integral of past outputs and efforts.

The update on effort cost type is given by:

$$K_t = \sum_{s=0}^{t/l} \frac{\left(y_s - (m_s^L + \hat{a}_s^L)\right)^2}{2\left(\frac{1}{h_s} + \frac{l}{h_\varepsilon}\right)} - \sum_{s=0}^{t/l} \frac{\left(y_s - (m_s^H + \hat{a}_s^H)\right)^2}{2\left(\frac{1}{h_s} + \frac{l}{h_\varepsilon}\right)}$$

where $K_t = ln(\frac{\mathbb{P}_t^H}{\mathbb{P}_t^L})$. The continuous time limit of this learning process is given by equation 14 in the main text.

Claim 1: *Worker's action in the discrete time model converge to a continuous time optimal action as step size (l) converges to zero.*

Proof. The action chosen by the worker in any given period is her effort (a_t^i). In equilibrium, the action must be such that it satisfies the first-order condition. I show that for sufficiently small step size, the action that first-order condition on effort in discrete time is arbitrarily close to some action that satisfies the FOC on effort in discrete time.

In discrete time, at time t, the FOC on effort is:

$$g_i^{'}(a_t^i) = \sum_{s=t}^{\bar{T}/l} \left[\left(\frac{1}{1+\rho l} \right)^{s-t} \mathbb{E}\left[\left(\frac{\partial w_s}{\partial y_t} \right) \right] \right]$$

The continuous-time limit of this FOC is identical to equation 15. Therefore, any level of effort that satisfies the FOC in the discrete-time case for a sufficiently low step size (l) will also be near an optimal effort for the continuous-time limit.

□

Claim 2: *Firms' actions in the discrete time model converge to continuous time optimal actions as step size (l) converges to zero.*

Proof. The actions chosen by a firm k are ($m_{\bar{T},k}^H$, $m_{\bar{T},k}^L$, $\mathbb{P}_{\bar{T},k}^H$, $\hat{a}_{\bar{T},k}^H$, $\hat{a}_{\bar{T},k}^L$)$\in [\underline{m}, \overline{m}] \times [\underline{m}, \overline{m}] \times [0,1] \times [0, \bar{A}] \times [0, \bar{A}]$. That is, each of the $N(>1)$ firms must choose a conjecture on a posterior mean for each type of agent, a probability that the agent is hard-working, and the action that is expected from each type of agent. The $m_{\bar{T},k}^H$, $m_{\bar{T},k}^L$, and $\mathbb{P}_{\bar{T},k}^H$ chosen in the discrete time equilibrium will converge to the continuous time actions. This is because, as we saw above, the Bayesian updating process converges.

As the optimal effort choice of the workers in the discrete-time model converges to the continuous-time optimal choice as $l \to 0$, the firms' equilibrium conjectures will also converge.

$\square$

$\square$

A.3 Proof of lemma 1

Proof. Using basic algebra, equation 14 can be simplified to:

$$\frac{\mathbb{P}_t^H}{\mathbb{P}_t^L} = e^{K_t} = exp\left[\int_0^t \frac{\left(a_s^{L\star} + m_t^L - a_s^{H\star} - m_t^H\right)\left(\left(a_s^{L\star} + m_t^L + a_s^{H\star} + m_t^H\right) ds - 2dY_s\right)}{2\left(\frac{1}{h_s} + \frac{1}{h_\varepsilon}\right)}\right] \tag{22}$$

From the above equation, it is clear that the change in market perception about a worker's effort cost with respect to a change in output is:

$$\frac{\partial k_t}{\partial y_t} = \frac{\left(a_t^{H\star} + m_t^H - a_t^{L\star} - m_t^L\right)}{\left(\frac{1}{h_t} + \frac{1}{h_\varepsilon}\right)} \tag{23}$$

where k_t is defined by $K_t = \int_0^t k_s ds$. Therefore, there is no learning about type from a change in dY_t if (and only if) $a_t^{H\star} + m_t^H = a_t^{L\star} + m_t^L$. When $a_t^{H\star} + m_t^H = a_t^{L\star} + m_t^L$ holds, then equation 16 simplifies to $\frac{\partial w_s^\star}{\partial Y_t} = \frac{h_\varepsilon}{h_s}$

That is, the only effect that changing effort will have is on the market perception on ability. This is because here, the effect on market belief regarding cost-type is not present. Also, since no change can be made in $m_t^L - m_t^H$ or in $\frac{1}{h_t}$ from choosing a different level of effort, no change in future efforts can be effected from the current choice of effort. This implies that:

$$\Rightarrow g_i'(a_t^i) = \left[\int_t^\infty e^{-\rho(s-t)} \frac{h_\varepsilon}{h_s} ds\right]$$

The equation above is exactly the first order condition for effort in a continuous time analougue of Holmström (1999). This also implies that if $a_t^{H\star} + m_t^H = a_t^{L\star} + m_t^L$, then: $a_t^{L\star} < a_t^{H\star}$. $\square$

A.4 Proof of lemma 2

Proof. To prove the above, I need to show that each of the following hold:

1. The persistent private information created by the agent by off-path play is only regarding her ability.

2. The same strategy is optimal on and off the path of play.

Private beliefs: Suppose that the market conjectures that the agent's output will follow the effort stream $(\hat{a}_i)_{t\geq 0}$. Any deviation from it changes the instantaneous output y_t. This impacts the market's beliefs m_t^H, m_t^L, and $\mathbb{P}_t^H$. However, the precision on ability conditioned on effort cost type (h_t) and the difference in conditional ability posteriors ($m_t^L - m_t^H$) remains unchanged. This is because

$$m_t^L - m_t^H = \frac{h_\varepsilon}{h_t}\left(\hat{A}_t^H - \hat{A}_t^L\right)$$

Therefore, the private information created by the deviation is in the level of m_t^i - with both m_t^H and m_t^L pushed up or down by the same amount.

Optimal action: After any deviation, the optimal action by the agent is to return to the on-path equilibrium. The private information created regarding the level of ability is not relevant to the optimal action.

In any MPE, the states that determine the optimal effort choice by the agent are: $m_t^L - m_t^H$, $\mathbb{P}_t^H$, and h_t. The only other state that could matter is m_t^i for either agent. However, the level of ability doesn't enter the first-order condition at all, only the difference $m_t^L - m_t^H$ enters the FOC that determines optimal effort.

I show that anticipated future changes in the market's posterior on the agent's effort cost type are independent of the level of ability posteriors.

Note that:

$$ln(\frac{\mathbb{P}_t^H}{\mathbb{P}_t^L}) = K_t = \left[\int_0^t \frac{\left(a_s^{L\star} + m_t^L - a_s^{H\star} - m_t^H\right)\left(\left(a_s^{L\star} + m_t^L + a_s^{H\star} + m_t^H\right)ds - 2dY_s\right)}{2\left(\frac{1}{h_s} + \frac{1}{h_\varepsilon}\right)}\right] \tag{24}$$

for a hard-working agent, $\mathbb{E}_t[dY_s] = m_t^H + a_s^{H\star} + \frac{1}{\sqrt{h_\varepsilon}}dB_s$, therefore, any future evolution of K_t will depend on $m_t^H - m_t^L$ and not on their levels.

$$\mathbb{E}^H[K_t] = \mathbb{E}^H\left[\int_0^t \frac{\left(a_s^{L\star} + m_t^L - a_s^{H\star} - m_t^H\right)\left(\left(a_s^{L\star} + m_t^L - a_s^{H\star} - m_t^H\right) ds - \frac{2}{\sqrt{h_\varepsilon}} dB_s\right)}{2\left(\frac{1}{h_s} + \frac{1}{h_\varepsilon}\right)}\right]$$

for a lazy agent, $\mathbb{E}_t[dY_s] = m_t^H + a_s^{H\star} + \frac{1}{\sqrt{h_\varepsilon}} dB_s$, therefore, any future evolution of K_t will depend on $m_t^H - m_t^L$ and not on their levels.

$$\mathbb{E}^L[K_t] = \mathbb{E}^L\left[\int_0^t \frac{\left(a_s^{L\star} + m_t^L - a_s^{H\star} - m_t^H\right)\left(\left(a_s^{H\star} + m_t^H - a_s^{L\star} - m_t^L\right) ds - \frac{2}{\sqrt{h_\varepsilon}} dB_s\right)}{2\left(\frac{1}{h_s} + \frac{1}{h_\varepsilon}\right)}\right]$$

The level of ability, while not relevant for effort choice, is relevant to wages offered by firms in the market. A deviation to a higher level of effort by the agent would create an incorrect posterior on ability in the market. This would result in higher wages, but the cost of this deviation will exceed its benefits. Otherwise, the on-path play would have not satisfied the first-order conditions. The same holds for any deviation that reduces effort. Therefore, there is no gain possible from any sustained deviation.

□

A.5 Proof of proposition 3

Now that I have shown that a pure strategy Nash equilibrium exists, I argue that $a_t^{i\star}(m_t^H, m_t^L, \mathbb{P}_t^H, \frac{1}{h_t})$ must be a continuous function of the state variables in any Markov Perfect Equilbrium with pure strategies.

Lemma 3. *The optimal action $a_t^{i\star}(m_t^H, m_t^L, \mathbb{P}_t^H, \frac{1}{h_t})$ must be a continuous function of the state variables in this environment.*

Proof. The state variables $\mathbb{P}_t^H$, m_t^H, and m_t^L vary continuously with observed output. The evolution of $\frac{1}{h_t}$ is deterministic in time. There is a random noise present. Expectations over future states will, therefore, vary continuously with effort choice.

Even if future actions vary discontinuously in their states, expected future states are a continuous function of current actions. Therefore, for any two states of the world that are arbitrarily close to each other, the optimal actions for those states must also be arbitrarily close to each other.

□

Lemma 4. *The optimal action $a_t^{i\star}$ must change continuously with time. There must be no discrete jumps in optimal action over time.*

Proof. This is a direct corollary of lemma 3 and the fact that Brownian noise is continuous (though not differentiable). □

Lemma 5. *A hard-working agent's cumulative effort will always exceed a lazy worker's cumulative effort.* $A_t^H > A_t^L$

Proof. **Claim 1:** It is not possible to have $a_t^{H\star} \leq a_t^{L\star}\ \forall t \in [0, \bar{T}]$

Proof. I will prove this claim by showing that in case $a_t^{H\star} \leq a_t^{L\star}\ \forall t \in [0, \bar{T}]$ holds then there will exist at least one point where $a_t^{H\star} + m_t^H = a_t^{L\star} + m_t^L$, at that point, it must be that $a_t^{H\star} > a_t^{L\star}$

Suppose (for contradiction) that $a_t^{H\star} \leq a_t^{L\star}\ \forall t \in [0, \bar{T}]$

It can't be that $a_0^{H\star} = a_0^{L\star}$, because then we would have $a_0^{H\star} + m_0^H = a_0^{L\star} + m_0^L$, and there $a_0^{H\star}$ should strictly exceed $a_0^{L\star}$. Therefore, it must be that $a_0^{H\star} < a_0^{L\star}$

From equation 33

$$a_t^{H\star} + m_t^H - a_t^{L\star} - m_t^L = a_t^{H\star} - a_t^{L\star} - \frac{h_\varepsilon}{h_t}\left(A_t^{H\star} - A_t^{L\star}\right) \tag{25}$$

$$= \frac{1}{h_t}\left(h_0\left(a_t^{H\star} - a_t^{L\star}\right) - h_\varepsilon\left(\left(A_t^{H\star} - a_t^{H\star}t\right) - \left(A_t^{L\star} - a_t^{L\star}t\right)\right)\right) \tag{26}$$

The above expression has to start off negative:

- Because at time $t = 0$, $m_0^H = m_0^L = m_0$ and $a_0^{H\star} < a_0^{L\star}$, and $A_0^{H\star} = A_0^{L\star} = 0$

The expression above has to eventually end up positive:

- As time $t \to \bar{T}$, $a_t^{H\star} \to 0$ and $a_t^{L\star} \to 0$ while $A_t^{L\star} > A_t^{H\star} > 0$
- $a_t^{i\star}$ tends to diminish in t. This implies that $A_t^{i\star} - a_t^{i\star}t$ will need to be positive for large values of t
- Moreover, $\left(A_t^{H\star} - a_t^{H\star}t\right) - \left(A_t^{L\star} - a_t^{L\star}t\right)$ will need to be negative for large values of t

Therefore, there needs to be at least one point in between where the expression ($a_t^{H\star} + m_t^H - a_t^{L\star} - m_t^L$) is equal to zero. At that point, it must be that $a_0^{H\star} > a_0^{L\star}$

□

Claim 2: $a_0^{H\star} > a_0^{L\star}$

Proof. It can't be that $a_0^{H\star} = a_0^{L\star}$, because then we would have $a_0^{H\star} + m_0^H = a_0^{L\star} + m_0^L$, and there $a_0^{H\star}$ should strictly exceed $a_0^{L\star}$. Therefore, suppose (for contradiction) that $a_0^{H\star} < a_0^{L\star}$.

Claim 1 proves that it cannot be that the effort of the hard-working agent cannot forever be less than the effort of the lazy agent. That is $a_t^{H\star} \leq a_t^{L\star}\ \forall t \geq 0$ is not possible. Therefore, there must be some point $t_2 > 0$ such that it is the first time where $a_t^{H\star} \geq a_t^{L\star}$.

Since $a_t^{H\star} \leq a_t^{L\star}\ \forall t \leq t_2$, with $a_0^{H\star} < a_0^{L\star}$, it must be that $A_{t_2}^{H\star} < A_{t_2}^{L\star}$. Using equation 33, this implies that $m_{t_2}^H > m_{t_2}^L$.

The posterior on ability of the lazy type must be lower for all time before t_2, that is $m_t^H > m_t^L \forall t \in (0, t_2]$. Following from lemma 4, optimal actions must evolve continuously over time. Therefore, before t_2, there will exist t_1 s.t. $a_{t_1}^{H\star} + m_{t_1}^H = a_{t_1}^{L\star} + m_{t_1}^L$.

From lemma 1, at t_1, it must be that $a_{t_1}^{H\star} > a_{t_1}^{L\star}$. Therefore, contradiction.

Therefore, it must be that $a_0^{H\star} > a_0^{L\star}$.

□

Claim 2 implies that there can only be finite periods where $a_t^H \leq a_t^L$.

Claim 3: While $a_t^{H\star} \leq a_t^{L\star}$, it must still be that $A_t^{H\star} > A_t^{L\star}$

Let $a_t^{H\star} \leq a_t^{L\star}$ for all $t \in [t_1, t_2]$.

Suppose, for contradiction, that while $a_t^{H\star} \leq a_t^{L\star}$, we also have $A_t^{H\star} \leq A_t^{L\star}$, then $m_t^L \geq m_t^H$.

Then there exists a $\hat{t} \in (t_1, t_2)$ such that $a_{\hat{t}}^{H\star} + m_{\hat{t}}^H = a_{\hat{t}}^{L\star} + m_{\hat{t}}^L$. At $\hat{t}$, it must be that $a_{\hat{t}}^{H\star} > a_{\hat{t}}^{L\star}$ (from lemma 1). This leads us to a contradition.

This implies that whenever $a_t^{H\star} \leq a_t^{L\star}$, we also have $A_t^{H\star} > A_t^{L\star}$,

□

Lemma 8 must be modified to state that:

Lemma 6. *$a_t^{i\star} \to 0$ as $t \to \bar{T}$ Effort cannot remain bounded away from zero as we approach the terminal time.*

Proof. This holds because in finite time the benefit of building reputation diminishes (and approaches zero) as we approach the terminal time. Therefore, equilibrium effort must also tend to zero as we approach time $\bar{T}$. □

proposition 3 states:

Proposition 5. *Based on the primitives (i, m_0, h_0, ρ, h_ε, $g_L(\cdot)$, $g_H(\cdot)$), there exists a unique ex-ante exptected cutoff time T. Before that time, a higher output makes the agent seem hardworking. After time T a higher output makes the agent seem lazy. Ex-ante, the worker expects that:*

$$\begin{array}{lcll} \mathbb{E}_{t=0}[\frac{\partial k_t}{\partial Y_t}] > 0 & \Leftrightarrow & \mathbb{E}_{t=0}[a_t^{H\star} + m_t^H] > \mathbb{E}_{t=0}[a_t^{L\star} + m_t^L] & \forall \quad t < T \\ \mathbb{E}_{t=0}\left[\frac{\partial k_t}{\partial Y_t}\right] < 0 & \Leftrightarrow & \mathbb{E}_{t=0}\left[a_t^{H\star} + m_t^H\right] < \mathbb{E}_{t=0}\left[a_t^{L\star} + m_t^L\right] & \forall \quad t > T \end{array}$$

Proof. Equation 23 states that:

$$\frac{\partial k_t}{\partial Y_t} = \frac{\left(a_t^{H\star} + m_t^H - a_t^{L\star} - m_t^L\right)}{\left(\frac{1}{h_t} + \frac{1}{h_\varepsilon}\right)}$$

I need to show that the above rate of change starts positive, but ends up negative.

- $\left(\frac{1}{h_t} + \frac{1}{h_\varepsilon}\right)$ is positive

From equation 33

$$\left(a_t^{H\star} + m_t^H - a_t^{L\star} - m_t^L\right) = a_t^{H\star} - a_t^{L\star} - \frac{h_\varepsilon}{h_t}\left(A_t^{H\star} - A_t^{L\star}\right) \tag{27}$$

$$= \frac{1}{h_t}\left(h_0\left(a_t^{H\star} - a_t^{L\star}\right) - h_\varepsilon\left(\left(A_t^{H\star} - a_t^{H\star}t\right) - \left(A_t^{L\star} - a_t^{L\star}t\right)\right)\right) \tag{28}$$

The above expression has to start off positive:

- Because at time $t = 0$, $m_0^H = m_0^L = m_0$ and $a_0^{H\star} > a_0^{L\star}$, and $A_0^{H\star} = A_0^{L\star} = 0$
- The expression being positive implies that a higher observed output makes the market believe that the agent is more likely to be hard-working

The expression above has to eventually end up negative:

- As time $t \to \bar{T}$, $a_t^{H\star} \to 0$ and $a_t^{L\star} \to 0$ while $A_t^{H\star} > A_t^{L\star} > 0$
- $a_t^{i\star}$ tends to diminish in t. This implies that $A_t^{i\star} - a_t^{i\star}t$ will need to be positive for large values of t
- Moreover, $\left(A_t^{H\star} - a_t^{H\star}t\right) - \left(A_t^{L\star} - a_t^{L\star}t\right)$ will need to be positive for large values of t

- This means that the higher the output observed, the greater is the likelihood placed by the market on the agent being lazy

Ex-ante, agents can calculate their an expected effort stream, as well as the market's expectations on the effort for each type, based on the primitives. Ex-ante, agents expect uncertainty to resolve over time. Therefore, the optimal effort is ex-ante expected to diminish over time. Given that this ex-ante expectation is fixed, the cutoff point T exists and is unique ex-ante.

□

A.6 Proof of proposition 4

Proof. **Proof part 1:** *Claim:* $\tau < T$

I need to show that the rate of change of the agent's continuation value with respect to the probability that is placed on her being hardworking (K_t) is positive initially and later becomes negative.

Envelope Theorem guarantees that this rate of change becomes negative at a time that it between 0 and T

According to the Envelope theorem:
$\frac{dv_t^i}{dK_t} = \frac{\partial v_t^i}{\partial K_t}|_{a=a^\star}$
Suppose we are at time $t = T$

- According to proposition 3, the lazy agent is expected to produce a greater output at all times $t \geq T$

- Since wages are competitive, being thought of as lazy increases the expected wage

- According to the Envelope theorem, I can evaluate the change brought about by an increase in the probability of being thought of as hard-working on the continuation value of the agent by partially differentiating it with respect to the change in probability, holding constant, the optimal actions $a^\star$

- At time T, $\frac{dv_T^i}{dK_T}$ is expected to be strictly negative.

This means that if we are at a time slightly to the left of T, say, at $T - \varepsilon$, then also the value of $\frac{dv_t^i}{dK_t}$ will be negative.

Since τ is the point at which this derivative turns negative, I have shown that τ is to the left of T.

Proof part 2: *Claim:* $\tau > 0$

Suppose time is $t = 0$, and no output has been observed yet. The market has the same priors on either type of agent $m_0^H = m_0^L = m_0$. Ex-ante, the market expects the hard-working agent to produce a higher output. Moreover, we can see from lemma 7 that the cumulative effort of hard-workers always exceeds the cumulative effort of lazy workers. While lemma 7 is retrospective in nature, its conclusion is valid in setting the expectation for the future as well.

At time $t = 0$, the market expects an average hard-working agent to make a higher output than an average lazy agent at all future times. Therefore, the agent would be paid a higher wage in the future if she were to look hard-working. Thus, the agent's continuation value is increasing in the market's posterior on her being hard-working. This implies that $\tau > 0$

$\square$

Appendix B Extensions

B.1 Infinite time model

In this section, I consider the same environment described in section 3, but with an infinite time horizon. A worker with an infinite lifespan has a symmetrically unknown ability, and privately known cost of effort. Both, the worker, and the market learn about the worker's ability over time. The market must also learn about the worker's effort cost, while the worker already knows her effort cost. The only observable signal is output, and I follow the career concerns literature in assuming that wages cannot be conditioned on output.

As is shown below, all of the results from section 3 can be reproduced when time is infinite.

In this section, I can still use equations 8, 9, 10, 11, 14, 16, 23, and 33 in their existing forms

- Equation 12 is updated to:

$$U_i = \mathbb{E}_i^{a^\star}\left[\int_0^\infty e^{-\rho t}\left(w_t^\star - g_i(a_t)\right) dt\right] \tag{29}$$

- Equation 13 is updated to:

$$v_t^i = \mathbb{E}_i^{a^\star}\left[\int_t^\infty e^{-\rho(s-t)}\left(w_s^\star - g_i(a_s)\right) ds\right] \tag{30}$$

Lemma 1 remains the same, and holds for the same reasons.

Lemma 7 remains the same, but I should re-work the proof. The lemma states that:

Lemma 7. *A hard-working agent's cumulative effort will always exceed a lazy worker's cumulative effort.* $A_t^H > A_t^L$

B.1.1 Proof of Lemma 7

Proof. **Claim 1:** It is not possible to have $a_t^{H\star} \leq a_t^{L\star}\ \forall t \geq 0$

Proof. I will prove this claim by showing that in case $a_t^{H\star} \leq a_t^{L\star}\ \forall t \geq 0$ holds then there will exist at least one point where $a_t^{H\star} + m_t^H = a_t^{L\star} + m_t^L$, at that point, it must be that $a_t^{H\star} > a_t^{L\star}$

Suppose (for contradiction) that $a_t^{H\star} \leq a_t^{L\star}\ \forall t \geq 0$

It can't be that $a_0^{H\star} = a_0^{L\star}$, because then we would have $a_0^{H\star} + m_0^H = a_0^{L\star} + m_0^L$, and there $a_0^{H\star}$ should strictly exceed $a_0^{L\star}$. Therefore, it must be that $a_0^{H\star} < a_0^{L\star}$

Note that

$$\begin{aligned} m_t^i &= \frac{h_0 m_0 + h_\varepsilon\left((m_0+\varepsilon_0)t + \int_0^t a_s ds + \int_0^t \frac{1}{\sqrt{h_\varepsilon}} dB_s - A_t^{i\star}\right)}{h_t} && (31) \\ &= m_0 + \frac{h_\varepsilon}{h_t}\left(\varepsilon_0 t + \int_0^t a_s ds + \int_0^t \frac{1}{\sqrt{h_\varepsilon}} dB_s - A_t^{i\star}\right) && (32) \end{aligned}$$

This implies that

$$m_t^L - m_t^H = \frac{h_\varepsilon}{h_t}\left(A_t^{H\star} - A_t^{L\star}\right) \tag{33}$$

From equation 33

$$\begin{aligned} a_t^{H\star} + m_t^H - a_t^{L\star} - m_t^L &= a_t^{H\star} - a_t^{L\star} - \frac{h_\varepsilon}{h_t}\left(A_t^{H\star} - A_t^{L\star}\right) && (34) \\ &= \frac{1}{h_t}\left(h_0\left(a_t^{H\star} - a_t^{L\star}\right) - h_\varepsilon\left(\left(A_t^{H\star} - a_t^{H\star}t\right) - \left(A_t^{L\star} - a_t^{L\star}t\right)\right)\right) && (35) \end{aligned}$$

The above expression has to start off negative:

- Because at time $t = 0$, $m_0^H = m_0^L = m_0$ and $a_0^{H\star} < a_0^{L\star}$, and $A_0^{H\star} = A_0^{L\star} = 0$

The expression above has to eventually end up positive:

- As time $t \to \infty$, $a_t^{H\star} \to 0$ and $a_t^{L\star} \to 0$ while $A_t^{L\star} > A_t^{H\star} > 0$

- $a_t^{i\star}$ tends to diminish in t. This implies that $A_t^{i\star} - a_t^{i\star} t$ will need to be positive for large values of t

- Moreover, $\left(A_t^{H\star} - a_t^{H\star} t\right) - \left(A_t^{L\star} - a_t^{L\star} t\right)$ will need to be negative for large values of t

Therefore, there needs to be at least one point in between where the expression $(a_t^{H\star} + m_t^H - a_t^{L\star} - m_t^L)$ is equal to zero. At that point, it must be that $a_0^{H\star} > a_0^{L\star}$

$\square$

Claim 2: $a_0^{H\star} > a_0^{L\star}$

Proof. It can't be that $a_0^{H\star} = a_0^{L\star}$, because then we would have $a_0^{H\star} + m_0^H = a_0^{L\star} + m_0^L$, and there $a_0^{H\star}$ should strictly exceed $a_0^{L\star}$. Therefore, suppose (for contradiction) that $a_0^{H\star} < a_0^{L\star}$.

Claim 1 proves that it cannot be that the effort of the hard-working agent cannot forever be less than the effort of the lazy agent. That is $a_t^{H\star} \leq a_t^{L\star}\ \forall t \geq 0$ is not possible. Therefore, there must be some point $t_2 > 0$ such that it is the first time where $a_t^{H\star} \geq a_t^{L\star}$.

Since $a_t^{H\star} \leq a_t^{L\star}\ \forall t \leq t_2$, with $a_0^{H\star} < a_0^{L\star}$, it must be that $A_{t_2}^{H\star} < A_{t_2}^{L\star}$. Using equation 33, this implies that $m_{t_2}^H > m_{t_2}^L$.

The posterior on ability of the lazy type must be lower for all time before t_2, that is $m_t^H > m_t^L \forall t \in (0, t_2]$. Following from lemma 4, optimal actions must evolve continuously over time. Therefore, before t_2, there will exist t_1 s.t. $a_{t_1}^{H\star} + m_{t_1}^H = a_{t_1}^{L\star} + m_{t_1}^L$.

From lemma 1, at t_1, it must be that $a_{t_1}^{H\star} > a_{t_1}^{L\star}$. Therefore, contradiction.

Therefore, it must be that $a_0^{H\star} > a_0^{L\star}$.

$\square$

Claim 2 implies that there can only be finite periods where $a_t^H \leq a_t^L$.

Claim 3: While $a_t^{H\star} \leq a_t^{L\star}$, it must still be that $A_t^{H\star} > A_t^{L\star}$

Let $a_t^{H\star} \leq a_t^{L\star}$ for all $t \in [t_1, t_2]$.

Suppose, for contradiction, that while $a_t^{H\star} \leq a_t^{L\star}$, we also have $A_t^{H\star} \leq A_t^{L\star}$, then $m_t^L \geq m_t^H$.

Then there exists a $\hat{t} \in (t_1, t_2)$ such that $a_{\hat{t}}^{H\star} + m_{\hat{t}}^H = a_{\hat{t}}^{L\star} + m_{\hat{t}}^L$. At $\hat{t}$, it must be that $a_{\hat{t}}^{H\star} > a_{\hat{t}}^{L\star}$ (from lemma 1). This leads us to a contradition.

This implies that whenever $a_t^{H\star} \leq a_t^{L\star}$, we also have $A_t^{H\star} > A_t^{L\star}$,

□

Lemma 7 views the problem from the perspective of the market. The market observes a history of output, based on which, it must learn about the worker. Given any output history, the lemma argues that if that output were produced by a hard-working employee, then the effort component of that output must be higher than if the same output history had been produced by a lazy worker. This result will be crucial to the market's assessment of the worker's ability.

In the next lemma, I show that as reputation solidifies, eventually no effort is made.

Lemma 8. $a_t^{i\star}(Y^t) \to 0$ *as* $t \to \infty$ *Effort cannot remain bounded away from zero forever*

Proof. It is clear from equation 10 that the precision on the posterior of the worker's ability tends to infinity as time tends to infinity. Learning about cost type also takes place at all instances where $a_t^{H\star} + m_t^H \neq a_t^{L\star} + m_t^L$. As time goes to infinity, the sum total amount of time such that $a_t^{H\star} + m_t^H \neq a_t^{L\star} + m_t^L$ also goes to infinity. Therefore, eventually, the worker's cost-type is also learned to an arbitrarily large precision by the market.

Future wages are determined by expected ability and effort, the rate of change in those expectations with respect to output approaches zero as time approaches infinity and precisions on ability and cost-type become arbitrarily high.

Furthermore, if I suppose that $a_t^{i\star} > \varepsilon > 0$ as time approaches infinity, we run into a contradiction. The worker will choose a lower a_t^i than the market expects since her savings in effort cost will outweigh the expected wage-loss from reducing effort.

□

From lemma 8, it is clear that eventually, costly effort will be a minuscule component of total output. This implies that a worker's wage will depend more on her perceived ability, rather than effort levels. I had shown in lemma 7 that the cumulative effort of workers with low effort costs always exceeds that of workers with high effort costs. This implies that the posterior on the ability of a worker with a given history of outputs is always higher if the market believes that the worker has high effort costs.

I use lemmas 7 and 8 to prove the surprising result that after a particular cut-off time, a lazy worker will be valued more highly by the market than a hard-working worker with the same output history.

proposition 3 states:

Proposition 6. *Based on the primitives (i, m_0, h_0, ρ, h_ε, $g_L(\cdot)$, $g_H(\cdot)$), there exists a unique ex-ante exptected cutoff time T. Before that time, a higher output makes the agent seem hardworking. After time T a higher output makes the agent seem lazy. Ex-ante, the worker expects that:*

$$\begin{array}{rcl} \mathbb{E}_{t=0}[\frac{\partial k_t}{\partial Y_t}] > 0 & \Leftrightarrow & \mathbb{E}_{t=0}[a_t^{H\star} + m_t^H] > \mathbb{E}_{t=0}[a_t^{L\star} + m_t^L] \quad \forall \quad t < T \\ \mathbb{E}_{t=0}\left[\frac{\partial k_t}{\partial Y_t}\right] < 0 & \Leftrightarrow & \mathbb{E}_{t=0}\left[a_t^{H\star} + m_t^H\right] < \mathbb{E}_{t=0}\left[a_t^{L\star} + m_t^L\right] \quad \forall \quad t > T \end{array}$$

The proof needs to be modified only slightly.

Proof. Equation 23 states that:

$$\frac{\partial k_t}{\partial Y_t} = \frac{\left(a_t^{H\star} + m_t^H - a_t^{L\star} - m_t^L\right)}{\left(\frac{1}{h_t} + \frac{1}{h_\varepsilon}\right)}$$

I need to show that the above rate of change starts off positive, but ends up negative.

- $\left(\frac{1}{h_t} + \frac{1}{h_\varepsilon}\right)$ is positive

From equation 33

$$\left(a_t^{H\star} + m_t^H - a_t^{L\star} - m_t^L\right) = a_t^{H\star} - a_t^{L\star} - \frac{h_\varepsilon}{h_t}\left(A_t^{H\star} - A_t^{L\star}\right) \tag{36}$$

$$= \frac{1}{h_t}\left(h_0\left(a_t^{H\star} - a_t^{L\star}\right) - h_\varepsilon\left(\left(A_t^{H\star} - a_t^{H\star}t\right) - \left(A_t^{L\star} - a_t^{L\star}t\right)\right)\right) \tag{37}$$

The above expression has to start off positive:

- Because at time $t = 0$, $m_0^H = m_0^L = m_0$ and $a_0^{H\star} > a_0^{L\star}$, and $A_0^{H\star} = A_0^{L\star} = 0$
- The expression being positive implies that a higher observed output makes the market believe that the agent is more likely to be hard-working

The expression above has to eventually end up negative:

- As time $t \to \infty$, $a_t^{H\star} \to 0$ and $a_t^{L\star} \to 0$ while $A_t^{H\star} > A_t^{L\star} > 0$
- $a_t^{i\star}$ tends to diminish in t. This implies that $A_t^{i\star} - a_t^{i\star}t$ will need to be positive for large values of t
- Moreover, $\left(A_t^{H\star} - a_t^{H\star}t\right) - \left(A_t^{L\star} - a_t^{L\star}t\right)$ will need to be positive for large values of t

- This means that the higher the output observed, the greater is the likelihood placed by the market on the agent being lazy

Ex-ante, agents can calculate their an expected effort stream, as well as the market's expectations on the effort for each type, based on the primitives. Ex-ante, agents expect uncertainty to resolve over time. Therefore, the optimal effort is ex-ante expected to diminish over time. Given that this ex-ante expectation is fixed, the cutoff point T exists and is unique ex-ante.

□

proposition 4 states that:

Proposition 7. *Based on the primitives (i, m_0, h_0, ρ, h_ε, $g_L(\cdot)$, $g_H(\cdot)$), there exists a unique ex-ante expected cutoff time $\tau \in (0, T)$ such that before that time, the continuation value of the agent is expected to increase with her being perceived as hardworking. After time τ, her continuation value (in expectation) increases when the market places a higher probability on her being lazy.*

The statement of the proposition does not change when I consider an infinite time environment. The proof is also exactly correct and needs no modification: