On the Measurement of Electoral Volatility

Sandip Sarkar[1*] and **Bharatee Bhusana Dash**[2]

[1,2] *Xavier School of Economics, Xavier University Bhubaneswar*

Abstract

The linear electoral volatility measure introduced in an influential article by Pedersen (1979), is not always responsive to certain changes of party system like increase or decrease in the number of parties, and/or relative strength of competing parties. This is against the hypothesis formulated in the same article. We address this issue by introducing a class of strictly concave additive separable electoral volatility measures. We axiomatically characterize this class following a set of independent axioms. We also show that this class of electoral volatility measures can be used in comparing any two arbitrary party systems. Finally, we discuss two sets of quasi orderings, following which a binary relation on electoral volatility ordering can be formulated. We also extend both the quasi approaches in the context of electoral volatility orderings for party systems, with different number of political parties.

Key words: Pedersen's Measure; Electoral Volatility; Party System Stability; Axioms.

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[*] *Corresponding author. Email id: sandip.isi.08@gmail.com*

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1 Introduction

Political scientists have long been interested in understanding party system (in)stability. A stable party system is considered as a hallmark of a matured democracy (Verba and Almond, 1963; Putnam et al., 1994; Inglehart, 1997; Przeworski et al., 2000). An unstable party system can lead to the implementation of short-term objective centric policies, open doors to unconventional candidates and political parties, expose the national security system to both internal and external threats, and weaken the fundamental institutions of democracy. Tóka (1997) argues that electoral volatility is the single most important indicator of presence or absence of party system stabilization. Pedersen (1979) defines electoral volatility as the change of relative strength of political party from one election to other. In order to measure electoral volatility he introduced the following measure:[1]

$$EVM = \frac{\sum_{i=1}^{n} |v_{0i} - v_{1i}|}{2} \tag{1}$$

where n is the number of parties, and for any party i v_{0i} and v_{1i} denotes the vote shares at time 0 and 1, respectively.

In order to measure electoral volatility, one of the main arguments put forward by Pedersen (1979) is that: *"electoral volatility would increase if there is a change in the number of parties, and in the relative distribution of electoral strength among the competing parties"* (Pedersen, 1979, pp 3).[2] However, the linearity imposed on the functional form of the electoral volatility index fails to capture this phenomena. We use a numerical example with vote shares of parties to demonstrate this. Let's consider three vote distribution of parties V_0 = (0.3, 0.3, 0.4), V_1 = (0.3, 0.3, 0.2, 0.2), and V_2 = (0.3, 0.3, 0.2, 0.1, 0.1), where V_0 being the outcome of initial election and V_1 and V_2 are two possible outcomes in next election.[3] It is quite straightforward to figure out that the number of

[1]Though the functional form 1 proposed in Pedersen (1979) is used to measure electoral volatility in the literature, in earlier studies similar measures were proposed and applied to study other features of party system closely related to party system (in)stability. Przeworski and Sprague (1971) and Przeworski (1975) have used the functional form 1, but to study the impacts of changes in voting behavior of relevant voters on party system institutionalization between elections. Ascher and Tarrow (1975) define volatility as the net change within the party system resulting from individual voters switching parties, but they measure it as the sum of the net gains and losses between all the elections analyzed, divided by the number of electoral intervals (i.e., number of elections under consideration minus one).

[2]A more detailed and formal discussion on this is available in section 4 of the paper (Pedersen, 1979, pp 12-17).

[3]Throughout the paper, we express vote distributions among competing parties in vectors. All parties vote shares add to 1, and the number of parties with positive vote shares determine the type of party

political parties has increased from 3 to 4 from V_0 to V_1. On the other hand, considering V_0 as the initial election and V_2 as the final election the number of political parties has increased from 3 to 5. In fact it is also quite straightforward to figure out that change of relative strength of political parties is quite different from V_0 to V_1 and V_0 to V_2. From this example it is easy to figure out that Pederson's volatility measure is not responsive to these changes and eventually the degree of volatility from V_0 to V_1 and from V_0 to V_2 both are equal to 0.2.

So far we have surveyed, no paper addresses the methodological issues of electoral volatility measurement discussed in the previous paragraph. As the first step towards this direction, we propose a class of strictly concave additive separable volatility measure that addresses all the issues related to measurement of electoral volatility discussed above. Furthermore, we also axiomatically characterize this class of electoral volatility measures.[4] The main axiom that is violated by $EVM(\cdot)$ is Vote transfer sensitivity (VTS). This axiom justifies the rationality of moving from a linear to a strictly concave functional form. If we relax this axiom, and consider only weakly concave volatility measures, then Pederson's measure also becomes a member of the extended class introduced in this paper.

We also do not find any paper that discusses the theoretical issues related to the comparisons of party systems with different number of parties. Nevertheless, empirical papers on electoral volatility even in the recent decades Bischoff (2013) do compare electoral volatility with different number of parties. We introduce an axiom named "*Strong Proportional Increments*" following which one can evaluate the volatility ordering of party systems with different number of political parties. The philosophy of this axiom is very much similar to the well known replication invariance used in welfare ordering of income (or related metrics of welfare) with different population sizes. We show that the proposed class of measure along with the Pedersen (1979) measure, can be used in comparing of any arbitrary party systems.

We also address electoral volatility ordering following a quasi approach.[5] We establish

system. Measuring volatility requires outcomes of two elections and all parties with positive vote shares in both the elections are considered for this. If a party is present only in one election, its other election's vote share is 0 and we interpret this as the party had a pseudo presence in the election.

[4]Axiomatic characterization is quite common in economics. Sen (1976) was the first to characterize the well known Sen's Poverty measure.

[5]Many seminal contributions exists in the literature on axiomatic social choice and welfare following quasi ordering approaches. For example, the equivalence between Lorenz dominance and S concave social welfare functions by Atkinson (1970); Dasgupta et al. (1973). Partial orderings of poverty with respect to poverty measures has been formulated initially by Atkinson (1987). This approach of poverty orderings

equivalence between some ranking conditions of distributions following a class of electoral volatility measure. We also extend both these quasi approaches in the context of party systems with different number of political parties.

In section 2 we discuss some prelimary issues related to electoral volatility measure. In section 3 we introduce the axioms and present the characterization result. In section 4 we present the results associated with the quasi ordering approaches. The final section presents our concluding remarks.

2 Preliminaries

Let in a society there are n $(n > 1)$ political parties. Let $\mathbb{P} = \{1, 2, .., n\}$ denotes the set of political parties. We denote the initial and final elections by 0 and 1, respectively. Let $V_t = \{(v_{t1}, v_{t2}, ..., v_{tn}) \in \mathbb{R}^n_+ | \forall i \in \mathbb{P},\ v_{ti} \geq 0 \text{ and } \sum_{i=1}^{n} v_{ti} = 1\}$ denotes the vector of vote shares at time point t, where $t \in \{0, 1\}$. $\mathbb{V}^n$ denotes the set of all vote share vectors. The ordered pair (V_0, V_1) denotes a party system. The set of all party system is denoted by: $\mathfrak{T}^n = \{V_0 \times V_1 \mid V_0, V_1 \in \mathbb{V}^n\}$. We now define two types of transfer that we use throughout this paper.

Definition 1. ***V-Transfer***: *For any $(V_0, V_1) \in \mathfrak{T}^n$, by V-transfer we mean a transfer of vote from some distinct party i and j, there exists a transfer of votes from i to party j from the initial to final election such that $v_{1i} = v_{0i} - \epsilon$, $v_{1j} = v_{0j} + \epsilon$ and $v_{1k} = v_{0k}$ where $k \in \{\mathbb{P} \backslash i, j\}$.*

Definition 2. ***VT-Transfer***: *For any $(V_0, V_1) \in \mathfrak{T}^n$, by VT-transfer we mean a transfer of vote between some distinct parties i, j and l. Party i losses (gains) some votes in the initial election which is divided between party j and l in the final election, such that $v_{1i} = v_{0i} - \epsilon$, $v_{1j} = v_{0j} + \epsilon_1$, $v_{1l} = v_{0l} + \epsilon_2$ and $v_{1k} = v_{0k}$ where $k \in \{\mathbb{P} \backslash i, j, l\}$ and $\epsilon = \epsilon_1 + \epsilon_2$.*

We now define replicated party systems which is useful in the comparison of two party systems with different number of political parties.

Definition 3. ***Replicated Party Systems***: *For any $(V_0, V_1) \in \mathfrak{T}^n$ a party system is said to be replicated k times which is denoted by $(V_0^k, V_1^k) \in \mathfrak{T}^{nk}$, where $\forall s \in \{0, 1\} :\ V_s^k = (1_k \frac{v_{s1}}{k}, 1_k \frac{v_{s2}}{k}, .., 1_k \frac{v_{s1}}{k})$ and $1_k = (1, 1, ..., 1)_{k \times 1}$.*

was further extended by Foster and Shorrocks (1988a,b).

Electoral volatility following Pedersen (1979) is the change in the electoral strength among competing political parties. Furthermore, following the author an electoral volatility measure can also be considered as a function as net change of electoral strength across political parties. Formally:

Definition 4. ***Electoral Volatility Measure****: For any* $(V_0, V_1) \in \mathfrak{T}^n$ *an electoral volatility measure is defined as follows:* $E(V_0, V_1) : D^n \mapsto \mathbb{R}$*, where* $D^n = (d_1, d_2, .., d_n)$*, and* $d_i = |v_{0i} - v_{1i}| \forall\ i \in \mathbb{P}$*.*

Let $\mathfrak{E}$ denotes the set of all electoral volatility measures that is defined in 4. Throughout this paper we restrict our attention only on the class of electoral volatility measure in $\mathfrak{E}$. The main result of this paper is on characterizing the following class of electoral volatility measures which is a subset of $\mathfrak{E}$.

Definition 5. *For any* $(V_0, V_1) \in \mathbb{V}^n$ *by Generalized Electoral Volatility Measure we mean the following class of measures which is a subset of* $\mathfrak{E}$*.*

$$GEV(V_0, V_1) = \frac{\sum_{i=1}^{n} |v_{1i} - v_{0i}|^{\alpha}}{2^{\alpha} n^{1-\alpha}} \tag{2}$$

where α *is the degree of number of parties aversion parameter.*

Note that if $\alpha = 1$ then GEV() corresponds to the original Pedersen (1979) measure defined in equation 1.

3 Main Results

We introduce the following axioms, which we believe any Electoral Volatility Measure must satisfy. We assume that $EV()$ is a member of $\mathfrak{E}$.

Axiom 1. ***Increased Volatility (IV)****: For any* $(V_0, V_1) \in \mathfrak{T}^n$*, the electoral volatility measure* $EV(V_0, V_1)$ *must be strictly increasing in its arguments.*

Axiom 2. ***Zero Volatility (ZV)****:* $EV(V_0, V_1) = 0$ *if and only if* $v_{0i} = v_{1i} \forall i \in \mathbb{P}$*.*

Axiom 3. ***Maximum Volatility (MV)****:* $EV(V_0, V_1) \leq 1$*.*

Axiom 4. ***Additive Separability (AS)****:* $E(V_0, V_1) = \sum_{i=1}^{n} f\big(|v_{1i} - v_{0i}|\big)$ *where* $f : (0, 1]^n \mapsto \mathbb{R}$.

Axiom 5. ***Proportional Increments (PI)****: Let* (V_0^k, V_1^k), (V_2^k, V_3^k), (V_4^k, V_5^k) *and* $(V_6^k, V_7^k) \in \mathfrak{T}^{nk}$ *be the k fold replicated party systems (following Definition 3) of* (V_0, V_1), (V_2, V_3), (V_4, V_5), *and* $(V_6, V_7) \in \mathfrak{T}^n$*, respectively. Following this axiom* $EV(V_0, V_1) - EV(V_2, V_3) \lesseqqgtr EV(V_4, V_5) - EV(V_6, V_7) \iff EV(V_0^k, V_1^k) - EV(V_2^k, V_3^k) \lesseqqgtr EV(V_4^k, V_5^k) - EV(V_6^k, V_7^k)$.

Axiom 6. ***Vote Transfer Sensitivity (VTS)****: Let* $(V_0, V_1), (V_2, V_3) \in \mathfrak{T}^n$ *such that* $D_{01} = (d_{011}, d_{012}, .., d_{01n})$ *and* $D_{23} = (d_{231}, d_{232}, ..., d_{23n})$*, where* $d_{01i} = |v_{0i} - v_{1i}|$ *and* $d_{23i} = |v_{2i} - v_{3i}|$*. There exists some distinct* $\{a, b\} \in \mathbb{P}$ *such that* $d_{01a} = \epsilon$, $d_{01b} = \epsilon$ *and* $d_{01i} = 0 \ \forall i \in \{P | i \neq a, b\}$*. For some distinct* $\{l, j, k\} \in \mathbb{P}$ *such that* $d_{23l} = \epsilon$, $d_{23j} = \epsilon_1$, $d_{23k} = \epsilon_2$ *and* $d_{23i} = 0 \forall i \in \{P | i \neq l, j, k\}$*, where* $\epsilon = \epsilon_1 + \epsilon_2$*. Then following this axiom:* $EV(V_2, V_3) > EV(V_0, V_1)$.

IV states that any measure of electoral volatility must be strictly increasing in its arguments. Thus if there is a transfer of vote from one political party to other, then higher the volume of transfer, higher is the electoral volatility. To cite an example consider the following two party systems $(V_0, V_1), (V_2, V_3) \in \mathfrak{T}^3$, where $V_0 = (0.1, 0.4, 0.5)$, $V_1 = (0.1, 0.3, 0.6)$, $V_2 = (0.1, 0.7, 0.2)$ and $V_3 = (0.1, 0.5, 0.4)$. We can say that V_1 is obtained from V_0 by a vote transfer of 0.1 from party 2 to 3. On the other hand V_3 is obtained from V_2 by a vote transfer of 0.2 from party 2 to 3. Following IV, the party system (V_2, V_3) must exhibit higher volatility.

ZV, guarantees that the lower bound of electoral volatility function must be 0. Furthermore, this value is achieved only when the initial and final vote share vectors are identical. Another implication of this axiom is that the underlying measure will never exhibit degree of volatility which is negative. MV ensures that the maximum level of volatility is 1.

AS ensures that the contribution of political parties are additive separable. This also implies that except for the zero sum game set up there is no interdependence between political parties. Also note that another inbuilt assumption hidden in this axiom is Party anonymity. The fact that the same function $f()$ attached to all political parties also implies that except for the vote shares nothing else will matter in the electoral volatility ordering.

In order to illustrate PI, consider four sets of party systems such that difference in electoral volatility form the first two sets is equal to that of third and fourth. Now suppose that each political party is replicated k times, such that their vote shares (both for initial and final election) is also is divided by k. Following this axiom if electoral volatility difference for the first two party systems is same as that of third and fourth, then replicated profiles will also exhibit similar patterns. In the next section we use a stronger version of this axiom, that enables us to compare volatility of two party systems with different number of parties.

VTS is the main axiom in our analysis. VTS ensures the necessity of departing from a linear to a strictly concave function. This axiom is applicable in the context of comparison between two independent party systems, with same set of political parties. In order to illustrate VTS, consider the following two party systems: $\mathfrak{P}_1 = (V_0, V_1) \in \mathfrak{T}^4$ and $\mathfrak{P}_2 = (V_0, V_2) \in \mathfrak{T}^4$, where $V_0 = (v_{01}, v_{02}, v_{03}, v_{04})$, $V_1 = (v_{01}, v_{02} - \epsilon, v_{03} + \epsilon, v_{04})$ and $V_2 = (v_{01}, v_{02} - \epsilon, v_{03} + \epsilon_3, v_{04} + \epsilon_4)$, where $\epsilon = \epsilon_3 + \epsilon_4$. In $\mathfrak{P}_1$, party 2 losses vote share ϵ and that goes to party 3. In $\mathfrak{P}_2$, party 2 losses vote share ϵ which is divided in between party 3 and 4. Following VTS, volatility in $\mathfrak{P}_1$ is less than $\mathfrak{P}_2$. Our main logic here is that in $\mathfrak{P}_1$ the vote share changes effects only 3 political parties, whereas 4 in the second case. Thus though the volume of transfer is same, $\mathfrak{P}_2$ must exhibit higher volatility than $\mathfrak{P}_1$. Some redistribution might be such that it changes the number of parties in a party system. For an illustration consider the following three vote share vectors: A_0=(0.3,0.2,0.5,0,0), A_1=(0.3,0.2,0.3,0.2,0) and A_3=(0.3,0.2,0.5,0.1,0.1). Any measure that satisfies VTS will exhibit higher volatility in the party system (A_0, A_2) compared to that of (A_0, A_1). Here the number of parties with non-zero votes in at least one election is 1 in the party system is (A_0, A_1) and (A_0, A_2), is 1 and 2, respectively. Following Pedearson's hypothesis (Hypothesis 2, pp) this must lead to an increase volatility. However, Pedearson's original measure violates this axiom. Thus in this context following equation 1: $EVM(A_0, A_1) = EVM(A_0, A_2) = 0.2$.

Another problem of Pedearson's measure is non-unique boundary points. To cite an example consider: B_0=(1,0,0,0), B_1=(0,1,0,0), B_2=(0,0.3,0.4,0.3) etc. It can be observed that $EVM(B_0, B_1) = EVM(B_0, B_2) = 1$. Both the party systems attains maximum volatility. VTS also ensures a unique upper bound, follows from the well known Jensen's inequality. We address this issue in Lemma 1 and Corollary 1.

Lemma 1. *For all* $V_0, V_1 \in \mathbb{V}^n$*, and* $\forall \alpha \in (0, 1)$ $\sum_{i=1}^{n} |v_{0i} - v_{1i}|^\alpha \leq 2^\alpha n^{1-\alpha}$.

Proof: Define $d_i = |v_{0i} - v_{1i}|$. Now $\sum_{i=1}^{n} d_i \leq \sum_{i=1}^{n} v_{0i} + \sum_{i=1}^{n} v_{1i} = 2$. Given $\alpha \in (0,1) \implies d_i^{\alpha}$ is a strictly concave function. Hence by Jensen's inequality:

$$\frac{\sum_{i=1}^{n} d_i^{\alpha}}{n} \leq \left(\frac{\sum_{i=1}^{n} d_i}{n}\right)^{\alpha} \tag{3}$$

Since $\sum_{i=1}^{n} d_i \leq 2 \implies \sum_{i=1}^{n} d_i^{\alpha} \leq 2^{\alpha} n^{1-\alpha}$. **Q.E.D.**

The next result is on characterizing a unique point at which GEV attains the maximum value. The volatility becomes maximum if and only if at the initial time point the number of parties is even and half of the parties enjoys equally of the total votes. On the other hand, in the next time point the scenario is exactly the reverse. For example, if $n = 8$ then the maximum volatility is attained whenever V_0=(.25, .25, .25, .25, 0,0,0,0) and V_1=(0,0,0,0,.25, .25, .25, .25). Formally:

Corollary 1. *For any $V_0, V_1 \in \mathbb{V}^n$, $GEV(V_0, V_1) = 1$ (defined in 2) if and only if the following conditions is satisfied: 1) n is an even 2) $V_t = (0_{\bar{n}}, a_{\bar{n}})$, $V_s = (a_{\bar{n}}, 0_{\bar{n}})$ where $0_n = (0, 0, .., 0)_{\bar{n}\times 1}$ $a_n = (a, a, .., a)_{\bar{n}\times 1}$, a=n/2 and $t, s \in \{0, 1\}$ such that $t \neq s$.*

Proof: $GEV(V_0, V_1) = 1 \iff \frac{\sum_{i=1}^{n} d_i^{\alpha}}{n} = \left(\frac{\sum_{i=1}^{n} d_i}{n}\right)^{\alpha}$. Since $\alpha \in (0, 1)$ like the case of any strictly concave function this equality holds if and only if $d_i = \bar{d} = 2/n \ \forall i \in \mathbb{P}$. Now if n is odd then $d_i = \bar{d}$ can not hold because at least one of the following two necessary conditions is violated: a) $\sum_{i=1}^{n} v_{0i} = 1$ and b) $\sum_{i=1}^{n} v_{1i} = 1$. Suppose n is even but there exists any $(V_3, V_4) \in \mathfrak{T}^n$, such that (V_3, V_4) is different from (V_0, V_1) and $GEV(V_3, V_4) = 1$. This implies that $\forall i \in \mathbb{P}$, $\hat{d}_{3i} = |v_{3i} - v_{4i}| = 2/n$. Since the ordered pair (V_3, V_4) and (V_0, V_1) are different this implies there $\exists k \in \mathbb{P}$ such that $V_{3k} = 2/n + \epsilon$ where $\epsilon > 0$. Given that $\sum_{i=1}^{n} v_{3i} = 1$ this implies that there must exist some $l \in \mathbb{P}$ $(l \neq k)$ such that $V_{3l} = 2/n - \epsilon_1$ where $\epsilon_1 \in (0, \epsilon]$. Since $\hat{d}_{3l} = 2/n \implies V_{4l} = -\epsilon_1$ which is a contradiction. **Q.E.D.**

We now present the main result of this paper. We show that the axioms proposed above (axioms 1-6) restrict only one class of Generalized Electoral Volatility Measures. Formally:

Theorem 1. *For all $V_0, V_1 \in \mathfrak{T}^n$, the only volatility measure in $\mathfrak{E}$ that satisfies axioms IV, ZV, MV, AS, PI and VTS is $GEV(V_0, V_1)$ defined in equation 2.*

Proof: Sufficiency: This part is easy to check and is thus omitted.

Necessary: Following axiom AS: $EV(V_0,V_1)=\sum_{i=1}^{n} f(|v_{1i}-v_{0i}|)$. Following ZV, the lower bound is obtained only when $V_0=V_1$ which implies $f(0)=0$. In order to apply PI, consider $(V_0,V_1),(V_2,V_3),\ (V_4,V_5),(V_6,V_7)\in\mathfrak{T}^n$ and k fold replicated vote share vectors (following Definition 3) as $(V_0^k,V_1^k),\ (V_2^k,V_3^k),\ (V_4^k,V_5^k),(V_6^k,V_7^k)\in\mathfrak{T}^{nk}$, respectively. Following this axiom $EV(V_0,V_1)-EV(V_2,V_3)=EV(V_4,V_5)-EV(V_6,V_7)\iff EV(V_0^k,V_1^k)-EV(V_2^k,V_3^k)=EV(V_4^k,V_5^k)-EV(V_6^k,V_7^k)$. Applying some algebra the additive separable functional form turns out to be: $\sum_{i=1}^{n} f(|v_{0i}-v_{1i}|)-\sum_{i=1}^{n} f(|v_{2i}-v_{3i}|)=\sum_{i=1}^{n} f(|v_{4i}-v_{5i}|)-\sum_{i=1}^{n} f(|v_{6i}-v_{7i}|)\iff f(\gamma|v_{0i}-v_{1i}|)-\sum_{i=1}^{n} f(\gamma|v_{2i}-v_{3i}|)=\sum_{i=1}^{n} f(\gamma|v_{4i}-v_{5i}|)-\sum_{i=1}^{n} f(\gamma|v_{6i}-v_{7i}|)$ where $\gamma=1/k$. This also implies that f() is either in interval scale or in log-interval scale (see Aczél and Roberts, 1989). Hence $f(.)$ satisfies PI if and only if for any $t\in[0,1]: f(t)=\theta_1+\theta_2 d_i^{\alpha}$ or $f(t)=\kappa_1+\kappa_2 log(d_i)$ (Aczél and Roberts, 1989; Chakraborty et al., 2008, see). Given $f(0)=0$, thus logarithmic functional form is ruled out. Hence $f(t)=\theta_1+\theta_2 t^{\alpha}$. Now $f(0)=0\implies\theta_1=0$. By ZV, IV and MV, it can be ensured that $\theta_2>0$. Further, by IV $f(t)$ must be an increasing function. Since, $f(t)=\theta_2 t^{\alpha}$ is differentiable this implies that $\theta_2\alpha t^{\alpha-1}>0\implies\alpha>0$. In order to apply TS, consider the following two party systems $(R_0,R_1),(R_0,R_2)\in\mathfrak{T}^3$ where $R_0=(r_{01},r_{02},r_{03}),R_1=(r_{01}-2\epsilon,r_{02}+2\epsilon,r_{03}),\ R_2=(r_{01}-2\epsilon,r_{02}+\epsilon,r_{03}+\epsilon)$. Following VTS: $EV(R_0,R_2)>EV(R_0,R_1)\implies 2\theta_2\epsilon^{\alpha}>\theta_2(2\epsilon)^{\alpha}\implies\alpha<1$. Hence, the functional form turns out to be: $EV(V_0,V_1)=\theta_2\sum_{i=1}^{n} d_i^{\alpha}$, where $d_i=|v_{1i}-v_{0i}|$. By MV, $\theta_2\sum_{i=1}^{n} d_i^{\alpha}\le 1$. Following Lemma 1, $max\left(\sum_{i=1}^{n} d_i^{\alpha}\right)=2^{\alpha}n^{1-\alpha}$. This also implies that MV is satisfied whenever: $\theta_2=\frac{1}{2^{\alpha}n^{1-\alpha}}$. Hence, $EV=\theta_2\sum_{i=1}^{n} d_i^{\alpha}=\frac{\sum_{i=1}^{n} d_i^{\alpha}}{2^{\alpha}n^{1-\alpha}}$ **Q.E.D.**

We now show that all the axioms discussed above (i.e., axioms 1-6) are consistent with one another. This also implies that they are not contradictory, and sufficient to characterize exactly one measure. Formally we establish the independence of the axioms in the following fashion:

Theorem 2. *The following axioms IV, ZV, MV, AS, PI and VTS are independent.*

Proof: We prove this theorem by providing counter examples. We show that there exists one measure which satisfies all but not one.

1) **Not IV**: $EV(V_0.V_1)=\frac{\sum_{i=1}^{n} 1(d_i\neq 0)}{n}$, where $1(d_i\neq 0)$ is an indicator function which takes value 1 if the statement in the parenthesis is true and 0 otherwise.

2) **Not ZV**: $EV(V_0, V_1) = \frac{\sum_{i=1}^{n} 10+|d_i|^{\sqrt{2}}}{10+\sqrt{2n}}$.

3) **Not MV**: $EV(V_0, V_1) = \sum_{i=1}^{n} |d_i|^{\sqrt{2}}$.

4) **Not AS**: $EV(V_0, V_1) = \left(\frac{\sum_{i=1}^{n} |v_{0i}-v_{1i}|^{\sqrt{2}}}{\sqrt{2n}}\right)^2$.

5) **Not PI**: $EV(V_0, V_1) = \frac{e^{\sum_{i=1}^{n} |d_i|^{\sqrt{2}}}-e}{e^{\sqrt{2n}}-e}$

6) **Not VTS**: $EV(V_0, V_1) = \frac{\sum_{i=1}^{n} d_i}{2}$

Q.E.D.

The above theorem also justifies that none of the axioms is a strong requirement. Also note following condition 7 it is quite straightforward to figure out that Pederason's Volatility measure satisfies all the axioms except VTS.

So far we have restricted our attention in the comparison of electoral volatility with party systems consisting of equal number of parties. We now relax this assumption to compare party systems with different party numbers. In order to do this we propose the following axiom:

Axiom 7. ***Strong Proportional Increments (SPI)***: *$(V_0, V_1) \in \mathfrak{T}^n$ and (V_0^k, V_1^k) is the k fold replicated vote share vectors (following Definition 3). Following this axiom $EV(V_0, V_1) = EV(V_0^k, V_0^k)$.*[6]

It can be easily figured out that GEV defined in 2 satisfies this axiom for any finite α. In order to illustrate how SPI can be applied in comparing party systems with different party sizes consider: $\mathfrak{P}_1 = (V_0, V_1) \in \mathfrak{T}^{n_1}$ and $\mathfrak{P}_2 = (V_2, V_3) \in \mathfrak{T}^{n_2}$, such that $n_1 \neq n_2$. Now apply 3 to construct the following replicated party systems: $\mathfrak{P}_3 = (V_0^{n_2}, V_1^{n_2}) \in \mathfrak{T}^{n_1 n_2}$ and $\mathfrak{P}_4 = (V_2^{n_1}, V_3^{n_1}) \in \mathfrak{T}^{n_1 n_2}$. That is we have replicated all parties in $\mathfrak{P}_1$ and $\mathfrak{P}_2$, n_2 and n_1 times, in order to form $\mathfrak{P}_3$ and $\mathfrak{P}_4$, respectively. Both the constructed party systems (i.e., $\mathfrak{P}_3$ and $\mathfrak{P}_4$) have number of parties equals to $n_1 n_2$. Following axiom SPI, $EV(\mathfrak{P}_3) = EV(\mathfrak{P}_1)$ and $EV(\mathfrak{P}_2) = EV(\mathfrak{P}_4)$. Thus $EV(\mathfrak{P}_3) \lesseqqgtr EV(\mathfrak{P}_4) \iff EV(\mathfrak{P}_1) \lesseqqgtr EV(\mathfrak{P}_2)$.[7]

[6] Throughout this paper we are interested only to figure out, how the absolute strength of political parties are distributed in different party systems. In a different context the well known Lorenz curve also does a kind of similar exercise. It can be readily observed that in this context the Lorenz curve of the original and replicated vote share profiles are same. This perhaps justifies the property that since the two distributions are same, then their must not be any difference in their volatility.

[7] The philosophy of this approach is closely related to the replication invariance axiom introduced in Dasgupta et al. (1973). This axiom is well known in the axiomatic social choice and welfare literature and is often used for welfare ordering of two distributions with different population size.

Hence, the ordering of the original party systems actually follows from the constructed replicated party systems.

4 Quasi Approaches

Conclusions of volatility ordering following the GEV() measure, may depend on the choice of the aversion parameter α. To cite an example, consider the following two party systems $(V_0, V_1), (V_2, V_3) \in \mathfrak{T}^4$, where V_0=(.2, .5 .15, .15) V_1=(.6,.1,.2,.1) V_2=.2,.6,.1,.1) and V_3 =(0.1,0.1,0.7,0.1). It can be checked that $GEV(V_0, V_1, \alpha = 0.1) = 0.886 > GEV(V_1, V_2, \alpha = 0.1) = 0.717.$[8] Furthermore, $GEV(V_0, V_1, \alpha = 0.9) = 0.472 < GEV(V_1, V_2, \alpha = 0.9) = 0.603$. Hence, the volatility ordering are completely different in the two cases. In this section, we focus on quasi orderings of electoral volatility. To be precise, we are interested in the of electoral volatility ordering following a class of electoral volatility measures. The end result is binary i.e., either $\geq$ or $<$. A quasi approach satisfies transitivity and reflexivity but violates completeness. This implies that in some cases volatility orderings may be inconclusive. This also implies that in some cases the electoral volatility may be higher for one party system following a certain volatility measure.

For any two party systems $(V_0, V_1), (V_2, V_3) \in \mathfrak{T}^n$ define $D_{01} = (d_{011}, d_{012}, .., d_{01n})$, $D_{23} = (d_{231}, d_{232}, .., d_{23n})$, where $\forall i \in \mathbb{P}$ $d_{01i} = |v_{0i} - v_{1i}|$ and $d_{23i} = |v_{2i} - v_{3i}|$. We define $D^*_{01} = \Pi_n D_{01} = (d^*_{011}, d^*_{012}, .., d^*_{01n})$ and $D^*_{23} = \hat{\Pi}_n D_{23} = (d^*_{231}, d^*_{232}, .., d^*_{23n})$, where Π_n and $\hat{\Pi}_n$ are permutation matrices of order n such that $d^*_{01i} > d^*_{01j}$ and $d^*_{23i} > d^*_{23j}$ for any distinct $\{i, j\} \in \mathbb{P}$ such that $i \neq j$.[9]

Definition 6. ***First Order Dominance:*** *D^*_{01} first order dominates D^*_{23} which is denoted by $D^*_{01} \succ_1 D^*_{23}$ if and only if $\forall i \in \mathbb{P}$: $d^*_{01i} \geq d^*_{23i}$, with $>$ in at least one case.*

Definition 7. ***Second Order Dominance:*** *D^*_{01} first order dominates D^*_{23} which is denoted by $D^*_{01} \succ_2 D^*_{23}$ if and only if $\forall i \in \mathbb{P}$: $\sum_{k=1}^{i} d^*_{01k} \geq \sum_{k=1}^{i} d^*_{23k}$, with $>$ in at least one case.*

Note that both $\succ_1$ and $\succ_2$ are partial in nature. That is it violates the completeness property in the sense some party system can not be ordered. An example includes the

[8] By $GEV(V_1, V_2, \alpha = 0.1)$ we mean the generalized electoral volatility measure that is defined in 2, for the party system (V_0, V_1) and $\alpha = 0.1$.

[9] A permutation matrix of order n is a square matrix with all its entires being either 0 or 1, such that sum of each row and column is 1.

two party systems presented in the first paragraph of this section.

We first establish equivalence between first order dominance and electoral volatility ordering following a class of measures which satisfies IV and AS.

Theorem 3. *For any two party systems $X = (V_0, V_1), Y = (V_2, V_3) \in \mathfrak{T}^n$, the following conditions are equivalent:*

1) $D^*_{01} \succ_1 D^*_{23}$

2) For any $EV \in \mathfrak{E}$ that satisfies AS and IV we have $EV(X) > EV(Y)$.

Proof: $1 \implies 2$

It is quite straightforward to figure out that there exists $a \in \mathbb{R}^n$, where $a = (|a_1|, |a_2|, .., |a_n|)$, such that $D^*_{01} - D^*_{23} = a$. By IV, EV() is increasing in its arguments. Hence $EV(X) > EV(Y)$.

$2 \implies 1$

Assume that $D^*_{01} \not\succ_1 D^*_{23}$, then $\exists j \in \mathbb{P}$ such that $d_{01j} < d_{23j}$. Choose the following function $EV = \sum_{i=1}^{n} d_{01i} w_i$. We can always choose w_j high enough in order to get a contradiction. **Q.E.D.**

We now establish the equivalence between $\succ_2$ and electoral volatility measures that satisfies IV, AS and VTS.

Theorem 4. *For any two party systems $X = (V_0, V_1), Y = (V_2, V_3) \in \mathfrak{T}^n$, the following conditions are equivalent:*

1) $D^{X^*} \succ_2 D^{Y^*}$

2) D^{X^*} can be obtained from D^{Y^*} by a finite sequence of V-transfer and VT-transfer.

3) For any $EV \in \mathfrak{E}$ that satisfies AS, IV and VTS we have $EV(X) > EV(Y)$.

Proof:

$1 \implies 2$

Following Marshall and Olkin (1979) $D^{X^*} \succ_2 D^{Y^*} \implies$ there exits a biostatistic matrix Q, such that $D^{X^*} \geq QD^{Y^*}$.[10] The rest of the proof is in line with Chakravarty and Zoli (2012) following which intermediate V-sequences and VT-sequences can be constructed.

[10] A bi-stochastic matrix is not a permutation matrix such that all the entries of the matrix lies in the closed set $[0, 1]$ and sum of all columns and rows is 1.

$2 \implies 3$

Applying transitivity we can establish this part.

$3 \implies 1$

For all $i \in \mathbb{P}$ define $CS(d^*_{01i}) = \sum_{j=1}^{i} d^*_{01j}$ and $CS(d^*_{23i}) = \sum_{j=1}^{i} d^*_{23j}$.

We begin with the assumption that $D^{X*} \not\succ_2 D^{Y*}$. Hence there exists $k \in \mathbb{P}$ such that $CS(d^*_{01k}) < CS(d^*_{23k})$. In order to get a contradiction we choose $F \in \mathfrak{E}$ such that $F(D^{X^*}) = \sum_{i=1}^{n} d^*_{01i}\beta_i$ and similarly $F(D^{Y^*}) = \sum_{i=1}^{n} d^*_{23i}\beta_i$ where $\beta_j = \sum_{j=1}^{i} w_i$ and $w_i > 0$. It can be readily figured out that $F(\cdot)$ is increasing and strictly concave, hence satisfies both IS and VTS. Applying some algebra we can write $F(D^{X^*}) = \sum_{i=1}^{n} w_i CS(d^*_{01i})$ and $F(D^{Y^*}) = \sum_{i=1}^{n} w_i CS(d^*_{23i})$. It is quite straightforward to figure out that we can choose w_k high enough in order to get a contradiction. **Q.E.D.**

The next two results extends the partial orderings in the context of party systems, with different number of parties.

Corollary 2. *For any two party systems $X = (V_0, V_1)\mathfrak{T}^{n_1}, Y = (V_2, V_3) \in \mathfrak{T}^{n_2}$, such that $n_1 \neq n_2$ the following conditions are equivalent:*

1) $D^{X^{n_2^*}} \succ_1 D^{Y^{n_1^*}}$

2) For any $EV \in \mathfrak{E}$ that satisfies AS , SPI and IV we have $EV(X) > EV(Y)$.

Proof: This proof is a trivial extension of Theorem 3.

Corollary 3. *For any two party systems $X = (V_0, V_1)\mathfrak{T}^{n_1}, Y = (V_2, V_3) \in \mathfrak{T}^{n_2}$, such that $n_1 \neq n_2$ the following conditions are equivalent:*

1) $D^{X^{n_2^*}} \succ_2 D^{Y^{n_1^*}}$

2) For any $EV \in \mathfrak{E}$ that satisfies AS, IV, SPI and VTS we have $EV(X) > EV(Y)$.

Proof: This proof is a trivial extension of Theorem 4.

5 Conclusion

The linear electoral volatility measure introduced in an influential article by Pedersen (1979) is not always responsive to certain changes of party system like changes in number of parties, and/or relative strength of existing parties. This is against the hypothesis cited in the same article by Pedersen (1979). In this paper we address this issue by introducing a class of strictly concave additive separable electoral volatility measures. We axiomatically characterize this class following a set of independent axioms.

So far we surveyed, this literature is also mute on the theoretical issues associated with the comparison of party systems with different party sizes. Nevertheless, empirical papers on electoral volatility even in the recent decades Bischoff (2013) do compare electoral volatility with different number of parties. We introduce an axiom very much similar to the well known replication invariance in the context of economic inequality literature. We show that the proposed class of measure including the Pedersen (1979) measure which is a special class in our class of index (relaxing strict concavity) satisfies this axiom, and thus can be compared in the context of party systems with different cardinalities.

As a future research agenda, we plan to apply our electoral volatility measure in the context of comparative party system analysis of India.

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