Unraveling of value-rankings in auctions with resale

Sanyyam Khurana*

November 17, 2019

Abstract

In this paper, we consider auctions with resale opportunities when the value-rankings are revealed to the bidders. There are two risk neutral bidders and the probability distributions are independently and asymmetrically distributed. The bidder with the highest valuation and lowest valuation is termed as "high-type" and "low-type" respectively. We show that, with the revelation of value-rankings, the classic result of "bid symmetrization" does not hold. Surprisingly, the low-type bidder produces a stronger bid-distribution than the high-type bidder. We also show that the revelation of value-rankings in auctions with resale *asymmetrizes* the bidding strategies. Next, we show that the high-type bidder bids more aggressively when the distribution function of the low-type bidder improves. Finally, we compare the bidders' preferences for a first-price and a second-price auction.

JEL classification: D44, D82

Keywords: First-price auction, resale, asymmetry, private value auctions, value-rankings

1 Introduction

There are various kinds of asymmetry that arises in auctions such as heterogeneous value-distributions, heterogeneous utility functions, asymmetric information sets and so on. In this paper, we consider a different kind of asymmetry in the form of revelation of value-rankings[1]. This means that, in addition to knowing their own valuations, all the bidders know the ordinal rankings of the valuations. Revelation of value-rankings can be naturally seen at various places in auctions. The knowledge about the ordinal value-rankings may be due to interactions in the past, knowledge about the financial wealth of the bidders, etc. This kind of asymmetry leads to the inefficiency of a first-price auction, i.e., the low valuation bidder wins the auction with positive probability. This creates opportunity for resale of the object. Resale

*Department of Economics, Delhi School of Economics, University of Delhi, Delhi-110007, India. Email: sanyyam@econdse.org

[1]This kind of asymmetry was first considered by Landsberger et al. [14].

opportunities affect the bidding strategies, bid distributions, seller's revenue and bidders' preferences. In this paper, we study the bidding behavior in a first-price auction when there are resale opportunities and the value-rankings are revealed to the bidders.

Consider a two-stage game with two risk neutral bidders. The valuations are drawn from independently and asymmetrically distributed probability functions. Additionally, value-rankings are revealed to both the bidders. The bidder with the highest valuation and lowest valuation is termed as *high-type* and *low-type* respectively. The structure of the game is as follows. In stage 1, the seller conducts a first-price auction. The losing bid is not revealed after the end of the auction. In stage 2, the winner of the auction *may* make a single offer to the loser of the auction. Resale takes place in the form of a *monopoly mechanism*, i.e., the winner of the auction makes a single offer to the loser. In Appendix A, we consider the case of a *monopsony mechanism* where the loser of the auction makes a single offer to the winner.

The aim of this paper is to determine the bidding behavior when there are resale opportunities and the value-rankings are revealed to the bidders. At first glance, one may expect that the high-type bidder produces a stronger bid distribution than the low-type bidder. We, however, show that this intuition is incorrect.

It is well known that, when there are no resale opportunities, the "weak" bidder produces a *weaker* bid distribution than the "strong" bidder[2] ([18]). The intuition behind this result is that the "marginal profit" of the strong bidder is more than that of the weak bidder. This is because of the fact that the weak bidder bids more aggressively than the strong bidder. Since the marginal profit of the strong and the weak bidder is equal to the inverse of reverse hazard rate of bids for the weak and the strong bidder respectively, the weak bidder produces a weaker bid distribution than the strong bidder.

When there are resale opportunities, both the strong and the weak bidder produce the same bid distribution ([8]). This fact is known as *bid symmetrization*. The intuition is that, unlike the no resale case, the marginal profits are same for both the bidders. To see this, notice that since the weak bidder bids more aggressively than the strong bidder, the weak bidder makes a single offer with positive probability and the strong bidder never makes a resale offer. If the weak bidder wins the auction by increasing his bid marginally, then he will make a resale offer to the strong bidder which will be accepted with certainty. Thus, his marginal profit is the resale price net of the bid. On the other hand, the strong bidder loses the auction by a marginal amount and therefore buys the object in the resale stage with certainty. Thus, his marginal profit is the resale price net of the bid. Since the marginal profits are same for both the bidders and they are equal to the reverse hazard rate of bids, the bid symmetrization holds.

[2]A bidder is "weak" and "strong" in the sense that the latter is more likely to get a high valuation than the former.

When there is one strong bidder and a finite number of weak bidders, the bid symmetrization fails to hold ([23]). Specifically, each of the weak bidder produces a *weaker* bid distribution than the strong bidder. The intuition is as follows. If the weak bidder wins the auction against the strong bidder by increasing his bid marginally, then his resale offer is accepted by the strong bidder with certainty. If the weak bidder wins the auction against another weak bidder by increasing his bid marginally, then his resale offer is accepted with positive probability but not with certainty. Thus, his marginal profit is strictly less than the resale price net of bid. On the other hand, the strong bidder loses the auction by a marginal amount and therefore buys the object in the resale stage with certainty. Thus, his marginal profit is the resale price net of the bid. Since the marginal profit of the weak bidder is less than that of the strong bidder and the marginal profit of the weak and the strong bidder is equal to the inverse of reverse hazard rate of the bids for the strong and the weak bidder respectively, the weak bidder produces a weaker bid distribution than the strong bidder.

Our main result is that bid symmetrization fails to hold even with two risk neutral bidders if some extra information is revealed in the form of value-rankings. Specifically, we get the striking result that the low-type bidder produces a *stronger* bid distribution than the high-type bidder. The intuition is as follows. Whenever the low-type bidder wins the auction by increasing his bid marginally, the high-type bidder accepts his offer with probability one. On the contrary, whenever the high-type bidder loses the auction by a marginal amount, he gets the object in the resale with probability one. Therefore, the marginal profit is same for both the bidders. However, the marginal profit is not equal to the inverse of reverse hazard rate of the bids for both the bidders. The marginal profit is equal to the inverse of reverse hazard rate of the bids for the low-type bidder but less than the inverse of reverse hazard rate of the bids for the high-type bidder. Thus, the bid distribution of the low-type bidder dominates that of the high-type bidder.

1.1 Contribution

Consider the following assumptions: (a) symmetric bidders, (b) private values, (c) no revelation of value-rankings, (d) risk neutrality, and (e) no resale of the object. When assumptions (a)-(e) are satisfied, then (i) revenue equivalence theorem holds, and (ii) an explicit expression for the bidding strategy in a first-price auction exists (Myerson [20]). When assumption (a) is relaxed, then (i) the "weak" bidder bids more aggressively and produces a weaker bid distribution than the "strong" bidder, and (ii) the revenue rankings for the first-price and the second-price auctions cannot be generalized (Maskin and Riley [18]). When assumptions (a) and (e) are relaxed, then (i) the weak bidder bids more aggressively than the strong bidder and there is *bid symmetrization*, and (ii) the first-price auction is revenue superior to a second-price auction (Hafalir and Krishna [8]). When assumption (c) is relaxed, then (i) revenue equivalence theorem does not hold, (ii) analytical solutions do not exists for uniform distributions. In this paper, we relax assumptions (a), (c) and (e).

In our environment, the classic result of bid symmetrization does not hold even with two risk neutral bidders. This is due to the revelation of value-rankings.

The following are the already existing results in the literature:
(1) When there are two risk neutral bidders, value-rankings are not revealed and no resale opportunities, then the weak bidder bids more aggressively and produces a weaker bid distribution than the strong bidder. Moreover, stochastic orders are necessary to unambiguously compare the bidding strategy and bid distribution ([16],[18], [11]).
(2) When there are two risk neutral bidders, the value-rankings are revealed and no resale opportunities, then the low-type bidder bids more aggressively than the high-type bidder. Moreover, stochastic orders are *not* necessary to unambiguously rank the bidding strategy. But, the bid distributions cannot *always* be compared even if we assume stochastic orders on the value-distribution ([14]).
(3) Consider an auction setting with resale opportunities and two risk neutral bidders. The value-rankings are not revealed. Then, the weak bidder bids more aggressively than the strong bidder and both the bidders produce the same bid distribution. Moreover, stochastic orders are necessary to compare the bidding strategy but are *not* necessary to compare the bid distributions. It may be noted that heterogeneous distribution functions lead to homogeneous bid distributions ([8]).
(4) Consider an auction setting with resale opportunities and $n > 2$ bidders (one strong and $n - 1$ weak bidders). The value-rankings are not revealed. Then, each of the weak bidder bids more aggressively and produces a weaker bid distribution than the strong bidder. Moreover, stochastic orders are necessary to compare the bidding strategy and bid distribution ([23]).

Our result conveys that the low-type bidder bids more aggressively and produces a *stronger* bid distribution than the high-type bidder. The main observation is that stochastic orders are *not* necessary to compare the bidding strategy and bid distribution. Notice that our result holds for homogeneous distribution functions as well. It conveys the fact that homogeneous distribution functions lead to heterogeneous bid distributions. In contrast to the literature, on one hand, with the introduction of resale opportunities, we can compare the bid distribution even without any stochastic order on the value-distribution. However, on the other hand, with the revelation of value-rankings, the bid symmetrization result no longer holds. Notice that when the value-rankings are revealed and there is no resale, the bid distributions cannot be compared. Our result states that, with the introduction of resale opportunities, we can compare the bid distributions.

Further, our environment allows us to analyze some results which are new in the auction with resale literature. First, we compare the bidding behavior when the distribution function of the low-type bidder changes stochastically. We show that, when the distribution function of the low-type bidder "improves" stochastically, then the high-type bidder starts behaving more aggressively. Moreover, when the valuation of the low-type bidder is high enough, then he produces a stronger bid distribution. Second, we compare the bidding behavior of "symmetric" and "asymmetric" bidders. Bidders are symmetric when the valuations are not correlated and are drawn from the same distribution function. We show that the low-type bidder whose valuation is correlated with the other bidder bids more aggressively than the low-type bidder whose valuation is not correlated with the other bidder.

1.2 The literature

The literature provides some strong results when bidders are symmetric such as revenue equivalance theorem, efficiency and explicit expression of bidding strategy. These results do not hold when bidders are asymmetric. More specifically, revenue rankings cannot be generalized, first-price auction is inefficient and analytical solution does not always exists. Inefficiency opens up the possibility of resale. Since the winner of the auction can be the bidder who does not have the highest valuation, it is profitable for him to resell the object. The literature on *auctions with resale* is relatively small. Some of the papers are Gupta and Lebrun [7]; Hafalir and Krishna [8, 9]; Lebrun [17]; Cheng and Tan [5]; and Virág [23].

Revelation of bids after the bidding stage plays a key role in determining the equilibrium bidding behavior. Whenever both the winning bid and the losing bid is revealed after the auction, the resale stage becomes game of complete information. The winner of the auction simply makes the resale offer equal to the valuation of the loser which will be accepted with certainty. Thus, efficiency is always attained in this scenario. With complete information in the resale stage, Gupta and Lebrun [7] derives an explicit expression of the bidding strategies for the asymmetric bidders in a first-price auction. However, the revelation of both bids is too strong assumption since in real world auctions the losing bid is not revealed. Lebrun [17] studies the effect of revelation of bids after the bidding stage on equilibrium behavior. He constructs a behavioral equilibrium when the bids are not revealed and shows that this equilibrium is equivalent to a separating equilibrium where the bids are revealed. Garratt and Tröger [6] characterizes the equilibrium by considering one speculator—a bidder who has zero value for the object—and one private valuation bidder. In their analysis, the losing bid is not announced in a first-price auction and the winning bid is not announced in a second-price auction[3]. Cheng and Tan [5] shows that a common value auction without

[3]In a first-price auction whenever the losing bid is not announced, it does not make any

resale is bid equivalent to a first-price auction with resale. He generalizes the revenue rankings established by H-K. Cheng [3] considers sequential bargaining offers by the seller. He shows that seller's revenue reduces in a first-price auction when the bargaining power of the weak bidder is reduced.

Maskin and Riley [18] and Landsberger et al. [14] studies asymmetric auctions without resale. The crucial difference between these two papers is that the value-rankings is not common knowledge in the former and it is common knowledge in the latter. The asymmetry in the former is due to heterogeneous distribution functions which are stochastically ranked; and the asymmetry in the latter is due to the revelation of value-rankings. The bidders in the former paper are distinguished as *strong* and *weak*; and in the latter as *high-type* and *low-type*. The former shows that the weak bidder bids more aggressively and produces a weaker bid distribution than the strong bidder. Moreover, the revenue rankings for the first-price and the second-price auction cannot be generalized. The latter shows that the low-type bidder bids more aggressively than the high-type bidder. They also show that, with uniform distributions, a first-price auction is revenue superior to a second-price auction. Hafalir and Krishna [8] (henceforth, H-K) studies asymmetric auctions with resale opportunities. They show that the weak bidder bids more aggressively than the strong bidder and there is bid symmetrization, i.e., the equilibrium winning probability is same for both the bidders. They also show that a first-price auction is revenue superior to a second-price auction. Virág [23] extends the work of H-K by considering $n > 2$ bidders. He shows that the weak bidder produces a weaker bid distribution than the strong bidder. He also shows that the weak bidder produces a stronger bid distribution with resale opportunities as compared to the case when there are no resale opportunities. Virág [24] considers H-K model by taking a reserve price. He shows that, with the introduction of reserve price, the strong bidder wins the auction more often than the weak bidder. Moreover, the second-price auction may yield higher revenue for the seller than the first-price auction.

Milgrom and Weber [19] considers a common-value auction with interdependent and correlated valuations. They derived explicit expressions for bidding strategies in a first-price and a second-price auction. They also show that a second-price auction is weakly revenue superior to a first-price auction. Krishna and Morgan [13] considers an all-pay auction with interdependent valuations and affiliated signals. They derived explicit expressions of bidding strategy in a first-price and a second-price all-pay auction[4]. They also show that the seller's revenue from a second-price all-pay auction is at least as large as from a first-price all-pay auction. In contrast, Siegel [21] considers a first-price all-pay auction with interdependent valuations and discrete correlated

difference whether the winning bid is announced or not. This is true only when the winner of the auction makes a single offer to the loser.

[4]In auction literature, a second-price all-pay auctions is generally refereed as a *war of attrition* and a first-price all-pay auction is simply refereed as an *all-pay auction*.

signals. He shows the existence of an equilibrium. Syrgkanis et al. [22] considers a hybrid[5] auction with interdependent valuations and discrete affiliated signals. They show the existence of a unique mixed strategy equilibrium.

1.3 Outline

The outline of the paper is as follows. In section 2, we formalize the model and describe the equilibrium. In section 4, we compare the bidding behavior for different degrees of asymmetry. In section 5, we compare the bidding behavior of symmetric and asymmetric bidders. Section 7 concludes the paper. In Appendix A, we describe the equilibrium for the *monopsony mechanism*. The proofs are collected in Appendix B.

2 The environment

Consider a first-price sealed bid auction for an indivisible object. There are two bidders with risk neutral preferences. The set of bidders is denoted by $N = \{h, l\}$. The type space is same for both the bidders and is given by $T = [0, \bar{a}] \subset \Re$. The random variables are given by $\tilde{t}_h$ and $\tilde{t}_l$. The distribution functions of $\tilde{t}_h$ and $\tilde{t}_l$ are given by F_h and F_l respectively. We assume that the distribution functions are twice continuously differentiable and are independently distributed. Moreover, the density functions, denoted by f_h and f_l, are positive and always bounded away from zero. An important feature of our framework is that the ranking of valuations is common knowledge among the bidders at the time of bidding. This means that each bidder knows that his realized valuation is either more or less than that of the other bidder but does not know the magnitude of the difference between the two valuations. Therefore, a bidder knows his own valuation, the distribution function of the other bidder as well as the ranking of the valuations.

Consider the following two stage game. In stage 1, the seller of the object conducts a first-price sealed bid auction. The losing bid is not revealed after the auction. In stage 2, the winner of the auction may make a single offer to the loser. The game ends after stage 2 and there is no further resale of the object. Bidders get their realized payoffs.

F_i is said to be **regular** if the *virtual valuation*, given by

$$t - \frac{1 - F_i(t)}{f_i(t)}$$

is strictly increasing in t. The definition of regularity of the distribution function is due to Myerson [20].

We make the following assumption on the distribution functions.

[5] In a hybrid auction, the winning bidder pays a bid which is equal to a linear combination of a first-price and a second-price auction.

Assumption 1. *F_i is regular for every $i \in N$.*

We assume that bidding strategies are strictly increasing and continuous. The bidding and inverse bidding strategy are denoted by β_i and ϕ_i respectively. In the following Lemma, we show that if the low-type bidder bids more aggressively than the high-type bidder, then the low-type bidder makes a resale offer with positive probability while the high-type bidder never makes a resale offer.

Lemma 1. *If $\beta_l(t) > \beta_h(t)$ for every $t \in T$, then the low-type bidder makes a resale offer with positive probability whereas the high-type bidder never makes a resale offer.*

To see why the above result is true, first consider the low-type bidder. Since the low-type bidder bids more aggressively than the high-type bidder, the type of the low-type bidder is less than the type required by the high-type bidder to match the bid made by the low-type bidder. Therefore, there are potential gains from trade if the low-type bidder makes a resale offer. Now consider the high-type bidder. Suppose the high-type bidder makes a resale offer. Then, the resale offer has to be greater than the valuation of the high-type bidder. Since the valuation of the low-type bidder is less than that of the high-type bidder, the low-type bidder will never accept the resale offer.

We begin the analysis by assuming that the low-type bidder bids more aggressively than the high-type bidder and later show that this is indeed the case. We require this assumption before setting up the optimization problems of both the bidders because we need to know the direction of the trade.

2.1 Resale Stage (Stage 2)

In stage 2, we set up the optimization problems for the high-type and the low-type bidder. From Lemma 1, the high-type bidder does not make any resale offer. Therefore, there is no optimization problem for the high-type bidder. Consider the low-type bidder with type t_l. Suppose he bids b, chooses a resale price p and the high-type bidder follows his equilibrium bidding strategy β_h. Thus, the optimization problem of the low-type bidder is

$$\max_p \Pr(p < \tilde{t}_h < \phi_h(b) | \tilde{t}_h > t_l)(p - b) + \Pr(\tilde{t}_h < p | \tilde{t}_h > t_l)(t_l - b)$$

The first-term in the optimization problem is the expected utility of the low-type bidder when his offer is accepted, and the second term is the expected utility when his offer is rejected. We can rewrite the first probability expression in the following manner:

$$\begin{aligned}\Pr(p < \tilde{t}_h < \phi_h(b)|\tilde{t}_h > t_l) &= \frac{\Pr(p < \tilde{t}_h < \phi_h(b))}{\Pr(\tilde{t}_h > t_l)} \\ &= \frac{F_h \circ \phi_h(b) - F_h(p)}{1 - F_h(t_l)}\end{aligned}$$

In deriving the above expression we made use the fact that $t_l \leq \phi_h(b)$. To see this, suppose $t_l > \phi_h(b)$. Then, low-type bidder will win the auction with probability 0. Hence, it is profitable to raise his bid. Similarly, we rewrite the second probability expression in the following way:

$$\begin{aligned}\Pr(\tilde{t}_h < p|\tilde{t}_h > t_l) &= \frac{\Pr(t_l < \tilde{t}_h < p)}{\Pr(\tilde{t}_h > t_l)} \\ &= \frac{F_h(p) - F_h(t_l)}{1 - F_h(t_l)}\end{aligned}$$

Thus, the optimization problem of the low-type bidder can be rewritten as

$$\max_p \frac{F_h \circ \phi_h(b) - F_h(p)}{1 - F_h(t_l)}(p - b) + \frac{F_h(p) - F_h(t_l)}{1 - F_h(t_l)}(t_l - b)$$

The first-order condition leads to the following equation

$$t_l = p - \frac{F_h \circ \phi_h(b) - F_h(p)}{f_h(p)} \tag{1}$$

From Assumption 1, the right hand side of the above equation is strictly increasing in the resale price. Hence, a unique p exists. Thus,

$$p(t_l, b) = \arg\max_p \frac{F_h \circ \phi_h(b) - F_h(p)}{1 - F_h(t_l)}(p - b) + \frac{F_h(p) - F_h(t_l)}{1 - F_h(t_l)}(t_l - b)$$

Furthermore, the regularity condition ensures that (1) is a sufficient condition. Notice that

$$t_l < p(t_l, b) < \phi_h(b)$$

Moreover, $p(t_l, b)$ is increasing in t_l and $\phi_h(b)$. Since the high-type bidder does not make a resale offer, we write $p(b) := p(t_l, b)$. We now turn to stage 1 problem.

2.2 Bidding stage (Stage 1)

Consider the low-type bidder with type t_l. Suppose he bids b and the high-type bidder follows his equilibrium bidding strategy. The optimization problem of the low-type bidder is

$$\max_b \frac{F_h \circ \phi_h(b) - F_h(p)}{1 - F_h(t_l)}(p - b) + \frac{F_h(p) - F_h(t_l)}{1 - F_h(t_l)}(t_l - b)$$

Using Envelope Theorem, we have the following first-order condition

$$F_h \circ \phi_h(b) = \mathrm{D}F_h \circ \phi_h(b)(p(b) - b) + F_h \circ \phi_l(b) \tag{2}$$

Notice that

$$\frac{F_h \circ \phi_h(b)}{\mathrm{D}F_h \circ \phi_h(b)} > p(b) - b \tag{3}$$

The numerator of the left hand side is the bid distribution of the high-type bidder and the denominator is its corresponding density. This ratio is the inverse of the reverse hazard rate of the bid. We interpret the reverse hazard rate of the bid as the probability that a bidder will bid around a neighborhood of b conditional on the fact that he will not bid more than b. On the other hand, the right hand side is the "marginal profit" of the low-type bidder. So this inequality states that the inverse of the reverse hazard rate of the bid for the high-type bidder is greater than the marginal profit of the low-type bidder.

Now, consider the high-type bidder with type t_h. Suppose he bids b and the low-type bidder follows his equilibrium bidding strategy. Thus, the optimization problem of the high-type bidder is

$$\max_b \Pr(\tilde{t}_l < \phi_l(b) | \tilde{t}_l < t_h)(t_h - b) + \{1 - \Pr(\tilde{t}_l < \phi_l(b) | \tilde{t}_l < t_h)\} \max\{t_h - p, 0\}$$

The first-term in the optimization problem is the expected utility of the high-type bidder when he wins the auction, and the second-term is the expected utility when he loses the auction and buys the object in the resale stage. The probability expression can be rewritten in the following way:

$$\begin{aligned} \Pr(\tilde{t}_l < \phi_l(b) | \tilde{t}_l < t_h) &= \frac{\Pr(\tilde{t}_l < \min\{\phi_l(b), t_h\})}{\Pr(\tilde{t}_l < t_h)} \\ &= \frac{F_l \circ \phi_l(b)}{F_l(t_h)} \end{aligned}$$

In deriving this expression, we made use of the fact that $\phi_l(b) \leq t_h$. To see this, suppose $\phi_l(b) > t_h$. Then, the high-type bidder can reduce his bid slightly and still win with probability 1. Thus, the optimization problem can be rewritten as

$$\max_b \frac{F_l \circ \phi_l(b)}{F_l(t_h)}(t_h - b) + \frac{F_l(t_h) - F_l \circ \phi_l(b)}{F_l(t_h)} \max\{t_h - p, 0\}$$

The first-order condition leads to the following differential equation

$$F_l \circ \phi_l(b) = \mathrm{D}F_l \circ \phi_l(b)(p(b) - b) \tag{4}$$

The above equation states that the inverse of reverse hazard rate of the bid for the low-type bidder is equal to the marginal profit of the high-type bidder. In the following Proposition, we describe the equilibrium.

Proposition 1. (ϕ_h, ϕ_l, p) *is an equilibrium profile if and only if it solves the following system*

$$\begin{aligned}
&F_h \circ \phi_h(b) = \mathrm{D}F_h \circ \phi_h(b)(p(b) - b) + F_h \circ \phi_l(b) \\
&F_l \circ \phi_l(b) = \mathrm{D}F_l \circ \phi_l(b)(p(b) - b) \\
&\phi_l(b) = p(b) - \frac{F_h \circ \phi_h(b) - F_h \circ p(b)}{f_h \circ p(b)} \\
&\phi_h(0) = \phi_l(0) = 0 \quad \& \quad \phi_h(\bar{b}) = \phi_l(\bar{b}) = \bar{a} \quad \exists \quad \bar{b} \in \Re_{++}
\end{aligned} \tag{5}$$

The above result conveys that the first-order differential equations are both necessary and sufficient for an equilibrium. The sufficiency part states that local deviations are not profitable for the high-type and the low-type bidder. Notice that, for the low-type bidder, local deviations are strictly worse off and, for the high-type bidder, local deviations are weakly worse off.

Until now, we have assumed that the low-type bidder bids more aggressively than the high-type bidder. In what follows, we show that this is indeed the case.

Proposition 2. *Suppose Assumption 1 is satisfied. Then, the low-type bidder bids more aggressively than the high-type bidder, i.e.,*

$$\beta_l(t) > \beta_h(t)$$

for every $t \in (0, \bar{a})$.

Let us gain some intuition of the above result. To see why the low-type bidder bids more aggressively than the high-type bidder, suppose this is not true. Instead, suppose that the high-type bidder bids more aggressively than the low-type bidder. Then, for any particular valuation, the low-type bidder bids less than the high-type bidder, and as we know that the realized valuation of the high-type bidder is more than that of the low-type bidder, the low-type bidder will always lose the auction. Moreover, the high-type bidder will never make a resale offer to the low-type bidder. Therefore, the low-type bidder bids more aggressively than the high-type bidder.

The following is the main result of our paper.

Theorem 1. *Suppose Assumption 1 is satisfied. Then, the low-type bidder produces a stronger bid-distribution than the high-type bidder, i.e.,*

$$F_l \circ \phi_l(b) < F_h \circ \phi_h(b)$$

for every $b \in (0, \bar{b})$.

We now explore the reasons why the low-type bidder produces a stronger bid distribution than the high-type bidder. Suppose the low-type bidder bids b. Then, $t_l = \phi_l(b)$. We are interested in calculating the profits from winning at the margin, i.e., where the bidder loses the auction by bidding b and wins the auction by marginally increasing his bid. Suppose the marginal increase is given by ϵ. Hence, $\phi_h(b) < t_h < \phi_h(b+\epsilon)$ must be true. Whenever, the low-type bidder bids b, he loses the auction and cannot buy the object in the resale market. Therefore, his profits are zero. On the other hand, whenever he bids $b+\epsilon$, he wins the object and resells to the high-type bidder. Therefore, his profits are $p(b) - b$. Thus, the marginal profits are $p(b) - b$.

Now suppose the high-type bidder bids b. Then, $t_h = \phi_h(b)$. Suppose he increases his bid marginally by ϵ. Again, we are interested in calculating the profits from winning at the margin. Hence, $\phi_l(b) < t_l < \phi_l(b+\epsilon)$ must be true. Whenever $t_l < \phi_l(b)$, he loses the auction and buys the object in the resale market. Therefore, his profits are $t_h - p(b)$. On the other hand, whenever $\phi_l(b+\epsilon) < t_l$, he wins the object but do not resell in the resale stage. Therefore, his profits are $t_h - b$. Thus, his marginal profits are $(t_h - b) - (t_h - p) = p(b) - b$.

Notice that the marginal profits are same for both the bidders. From (3), the marginal profits are less than the reverse hazard rate of the high-type bidder. From (4), the marginal profits are equal to the reverse hazard rate of the low-type bidder. Thus, the bid distribution of the low-type bidder dominates that of the high-type bidder in terms of reverse hazard rate. Since reverse hazard rate dominance implies first-order stochastic dominance, the low-type bidder produces a stronger bid distribution than the high-type bidder.

We shall now discuss how our results are related to the already existing results in the literature. Recall that a bidder is **strong** if the probability distribution of his type dominates that of the other bidder. A **weak** bidder is defined in a similar way. In our framework, we have considered a high-type and a low-type bidder. The following two potential cases are of interest: (a) the "strong" bidder is a "high-type" bidder and the "weak" bidder is a "low-type" bidder; and (b) the "strong" bidder is a "low-type" bidder and the "weak" bidder is a "high-type" bidder. We develop some new terminologies. If the distribution function of the high-type bidder is dominant to that of the low-type bidder, we say the former is a **super-strong** bidder and the latter is a **super-weak** bidder. On the contrary, if the distribution function of the low-type bidder is dominant to that of the high-type bidder, we say the former is a **not-so-strong** bidder and the latter is a **not-so-weak** bidder. We note the following observations.

Remark 1. *Suppose the distribution function of the high-type bidder dominates that of the low-type bidder. Then, the super-weak bidder bids more aggressively and produces a stronger bid distribution than the super-strong bidder. In this scenario, the dominance of the super-strong bidder in terms of value-distribution leads to his dominance in terms of bid distribution.*

Remark 2. *Suppose the distribution function of the low-type bidder dominates that of the high-type bidder. Then, the not-so-strong bidder bids more aggressively and produces a stronger bid distribution than the not-so-weak bid-*

der. In this scenario, the dominance of the not-so-weak bidder in terms of value-distribution leads to the dominance of not-so-strong bidder in terms of bid distribution.

Remark 3. *Our result still holds even if the distribution functions are identical. In H-K framework, heterogeneous value-distributions lead to identical bid-distributions. In contrast to the literature, our result conveys that identical value-distributions lead to heterogeneous bid-distributions.*

Remark 4. *We have ranked the bidding strategies and bid-distributions unanimously without any stochastic ranking of the value-distributions. In other words, stochastic orders are not necessary to unambiguously rank the bidding strategies and bid-distributions.*

Remark 5. *With the introduction of resale opportunities, the bid-distributions can be unanimously ranked which is not possible otherwise.*

3 Connecting function

In this section, we compare the "connecting function" of the high-type bidder under different environments. The benchmark environment is that the value-rankings are revealed and there are resale opportunities. The two environments with which we will compare our benchmark environment are (a) the value-rankings are not revealed but there are resale opportunities and (b) the value-rankings are not revealed and there are no resale opportunities. In subsection 3.1, we compare the connecting function of the benchmark case with that of (a). In subsection 3.2, we compare the connecting function of the benchmark case with that of (b).

3.1 Presence of resale

First, consider the benchmark case when the value-rankings are revealed. The **connecting function** $\Omega_i : T \to T$ is defined as

$$\Omega_i(t) := \phi_i \circ \beta_j(t)$$

for every $i \in N$. We interpret Ω_i as the type required by bidder i in order to match the bid made by bidder j.

We describe the equilibrium in the following manner:

$$\begin{aligned} \mathrm{D}\Omega_l(t) &= \frac{F_l \circ \Omega_l(t)}{f_l \circ \Omega_l(t)} \frac{f_h(t)}{F_h(t) - F_h \circ \Omega_l(t)} \\ \mathrm{D}\Omega_h(t) &= \frac{F_h \circ \Omega_h(t) - F_h(t)}{f_h \circ \Omega_h(t)} \frac{f_l(t)}{F_l(t)} \\ \Omega_l(0) = \Omega_h(0) = 0 \quad &\& \quad \Omega_l(\bar{a}) = \Omega_h(\bar{a}) = \bar{a} \end{aligned} \tag{6}$$

Now consider a scenario when the value-rankings are not revealed. Suppose the bidding strategy, inverse bidding strategy and resale price are denoted by ψ_i, μ_i and r respectively. The characterization of inverse bidding strategy is given by[6]

$$
\begin{aligned}
\mathrm{D}\mu_l(b) &= \frac{F_l \circ \mu_l(b)}{f_l \circ \mu_l(b)} \frac{1}{r(b) - b} \\
\mathrm{D}\mu_h(b) &= \frac{F_h \circ \mu_h(b)}{f_h \circ \mu_h(b)} \frac{1}{r(b) - b} \\
\mu_l(0) = \mu_h(0) = 0 \quad &\& \quad \mu_l(\hat{b}) = \mu_h(\hat{b}) = \bar{a} \quad \exists \quad \hat{b} \in \Re_{++}
\end{aligned}
$$

We define a **connecting function** $\Theta_i : T \to T$ as

$$\Theta_i(t) := \mu_i \circ \psi_j(t)$$

for every $i \in N$.

We describe the equilibrium in the following manner:

$$
\begin{aligned}
\mathrm{D}\Theta_l(t) &= \frac{F_l \circ \Theta_l(t)}{f_l \circ \Theta_l(t)} \frac{f_h(t)}{F_h(t)} \\
\mathrm{D}\Theta_h(t) &= \frac{F_h \circ \Theta_h(t)}{f_h \circ \Theta_h(t)} \frac{f_l(t)}{F_l(t)} \\
\Theta_l(0) = \Theta_h(0) = 0 \quad &\& \quad \Theta_l(\bar{a}) = \Theta_h(\bar{a}) = \bar{a}
\end{aligned}
\tag{7}
$$

In the following result, we compare the connecting function of the high-type bidder when the value-rankings are revealed with the case when they are not revealed.

Theorem 2. *Suppose (ϕ_l, ϕ_h, p) is an equilibrium profile and (Ω_l, Ω_h) is the corresponding connecting function when the value-rankings are revealed. Suppose (μ_l, μ_h, r) is an equilibrium profile and (Θ_l, Θ_h) is the corresponding connecting function when the value-rankings are not revealed. Then,*

$$\Omega_h(t) > \Theta_h(t)$$

for every $t \in (0, \bar{a})$.

The type required by the high-type bidder in order to match the bid made by the low-type bidder is more when value-rankings are revealed as compared to the case when value-rankings are not revealed. Notice that $\Omega_h(t) > t$ since the low-type bidder bids more aggressively than the high-type bidder (Proposition 2). This means that the (absolute) difference in bid functions is more when value-rankings are revealed. Thus, the revelation of value-rankings

[6]For derivation, see Hafalir and Krishna [8].

in auctions with resale *asymmetrizes* the bidding strategies. However, $\Theta_h(t)$ can be either $>$ or $< t$. Whenever F_h dominates F_l in terms of reverse hazard rate, $\Omega_h(t) > \Theta_h(t) > t$ holds. On the other hand, whenever F_l dominates F_h in terms of reverse hazard rate, $\Omega_h(t) > t > \Theta_h(t)$ holds. We consider both the cases.

Whenever $\Omega_h(t) > \Theta_h(t) > t$ is true, then (a) the low-type bidder bids more aggressively than the high-type bidder in both the scenarios, i.e., with and without revelation of value-rankings, and (b) the low-type bidder increases his level of aggression against the high-type bidder when the value-rankings are revealed.

Whenever $\Omega_h(t) > t > \Theta_h(t)$ is true, then (a) the low-type bidder bids more aggressively than the high-type bidder when the value-rankings are revealed and the high-type bidder bids more aggressively than the low-type bidder when the value-rankings are not revealed, and (b) the level of aggression of the low-type bidder against the high-type bidder—when the value-rankings are revealed—is more than the level of aggression of the high-type bidder against the low-type bidder—when the value-rankings are not revealed.

3.2 Absence of resale

Now consider a scenario when there are no resale opportunities and value-rankings are not revealed. Suppose the bidding strategy and inverse bidding strategy is given by α_i and σ_i respectively. The characterization of inverse bidding strategy is given by[7]

$$\begin{aligned}
\mathrm{D}\sigma_l(b) &= \frac{F_l \circ \sigma_l(b)}{f_l \circ \sigma_l(b)} \frac{1}{\sigma_h(b) - b} \\
\mathrm{D}\sigma_h(b) &= \frac{F_h \circ \sigma_h(b)}{f_h \circ \sigma_h(b)} \frac{1}{\sigma_l(b) - b} \\
\sigma_l(0) &= \sigma_h(0) = 0 \quad \& \quad \sigma_l(\underline{b}) = \sigma_h(\underline{b}) = \bar{a} \quad \exists \quad \underline{b} \in \Re_{++}
\end{aligned}$$

The connecting function $\Lambda_i : T \to T$ is given by

$$\Lambda_i(t) := \sigma_i \circ \alpha_j(t)$$

for every $i \in N$. The description of equilibrium is given by[8]

$$\begin{aligned}
\mathrm{D}\Lambda_l(t) &= \frac{F_l \circ \Lambda_l(t)}{f_l \circ \Lambda_l(t)} \frac{f_h(t)}{F_h(t)} \frac{\Lambda_l(t) - \alpha_h(t)}{t - \alpha_h(t)} \\
\mathrm{D}\Lambda_h(t) &= \frac{F_h \circ \Lambda_h(t)}{f_h \circ \Lambda_h(t)} \frac{f_l(t)}{F_l(t)} \frac{\Lambda_h(t) - \alpha_l(t)}{t - \alpha_l(t)} \\
\Lambda_l(0) &= \Lambda_h(0) = 0 \quad \& \quad \Lambda_l(\bar{a}) = \Lambda_h(\bar{a}) = \bar{a}
\end{aligned} \tag{8}$$

[7]The derivation can be found in Maskin and Riley [18] and Lebrun [16].

[8]This can also be found in Lebrun [16].

In the following result, we compare the connecting function of the high-type bidder when there are resale opportunities and value-rankings are revealed with the case when there are no resale opportunities and value-rankings are not revealed.

Theorem 3. *Suppose (ϕ_l, ϕ_h, p) is an equilibrium profile and (Ω_l, Ω_h) is the corresponding connecting function when there are resale opportunities and value-rankings are revealed. Suppose (σ_l, σ_h) is an equilibrium profile and (Λ_l, Λ_h) is the corresponding connecting function when there are no resale opportunities and the value-rankings are not revealed. Then,*

$$\Omega_h(t) > \Lambda_h(t)$$

for every $t \in (0, \bar{a})$.

The above result conveys that the type required by high-type bidder to match the bid of low-type bidder is more when there are resale opportunities and value-rankings are revealed as compared to the case when there are no resale opportunities and value-rankings are not revealed. Since $\Omega_h(t) > t$, it follows that the revelation of value-rankings in auctions with resale *asymmetrizes* the bidding strategies. Note that whenever F_h dominates F_l in terms of reverse hazard rate, $\Omega_h(t) > \Lambda_h(t) > t$ and whenever F_l dominates F_h in terms of reverse hazard rate, $\Omega_h(t) > t > \Lambda_h(t)$. Whenever $\Omega_h(t) > \Lambda_h(t) > t$, the low-type bidder increases his level of aggression against the high-type bidder with the introduction of resale opportunities and the revelation of value-rankings. On the other hand, whenever $\Omega_h(t) > t > \Lambda_h(t)$, the level of aggression of the low-type bidder against the high-type bidder—when there are resale opportunities and the value-rankings are revealed—is more than the level of aggression of the high-type bidder against the low-type bidder—when there are no resale opportunities and the value-rankings are not revealed.

4 Comparative statics

In this section, we study the bidding behavior when the distribution function of the low-type bidder changes stochastically. Fix the distribution function of the high-type bidder and change the distribution function of the low-type bidder in a manner that the new distribution function is dominant to the old distribution in terms of reverse hazard rate. We ask the following questions:

Formally, suppose the distribution function of the low-type bidder changes from F_l to G_l such that G_l is conditional stochastic dominant to F_l. When the distribution functions are F_h and F_l, the characterization of inverse bidding strategy is

$$
\begin{aligned}
&\mathrm{D}\phi_l(b) = \frac{F_l \circ \phi_l(b)}{f_l \circ \phi_l(b)} \frac{1}{p(b) - b} \\
&\mathrm{D}\phi_h(b) = \frac{F_h \circ \phi_h(b) - F_h \circ \phi_l(b)}{f_h \circ \phi_h(b)} \frac{1}{p(b) - b} \\
&\phi_l(b) = p(b) - \frac{F_h \circ \phi_h(b) - F_h \circ p(b)}{f_h \circ p(b)} \\
&\phi_h(0) = \phi_l(0) = p(0) = 0 \\
&\phi_h(\bar{b}) = \phi_l(\bar{b}) = \bar{a} \quad \exists \quad \bar{b} \in \Re_{++}
\end{aligned} \tag{9}
$$

When the distribution functions are G_l and F_h, we denote the bidding strategy, inverse bidding strategy and resale price by ψ_i, λ_i and q respectively for every $i \in N$. The characterization of inverse bidding strategy after the distribution change is given by

$$
\begin{aligned}
&\mathrm{D}\lambda_l(b) = \frac{G_l \circ \lambda_l(b)}{g_l \circ \lambda_l(b)} \frac{1}{q(b) - b} \\
&\mathrm{D}\lambda_h(b) = \frac{F_h \circ \lambda_h(b) - F_h \circ \lambda_l(b)}{f_h \circ \lambda_h(b)} \frac{1}{q(b) - b} \\
&\lambda_l(b) = q(b) - \frac{F_h \circ \lambda_h(b) - F_h \circ q(b)}{f_h \circ q(b)} \\
&\lambda_h(0) = \lambda_l(0) = q(0) = 0 \\
&\lambda_h(\tilde{b}) = \lambda_l(\tilde{b}) = \bar{a} \quad \exists \quad \tilde{b} \in \Re_{++}
\end{aligned} \tag{10}
$$

Theorem 4. *Suppose (ϕ_h, ϕ_l, p) and $(\lambda_h, \lambda_l, q)$ are an equilibrium profile when the distribution functions are (F_h, F_l) and (F_h, G_l) respectively. Suppose Assumption 1 is satisfied and G_l dominates F_l in terms of reverse hazard rate. Let $F_l(0) > 0$ and $G_l(0) > 0$. Then,*

$$\phi_h(b) > \lambda_h(b)$$

for every $b \in (0, \bar{b}]$.

The high-type bidder bids more aggressively than before when the distribution function of the low-type bidder improves stochastically.

The idea of the proof is to show that $\phi_l \leq \lambda_l$ and $\phi_h \leq \lambda_h$ cannot hold simultaneously. This would imply that $\bar{b} < \tilde{b}$. Thus, $\phi_h > \lambda_h$ around some neighborhood of $\bar{b}$. We then show that these two functions cannot intersect.

The next result compares the bid distribution of the low-type bidder before and after the distribution change.

Proposition 3. *Suppose Assumption 1 is satisfied and G_l dominates F_l in terms of reverse hazard rate. Let $F_l(0) > 0$ and $G_l(0) > 0$. Then, for high enough valuation,*

$$G_l \circ \lambda_l(b) < F_l \circ \phi_l(b)$$

The above result tells that, for a high enough valuation, the low-type bidder produces a stronger bid distribution after the change of the distribution function.

5 Symmetric bidders

Consider the following two environments: (a) the value-rankings are revealed and one bidder's distribution function is F_h and the other bidder's is F_l, and (b) the valu-rankings are not revealed and both the bidders distribution function is either F_h or F_l. In this section, we compare the bidding distributions between (a) and (b). We refer to bidders in (a) as *asymmetric bidders* and bidders in (b) as *symmetric bidders.*

The bidding strategy and inverse bidding strategy of symmetric bidders are denoted by (Γ_k, Γ_k) and (θ_k, θ_k) for every $k \in N$ respectively. The characterization of inverse bidding strategy is given by

$$\begin{aligned} \mathrm{D}\theta_k(b) &= \frac{F_k \circ \theta_k(b)}{f_k \circ \theta_k(b)} \frac{1}{\theta_k(b) - b} \\ \theta_k(0) = 0 \quad &\& \quad \theta_k(\bar{b}_k) = \bar{a} \quad \exists \quad \bar{b}_k \in \Re_{++} \end{aligned} \tag{11}$$

for every $k \in N$.

It is worthwhile to note that, when bidders are symmetric, then there will be no resale. This is because symmetric auctions are always efficient. In the following result, we compare the bidding strategy of a bidder with distribution function F_l in both environments.

Theorem 5. *Suppose (θ_k, θ_k) is an equilibrium profile, when bidders are symmetric, for every $k \in N$. Suppose (ϕ_h, ϕ_l) is an equilibrium profile when bidders are asymmetric. Then,*

$$\theta_l(b) > \phi_l(b)$$

for every $b \in (0, \bar{b})$.

The above result conveys that the bidder with distribution function F_l bids more aggressively in environment (a) than in environment (b). The idea of the proof is to show that $\theta_l > \phi_l$ around the neighborhood of 0. After establishing this fact, we are left to show that the two functions do not intersect.

In the next result, we compare the bidding strategy of asymmetric high-type bidder with symmetric strong bidder.

Proposition 4. *For small enough type, $\Gamma_h(t) > \beta_h(t)$.*

The above result tells that, for a small enough type, the bidder with distribution function F_s bids less aggressively in environment (a) than in environment (b).

Remark 6. *The comparison of bidding strategy between environment (a) and (b) requires no stochastic ranking on F_h and F_l.*

6 Bidders' preferences

In this section, we compare the bidders' preferences for a first-price and a second-price auction. In a second-price auction with resale and presence of value-rankings, truth-telling strategy is *not* a dominant strategy. Nonetheless, truth-telling strategy is still an equilibrium strategy.

Proposition 5. *Suppose the auction format is a second-price auction. Then, truth-telling strategy is an equilibrium strategy.*

We make the following assumption on the parameters of the model.

Assumption 2. *Suppose $F_l(0) > 0$ and*

$$1 - \frac{f_l(t_h)}{F_l(t_h)} \int_0^{t_h} \mathrm{d}z f_l(z)(t_h - z) > F_l(t_h) - F_l(0)$$

Consider the high-type bidder with type t_h. Suppose b^* is the optimal bid made by the high-type bidder and p^* is the optimal resale offer made by the low-type bidder. Then, the value function of the high-type bidder is

$$V_h^I(t_h) = \frac{F_l \circ \phi_l(b^*)}{F_l(t_h)}(t_h - b^*) + \frac{F_l(t_h) - F_l \circ \phi_l(b^*)}{F_l(t_h)} \max\{t_h - p^*, 0\}$$

Notice that $V_h^I(0) = 0$. Using Envelope Theorem, we have

$$\mathrm{D}V_h^I(t_h) = 1 - \frac{F_l \circ \phi_l(b^*) f_l(t_h)}{(F_l(t_h))^2}(t_h - b^* - \max\{t_h - p^*, 0\})$$

Notice that $\mathrm{D}V_h^I(0) = 1$. Now, consider the second-price auction. Since truth-telling strategy is an equilibrium strategy and $t_l < t_h$, the low-type bidder always loses the auction and cannot buy the object in the resale stage. Therefore, the value function of the high-type bidder with type t_h is

$$V_h^{II}(t_h) = \int_0^{t_h} f_l(\mathrm{d}z)(t_h - z)$$

Notice that $V_h^{II}(0) = 0$. The derivative of the value function is

$$\mathrm{D}V_h^{II}(t_h) = F_l(t_h) - F_l(0)$$

Notice that $\mathrm{D}V_h^{II}(0) = 0$. Since $V_h^I(0) = V_h^{II}(0) = 0$ and $\mathrm{D}V_h^I(0) > \mathrm{D}V_h^{II}(0)$, it follows that $V_h^I > V_h^{II}$ around some neighborhood of 0. To show that $V_h^I(t_h) > V_h^{II}(t_h)$ for every type, it suffices to show that if there exists any $t_h^* > 0$ such that $V_h^I(t_h^*) = V_h^{II}(t_h^*)$, then $\mathrm{D}V_h^I(t_h^*) > \mathrm{D}V_h^{II}(t_h^*)$.

We state the following result.

Theorem 6. *(A) The low-type bidder always prefers a first-price auction over a second-price auction.*
(B) Suppose Assumption 2 is satisfied. Then, the high-type bidder prefers a first-price auction over a second-price auction.

7 Conclusion

We have shown that the classic result of bid symmetrization, as shown by Hafalir and Krishna [8], does not hold even with two risk neutral bidders if the value-rankings are common knowledge among the bidders. Specifically, the low-type bidder bids so aggressively that he produces a *stronger* bid distribution than the high-type bidder. The stochastic ranking of value-distribution is *not* necessary to unambiguously rank the bidding strategy and bid distribution. The introduction of resale possibilities allow us to unanimously rank the bid distributions which were otherwise not possible. We have also shown that the presence of value-rankings in auctions with resale *asymmetrizes* the bid functions. We have also compared the bidding strategy when the distribution function of the low-type bidder improves. We have shown that the high-type bidder bids more aggressively when the low-type bidder's value-distribution improves. Finally, we have shown that the low-type bidder always prefers a first-price auction over a second-price auction and under parametric conditions, the high-type bidder also prefers a first-price auction over a second-price auction.

Appendix

A Monopsony mechanism

In this section, we describe the equilibrium when the loser makes a single offer to the winner of the auction. We present the results without providing the proofs. They are analogous to the proofs of the results presented in the main body of the paper.

A.1 Resale stage

Suppose the low-type bidder bids more aggressively than the high-type bidder. It can be easily verified that the high-type bidder makes a resale offer with positive probability whereas the low-type bidder never makes a resale offer.

Consider the high-type bidder with type t_h. The optimization problem is

$$\max_r \Pr(\tilde{t}_l < \phi_l(b)|\tilde{t}_l < t_h)(t_h - b) + \Pr(\phi_l(b) < \tilde{t}_l < p|\tilde{t}_l < t_h) \max\{t_h - p, 0\}$$

The first-order condition is

$$t_h = p - \frac{F_l \circ \phi_l(b) - F_l(p)}{f_l(p)}$$

A.2 Bidding stage

Consider the high-type bidder with type t_h. The optimization problem is

$$\max_b \Pr(\tilde{t}_l < \phi_l(b)|\tilde{t}_l < t_h)(t_h - b) + \Pr(\phi_l(b) < \tilde{t}_l < p|\tilde{t}_l < t_h) \max\{t_h - p, 0\}$$

Using envelope theorem, the first-order condition is

$$F_l \circ \phi_l(b) = \mathrm{D}F_l \circ \phi_l(b)(p(b) - b) \tag{12}$$

Now consider the low-type bidder with type t_l. The optimization problem is

$$\max_b \Pr(\tilde{t}_h < \phi_h(b)|\tilde{t}_h > t_l)(p - b)$$

The first-order condition is

$$F_h \circ \phi_h(b) = \mathrm{D}F_h \circ \phi_h(b)(p(p) - b) + F_h \circ \phi_l(b) \tag{13}$$

We describe the equilibrium in the following proposition.

Proposition A.1. *(ϕ_h, ϕ_l, p) is an equilibrium profile if and only if it solves the following system*

$$\begin{aligned}
&F_l \circ \phi_l(b) = \mathrm{D}F_l \circ \phi_l(b)(p(b) - b)\\
&F_h \circ \phi_h(b) = \mathrm{D}F_h \circ \phi_h(b)(p(b) - b) + F_h \circ \phi_l(b)\\
&\phi_h(b) = p - \frac{F_l \circ \phi_l(b) - F_l(p)}{f_l(p)}\\
&\phi_h(0) = \phi_l(0) = 0 \quad \& \quad \phi_h(\bar{b}) = \phi_l(\bar{b}) = \bar{a} \quad \exists \quad \bar{b} \in \Re_{++}
\end{aligned} \tag{14}$$

The following theorem states that the low-type bidder bids more aggressively and produces a stronger bid distribution than the high-type bidder.

Theorem A.1. *Suppose Assumption 1 is satisfied. Then,*

(A) $\beta_l(t) > \beta_h(t)$ *for every* $t \in T_l$*,*

(B) $F_l \circ \phi_l(b) < F_h \circ \phi_h(b)$ *for every* $b \in (0, \bar{b})$*.*

B Proofs

Proof of Lemma 1. Consider the low-type bidder with type t_l. Suppose the low-type bidder wins the auction by bidding $\beta_l(t_l)$. Then, $\beta_l(t_l) > \beta_h(t_h)$, and thus, $t_h < \beta_h^{-1} \circ \beta_l(t_l)$. This means that the type of the high-type bidder is less than the type required to bid the same as the low-type bidder. Since $\beta_l(t_l) > \beta_h(t_l)$, we have $t_l < \beta_h^{-1} \circ \beta_l(t_l)$. This means that the type of the low-type bidder is less than the type required by the high-type bidder to bid the same as the low-type bidder. Therefore, the low-type bidder will make a resale offer with price between t_l and $\beta_h^{-1} \circ \beta_l(t_l)$.

Now, consider the high-type bidder with type t_h. Suppose the high-type bidder wins the auction by bidding $\beta_h(t_h)$. Since $\beta_l(t_h) > \beta_h(t_h)$, we have $t_h > \beta_l^{-1} \circ \beta_h(t_h)$. Since the type of the high-type bidder is less than the type required by the low-type bidder to bid the same as the high-type bidder. Since $t_h > \beta_l^{-1} \circ \beta_h(t_h)$ and given that the high-type bidder wins the auction, it must be true that $t_h > \beta_l^{-1} \circ \beta_h(t_h) > t_l$. Hence, the high-type bidder does not make a resale offer. ■

Proof of Proposition 1. Suppose (ϕ_h, ϕ_l) is an equilibrium profile. We show $\beta_l(0) = \beta_h(0) = 0$. Consider the low-type bidder. Suppose $\beta_l(0) > \beta_h(0) \geq 0$. Then, the low-type bidder makes a resale offer $p \circ \beta_l(0)$ such that $p \circ \beta_l(0) > \beta_l(0) > \beta_h(0)$. Then, it is profitable for the high-type bidder to deviate and bid in $(\beta_l(0), p \circ \beta_l(0))$. This implies $\beta_l(0) > \beta_h(0)$ cannot be the case. Thus, $\beta_l(0) = \beta_h(0) \geq 0$.

Suppose $\beta_l(0) = \beta_h(0) > 0$. Consider a sequence $(t^n)_{n=1}^{\infty}$ such that $t^n \downarrow 0$. Then, $\beta_l(t^n) \geq \beta_h(t^n)$ for every $n \in \mathcal{N}$. For large enough n, $\beta_l(t^n) > t^n$. If the low-type bidder wins, then he makes a resale offer which is lower than $\beta_l(t^n)$, and thus, payoffs are negative. On the other hand, if the low-type bidder loses, then the high-type bidder will not make a resale offer, and hence, payoffs are zero. Therefore, $\beta_l(0) > 0$ is not profitable. Hence, $\beta_l(0) = \beta_h(0) = 0$.

We show there exist a common upper bound on the bidding space. Suppose there exists $\bar{b}_l, \bar{b}_h > 0$ such that $\bar{b}_l \neq \bar{b}_h$, $\beta_l(a_l) = \bar{b}_l$ and $\beta_h(a_h) = \bar{b}_h$. Since the low-type bidder bids more aggressively, we have $\bar{b}_l \geq \bar{b}_h$. If $\bar{b}_l = \bar{b}_h$, then the result holds trivially. Suppose $\bar{b}_l > \bar{b}_h$. Then, the low-type bidder willl make a resale offer of $p \circ \beta_l(a_l)$ such that $p \circ \beta_l(a_l) > \beta_l(a_l) > \beta_h(a_h)$. Then, it is profitable for the high-type bidder to deviate and bid in $(\beta_l(a_l), p \circ \beta_l(a_l))$. Hence, $\bar{b}_l > \bar{b}_h$ cannot be true.

Conversely, suppose (ϕ_h, ϕ_l) solves the system given by (5). Consider the low-type bidder. The value function is

$$V_l(t_l, b) = \frac{F_h \circ \phi_l(b) - F_h(p)}{1 - F_h(t_l)}(p - b) + \frac{F_h(p) - F_h(t_l)}{1 - F_h(t_l)}(t_l - b)$$

The first-order derivative is

$$\mathrm{D}V_l(t_l, b) = \mathrm{D}F_h \circ \phi_l(b)(p - b) - F_h \circ \phi_l(b) + F_h(t_l)$$

Suppose the low-type bidder over bids by choosing b' such that $\phi_l(b') > t_l$. Then,

$$\begin{aligned}
\mathrm{D}V_l(t_l, b') &= \mathrm{D}F_h \circ \phi_l(b')(p - b') - F_h \circ \phi_l(b') + F_h(t_l) \\
&< \mathrm{D}F_h \circ \phi_l(b')(p - b') - F_h \circ \phi_l(b') + F_h \circ \phi_l(b') \\
&= 0
\end{aligned}$$

Hence, it is not profitable for the low-type bidder to deviate.

On the other hand, suppose the super-weak bidder under bids by choosing b'' such that $\phi_l(b'') < t_l$. Then,

$$\begin{aligned}
\mathrm{D}V_l(t_l, b'') &= \mathrm{D}F_h \circ \phi_l(b'')(p - b'') - F_h \circ \phi_l(b'') + F_h(t_l) \\
&> \mathrm{D}F_h \circ \phi_l(b'')(p - b'') - F_h \circ \phi_l(b'') + F_h \circ \phi_l(b'') \\
&= 0
\end{aligned}$$

Hence, it is not profitable for the super-weak bidder to deviate.

Now consider the high-type bidder. The value function is

$$V_h(t_h, b) = \frac{F_l \circ \phi_l(b)}{F_l(t_h)}(t_h - b) + \frac{F_h(t_h) - F_l \circ \phi_l(b)}{F_l(t_h)}(t_h - p)$$

The first-order derivative is

$$\begin{aligned} \mathrm{D}V_h(t_h, b) &= \mathrm{D}F_l \circ \phi_l(b)(p - b) - F_l \circ \phi_l(b) \\ &= 0 \end{aligned}$$

Hence, it is not profitable for the high-type bidder to deviate. Therefore, (ϕ_h, ϕ_l) is an equilibrium. ■

Proof of Propositon 2. We show the low-type bidder bids more aggressively than the high-type bidder. Since $\phi_h'(\bar{b}) = 0 < \phi_l'(\bar{b})$, it follows that there exists $\epsilon > 0$ such that $\phi_h(b) > \phi_l(b)$ for every $b \in (\bar{b} - \epsilon, \bar{b})$. Suppose there exists b^* such that $\phi_h(b^*) = \phi_l(b^*)$ and $\phi_h(b) > \phi_l(b)$ for every $b \in (b^*, \bar{b})$. Then, $p(b^*) = \phi_h(b^*) = \phi_l(b^*)$. From (9), we have

$$\phi_l'(b^*) > \phi_h'(b^*)$$

This implies that there exists $\delta > 0$ such that $\phi_l(b^* + \delta) > \phi_h(b^* + \delta)$, which is a contradiction. Hence, $\phi_h(b) > \phi_l(b)$ for every $b \in (0, \bar{b})$. ■

Proof of Theorem 1. We show the low-type bidder produces a stronger bid distribution than the high-type bidder. From (3) and (4), we have

$$\frac{F_h \circ \phi_h(b)}{\mathrm{D}F_h \circ \phi_h(b)} > \frac{F_l \circ \phi_l(b)}{\mathrm{D}F_l \circ \phi_l(b)}$$

Thus,

$$\mathrm{D}\left(\frac{F_l \circ \phi_l(b)}{F_h \circ \phi_h(b)}\right) > 0$$

Since $F_h \circ \phi_h(\bar{b}) = F_l \circ \phi_l(\bar{b}) = 1$, we have $F_l \circ \phi_l(b) < F_h \circ \phi_h(b)$ for every $b \in (0, \bar{b})$. ■

Proof of Theorem 2. Notice that $\Omega_h'(\bar{a}) = 0$ and $\Theta_h'(\bar{a}) > 0$. Since $\Omega_h'(\bar{a}) < \Theta_h'(\bar{a})$, it follows that there exists $\epsilon > 0$ such that $\Omega_h(t) > \Theta_h(t)$ for every $t \in (\bar{a} - \epsilon, \bar{a})$. Suppose there exists $t^* > 0$ such that $\Omega_h(t^*) = \Theta_h(t^*)$ and $\Omega_h(t) > \Theta_h(t)$ for every $t \in (t^*, \bar{a})$. Then, from (6) and (7), we have

$$\Omega_h'(t^*) < \Theta_h'(t^*)$$

Thus, there exists $\delta > 0$ such that $\Omega_h(t^* + \delta) < \Theta_h(t^* + \delta)$, a contradiction. Hence, no such t^* exists. Therefore, $\Omega_h(t) > \Theta_h(t)$ for every $t \in (0, \bar{a})$. ■

Proof of Theorem 3. Notice that $\mathrm{D}\Omega_h(\bar{a}) = 0$ and $\mathrm{D}\Lambda_h(\bar{a}) > 0$. Since $\mathrm{D}\Omega_h(\bar{a}) < \mathrm{D}\Lambda_h(\bar{a})$, it follows that there exists $\epsilon > 0$ such that $\Omega_h(t) > \Lambda_h(t)$ for every $t \in (\bar{a} - \epsilon, \bar{a})$. Suppose there exists $t^* > 0$ such that $\Omega_h(t^*) = \Lambda_h(t^*)$ and $\Omega_h(t) > \Lambda_h(t)$ for every $t \in (t^*, \bar{a})$. Then, from (6), (8) and the fact that $\Lambda_h(t) > t$ for every t, we have

$$\begin{aligned}
\Lambda_h'(t^*) &= \frac{F_h \circ \Lambda_h(t^*)}{f_l \circ \Lambda_h(t^*)} \frac{f_l(t^*)}{F_l(t^*)} \frac{\Lambda_h(t^*) - \alpha_l(t^*)}{t^* - \alpha_l(t^*)} \\
&> \frac{F_h \circ \Lambda_h(t^*)}{f_l \circ \Lambda_h(t^*)} \frac{f_l(t^*)}{F_l(t^*)} \\
&= \frac{F_h \circ \Omega_h(t^*)}{f_l \circ \Omega_h(t^*)} \frac{f_l(t^*)}{F_l(t^*)} \\
&> \frac{F_h \circ \Omega_h(t^*) - F_h(t^*)}{f_l \circ \Omega_h(t^*)} \frac{f_l(t^*)}{F_l(t^*)} \\
&= \Omega_h(t^*)
\end{aligned}$$

Thus, there exists $\delta > 0$ such that $\Lambda_h(t^* + \delta) > \Omega_h(t^* + \delta)$, a contradiction. Hence, no such t^* exists. Therefore, $\Omega_h(t) > \Lambda_h(t)$ for every $t \in (0, \bar{a})$. ■

Proof of Theorem 4. Suppose $\phi_l(c) \le \lambda_l(c)$ and $\phi_h(c) \le \lambda_h(c)$ for every $c \in (0, \min\{\bar{b}, \tilde{b}\})$. Then, $p(c) \le q(c)$.

We show that there exists $\epsilon > 0$ such that

$$\phi_h(b) < \lambda_h(b) \quad \& \quad \frac{F_l \circ \phi_l(b)}{G_l \circ \lambda_l(b)} < \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)}$$

for every $b \in (c - \epsilon, c)$.

Since $\phi_l \le \lambda_l$, we have

$$\frac{F_l \circ \phi_l(b)}{G_l \circ \lambda_l(b)} < \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(b)}$$

for every $b < c$.

Case 1: $\phi_l(c) < \lambda_l(c)$ and $\phi_h(c) < \lambda_h(c)$ for every $c \in (0, \min\{\bar{b}, \tilde{b}\})$

Then, $p(c) < q(c)$. It is straightforward to see that there exists $\epsilon > 0$ such that

$$\phi_h(b) < \lambda_h(b) \quad \& \quad \frac{F_l \circ \phi_l(b)}{G_l \circ \lambda_l(b)} < \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)}$$

for every $b \in (c - \epsilon, c)$.

Case 2: $\phi_l(c) = \lambda_l(c)$ and $\phi_h(c) < \lambda_h(c)$ for every $c \in (0, \min\{\bar{b}, \tilde{b}\})$

Then, $p(c) < q(c)$. From the system given by (9) and (10), we have

$$\phi_l'(c) > \lambda_l'(c)$$

Then, there exists $\epsilon > 0$ such that

$$\phi_h(b) < \lambda_h(b) \quad \& \quad \frac{F_l \circ \phi_l(b)}{G_l \circ \lambda_l(b)} < \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)}$$

for every $b \in (c - \epsilon, c)$.

Case 3: $\phi_l(c) < \lambda_l(c)$ and $\phi_h(c) = \lambda_h(c)$ for every $c \in (0, \min\{\bar{b}, \tilde{b}\})$

Then, $p(c) < q(c)$. From the system given by (9) and (10), we have

$$\phi_h'(c) > \lambda_h'(c)$$

Then, there exists $\epsilon > 0$ such that

$$\phi_h(b) < \lambda_h(b) \quad \& \quad \frac{F_l \circ \phi_l(b)}{G_l \circ \lambda_l(b)} < \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)}$$

for every $b \in (c - \epsilon, c)$.

Case 4: $\phi_l(c) = \lambda_l(c)$ and $\phi_h(c) = \lambda_h(c)$ for every $c \in (0, \min\{\bar{b}, \tilde{b}\})$

Then, $p(c) = q(c)$. From the system given by (9) and (10), we have

$$\phi_l'(c) > \lambda_l'(c)$$

The second differential equation of the system given by (9) and (10) can be rewritten as

$$\begin{aligned}
\mathrm{D} \log F_h \circ \phi_h(b) &= \frac{F_h \circ \phi_h(b) - F_h \circ \phi_l(b)}{F_h \circ \phi_h(b)} \frac{1}{p(b) - b} \\
\mathrm{D} \log F_h \circ \lambda_h(b) &= \frac{F_h \circ \lambda_h(b) - F_h \circ \lambda_l(b)}{F_h \circ \lambda_h(b)} \frac{1}{q(b) - b}
\end{aligned}$$

Taking the derivative of the above equations, we have

$$\begin{aligned}
\mathrm{D}^2 \log F_h \circ \phi_h = {} & \frac{F_h \circ \phi_h - F_h \circ \phi_l}{F_h \circ \phi_h} \frac{1 - \mathrm{D}p}{(p - b)^2} + \\
& \frac{1}{p - b} \left\{ \frac{\mathrm{D}\phi_h f_h \circ \phi_h F_h \circ \phi_l - \mathrm{D}\phi_l F_h \circ \phi_h f_h \circ \phi_l}{(F_h \circ \phi_h)^2} \right\} \\
\mathrm{D}^2 \log F_h \circ \lambda_h = {} & \frac{F_h \circ \lambda_h - F_h \circ \lambda_h}{F_h \circ \lambda_h} \frac{1 - \mathrm{D}q}{(q - b)^2} + \\
& \frac{1}{q - b} \left\{ \frac{\mathrm{D}\lambda_h f_h \circ \lambda_h F_h \circ \lambda_l - \mathrm{D}\lambda_l F_h \circ \lambda_h f_h \circ \lambda_l}{(F_h \circ \lambda_h)^2} \right\}
\end{aligned}$$

The above two expressions are strictly decreasing in $\mathrm{D}\phi_l$ and $\mathrm{D}\lambda_l$ respectively. Taking the derivatives of third equation for the system given by (9) and (10), we have

$$\begin{aligned}
\mathrm{D}\phi_l &= \mathrm{D}p - \frac{f_h \circ p(\mathrm{D}\phi_h f_h \circ \phi_h - \mathrm{D}p f_h \circ p) - (F_h \circ \phi_h - F_h \circ p)\mathrm{D}p f_h' \circ p}{(f_h \circ p)^2} \\
\mathrm{D}\lambda_l &= \mathrm{D}q - \frac{f_h \circ q(\mathrm{D}\lambda_h f_h \circ \lambda_h - \mathrm{D}q f_h \circ q) - (F_h \circ \lambda_h - F_h \circ q)\mathrm{D}q f_h' \circ q}{(f_h \circ q)^2}
\end{aligned}$$

Since $\mathrm{D}\phi_l > \mathrm{D}\lambda_l$, $\mathrm{D}\phi_h = \mathrm{D}\lambda_h$, $\phi_h = \lambda_h$, $\phi_l = \lambda_l$ and $p = q$, comparing the above two expressions, we have $\mathrm{D}p > \mathrm{D}q$. Using this fact in the expressions of $\mathrm{D}^2 \log F_h \circ \phi_h$ and $\mathrm{D}^2 \log F_h \circ \lambda_h$, we have

$$\mathrm{D}^2 \log F_h \circ \phi_h < \mathrm{D}^2 \log F_h \circ \lambda_h$$

Then, there exists $\epsilon > 0$ such that

$$\phi_h(b) < \lambda_h(b) \quad \& \quad \frac{F_l \circ \phi_l(b)}{G_l \circ \lambda_l(b)} < \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)}$$

for every $b \in (c - \epsilon, c)$.

Let

$$M := \inf \Bigg\{ x \in [0, c - \epsilon] : \quad \phi_h(b) < \lambda_h(b) \quad \wedge \\ \frac{F_l \circ \phi_l(b)}{G_l \circ \lambda_l(b)} < \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)} \text{ for every } b \in (c - \epsilon, c) \Bigg\}$$

We show $M = 0$. We show by contradiction. Suppose $M > 0$. Then, either

$$\phi_h(M) = \lambda_h(M) \quad \text{or} \quad \frac{F_l \circ \phi_l(M)}{G_l \circ \lambda_l(M)} = \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)}$$

Since $\phi_h(b) < \lambda_h(b)$ and $\phi_l(b) \leq \lambda_l(b)$ for every $b \in (M, c - \epsilon)$, we have $p(b) < q(b)$. From the system given by (9) and (10), we have

$$\frac{F_l \circ \phi_l(b)}{\mathrm{D}F_l \circ \phi_l(b)} < \frac{G_l \circ \lambda_l(b)}{\mathrm{D}G_l \circ \lambda_l(b)}$$

This implies

$$\mathrm{D} \log F_l \circ \phi_l(b) > \mathrm{D} \log \left\{ \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)} G_l \circ \lambda_l(b) \right\}$$

Since $M < c - \epsilon$, we have

$$\log F_l \circ \phi_l(c - \epsilon) - \log F_l \circ \phi_l(M) > \log \left\{ \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)} G_l \circ \lambda_l(c - \epsilon) \right\} - \\ \log \left\{ \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)} G_l \circ \lambda_l(M) \right\}$$

Rearranging the above expression, we have

$$\log \left\{ \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)} G_l \circ \lambda_l(M) \right\} - \log F_l \circ \phi_l(M) > \\ \log \left\{ \frac{F_l \circ \lambda_l(c)}{G_l \circ \lambda_l(c)} G_l \circ \lambda_l(c - \epsilon) \right\} - \log F_l \circ \phi_l(c - \epsilon)$$

From the definition of M, we have

$$\log\left\{\frac{F_l\circ\lambda_l(c)}{G_l\circ\lambda_l(c)}G_l\circ\lambda_l(c-\epsilon)\right\}-\log F_l\circ\phi_l(c-\epsilon)>0$$

Then,

$$\frac{F_l\circ\phi_l(M)}{G_l\circ\lambda_l(M)}<\frac{F_l\circ\lambda_l(c)}{G_l\circ\lambda_l(c)}$$

Therefore, $\phi_l(M)=\lambda_l(M)$ must be true. From the system given by (9) and (10), the definition of M and the assumption of conditional stochastic dominance, we have

$$\frac{F_l\circ\phi_l(M)}{G_l\circ\lambda_l(M)}<\frac{F_l\circ\lambda_l(c)}{G_l\circ\lambda_l(c)}<\frac{F_l\circ\lambda_l(M)}{G_l\circ\lambda_l(M)}$$

This implies $\phi_l(M)<\lambda_l(M)$. Since $\phi_h(M)=\lambda_h(M)$ and $\phi_l(M)<\lambda_l(M)$, we have $p(M)<q(M)$. Then $\phi_h'(M)>\lambda_h'(M)$. Thus, there exists $\delta>0$ such that $\phi_h(M+\delta)>\lambda_h(M+\delta)$, which is a contradiction. Therefore, $M=0$.

Hence,

$$\phi_h(b)<\lambda_h(b)\quad\&\quad\frac{F_l\circ\phi_l(b)}{G_l\circ\lambda_l(b)}<\frac{F_l\circ\lambda_l(c)}{G_l\circ\lambda_l(c)}$$

for every $b\in(0,c)$.

We show that, for every $c\in(0,\min\{\bar{b},\tilde{b}\})$, $\phi_l(c)\leq\lambda_l(c)$ and $\phi_h(c)\leq\lambda_h(c)$ cannot hold simultaneously. We show by contradiction. Suppose there exists c^* such that $\phi_l(c^*)\leq\lambda_l(c^*)$ and $\phi_h(c^*)\leq\lambda_h(c^*)$. Then,

$$\frac{F_l\circ\phi_l(b)}{G_l\circ\lambda_l(b)}<\frac{F_l\circ\lambda_l(c^*)}{G_l\circ\lambda_l(c^*)}$$

Taking the limit at $b\downarrow 0$, we have

$$\frac{F_l(0)}{G_l(0)}<\frac{F_l\circ\lambda_l(c^*)}{G_l\circ\lambda_l(c^*)}$$

Since $\lambda_l(c^*)>0$, it follows from the above expression that F_l/G_l is strictly increasing, which is a contradiction. Hence, $\phi_l(c)\leq\lambda_l(c)$ and $\phi_h(c)\leq\lambda_h(c)$ cannot hold simultaneously.

We show $\tilde{b}>\bar{b}$. We show by contradiction. Suppose $\tilde{b}\leq\bar{b}$. Then, $\phi_l(\tilde{b})\leq\lambda_l(\tilde{b})$ and $\phi_h(\tilde{b})\leq\lambda_h(\tilde{b})$, which is a contradiction as $\phi_l(c)\leq\lambda_l(c)$ and $\phi_h(c)\leq\lambda_h(c)$ cannot hold simultaneously. Hence, $\tilde{b}>\bar{b}$.

Finally, we show $\phi_h(b)>\lambda_h(b)$ for every $b\in(0,\bar{b}]$. Since $\phi_h(\bar{b})>\lambda_h(\bar{b})$, it implies that there exists $\epsilon>0$ such that $\phi_h(b)>\lambda_h(b)$ for every $b\in(\bar{b}-\epsilon,\bar{b}]$. Suppose there exists b^* such that $\phi_h(b^*)=\lambda_h(b^*)$ and $\phi_h(b)>\lambda_h(b)$ for every $b\in(b^*,\bar{b}]$. Then, $\phi_l(b^*)>\lambda_l(b^*)$. As $\phi_h(b^*)=\lambda_h(b^*)$ and $\phi_l(b^*)>\lambda_l(b^*)$, we have $p(b^*)>q(b^*)$. From the system given by (9) and (10), we have

$$\phi_h'(b^*)<\lambda_h'(b^*)$$

Then, there exists $\delta>0$ such that $\phi_h(b^*+\delta)<\lambda_h(b^*+\delta)$, which is a contradiction. Therefore, $\phi_h(b)>\lambda_h(b)$ for every $b\in(0,\bar{b}]$. ■

Proof of Proposition 3. Since $\tilde{b} > \bar{b}$, it follows $\phi_l(\bar{b}) > \lambda_l(\bar{b})$ and $\phi_h(\bar{b}) > \lambda_h(\bar{b})$. Then, there exists $\epsilon > 0$ such that $\phi_l(b) > \lambda_l(b)$ for every $b \in (\bar{b} - \epsilon, \bar{b})$. Since $\phi_h(b) > \lambda_h(b)$ for every $b \in (0, \bar{b})$, it follows that $p(b) > q(b)$ for every $b \in (\bar{b} - \epsilon, \bar{b})$.

From the system given by (9) and (10), we have

$$\frac{F_l \circ \phi_l(b)}{\mathrm{D}F_l \circ \phi_l(b)} < \frac{G_l \circ \lambda_l(b)}{\mathrm{D}G_l \circ \lambda_l(b)}$$

Thus,

$$\mathrm{D}\left(\frac{G_l \circ \lambda_l(b)}{F_l \circ \phi_l(b)}\right) > 0$$

Since $\tilde{b} > \bar{b}$ and $F_l \circ \phi_l(\bar{b}) = 1 > G_l \circ \lambda_l(\bar{b})$, we have $G_l \circ \lambda_l(b) < F_l \circ \phi_l(b)$ for some neighborhood around $\bar{b}$. ■

Proof of Theorem 5. Define $y_i : T_i \to \Re$ as

$$y_i(t) = t\frac{f_i(t)}{F_i(t)}$$

for every $i \in N$. Then,

$$y_i(0) = 1 \quad \& \quad y_i'(0) = \frac{f_i'(0)}{2f_i(0)}$$

So,

$$y_i \circ \phi_i(b) = \phi_i(b)\frac{f_i \circ \phi_i(b)}{F_i \circ \phi_i(b)}$$

Using (9) for $i = l$ in the above equation, we have

$$\phi_l'(b) y \circ \phi_l(b) = \frac{\phi_l(b)}{p(b) - b}$$

At $b = 0$, using L'Hôpital's rule, we have

$$\phi_l'(0) = \frac{\phi_l'(0)}{p'(0) - 1}$$

This implies $p'(0) = 2$. Similarly, for $i = s$, we have

$$y_h \circ \phi_h(b) = \frac{\phi_h(b) f_h \circ \phi_h(b)}{\phi_h'(b) F_h \circ \phi_h(b)(p(b) - b) + F_h \circ \phi_l(b)}$$

At $b = 0$, using L'Hôpital's rule, we have

$$\phi_l'(0) = 0$$

Differentiating third equation of (9) and calculating it at $b = 0$, we have

$$\phi_h'(0) = 4$$

Similarly, we have

$$\theta_l(0) = \theta_h(0) = 2$$

Since $\theta_l'(0) > \phi_l'(0)$, it follows that there exists $\epsilon > 0$ such that $\theta_l(b) > \phi_l(b)$ for every $b \in (0, \epsilon)$. Suppose there exists b^* such that $\theta_l(b^*) = \phi_l(b^*)$ and $\theta_l(b) > \phi_l(b)$ for every $b \in (0, b^*)$. Then, $p(b^*) > \phi_l(b^*) = \theta_l(b^*)$. From (9) and (11), we have

$$\theta_l'(b^*) > \phi_l'(b^*)$$

This implies that there exists $\delta > 0$ such that $\theta_l(b^* - \delta) < \phi_l(b^* - \delta)$, which is a contradiction. Hence,

$$\theta_l(b) > \phi_l(b)$$

for every $b \in (0, \bar{b})$. ■

Proof of Proposition 4. Since $\Gamma_h'(0) > \theta_h'(0)$, it follows that there exists $\delta > 0$ such that $\phi_h(b) > \theta_h(b)$ for every $b \in (0, \epsilon)$. ■

Proof of Proposition 5. Suppose the equilibrium bidding strategy in a second-price auction is denoted by κ_i for every $i \in N$. We show that $\kappa_l(t_l) = t_l$ and $\kappa_h(t_h) = t_h$ is an equilibrium profile. First, consider the low-type bidder with type t_l. Suppose the high-type bidder follows his equilibrium bidding strategy $\kappa_h(t_h) = t_h$. We show that unilateral deviation for the low-type bidder is not profitable. If the low-type bidder bids according to $\kappa_l(t_l) = t_l$, then he gets a payoff of 0. Suppose the low-type bidder under bids by bidding b' such that $b' < t_l$. Since we know that $t_l < t_h$, he loses the auction by bidding b' and cannot buy the object in the resale stage. Therefore, his payoff is 0. Thus, under bidding is not profitable. Now, suppose the low-type bidder over bids by bidding b'' such that $b'' > t_l$. Since $t_l < t_h$, then either $t_l < b'' < t_h$ or $t_l < t_h < b''$. Whenever $t_l < b'' < t_h$, the low-type bidder loses the auction and cannot buy the object in the resale stage thereby getting a payoff of 0. Whenever, $t_l < t_h < b''$, the low-type bidder wins the auction but cannot resell the object in the resale stage thereby getting a payoff of $t_l - t_h < 0$. Thus, in both cases, over bidding is not profitable.

Now consider the high-type bidder with type t_h. Suppose the low-type bidder follows his equilibrium bidding strategy $\kappa_l(t_l) = t_l$. We show that unilateral deviation for the high-type bidder is not profitable. If high-type bidder bids according to $\kappa_h(t_h) = t_h$, then he gets a payoff of $t_h - t_l$. Suppose the high-type bidder under bids by bidding b' such that $b' < t_h$. Since $t_l < t_h$, then either $t_l < b' < t_h$ or $b' < t_l < t_h$. Whenever $t_l < b' < t_h$, the high-type bidder wins the auction and does not sell the object in the resale stage thereby getting a payoff of $t_h - t_l$. Whenever, $b' < t_l < t_h$, the high-type bidder loses the auction and may be able to buy the object in the resale stage. If he is able to buy the object in the resale stage, he ends up paying weakly more than t_l, and, on the other hand, if he is not able to buy the object in the resale stage, he ends up getting a payoff of 0. Thus, in both the cases, under bidding is not profitable. Now, suppose the high-type bidder over bids by bidding b''

such that $b'' > t_h$. Since, $t_h > t_l$, he wins the auction and does not make any resale offer thereby getting a payoff of $t_h - t_l$. Thus, over bidding is not profitable. Therefore, truth-telling strategy is an equilibrium strategy. ■

Proof of Theorem 6. Since $\mathrm{D}V_h^I(0) > \mathrm{D}V_h^{II}(0)$, it follows that there exists $\epsilon > 0$ such that $V_h^I(t_h) > V_h^{II}(t_h)$ for every $t_h \in (0, \epsilon)$. Suppose there exists $t_h^* > 0$ such that $V_h^I(t_h^*) = V_h^{II}(t_h^*)$ and $V_h^I(t_h) > V_h^{II}(t_h)$ for every $t_h \in (0, t_h^*)$. Then, from the value functions, we have

$$t_h^* - b^* - \max\{t_h^* - p^*, 0\} = \frac{F_l(t_h^*)}{F_l \circ \phi_l(b^*)}\left(\int_0^{t_h^*} f_l(\mathrm{d}z)(t_h^* - z) - \max\{t_h^* - p^*, 0\}\right)$$

Using the above equation in the derivative equation of the value functions, we have

$$\mathrm{D}V_h^I(t_h^*) = 1 - \frac{f_l(t_h^*)}{F_l(t_h^*)}\int_0^{t_h^*} f_l(\mathrm{d}z)(t_h^* - z) + \frac{f_l(t_h^*)}{F_l(t_h^*)}\max\{t_h^* - p^*, 0\}$$

and

$$\mathrm{D}V_h^{II}(t_h^*) = F_l(t_h^*) - F_l(0)$$

From Assumption 2, it follows $\mathrm{D}V_h^I(t_h^*) > \mathrm{D}V_h^{II}(t_h^*)$. Therefore, $V_h^I(t_h) > V_h^{II}(t_h)$ for every $t_h \in (0, \bar{a})$. ■

References

[1] Brian Baisa and Justin Burkett. Discriminatory price auctions with resale and optimal quantity caps. 2017.

[2] Gorkem Celik and Okan Yilankaya. Resale in second-price auctions with costly participation. *International Journal of Industrial Organization*, 54:148–174, 2017.

[3] Harrison Cheng. Auctions with resale and bargaining power. *Journal of Mathematical Economics*, 47(3):300–308, 2011.

[4] Harrison Cheng. Revenue ranking in hybrid auctions with speculative resale. 2015.

[5] Harrison H Cheng and Guofu Tan. Asymmetric common-value auctions with applications to private-value auctions with resale. *Economic Theory*, 45(1-2):253–290, 2010.

[6] Rod Garratt and Thomas Tröger. Speculation in standard auctions with resale. *Econometrica*, 74(3):753–769, 2006.

[7] Madhurima Gupta and Bernard Lebrun. First price auctions with resale. *Economics Letters*, 64(2):181–185, 1999.

[8] Isa Hafalir and Vijay Krishna. Asymmetric auctions with resale. *American Economic Review*, 98(1):87–112, 2008.

[9] Isa Hafalir and Vijay Krishna. Revenue and efficiency effects of resale in first-price auctions. *Journal of Mathematical Economics*, 45(9-10): 589–602, 2009.

[10] Isa Hafalir and Musab Kurnaz. Discriminatory auctions with resale. *Economic Theory Bulletin*, pages 1–17, 2018.

[11] René Kirkegaard. Asymmetric first price auctions. *Journal of Economic Theory*, 144(4):1617–1635, 2009.

[12] Vijay Krishna. *Auction theory*. Academic press, 2009.

[13] Vijay Krishna and John Morgan. An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory*, 72(2):343–362, 1997.

[14] Michael Landsberger, Jacob Rubinstein, Elmar Wolfstetter, and Shmuel Zamir. First–price auctions when the ranking of valuations is common knowledge. *Review of Economic Design*, 6(3-4):461–480, 2001.

[15] Bernard Lebrun. Comparative statics in first price auctions. *Games and Economic Behavior*, 25(1):97–110, 1998.

[16] Bernard Lebrun. First price auctions in the asymmetric n bidder case. *International Economic Review*, 40(1):125–142, 1999.

[17] Bernard Lebrun. First-price auctions with resale and with outcomes robust to bid disclosure. *The RAND Journal of Economics*, 41(1):165–178, 2010.

[18] Eric Maskin and John Riley. Asymmetric auctions. *The Review of Economic Studies*, 67(3):413–438, 2000.

[19] Paul R Milgrom and Robert J Weber. A theory of auctions and competitive bidding. *Econometrica: Journal of the Econometric Society*, pages 1089–1122, 1982.

[20] Roger B Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981.

[21] Ron Siegel. Asymmetric all-pay auctions with interdependent valuations. *Journal of Economic Theory*, 153:684–702, 2014.

[22] Vasilis Syrgkanis, David Kempe, and Eva Tardos. Information asymmetries in common-value auctions with discrete signals. *Mathematics of Operations Research*, 2019.

[23] Gábor Virág. First-price auctions with resale: the case of many bidders. *Economic Theory*, 52(1):129–163, 2013.

[24] Gábor Virág. Auctions with resale: Reserve prices and revenues. *Games and Economic Behavior*, 99:239–249, 2016.