FURTHER RESULTS ON THE DIVISION PROBLEM*

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Abstract

We consider the problem of dividing one unit of an infinitely divisible object amongst some agents. First, we provide all single-peaked domains where a division rule satisfies efficiency, strategy-proofness, and equal treatment of equals if and only if it is the uniform rule. Our characterisation depends on the number of agents. We also provide a class of division rules satisfying these properties on the domains on which the said characterization does not hold. Second, we consider domains that are not necessarily single-peaked and provide a characterization of maximal non-single-peaked domains on which a division rule satisfies efficiency, strategy-proofness, and equal treatment of equals if and only if it is the uniform rule. Third, we consider the case where the shares of the agents are bounded and provide a characterisation of the division rules on single-peaked domains satisfying efficiency, strategy-proofness, and equal treatment of equals. Finally, we drop the requirement of equal treatment of equals and provide a functional form characterization of efficient and strategy-proof division rules under replacement monotonicity and some responsiveness assumption.

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1. INTRODUCTION

We consider the problem of dividing one unit of an infinitely divisible good amongst a number of agents. Each agent has a single-peaked preference over the possible shares. A division rule decides a share for each agent at every collection of preferences.

A division rule is efficient if it is not possible to give everybody a better share. It is strategy-proof if no agent can strictly benefit by misreporting his/her preferences. It is said to satisfy equal treatment of equals if whenever two agents have the same preferences, they get the same amount of shares.

Theorem 3.2 of this paper provides a characterization of all single-peaked domains on which a division rule satisfies efficiency, strategy-proofness, and equal treatment of equals if and only if it is the uniform rule. Sprumont (1991) shows the same result but only for single-peaked domains containing all continuous single-peaked preferences. It follows from our result that the uniform rule is the only division rule satisfying efficiency, strategy-proofness, and equal treatment of equals on single-peaked domains such as Euclidean, generalized Euclidean, etc., that arise in practical problems. Next, we consider domains that do not satisfy our characterization and provide a class of non-uniform rules on those that satisfy efficiency, strategy-proofness, and equal treatment of equals.

Next, we consider the case where preferences need not be single-peaked. While single-peakedness over share is plausible, non-single-peaked preferences can arise when agents have two or more favourite shares. For instance, if the total amount of administrative work has to be divided amongst the faculties, some faculty might prefer to have either no administrative work or a lot of administrative work in a semester. This is because, by doing a lot of administrative work a faculty may get the next semester free from administrative work, while doing moderate administrative work every semester might just cause disturbance for him/her to do research. We provide a characterization of maximal such non-single-peaked domains on which a division rule satisfies efficiency, strategy-proofness, and equal treatment of equals if and only if it is the uniform rule.

We consider the situation where there are restrictions on the shares of the agents. Such a situation arises when some government property (land, etc.) has to be divided amongst its citizens and for Egalitarian reasons the planner might like to ensure that nobody gets too small or too high shares. There can be other reasons for having such restrictions as well. We provide a characterization of the division rules on the single-peaked domain satisfying efficiency, strategy-proofness, and equal treatment of equals when shares of the agents are restricted.

Finally, we relax the assumption of equal treatment of equals. Although equal treatment of equals is a fairness property, the social planner might like to distinguish agents based on characteristics such as age, sex, financial strength, etc. We discuss that the structure of such rules are not quite tractable. In view of this, we impose two additional mild restrictions: replacement monotonicity and responsiveness. Replacement monotonicity says that if the share of an agent decreases (increases), then the shares of all other agents will increase (decrease). Responsiveness ensures some consistency property of the division rules. We provide a characterization of division rules that satisfy efficiency, strategy-proofness, replacement monotonicity, and responsiveness on the maximal single-peaked domain.

Barberà et al. (1997) provide a characterization of division rules that satisfy efficiency, strategy-proofness, and replacement monotonicity on the maximal single-peaked domain. Their characterization result involves some other function (that they denote by g) which they require to satisfy some additional properties. However, to the best of our understanding, characterizing g is a hard problem. The results in Ching (1994) follows as a corollary of our result.

2. Model

2.1 Domains and their properties

Let $N = \{1, \ldots, n\}$ be the set of agents who must share one unit of some perfectly divisible good. Each agent $i \in N$ has a preference R_i which is a complete and transitive binary relation on $[0, 1]$. For all $x, y \in [0, 1]$, xR_iy means consuming a quantity x of the good is, from i's viewpoint, at least as good as consuming a quantity y. Strict preference of R_i is denoted by P_i, indifference by I_i. A preference R_i is continuous if for each $x \in [0, 1]$, $\{y \in [0, 1] \mid yR_ix\}$ and $\{y \in [0, 1] \mid xR_iy\}$ are closed sets.

We assume preferences are single-peaked which is defined as follows. A preference R_i is single-peaked if there exists $\tau(R_i) \in [0, 1]$, called the peak of R_i, such that for all $x, y \in [0, 1]$

$$[\tau(R_i) < x < y] \text{ or } [y < x < \tau(R_i)] \implies [\tau(R_i) P_i x P_i y].$$

Thus, a preference is single-peaked if it decreases as one goes far from its peak (in one particular direction). Throughout this paper we denote by $\mathcal{S}$ a set of single-peaked preferences. Next, we introduce the notion of minimal richness for a single-peaked domain.

Definition 2.1. A single-peaked domain $\mathcal{S}$ is said to satisfy minimal richness if for all $x \in [0,1]$, there exists a continuous preference $R \in \mathcal{S}$ such that $\tau(R) = x$.

We let $R_N = (R_i)_{i \in N} \in \mathcal{S}^n$ denote the announced preferences (also called a profile) of all agents and R_{-i} denote $(R_j)_{j \in N \setminus i}$ for $i \in N$. For a profile R_N, we define $\tau(R_N) = (\tau(R_1), \ldots, \tau(R_n))$ as the collection of peaks at the profile R_N. We let $\mathcal{S}^n_+ = \{R_N \in \mathcal{S}^n \mid \sum\limits_{i=1}^{n} \tau(R_i) \geq 1\}$ denote the profiles where the total demand is at least 1 and let $\mathcal{S}^n_- = \{R_N \in \mathcal{S}^n \mid \sum\limits_{i=1}^{n} \tau(R_i) \leq 1\}$ denote the profiles where the total demand is at most 1.

2.2 Division rules and their properties

In this section, we introduce the notion of division rules and discuss their properties.

Let Δ_n be the set $\{(x_1, \ldots, x_n) \in [0,1]^n \mid \sum\limits_{i=1}^{n} x_i = 1\}$ of all divisions of the good amongst n agents. A division rule f is a function $f : \mathcal{S}^n \to \Delta_n$. In other words, a division rule decides a division of the good at every given profile. For a division rule f, a profile R_N, and an agent $i \in N$, we denote by $f_i(R_N)$ the share of agent i at the profile R_N by the rule f. Below, we mention some desirable properties of a division function.

Efficiency says that if the total demand at a profile, i.e., the sum of the peaks at that profile, is weakly less than the total available amount 1 (or weakly bigger than that), then each agent will receive a share that is weakly bigger than (or weakly lesser than) his/her peak. This ensures that the outcome is Pareto efficient, that is, cannot be modified in a way so that it is weakly better for everybody and strictly better for somebody.

Definition 2.2. A division rule $f : \mathcal{S}^n \to \Delta_n$ is **efficient** if for all $R_N \in \mathcal{S}^n$,

$$\left[R_N \in \mathcal{S}^n_-\right] \implies [f_i(R_N) \geq \tau(R_i) \text{ for all } i \in N], \text{ and}$$
$$\left[R_N \in \mathcal{S}^n_+\right] \implies [f_i(R_N) \leq \tau(R_i) \text{ for all } i \in N].$$

Strategy-proofness ensures that if an agent misreports his/her preferences, then he/she will not get a share that is strictly preferred for him/her.

Definition 2.3. A division rule $f : \mathcal{S}^n \to \Delta_n$ is **strategy-proof** if for all $i \in N$, all $R_N \in \mathcal{S}^n$, and all $R'_i \in \mathcal{S}$, we have

$$f_i(R_N) R_i f_i(R'_i, R_{-i}).$$

Equal treatment of equals says that if two agents have the same preference, they will get an equal share of the good.

Definition 2.4. A division rule $f : \mathcal{S}^n \to \Delta_n$ satisfies **equal treatment of equals** if for all $i, j \in N$ and all $R_N \in \mathcal{S}^n$, we have

$$[R_i = R_j] \implies [f_i(R_N) = f_j(R_N)].$$

Now, we introduce the notion of the uniform rule (Benassy (1982)).

Definition 2.5. A division rule $u : \mathcal{S}^n \to \Delta_n$ is called the **uniform rule** if for all $R_N \in \mathcal{S}^n$ and all $i \in N$,

$$u_i(R_N) = \begin{cases} \min\{\tau(R_i), \lambda(R_N)\} \text{ if } R_N \in \mathcal{S}^n_+, \text{ and} \\ \max\{\tau(R_i), \mu(R_N)\} \text{ if } R_N \in \mathcal{S}^n_-, \end{cases}$$

where $\lambda(R_N) \geq 0$ solves the equation $\sum_{i \in N} \min\{\tau(R_i), \lambda(R_N)\} = 1$ and $\mu(R_N) \geq 0$ solves the equation $\sum_{i \in N} \max\{\tau(R_i), \mu(R_N)\} = 1$.

In what follows, we explain the uniform rule by means of an example

3. A CHARACTERIZATION OF DOMAINS FOR UNIFORM RULE

In this section, we present a characterization of the domains on which every division rule satisfying efficiency, strategy-proofness, and equal treatment of equals is the uniform rule. Our characterization depends on the number of agents. We consider different cases based on this number.

3.1 THE CASE OF TWO AGENTS

First, we present a condition that we use in our characterization result. *Condition U for two agents* says that for every interval (x, y) not containing the point $\frac{1}{2}$, there is a preference with peak in that interval such that the boundary point of the interval that is closer to $\frac{1}{2}$ is strictly preferred to the other one, that is, if $(x,y) \subseteq [0, \frac{1}{2}]$ then y is preferred to x, and if $(x,y) \subseteq [\frac{1}{2}, 1]$ then x is preferred to y according to that preference.

Definition 3.1. A domain $\mathcal{S}$ is said to satisfy **Condition U for two agents** if

(i) for all intervals $(x,y) \subseteq [0, \frac{1}{2}]$ there exists $R \in \mathcal{S}$ with $\tau(R) \in (x,y)$ such that yPx, and

(ii) for all intervals $(x,y) \subseteq [\frac{1}{2}, 1]$ there exists $R \in \mathcal{S}$ with $\tau(R) \in (x,y)$ such that xPy.

Our next theorem provides a characterization of all domains on which every division rule for two agents satisfying efficiency, strategy-proofness, and equal treatment of equals is the uniform rule.

Theorem 3.1. *(i) Suppose a minimally rich single-peaked domain $\mathcal{S}$ satisfies Condition U for two agents. Then, a division rule $f : \mathcal{S}^2 \to \Delta_n$ satisfies efficiency, strategy-proofness, and equal treatment of equals if and only if it is the uniform rule.*
(ii) Suppose a single-peaked domain $\mathcal{S}$ does not satisfy Condition U for two agents. Then, there is a division rule $f : \mathcal{S}^2 \to \Delta_n$ other than the uniform rule that satisfies efficiency, strategy-proofness, and equal treatment of equals.

Proof. The proof of the theorem is relegated to Appendix A. ■

3.2 THE CASE OF MORE THAN TWO AGENTS

In this section, we provide a characterization of all domains on which every division rule for more that two agents satisfying efficiency, strategy-proofness, and equal treatment of equals is the uniform rule. We use the following condition in our characterization. *Condition U for n agents*, where $n > 2$, is a stricter version of Condition U for 2 agents. It imposes (i) and (ii) of the latter condition with the (obvious) modification that $\frac{1}{2}$ is now replaced by $\frac{1}{n}$. However, it additionally imposes two other conditions that are, in a sense, partial complements of (i) and (ii) in the Condition U for 2 agents. Recall that (i) of the said condition says that for every subset (x,y) of $[0, \frac{1}{2}]$, there is a preference with the peak in that interval according to which y is preferred to x. Part (iii) of Condition U for n agents requires that for such intervals (now subsets of $[0, \frac{1}{n}]$), there is a preference according to which x is preferred to y. In a similar manner, (iv) of Condition U for n agents is kind of the complement of (ii) of Condition U for two agents. However, in contrast to (ii), (iv) is imposed only on the intervals that are subsets of $[\frac{1}{n}, \frac{1}{2}]$. It requires for such intervals that there is a preference with peak in that interval according to which y is preferred to x.

Note that combining (i) and (iii), and (ii) and (iv) in Condition U for n agents, it follows that for every interval (x, y) such that either $(x, y) \subseteq [0, \frac{1}{n}]$ or $(x, y) \subseteq [\frac{1}{n}, \frac{1}{2}]$, there are two preferences with the peaks in that interval such that preference over x and y is reversed in those two preferences, that is, according to one of them, x is preferred to y, and according to the other, y is preferred to x. Apart from the said implication, Condition (ii) additionally imposes some restrictions on intervals that are subsets of $[\frac{1}{n}, 1]$.

Definition 3.2. A domain $\mathcal{S}$ is said to satisfy **Condition U for n agents**, where $n > 2$, if

(i) for all intervals $(x, y) \subseteq [0, \frac{1}{n}]$, there exists $R \in \mathcal{S}$ with $\tau(R) \in (x, y)$ such that yPx,

(ii) for all intervals $(x, y) \subseteq [\frac{1}{n}, 1]$, there exists $R \in \mathcal{S}$ with $\tau(R) \in (x, y)$ such that xPy,

(iii) for all intervals $(x, y) \subseteq [0, \frac{1}{n}]$, there exists $R \in \mathcal{S}$ with $\tau(R) \in (x, y)$ such that xPy, and

(iv) for all intervals $(x, y) \subseteq [\frac{1}{n}, \frac{1}{2}]$, there exists $R \in \mathcal{S}$ with $\tau(R) \in (x, y)$ such that yPx.

Our next theorem presents a characterization of all domains on which every division rule for more than two agents satisfying efficiency, strategy-proofness, and equal treatment of equals is the uniform rule.

Theorem 3.2. *(i) Suppose $n > 2$ and let a minimally rich single-peaked domain $\mathcal{S}$ satisfy Condition U for n agents. Then, a division rule $f : \mathcal{S}^n \to \Delta_n$ satisfies efficiency, strategy-proofness, and equal treatment of equals if and only if it is the uniform rule.*
(ii) Suppose $n > 2$ and let a single-peaked domain $\mathcal{S}$ do not satisfy Condition U for n agents. Then, there is a division rule $f : \mathcal{S}^n \to \Delta_n$ other than the uniform rule that satisfies efficiency, strategy-proofness, and equal treatment of equals.

Proof. The proof of the theorem is relegated to Appendix B. ■

4. VIOLATION OF CONDITION U

In this section, we investigate the structure of division rules that evolve if a domain fails to satisfy Condition U (for 2 of n agents). As before, we treat the cases of 2 agents and more than 2 agents separately.

4.1 The case of two agents

Note that a domain violates Condition U for two agents if there is an interval (x, y) not containing the point $\frac{1}{2}$ such that for each preferences with the peak in that interval, the boundary point $z \in \{x, y\}$ of the interval that is farther away from $\frac{1}{2}$ is weakly preferred to the other one. Below, we present this observation formally.

Observation 4.1. *A domain $\mathcal{S}$ is said to violate Condition U for 2 agents on an interval (x, y) with $\frac{1}{2} \notin (x, y)$ if for all $R \in \mathcal{S}$, $\tau(R) \in (x, y)$ implies*

(i) xRy *if* $(x, y) \subseteq [0, \frac{1}{2}]$, *and*

(ii) yRx *if* $(x, y) \subseteq [\frac{1}{2}, 1]$.

To ease our presentation, for two subsets A and B of $[0, 1]$, we write $A < B$ to mean that each element of A is less than each element of B, that is, $a < b$ for all $a \in A$ and all $b \in B$. Similarly, for a number x and an interval (a, b), we write $x < (a, b)$ to mean that $x < a$. We use similar notations without further explanation.

We say a domain violates Condition U for 2 agents on intervals $(x_1, y_1), \ldots, (x_{k_1}, y_{k_1})$ and on intervals $(w_1, z_1), \ldots, (w_{k_2}, z_{k_2})$, where $0 < (x_1, y_1) < \cdots < (x_{k_1}, y_{k_1}) < \frac{1}{2} < (w_1, z_1) < \cdots < (w_{k_2}, z_{k_2}) < 1$, if each interval in this collection satisfies the corresponding condition (based on whether it is less than or bigger than $\frac{1}{2}$) in Observation 4.1. In what follows, we present a class of division rules on such domains that are different from the uniform rule.

To help the reader, we first present this division rule for a domain that violates Condition U for 2 agents only on two intervals (x, y) and (w, z), where $(x, y) < \frac{1}{2} < (w, z)$. We call these rules adjusted uniform rules. These rules behave like the uniform rule at every profile except a few where they adjust the outcome of the uniform rule by giving some lesser preferred amount to some particular agent i. These profiles are those where (i) the total demand (that is, the sum of the peaks) is at least 1 and agent i's peak is in the interval $[x, y)$, or (ii) the total demand is at most 1 and agent i's peak is in the interval $(w, z]$. In Case (i), $x + \tau(R_j) \geq 1$ implies that agent i gets x and the other agent j gets the rest, and $x + \tau(R_j) < 1$ implies agent j gets his/her top and agent i gets the rest. In Case (ii), $z + \tau(R_j) \leq 1$ implies agent i gets z and agent j gets the rest, and $z + \tau(R_j) > 1$ implies agent j gets his/her top and agent i gets the rest. Note that in both Case (i) and Case (ii), agent i would get his/her peak and agent j would get the rest by the uniform rule. Thus, these rules are in a sense negatively biased towards the agent i in comparison with the uniform rule. For ease of presentation, we just mention the outcome share of one agent, that of the other agent is the remaining share.

Definition 4.1. A division rule $f : \mathcal{S}^2 \to \Delta_2$ is said to be an **adjusted uniform rule for 2 agents with respect to intervals** (x, y) and (w, z) if there exists an agent $i \in N$, such that

(i) for all $(R_1, R_2) \in \mathcal{S}^2_+$ with $\tau(R_i) \in [x, y)$, we have

  (a) $x + \tau(R_j) \geq 1 \implies f_j(R_N) = 1 - x$,

  (b) $x + \tau(R_j) < 1 \implies f_j(R_N) = \tau(R_j)$,

(ii) for all $(R_1, R_2) \in \mathcal{S}^2_-$ with $\tau(R_i) \in (w, z]$, we have

  (a) $z + \tau(R_j) \leq 1 \implies f_j(R_N) = 1 - z$,

  (b) $z + \tau(R_j) > 1 \implies f_j(R_N) = \tau(R_j)$, and

(iii) for all other profiles $(R_1, R_2) \in \mathcal{S}^2$, $f(R_1, R_2) = u(R_1, R_2)$.

We are now ready to present our general class of division rules on domains that violate Condition U for 2 agents on arbitrary class of intervals. These rules treat each interval below $\frac{1}{2}$ and each interval above $\frac{1}{2}$ in the same way as the adjusted uniform rule presented above treats the intervals (x, y) and (w, z), respectively.

Definition 4.2. Let $(x_1, y_1), \ldots, (x_{k_1}, y_{k_1})$ and $(w_1, z_1), \ldots, (w_{k_2}, z_{k_2})$ be a collection of intervals such that $0 < (x_1, y_1) < \cdots < (x_{k_1}, y_{k_1}) < \frac{1}{2} < (w_1, z_1) < \cdots < (w_{k_2}, z_{k_2}) < 1$. A division rule $f : \mathcal{S}^2 \to \Delta_2$ is said to be an **adjusted uniform rule for 2 agents with respect to intervals** $(x_1, y_1), \ldots, (x_{k_1}, y_{k_1})$ and $(w_1, z_1), \ldots, (w_{k_2}, z_{k_2})$ if there exists an agent $i \in N$, such that

(i) for all $(R_1, R_2) \in \mathcal{S}^2_+$ such that there exists $r \in \{1, \ldots, k_1\}$ with $\tau(R_i) \in [x_r, y_r)$, we have

  (a) $x_r + \tau(R_j) \geq 1 \implies f_j(R_N) = 1 - x_r$,

  (b) $x_r + \tau(R_j) < 1 \implies f_j(R_N) = \tau(R_j)$,

(ii) for all $(R_1, R_2) \in \mathcal{S}^2_-$ such that there exists $s \in \{1, \ldots, k_2\}$ with $\tau(R_i) \in (w_s, z_s]$, we have

  (a) $z_s + \tau(R_j) \leq 1 \implies f_j(R_N) = 1 - z_s$,

(b) $z_s + \tau(R_j) > 1 \implies f_j(R_N) = \tau(R_j)$, and

(iii) for all other profiles $(R_1, R_2) \in \mathcal{S}^2$, $f(R_1, R_2) = u(R_1, R_2)$.

Clearly, adjusted uniform rules are different from the uniform rule. Our next theorem says that $\star$ rules satisfy efficiency, strategy-proofness, and equal treatment of equals on a domain that violates Condition U for 2 agents on intervals $(x_1, y_1), \ldots, (x_{k_1}, y_{k_1})$ and $(w_1, z_1), \ldots, (w_{k_2}, z_{k_2})$, where $0 < (x_1, y_1) < \cdots < (x_{k_1}, y_{k_1}) < \frac{1}{2} < (w_1, z_1) < \cdots < (w_{k_2}, z_{k_2}) < 1$.

Theorem 4.1. *Let $(x_1, y_1), \ldots, (x_{k_1}, y_{k_1})$ and $(w_1, z_1), \ldots, (w_{k_2}, z_{k_2})$ be a collection of intervals such that $0 < (x_1, y_1) < \cdots < (x_{k_1}, y_{k_1}) < \frac{1}{2} < (w_1, z_1) < \cdots < (w_{k_2}, z_{k_2}) < 1$ and let $\mathcal{S}$ be a single-peaked domain that violates Condition U for 2 agents on these intervals. Then, every adjusted uniform rule for 2 agents $f : \mathcal{S}^2 \to \Delta_2$ satisfies efficiency, strategy-proofness, and equal treatment of equals.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.1. So we left it to the reader. ■

4.2 THE CASE OF n AGENTS

As we have mentioned earlier, (i) and (ii) of Condition U for n agents are suitable adaptations (with $\frac{1}{2}$ being replaced by $\frac{1}{n}$) of (i) and (ii) of Condition U for 2 agents. Thus, if a domain violates any of these conditions, then suitably modified (for n agents) versions of adjusted uniform rules will satisfy efficiency, strategy-proofness, and equal treatment of equals. For the sake of completeness, we present these rules below. As before, to help the reader, we first present these rules for the case where a domain violates (i) and (ii) on just two intervals (x, y) and (w, z) such that $0 < (x, y) < \frac{1}{n} < (w, z) < 1$.

To describe the rules formally, we first introduce a generalized version of the uniform rule. While the uniform rule divides 1 amount of the good amongst all the agents, a generalized uniform rule does the same for arbitrary amount of the good amongst arbitrary subsets of agents. It has a similar formulation as the uniform rule.

To ease the presentation, we introduce the following notations. For an amount $x \in [0, 1]$ of the good and a subset $\bar{N} = \{1, \ldots, |\bar{N}|\} \subseteq N$ of agents, we denote by $\Delta^x_{|\bar{N}|}$ the set of all divisions of the amount x amongst the agents in $\bar{N}$, that is, $\Delta^x_{|\bar{N}|} = \{(x_1, \ldots, x_{|\bar{N}|}) \in [0, 1]^{|\bar{N}|} \mid \sum_{j=1}^{|\bar{N}|} x_j = x\}$.

Definition 4.3. For $\bar{N} \subseteq N$ and $x \in [0,1]$, a division rule $u^{(x,\bar{N})} : \mathcal{S}^{|\bar{N}|} \to \Delta^{x}_{|\bar{N}|}$ is called the **generalized uniform rule for** $(x, \bar{N})$ if for all $R_{\bar{N}} \in \mathcal{S}^{|\bar{N}|}$ and all $i \in \bar{N}$,

$$u_i^{(x,\bar{N})}(R_{\bar{N}}) = \begin{cases} \min\{\tau(R_i), \lambda(R_{\bar{N}})\} \text{ if } \sum\limits_{i \in \bar{N}} \tau(R_i) \geq x, \text{ and} \\ \max\{\tau(R_i), \mu(R_{\bar{N}})\} \text{ if } \sum\limits_{i \in \bar{N}} \tau(R_i) < x, \end{cases}$$

where $\lambda(R_{\bar{N}}) \geq 0$ solves the equation $\sum\limits_{i \in \bar{N}} \min\{\tau(R_i), \lambda(R_{\bar{N}})\} = x$ and $\mu(R_{\bar{N}}) \geq 0$ solves the equation $\sum\limits_{i \in \bar{N}} \max\{\tau(R_i), \mu(R_{\bar{N}})\} = x$.

Note that when $x = 1$ and $\bar{N} = N$, the rule $u^{(x,\bar{N})}$ boils down to the uniform rule u.

As before, we only specify the shares of $n-1$ agents in an outcome, the remaining agent gets the remaining share. An adjusted uniform rule for n agents behaves in the same manner as an adjusted uniform rule for 2 agents with the modification that the shares of the agents other than the "particular agent" i are decided by using a generalized uniform rule.

Definition 4.4. A division rule $f : \mathcal{S}^n \to \Delta_n$ is said to be an **adjusted uniform rule for** n **agents with respect to intervals** (x, y) and (w, z), where $0 < (x, y) < \dfrac{1}{n} < (w, z) < 1$, if there exists an agent $i \in N$, such that

(i) for all $R_N \in \mathcal{S}^n_+$ with $\tau(R_i) \in [x, y)$ and $\tau(R_j) \geq y$ for all $j \neq i$, we have

   (a) $x + \sum\limits_{j \neq i} \tau(R_j) \geq 1 \implies f_j(R_N) = u_j^{1-x}(R_{N \setminus i})$ for all $j \neq i$,

   (b) $x + \sum\limits_{j \neq i} \tau(R_j) < 1 \implies f_j(R_N) = \tau(R_j)$ for all $j \neq i$,

(ii) for all $R_N \in \mathcal{S}^n_-$ with $\tau(R_i) \in (w, z]$ and $\tau(R_j) \leq w$ for all $j \neq i$, we have

   (a) $z + \sum\limits_{j \neq i} \tau(R_j) \leq 1 \implies f_j(R_N) = u_j^{1-z}(R_{N \setminus i})$ for all $j \neq i$,

   (b) $z + \sum\limits_{j \neq i} \tau(R_j) > 1 \implies f_j(R_N) = \tau(R_j)$ for all $j \neq i$, and

(iii) for all other profiles $R_N \in \mathcal{S}^n$, $f(R_N) = u(R_N)$.

We now present the adjusted uniform rules for n agents for the general case where a domain violates (i) and (ii) of Condition U for n agents on an arbitrary collection of intervals.

Definition 4.5. Let $(x_1, y_1), \ldots, (x_{k_1}, y_{k_1})$ and $(w_1, z_1), \ldots, (w_{k_2}, z_{k_2})$ be a collection of intervals such that $0 < (x_1, y_1) < \cdots < (x_{k_1}, y_{k_1}) < \frac{1}{n} < (w_1, z_1) < \cdots < (w_{k_2}, z_{k_2}) < 1$. A division rule $f : \mathcal{S}^n \to \Delta_n$ is said to be an **adjusted uniform rule for n agents with respect to intervals** $(x_1, y_1), \ldots, (x_{k_1}, y_{k_1})$ and $(w_1, z_1), \ldots, (w_{k_2}, z_{k_2})$ if there exists an agent $i \in N$, such that

(i) for all $R_N \in \mathcal{S}^n_+$ such that there exists $r \in \{1, \ldots, k_1\}$ with $\tau(R_i) \in [x_r, y_r)$ and $\tau(R_j) \geq y_r$ for all $j \neq i$, we have

   (a) $x_r + \sum_{j \neq i} \tau(R_j) \geq 1 \implies f_j(R_N) = u_j^{1-x_r}(R_{N \setminus i})$ for all $j \neq i$,

   (b) $x_r + \sum_{j \neq i} \tau(R_j) < 1 \implies f_j(R_N) = \tau(R_j)$ for all $j \neq i$,

(ii) for all $R_N \in \mathcal{S}^n_-$ such that there exists $s \in \{1, \ldots, k_2\}$ with $\tau(R_i) \in (w_s, z_s]$ and $\tau(R_j) \leq w_s$ for all $j \neq i$, we have

   (a) $z_s + \sum_{j \neq i} \tau(R_j) \leq 1 \implies f_j(R_N) = u_j^{1-z_s}(R_{N \setminus i})$ for all $j \neq i$,

   (b) $z_s + \sum_{j \neq i} \tau(R_j) > 1 \implies f_j(R_N) = \tau(R_j)$ for all $j \neq i$, and

(iii) for all other profiles $R_N \in \mathcal{S}^n$, $f(R_N) = u(R_N)$.

Our next theorem says that adjusted uniform rules satisfy efficiency, strategy-proofness, and equal treatment of equals on a domain that violates (i) and (ii) of Condition U for n agents.

Theorem 4.2. *Let $(x_1, y_1), \ldots, (x_{k_1}, y_{k_1})$ and $(w_1, z_1), \ldots, (w_{k_2}, z_{k_2})$ be a collection of intervals such that $0 < (x_1, y_1) < \cdots < (x_{k_1}, y_{k_1}) < \frac{1}{n} < (w_1, z_1) < \cdots < (w_{k_2}, z_{k_2}) < 1$ and let $\mathcal{S}$ be a single-peaked domain that violates (i) and (ii) of Condition U for n agents on these intervals. Then, every adjusted uniform rule for n agents $f : \mathcal{S}^n \to \Delta_n$ satisfies efficiency, strategy-proofness, and equal treatment of equals.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.2. So we left it to the reader. ∎

Next, we investigate what happens if a domain violates (iii) or (iv) of Condition U for n agents. Note that a domain violates (iii) or (iv) if either there is an interval $(x, y) \subseteq [0, \frac{1}{n}]$ such that y is weakly preferred to x for every preference with peak in that interval, or there is an interval $(x, y) \subseteq [\frac{1}{n}, \frac{1}{2}]$ such that x is weakly preferred to y for every preference with peak in that interval.

Observation 4.2. *A domain $\mathcal{S}$ is said to violate (iii) or (iv) of Condition U for n agents on an interval* $(x,y) \subseteq [0, \frac{1}{2}]$ *with* $\frac{1}{n} \notin (x,y)$ *if for all* $R \in \mathcal{S}$, $\tau(R) \in (x,y)$ *implies*

(i) yRx *if* $(x,y) \subseteq [0, \frac{1}{n}]$, *and*

(ii) xRy *if* $(x,y) \subseteq [\frac{1}{n}, \frac{1}{2}]$.

In what follows, we present a class of division rules on domains that violate (iii) and (iv) of Condition U for n agents. We call these rules *adjusted* uniform rule*. For simplicity, we present them for the case where there are exactly two intervals (x,y) and (w,z) with $0 < (x,y) < \frac{1}{n} < (w, z) < \frac{1}{2}$ on which (iii) or (iv) of Condition U for n agents is violated. Versions of these rules for other cases (that is, when the said condition is violated on multiple intervals) can be obtained in a similar way as we have done in Definition 4.5. Recall that an adjusted uniform rule for n agents is negatively biased towards a particular agent who we have denoted by i. Adjusted* uniform rules too are negatively biased towards some agent i, however for the ease of presentation, we present these rules for the case where $i = n-1$. Versions of these rules for arbitrary values of i can be constructed symmetrically.

We first explain an adjusted* uniform rule for the case when there is an interval (x,y) on which (iii) or (iv) of Condition U for n agents is violated. For ease of presentation, we use the notation (x_{n-1}, y_{n-1}) to denote the interval (x,y). We do this because, as we have mentioned, we present these rules such that agent $n-1$ is treated differently (in fact, negatively). Such a rule is based on a collection of parameters: $n-2$ points $x_1, \ldots, x_{n-2}$ and two intervals $(x_{n-1}, y_{n-1}), (x_n, y_n)$ such that $\{x_1, \ldots, x_{n-2}\} < (x_{n-1}, y_{n-1}) < (x_n, y_n)$. An adjusted* uniform rule coincides with the uniform rule at all profiles except a few as follows. Consider an arbitrary profile with total demand at most 1 such that agents $1, \ldots, n-2$ have peaks $x_1, \ldots, x_{n-2}$, respectively, and agent $n-1$ has peak in the interval (x_{n-1}, y_{n-1}). Adjusted* uniform rule says that (a) if agent n has peak in the interval (x_n, y_n), then everybody except agent $n-1$ will get their peaks, and (b) if agent n's peak is weakly less than x_n, then everybody except agents $n-1$ and n will get their peaks, and agent n will get x_n. Note that for the uniform rule, agent $n-1$ would get $\tau(R_{n-1})$ in case (a) and $\tau(R_{n-1})$ in case (b). Thus, adjusted* uniform rules are negatively biased towards some particular agents, who, in our case, is agent $n-1$. The behaviour of this rule with respect to an interval (w,z) (which we denote by (w_{n-1}, z_{n-1})) on which (iii) or (iv) of Condition U for n

agents is violated is symmetric. In what follows, we present a formal definition of these rules considering both the situations (that is, violation of both (iii) and (iv) of Condition U for n agents).

Definition 4.6. Let $\{x_1, \ldots, x_{n-2}\} < (x_{n-1}, y_{n-1}) < (x_n, y_n)$ and $\{w_1, \ldots, w_{n-2}\} < (w_{n-1}, z_{n-1}) < (w_n, z_n)$ be such that $\sum_{i=1}^{n-2} x_i + x_{n-1} + y_n = \sum_{i=1}^{n-2} x_i + y_{n-1} + x_n = 1$ and $\sum_{i=1}^{n-2} w_i + w + z_n = \sum_{i=1}^{n-2} w_i + z + w_n = 1$. A division rule $f : \mathcal{S}^n \to \Delta_n$ is said to be a an **adjusted* uniform rule**, if

(i) for all $R_N \in \mathcal{S}^n_-$ with $\tau(R_i) = x_i$ for all $i \in \{1, \ldots, n-2\}$ and $\tau(R_{n-1}) \in (x_{n-1}, y_{n-1})$, we have

  (a) $\tau(R_n) \in (x_n, y_n) \implies f_i(R_N) = \tau(R_i)$ for all $i \neq n-1$, and

  (b) $\tau(R_n) \leq x_n \implies f_i(R_N) = \tau(R_i)$ for all $i \in \{1, \ldots, n-2\}$ and $f_n(R_N) = x_n$,

(ii) for all $R_N \in \mathcal{S}^n_-$ with $\tau(R_i) = w_i$ for all $i \in \{1, \ldots, n-2\}$ and $\tau(R_{n-1}) \in (w_{n-1}, z_{n-1})$, we have

  (a) $\tau(R_n) \in (w_n, z_n) \implies f_i(R_N) = \tau(R_i)$ for all $i \neq n-1$, and

  (b) $\tau(R_n) \geq z_n \implies f_i(R_N) = \tau(R_i)$ for all $i \in \{1, \ldots, n-2\}$ and $f_n(R_N) = z_n$,

(iii) for all other $R_N \in \mathcal{S}^n$, we have $f(R_N) = u(R_N)$.

Our next theorem says that adjusted* uniform rules satisfy efficiency, strategy-proofness, and equal treatment of equals on a domain that violates (iii) and (iv) of Condition U for n agents.

Theorem 4.3. *Let (x_{n-1}, y_{n-1}) and (w_{n-1}, z_{n-1}) be two intervals such that $0 < (x_{n-1}, y_{n-1}) < \frac{1}{n} < (w_{n-1}, z_{n-1}) < \frac{1}{2}$ and let $\mathcal{S}$ be a single-peaked domain that violates (iii) and (iv) of Condition U for n agents on these intervals. Then, every adjusted* uniform rule $f : \mathcal{S}^n \to \Delta_n$ satisfies efficiency, strategy-proofness, and equal treatment of equals.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.2. So we left it to the reader. ∎

5. A NECESSARY CONDITION ON AN ARBITRARY DOMAIN FOR THE UNIFORM RULE

In Section , we have provided a necessary and sufficient condition on a single-peaked domain so that the uniform rule is the only division rule satisfying efficiency, strategy-proofness, and equal treatment of equals. Question arises as to whether single-peakedness is necessary for the uniform rule to satisfy these properties. It turns out that the answer is negative, that is, the uniform rule can satisfy these properties even on non-single-peaked domains. In what follows, we provide a necessary condition on a domain so that the uniform rule satisfies efficiency, strategy-proofness, and equal treatment of equals. We further show that under some richness condition, the uniform rule is the unique rule satisfying efficiency, strategy-proofness, and equal treatment of equals on domains satisfying our necessary condition.

A domain (not necessarily single-peaked) is said to satisfy *Condition N* if the following holds: if the peak of a preference is bigger than $\frac{1}{n}$, then preference declines as one moves from the top-ranked alternative towards $\frac{1}{n}$, and all the points that are less than $\frac{1}{n}$ are ranked below $\frac{1}{n}$.[1] Note that there is no restriction on the relative ordering of the points that are less than $\frac{1}{n}$ or bigger than the top-ranked alternative of the preference. Symmetrically opposite thing happens for a preference with top-ranked alternative smaller than $\frac{1}{n}$. Chatterji et al. (2013) introduce the notion of semi-single-peaked domains. It can be verified that Condition N is equivalent to semi single-peakedness with threshold $\frac{1}{n}$.

Definition 5.1. A domain $\mathcal{D}$ satisfies **Condition N** if for all $R \in \mathcal{D}$ the following holds:

(i) $\tau(R) \geq \frac{1}{n}$ implies that for all x, y, z with $0 \leq z < \frac{1}{n} \leq x < y \leq \tau(R) \leq 1$, we have $yPxPz$, and

(ii) $\tau(R) < \frac{1}{n}$ implies that for all x, y, z with $0 \leq \tau(R) \leq y < x \leq \frac{1}{n} < z \leq 1$ we have $yPxPz$.

Our next theorem says that a domain has to satisfy Condition N in order for the uniform rule to satisfy efficiency, strategy-proofness, and equal treatment of equals.

Theorem 5.1. *Let $\mathcal{D}$ be a domain such that the uniform rule $u : \mathcal{D}^n \to \Delta_n$ satisfies efficiency, strategy-proofness, and equal treatment of equals. Then, $\mathcal{D}$ must satisfy Condition N.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.1. So we left it to the reader. ■

Our following theorem asserts that if a domain contains all preferences satisfying Condition N, then the uniform rule is the only rule that satisfies efficiency, strategy-proofness, and equal

[1] By peak, we refer to the top-ranked alternative of a preference.

treatment of equals. Thus, Theorem 5.2 generalizes Theorem 3.2 on arbitrary (non-single-peaked) domains.

Theorem 5.2. *Let $\mathcal{D}$ be the domain containing all preferences satisfying Condition N. Then, every division rule $f : \mathcal{D}^n \to \Delta_n$ satisfying efficiency, strategy-proofness, and equal treatment of equals if and only if it is the uniform rule.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.2. So we left it to the reader. ■

6. Division problems with reservations

In this section, we consider situations where there are bounds on the shares that an agent can receive. For $0 \leq \alpha < \beta \leq 1$, a division rule $f : \mathcal{S}^n \to \Delta_n$ is said to be $[\alpha, \beta]$-restricted if $f_i(R_N) \in [\alpha, \beta]$ for all $i \in N$ and all $R_N \in \mathcal{S}^n$. Note that an $[\alpha, \beta]$-restricted division rule satisfies equal treatment of equals only if $\frac{1}{n} \in [\alpha, \beta]$. So, throughout this section we assume that $\frac{1}{n} \in [\alpha, \beta]$. In what follows, we introduce a particular type of $[\alpha, \beta]$-restricted division rule, which we call the $[\alpha, \beta]$-restricted uniform rule and denote by $u^{[\alpha,\beta]}$.

The $[\alpha, \beta]$-restricted uniform rule works in the following way. Consider an arbitrary profile R_N. Compute the outcome $u(R_N)$ of the uniform rule u at the profile. For each agent i whose share $u_i(R_N)$ is less than α, calculate the deficit $\alpha - u_i(R_N)$ of i's share from α. Finally, calculate the total deficit $\sum_{\{i \mid u_i(R_N) < \alpha\}} (\alpha - u_i(R_N))$ from α of all agents whose shares are less than α. Similarly, calculate the total excess $\sum_{\{i \mid u_i(R_N) > \beta\}} (u_i(R_N) - \beta)$ from β of all agents whose shares are bigger than β.

Consider the case where the total deficit from α is strictly bigger than the total excess from β at $u(R_N)$. The $[\alpha, \beta]$-restricted uniform rule $u^{[\alpha,\beta]}$ gives α to each agent i whose share $u_i(R_N)$ is smaller than α. For all other agents, it fixes a "cut-off" $\nu \in [\alpha, \beta]$ such that (a) if someone's share in $u(R_N)$ is smaller than ν, he/she continues to get the same share as in $u(R_N)$, and (b) if someone's share in $u(R_N)$ is bigger than ν, then he/she gets ν. The cut-off ν is chosen in a way that sum of all shares at $u^{[\alpha,\beta]}(R_N)$ is 1. The case where the total deficit from α is strictly smaller than the total excess from β at $u(R_N)$ is analogous (or, symmetric).

Consider the remaining case where the total deficit from α is equal to the total excess from β at $u(R_N)$. In this case, the $[\alpha, \beta]$-restricted uniform rule $u^{[\alpha,\beta]}$ gives (a) α to each agent i whose share

$u_i(R_N)$ is smaller than α, (b) β to each agent i whose share $u_i(R_N)$ is bigger than β, and (c) the same share as $u(R_N)$ to every other agent.[2]

Definition 6.1. An $[\alpha, \beta]$-restricted division rule $u^{[\alpha,\beta]} : \mathcal{S}^n \to \Delta_n$ is called the $[\alpha, \beta]$-restricted uniform rule if for all $R_N \in \mathcal{S}^n$,

(i) $\sum_{\{i \in N | u_i(R_N) < \alpha\}} \big(\alpha - u_i(R_N)\big) > \sum_{\{i \in N | u_i(R_N) > \beta\}} \big(u_i(R_N) - \beta\big)$ implies

(a) $u_i^{[\alpha,\beta]}(R_N) = \alpha$ for all $i \in N$ with $u_i(R_N) < \alpha$, and

(b) $u_i^{[\alpha,\beta]}(R_N) = \min\{\nu, u_i(R_N)\}$ for all $i \in N$ with $u_i(R_N) \geq \alpha$,

where $\nu \leq \beta$ is such that $\sum_{i=1}^{n} u_i^{[\alpha,\beta]}(R_N) = 1$,

(ii) $\sum_{\{i \in N | u_i(R_N) < \alpha\}} \big(\alpha - u_i(R_N)\big) < \sum_{\{i \in N | u_i(R_N) > \beta\}} \big(u_i(R_N) - \beta\big)$ implies

(a) $u_i^{[\alpha,\beta]}(R_N) = \beta$ for all $i \in N$ with $u_i(R_N) > \beta$, and

(b) $u_i^{[\alpha,\beta]}(R_N) = \max\{\mu, u_i(R_N)\}$ for all $i \in N$ with $u_i(R_N) \leq \beta$,

where $\mu \geq \alpha$ is such that $\sum_{i=1}^{n} u_i^{[\alpha,\beta]}(R_N) = 1$, and

(iii) $\sum_{\{i \in N | u_i(R_N) < \alpha\}} |u_i(R_N) - \alpha| = \sum_{\{i \in N | u_i(R_N) > \beta\}} |u_i(R_N) - \beta|$, then

(a) $u_i^{[\alpha,\beta]}(R_N) = \alpha$ for all $i \in N$ with $u_i(R_N) < \alpha$,

(b) $u_i^{[\alpha,\beta]}(R_N) = \beta$ for all $i \in N$ with $u_i(R_N) > \beta$, and

(c) $u_i^{[\alpha,\beta]}(R_N) = u_i(R_N)$ for all $i \in N$ with $\alpha \leq u_i(R_N) \leq \beta$.

It is left to the reader to verify that the $[\alpha, \beta]$-restricted uniform rule is unique.

Our next theorem provides a characterization of the division rules satisfying efficiency, strategy-proofness, and equal treatment of equals when shares of the agents are bounded. It says that a division rule satisfies the mentioned properties if and only if it is the $[\alpha, \beta]$-restricted uniform rule.

Theorem 6.1. *Let $\mathcal{S}$ be a minimally rich single-peaked domain. Suppose that the shares of the agents must lie in the interval $[\alpha, \beta]$, where $0 \leq \alpha \leq \frac{1}{n} \leq \beta \leq 1$. Then, a division rule satisfies efficiency, strategy-proofness, and equal treatment of equals if and only if it is the $[\alpha, \beta]$-restricted uniform rule.*

[2]This case can be considered as special case of the previous case by replacing the strict inequality between total deficit and total excess by a weak inequality. However, to make the presentation reader friendly, we make it a separate case.

Proof. The proof of this theorem is similar to the proof of Theorem 3.2. So we left it to the reader. ■

7. RELAXING EQUAL TREATMENT OF EQUALS

In this section, we explore what happens to the structure of the division rules if we drop equal treatment of equals. Such a rule can be characterized by means of a property called uncompromisingness. Uncompromisingness says that if an agent moves his/her peak closer or farther away from his/her share (without changing the side with respect to the share), then his/her share cannot be changed. For instance, if an agent's peak is x at a profile and his/her share at that profile is y where $x < y$, then his/her share will continue to be y as long as his/her peak is in the interval $[0, y]$ (while other agents do not change their preferences).

Definition 7.1. A division rule $f : \mathcal{S}^n \to \Delta_n$ satisfies uncompromisingness if for all $R_N \in \mathcal{S}^n$, all $i \in N$, and all $R'_i \in \mathcal{S}$

(i) $f_i(R_N) < \tau(R_i)$ and $f_i(R_N) \leq \tau(R'_i)$ imply $f_i(R_N) = f_i(R'_i, R_{-i})$, and

(ii) $f_i(R_N) > \tau(R_i)$ and $f_i(R_N) \geq \tau(R'_i)$ imply $f_i(R_N) = f_i(R'_i, R_{-i})$.

Our next theorem characterizes the efficient and strategy-proof division rules by means of uncompromisingness. This result can be found in Sprumont (1991) and Barberà et al. (1997).

Note that the above theorem is not so helpful in constructing a division rule that is efficient and strategy-proof. In view of this, we proceed to impose some additional (milder than ETE) restrictions on a division rule so that we can provide a functional form of the division rules satisfying those properties. Our next property is called replacement monotonicity (Barberà et al. (1997)). It says that if the share of an agent changes by some unilateral deviation of that agent, then the shares of every other agents will change in the same direction (that is, either all of them will crease or all of them will decrease).

Definition 7.2. A division rule $f : \mathcal{S}^n \to \Delta_n$ is said to satisfy **replacement monotonicity** if for all $R_N \in \mathcal{S}^n$, all $i \in N$, and all $R'_i \in \mathcal{S}$,

$$f_i(R'_i, R_{-i}) - f_i(R_N) \geq 0 \implies f_j(R'_i, R_{-i}) - f_j(R_N) \leq 0 \text{ for all } j \in N \setminus i.$$

Barberà et al. (1997) provide some structure of division rules that satisfy efficiency, strategy-proofness, and replacement monotonicity. However, they do not provide any functional form characterization, which makes it hard to use their rules in practical problems.

The main problem with characterizing division rules satisfying efficiency and strategy-proofness is that when an agent changes his/her preferences, efficiency and strategy-proofness do not have any control on how the shares of the other agents will change. Imposing non-bossiness does not help as it only takes care of situations when the share of the deviating agent does not change, but when that changes, then, again, one cannot track the shares of the other agents. Therefore, to have a clean characterization, we impose some restriction (together with replacement monotonicity) on how the shares of the other agents will change when that of some agent changes by some unilateral deviation.

In what follows, we present a verbal description of our restriction. Consider a division problem with at least three agents. Consider two arbitrary agents (need not be distinct), say 1 and 2. Consider two situations, say $S1$ and $S2$. In situation $S1$, agent 1 unilaterally changes his/her preference at some profile R_N to some preference R_1', and in situation $S2$, agent 2 unilaterally changes his/her preference at some profile Q_N to some preference Q_2'. Suppose that the situations are such that either both R_N and Q_N are in $\mathcal{S}_+^n$ or both R_N and Q_N are in $\mathcal{S}_-^n$. Suppose further that in situation Si, agent i's share changes by a 'small' (less than some prefixed small amount ϵ) amount by this deviation. Consider two other distinct agents, say 3 and 4. Note that in both the situations, they do not change their preferences. What we want to ensure is that if the shares of these agents change, then the relative changes in their shares will be the same in both the situations. For a clearer description, suppose that the change in the share of agent $j \in \{3, 4\}$ in situation $S \in \{S1, S2\}$ is δ_j^S. For instance, $\delta_3^{S1} = f_3(R_1', R_{-1}) - f_3(R_N)$. The *proportionally responsive to infinitesimal changes in shares* (PRICS) property says that $\frac{\delta_3^{S1}}{\delta_4^{S1}} = \frac{\delta_3^{S2}}{\delta_4^{S2}}$. In other words, whenever the share of some agent changes by a very small amount by his/her unilateral deviation, other agents are 'affected' with a fixed proportion.

To illustrate the idea PRICS, we present a numerical example. Suppose that there are 5 agents. For ease of presentation, we denote a profile by its top-ranked alternatives. For instance, we write $(0.4, 0.2, 0.1, 0.3, 0.2)$ to denote a profile where the peaks of agents 1, 2, 3, 4, and 5 are 0.4, 0.2, 0.1, 0.3, 0.2, respectively. Consider two profiles $(0.3, 0.2, 0.1, 0.1, 0.2)$ and $(0.15, 0.35, 0.1, 0.1, 0.2)$. Note that the total demand at both the profiles is at most 1. Suppose that the outcomes at these profiles by some division rule f are $(0.3, 0.2, 0.15, 0.15, 0.2)$ and $(0.15, 0.35, 0.15, 0.15, 0.2)$. Consider the following two situations. Situation 1: agent 1 changes his/her peak at the profile $(0.3, 0.2, 0.1, 0.1, 0.2)$ to 0.2 and his/her share changes to 0.2 by this unilateral deviation. Situation 2: agent 2 changes his/her peak at the profile $(0.15, 0.35, 0.1, 0.2, 0.1)$ to 0.31 and his/her share changes to 0.31 by this unilateral deviation. Note that the shares of both agents 1 and 2 decrease from the former profile to the latter in situations 1 and 2, respectively. The question is how that "excess" amount (0.1 in Situation 1 and 0.04 in Situation 2) will be distributed amongst the other agents. Loosely put, PRICS says that such a distribution should be done in a consistent manner over the two situations. More formally, if the shares of two agents change, then the relative change of their shares must be the same in both the situations. For instance, if $f(0.2, 0.2, 0.1, 0.1, 0.2) = (0.2, 0.2, 0.2, 0.2, 0.2)$, then the outcome $f(0.15, 0.31, 0.1, 0.1, 0.2)$ has to be $(0.15, 0.31, 0.17, 0.17, 0.2)$. Some other choices of outcomes at the two mentioned profiles are as follows: $((0.2, 0.2, 0.17, 0.23, 0.2)$ and $(0.15, 0.31, 0.16, 0.18, 0.2))$, or $((0.2, 0.2, 0.23, 0.17, 0.2)$ and $(0.15, 0.31, 0.18, 0.16, 0.2))$, etc.

Definition 7.3. A replacement monotonic division rule f is said to be **proportionally responsive to infinitesimal changes in shares** (PRICS) if for all $i, j \in N$ and all $R_N, Q_N \in \mathcal{S}^n$ there exist $R'_i, Q'_j \in \mathcal{S}$ and $\epsilon > 0$ with $0 < f_i(R'_i, R_{-i}) - f_i(R_N) < \epsilon$ and $0 < f_j(Q'_j, Q_{-j}) - f_j(Q_N) < \epsilon$ such that for all $\{k, l\} \subseteq N \setminus \{1, 2\}$ and all $s \in \{k, l\}$,

$$f_s(R_N) - f_s(R'_i, \bar{R}_{-i}) > 0 \text{ and } f_s(Q_N) - f_s(Q'_j, Q_{-j}) > 0$$

imply

$$\frac{f_k(R'_i, R_{-i}) - f_k(R_N)}{f_l(R'_i, R_{-i}) - f_l(R_N)} = \frac{f_k(Q'_j, Q_{-j}) - f_k(Q_N)}{f_l(Q'_j, Q_{-j}) - f_l(Q_N)}, \tag{1}$$

where either $\left[(R'_i, R_{-i}), (Q'_j, Q_{-j}) \in \mathcal{S}^n_+\right]$ or $\left[(R'_i, R_{-i}), (Q'_j, Q_{-j}) \in \mathcal{S}^n_-\right]$.

A special case of PRICS is *equally responsive to infinitesimal changes in shares* (ERICS), where, as the name suggests, the value of the expressions in Equation (1) is 1.[3] Roughly speaking, here non-deviating agents are equally "affected" (if at all they are affected) by unilateral deviations of the others.

[3]That is, $\frac{f_k(\bar{R}'_i, \bar{R}_{-i}) - f_k(\bar{R}_N)}{f_l(\bar{R}'_i, \bar{R}_{-i}) - f_l(\bar{R}_N)} = \frac{f_k(\hat{R}'_j, \hat{R}_{-j}) - f_k(\hat{R}_N)}{f_l(\hat{R}'_j, \hat{R}_{-j}) - f_l(\hat{R}_N)} = 1.$

7.1 A NEW CLASS OF DIVISION RULES

In this section, we define a particular class of allocation functions which we call parametrized uniform rules. These rules are generalizations of the uniform rule. Recall that the uniform rule determines the outcome at a profile R_N by the equation $\min\{\tau(R_i), \lambda(R_N)\}$ or $\min\{\tau(R_i), \mu(R_N)\}$ depending on whether $R_N \in \mathcal{S}^n_+$ or $R_N \in \mathcal{S}^n_-$, where $\lambda(R_N)$ and $\mu(R_N)$ are such that the total share becomes 1. Note that for the uniform rule the outcome at the profile where everybody's peak is 1 turns out to be $(\frac{1}{n}, \ldots, \frac{1}{n})$ and the profile where everybody's peak is 0 turns out to be $(\frac{1}{n}, \ldots, \frac{1}{n})$. For a parametrized uniform rule, these two outcomes are fixed at two prior divisions q^H and q^L. For the uniform rule, the outcomes at the other profiles are determined by giving all the agents equal weights (in a suitable sense), however, for a parametrized uniform rule, these weights are fixed a priori as γ and δ for profiles with total demand is at least 1 and at most 1, respectively.

Definition 7.4. (**Parametrized uniform rules**) An allocation rule f is called parametrized uniform rule with respect to a tuple $(q^H, q^L, \gamma, \delta)$, where $q^H, q^L \in \Delta$ and $\gamma, \delta \in \mathbb{R}^n_{++}$, if for all $i \in N$,

$$f_i(R_N) = \begin{cases} \min\{\tau(R_i), q_i^H + \gamma_i \lambda(R_N)\} \text{ if } R_N \in \mathcal{S}^n_+, \text{ and} \\ \max\{\tau(R_i), q_i^L - \delta_i \mu(R_N)\} \text{ if } R_N \in \mathcal{S}^n_- \end{cases}$$

where $\lambda(R_N) \geq 0$ solves the equation $\sum_{i=1}^{n} \min\{\tau(R_i), q_i^H + \gamma_i \lambda(R_N)\} = 1$ and $\mu(R_N) \geq 0$ solves the equation $\sum_{i=1}^{n} \max\{\tau(R_i), q_i^L - \delta_i \mu(R_N)\}$.

A special case of a parametrized uniform rule with respect to $(q^H, q^L, \gamma, \delta)$ is called *a uniform* rule* if $\gamma_i = \delta_i = 1$ for all $i \in N$. Also, a uniform* rule is called *the uniform rule* if $q_i^H = q_i^L = \frac{1}{n}$ for all $i \in N$. In what follows, we provide an example to illustrate how a parametrized uniform rule works.

Example 7.1. Let $N = \{1, 2, 3, 4, 5\}$. Consider the parametrized uniform rule f with respect to the parameters $(q^H, q^L, \gamma, \delta)$, where $q^H = q^L = (0.3, 0.1, 0.2, 0.4, 0)$ and $\gamma = \delta = (2, 1, 3, 1, 2)$. Take a profile R_N with $\tau(R_N) = (0.4, 0.2, 0.1, 0.3, 0.2)$. Since $R_N \in \mathcal{S}^n_+$, by the definition of parametrized uniform rule, $f_i(R_N) = \min\{\tau(R_i), q_i^H + \gamma_i \lambda(R_N)\}$ for all $i \in N$, where $\lambda(R_N) \geq 0$ solves the equation $\sum_{i=1}^{n} \min\{\tau(R_i), q_i^H + \gamma_i \lambda(R_N)\} = 1$. Take $i \in \{3, 4\}$. Note that $\tau(R_i) < q_i^H$. Therefore, $\min\{\tau(R_i), q_i^H + \gamma_i \lambda(R_N)\} = \tau(R_i)$, and hence $f_i(R_N) = \tau(R_i)$.

By using the values of $f_i(R_N)$ for $i \in \{3,4\}$, we get

$$\begin{aligned} &\min\{0.4, 0.3 + 2\lambda(R_N)\} + \min\{0.2, 0.1 + \lambda(R_N)\} + 0.1 + 0.3 + \min\{0.2, 2\lambda(R_N)\} = 1 \\ \implies &\min\{0.4, 0.3 + 2\lambda(R_N)\} + \min\{0.2, 0.1 + \lambda(R_N)\} + \min\{0.2, 2\lambda(R_N)\} = 0.6. \end{aligned} \tag{2}$$

It is easy to check that if $\lambda(R_N) > 0.05$, then $\min\{0.4, 0.3 + 2\lambda(R_N)\} = 0.4$, $\min\{0.2, 0.1 + \lambda(R_N)\} \geq 0.2$, and $\min\{0.2, 2\lambda(R_N)\} \geq 0.1$. However, then (2) cannot hold. So, it must be that $\lambda(R_N) \leq 0.05$. Note that in that case the minimum value of each of the expressions in the left hand side of (2) will be the term containing $\lambda(R_N)$. Therefore, (2) reduces to

$$\begin{aligned} &0.3 + 2\lambda(R_N) + 0.1 + \lambda(R_N) + 2\lambda(R_N) = 0.6 \\ \implies &0.4 + 5\lambda(R_N) = 0.6 \\ \implies &\lambda(R_N) = 0.04. \end{aligned}$$

Using the value of $\lambda(R_N)$, we obtain $f(R_N) = (0.38, 0.14, 0.1, 0.3, 0.08)$.

7.2 Results

Our first theorem characterizes the division rules satisfying efficiency, strategy-proofness, and PRICS as the parametrized uniform rules.

Theorem 7.1. *Let $\mathcal{S}$ be a single-peaked domain. Then, every division rule $f : \mathcal{S}^n \to \Delta_n$ satisfies efficiency, strategy-proofness, and PRICS if and only if it is a parametrized uniform rule.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.1. So we left it to the reader. ■

The following corollary characterizes the division rules satisfying efficiency, strategy-proofness, and ERICS as uniform* rules.

Corollary 7.1. *Let $\mathcal{S}$ be a single-peaked domain. Then, every division rule $f : \mathcal{S}^n \to \Delta_n$ satisfies efficiency, strategy-proofness, and ERICS if and only if it is a uniform* rule.*

We obtain the results in Sprumont (1991) and Ching (1994) as corollaries of Theorem 7.1.

Corollary 7.2. *Let $\mathcal{S}$ be a single-peaked domain. Then, every division rule $f : \mathcal{S}^n \to \Delta_n$ satisfies efficiency, strategy-proofness, and anonymity (or ETE) if and only if it is the uniform rule.*

A. Proof of Theorem 3.1

Proof. (i) (If part) Since $\mathcal{S}$ is a single-peaked domain, if part of the Theorem follows from Sprumont (1991).

(Only-if part) Let $f : \mathcal{S}^2 \to \Delta_2$ be a division rule satisfying efficiency, strategy-proofness, and equal treatment of equals. We show that for all $(R_1, R_2) \in \mathcal{S}^2$, $f(R_1, R_2) = g(R_1, R_2)$. Consider a profile $(R_1, R_2) \in \mathcal{S}^2$. We distinguish the following cases:

Case (i) $\max\{\tau(R_1), \tau(R_2)\} \leq \frac{1}{2}$ or $\min\{\tau(R_1), \tau(R_2)\} \geq \frac{1}{2}$.

We only prove the case when $\max\{\tau(R_1), \tau(R_2)\} \leq \frac{1}{2}$. The case $\min\{\tau(R_1), \tau(R_2)\} \geq \frac{1}{2}$ can be proved using similar arguments. By the definition of uniform rule, $g(R_1, R_2) = (\frac{1}{2}, \frac{1}{2})$. Note that if $\tau(R_1) = \tau(R_2) \leq \frac{1}{2}$ then by strategy-proofness, efficiency, and equal treatment of equals $f(R_1, R_2) = (\frac{1}{2}, \frac{1}{2})$. This means if $\tau(R_1) < \tau(R_2) = \frac{1}{2}$, $f(R_1, R_2) = (\frac{1}{2}, \frac{1}{2})$. Suppose not, then by efficiency $f_2(R_1, R_2) > \frac{1}{2}$. But $f_2(R_1, R_2) = \frac{1}{2}$. So, agent 2 will manipulate at (R_1, R_2) via R_1, a contradiction. Similarly, for $\tau(R_2) < \tau(R_1) = \frac{1}{2}$, $f(R_1, R_2) = (\frac{1}{2}, \frac{1}{2})$. Now suppose $\tau(R_1), \tau(R_2) < \frac{1}{2}$. If $f(R_1, R_2) \neq (\frac{1}{2}, \frac{1}{2})$, then there exists $i \in \{1, 2\}$ such that $f_i(R_1, R_2) > \frac{1}{2}$. WLG assume $i = 1$. But this is a contradiction to strategy-proofness since $f(R_2, R_2) = (\frac{1}{2}, \frac{1}{2})$ and by single-peakedness $\frac{1}{2} P_1 f_1(R_1, R_2)$.

Case (ii) $\max\{\tau(R_1), \tau(R_2)\} > \frac{1}{2}$ and $\min\{\tau(R_1), \tau(R_2)\} < \frac{1}{2}$.

WLG assume that $\max\{\tau(R_1), \tau(R_2)\} = \tau(R_1)$ and $\min\{\tau(R_1), \tau(R_2)\} = \tau(R_2)$. Suppose $\tau(R_1) + \tau(R_2) < 1$. By the definition of uniform rule $g(R_1, R_2) = (\tau(R_1), 1 - \tau(R_1))$. Assume for contradiction that $g(R_1, R_2) \neq f(R_1, R_2)$. This means by efficiency $f_1(R_1, R_2) > \tau(R_1)$. Note that by efficiency and strategy-proofness for all $y \in [0, 1]$ and all $R^y, R_1^y \in \mathcal{S}$, $f_1(R^y, R_2) = f_1(R_1^y, R_2)$. Consider the set $\{y \in [0, 1] \mid f_1(R_1, R_2) = f_1(R^y, R_2)\}$. Let $x = \inf\{y \in [0, 1] \mid f_1(R_1, R_2) = f_1(R^y, R_2)\}$. Note that by strategy-proofness for all $y \in (x, f_1(R_1, R_2)]$, $f_1(R^y, R_2) = f_1(R_1, R_2)$. Since $f_1(R_1, R_2) > \tau(R_1)$, we have $x < f_1(R_1, R_2)$ and by Case (i) $x > \frac{1}{2}$. Suppose $x \in \{y \in [0, 1] \mid f_1(R_1, R_2) = f_1(R^y, R_2)\}$, i.e., $f_1(R^x, R_2) = f_1(R_1, R_2)$. Note that by the minimal richness condition there exists a continuous single-peaked preference R with $\tau(R) = x$. Let $z < x$ be such that $zPf_1(R_1, R_2)$. By efficiency and $f_1(R^x, R_2) = f_1(R_1, R_2)$, we have $f_1(R^z, R_2) \in [z, x)$. This means agent 1 manipulates at (R, R_2) via R^z, a contradiction. Thus,

$x \notin \{y \in [0,1] \mid f_1(R_1, R_2) = f_1(R^y, R_2)\}$. By efficiency, we have $f_1(R^x, R_2) = x$. Since $x > \frac{1}{2}$ by Condition 2* there exists a R' with $\tau(R') \in (x, f_1(R_1, R_2))$ such that $xP' f_1(R_1, R_2)$. But this means f is manipulable at (R', R_2) via R^x as $f_1(R^y, R_2) = f_1(R_1, R_2)$ for all $y \in (x, \tau(R_1)]$. Thus $f_1(R_1, R_2) = \tau(R_1)$, and $f_2(R_1, R_2) = 1 - \tau(R_1)$, which in turn implies $f(R_1, R_2) = g(R_1, R_2)$. The case $\tau(R_1) + \tau(R_2) > 1$ can be proved similarly.

(ii) Suppose $\mathcal{S}$ does not satisfy Condition 2*. This means WLG there exists an interval $(x, y) \subseteq [\frac{1}{2}, 1]$ such that for all $R \in \mathcal{S}$ with $\tau(R) \in (x, y)$, yRx. Consider the division rule f given in the following

$$f(R_N) = g(R_N) \text{ if } \tau(R_N) \geq 1 \text{ or } \tau(R_i) \notin (x, y] \text{ for all } i \in \{1, 2\},$$
$$f(R_N) = (y, 1 - y) \text{ if } \tau(R_1) \in (x, y] \text{ and } \tau(R_2) \leq 1 - y$$
$$f(R_N) = (1 - \tau(R_2), \tau(R_2)) \text{ if } \tau(R_1) \in (x, y] \text{ and } 1 - \tau(R_1) \geq \tau(R_2) > 1 - y$$

We show that f satisfies equal treatment of equals, efficiency, and strategy-proofness. Since for all cases where $R_1 = R_2$ by definition of f, we have $f(R_1, R_2) = g(R_1, R_2)$ and g satisfies equal treatment of equals, f satisfies equal treatment of equals. To check efficiency, note that we need to check for only those $(R_1, R_2) \in \mathcal{S}^2$ where $f(R_1, R_2) \neq g(R_1, R_2)$. Suppose $\tau(R_1) \in (x, y]$ and $\tau(R_2) \leq 1 - y$. By definition of f, $f_1(R_1, R_2) = y$ and $f_2(R_1, R_2) = 1 - y$. Since preferences are single-peaked and $\tau(R_1) \leq f_1(R_1, R_2)$, $\tau(R_2) \leq f_2(R_1, R_2)$, this is efficient. Similarly, it can be shown for the case when $\tau(R_1) \in (x, y]$ and $1 - \tau(R_1) \geq \tau(R_2) > 1 - y$. This shows f satisfies efficiency.

To show strategy-proofness suppose $(R_1, R_2) \in \mathcal{S}^2$ such that $\tau(R_1, R_2) \geq 1$ and $\tau(R_1) \geq \tau(R_2)$. By the definition of f, $f_1(R_1, R_2) = 1 - \tau(R_2)$ and $f_2(R_1, R_2) = \tau(R_2)$. If $\tau(R_2) \leq 1 - y$, then since $\tau(R_1) + \tau(R_2) \geq 1$, $f_1(R_1, R_2) R_1 y$ and hence agent 1 would not manipulate at (R_1, R_2) via R_1' with $\tau(R_1') \in (x, y]$. If $\tau(R_2) \geq 1 - y$, then $f_1(R_1', R_2) = 1 - \tau(R_2)$ for all R_1' with $\tau(R_1') \in (x, y]$ and hence, agent 1 would not manipulate at (R_1, R_2) via R_1'. ■

B. Proof of Theorem 3.2

Proof. (i) Let $n \geq 3$ and $\mathcal{S}$ be a domain satisfying Condition $\star$ for n agents. We show that a division rule $f : \mathcal{S}^n \to \Delta_n$ satisfying efficiency, strategy-proofness, and equal treatment of equals if and only if it is the uniform rule.

(If part) Since $\mathcal{S}$ is single-peaked domain If part of theorem follows from the main Theorem of Sprumont (1991).

(Only-if part) We first prove a lemma.

Lemma B.1. *(i) Let $R_N \in \mathcal{S}^n$ and $i \in N$ be such that $f_i(R_N) < \tau(R_i)$. Further let $R_i' \in \mathcal{S}$ be such that $f_i(R_N) \leq \tau(R_i')$. Then, $f_i(R_i', R_{-i}) \leq \frac{1}{2}$ implies $f_i(R_N) = f_i(R_i', R_{-i})$.*

(ii) Let $R_N \in \mathcal{S}^n$ and $i \in N$ be such that $f_i(R_N) > \tau(R_i)$. Further let $R_i' \in \mathcal{S}$ be such that $f_i(R_N) \geq \tau(R_i')$. Then, $f_i(R_N) = f_i(R_i', R_{-i})$.

Proof. (i) Since $f_i(R_N) < \tau(R_i)$, by efficiency $f_j(R_N) \leq \tau(R_j)$ for all $j \in N$, and hence, $\sum_{j=1}^{n} \tau(R_j) > 1$. Similarly, as $f_i(R_N) \leq \tau(R_i')$ and $f_j(R_N) \leq \tau(R_j)$ for all $j \neq i$, we have $\tau(R_i') + \sum_{j \neq i} \tau(R_j) \geq \sum_{j=1}^{n} f_j(R_N) = 1$. This means by efficiency, $f_i(R_i', R_{-i}) \leq \tau(R_i')$.

Assume for contradiction $f_i(R_i', R_{-i}) \leq \frac{1}{2}$ but $f_i(R_i', R_{-i}) \neq f_i(R_N)$. This together with strategy-proofness imply, $f_i(R_N) < \tau(R_i) < f_i(R_i', R_{-i}) \leq \tau(R_i')$. Let $x = \sup\{y \in [0,1] \mid f_i(R^y, R_{-i}) = f_i(R_N)\}$. This is well defined since by strategy-proofness and efficiency, $f_i(R^y, R_{-i}) = f_i(\bar{R}^y, R_{-i})$ for all $y \in [0,1]$, and all $R^y, \bar{R}^y \in \mathcal{S}$. Note that since $f_i(R_N) < \tau(R_i) < f_i(R_i', R_{-i}) \leq \frac{1}{2}$, $f_i(R_N) < x < \frac{1}{2}$. Suppose $x \in \{y \in [0,1] \mid f_i(R^y, R_{-i}) = f_i(R_N)\}$. Let R be a continuous preference with $\tau(R) = x$ and $x < z$ be such that $zPf_i(R_N)$. Consider $R^z \in \mathcal{S}$. By efficiency, $f_i(R^z, R_{-i}) \in (x, z]$. This means agent i manipulates at (R, R_{-i}) via R^z, a contradiction, and hence, $x \notin \{y \in [0,1] \mid f_i(R^y, R_{-i}) = f_i(R_N)\}$. This together with efficiency imply $f_i(R, R_{-i}) = x$. Note that by strategy-proofness for all $y \in [f_i(R_N), x)$, $f_i(R^y, R_{-i}) = f_i(R_N)$. Since $\mathcal{S}$ satisfies Condition $\star$ for n agents and $x < \frac{1}{2}$, there exists a preference $\hat{R}$ with

- $\tau(\hat{R}) \in (f_i(R_N), x)$ if either $(f_i(R_N), x) \subseteq (0, \frac{1}{n})$ or $(f_i(R_N), x) \subseteq (\frac{1}{n}, \frac{1}{2})$ such that $x\hat{P}f_i(R_N)$, or
- $\tau(\hat{R}) \in (\frac{1}{n}, x)$ if $f_i(R_N) < \frac{1}{n} < x$ such that $x\hat{P}\frac{1}{n}$.

In both the cases agent i manipulates at R_N via R, a contradiction, and hence, $f_i(R_N) = f_i(R_i', R_{-i})$.

(ii) We can prove this using similar arguments as in the proof of (i). ∎

Let $R_N \in \mathcal{S}^n$. If $\sum_{i \in N} \tau(R_i) = 1$ then, by efficiency, $\tau(R_i) = f_i(R_N) = g_i(R_N)$ for all $i \in N$. Suppose $\sum_{i \in N} \tau(R_i) > 1$. Note that by efficiency, $f_i(R_N) \leq \tau(R)$ and $g_i(R_N) \leq \tau(R_i)$ for all $i \in N$. WLG assume that $\tau(R_1) \leq \cdots \leq \tau(R_n)$. Assume for contradiction $f(R_N) \neq g(R_N)$. If $R_N = (R_n, \ldots, R_n)$ then by equal treatment of equals $f_i(R_N) = g_i(R_N)$ for all $i \in N$, a contradiction. Therefore, assume $R_N \neq (R_n, \ldots, R_n)$. We proceed to Step 1.

Step 1. Since $f(R_N) \neq g(R_N)$, there exists $i \in N$ such that $f_i(R_N) < g_i(R_N) \leq \tau(R_i)$. Let $R'_i = R_n$. We show $f_i(R'_i, R_{-i}) < g_i(R'_i, R_{-i})$. If $i = n$, then there is nothing to show. Suppose $i \neq n$. First we show, $f_i(R'_i, R_{-i}) \leq \frac{1}{2}$. Since at (R'_i, R_{-i}), $R'_i = R_n$, by equal treatment of equals $f_i(R'_i, R_{-i}) = f_n(R'_i, R_{-i})$, which in turn implies $f_i(R'_i, R_{-i}) \leq \frac{1}{2}$. Thus applying Lemma B.1, we can say that $f_i(R_N) = f_i(R'_i, R_{-i})$. By the definition of uniform rule, $g_i(R_N) \leq g_i(R'_i, R_{-i})$ as $\tau(R_i) \leq \tau(R'_i)$. Combining all these observations we get $f_i(R'_i, R_{-i}) < g_i(R'_i, R_{-i})$. If $(R'_i, R_{-i}) = (R_n, \ldots, R_n)$ then by equal treatment of equals we have a contradiction. Suppose $(R'_i, R_{-i}) \neq (R_n, \ldots, R_n)$. We proceed to Step 2.

Step 2. Since $f_k(R'_i, R_{-i}) < g_k(R'_i, R_{-i})$ for all $k \in \{i, n\}$, there exists $j \notin \{i, n\}$ such that $g_j(R'_i, R_{-i}) < f_j(R'_i, R_{-i})$. By efficiency, $g_j(R'_i, R_{-i}) < f_j(R'_i, R_{-i}) \leq \tau(R_j)$. Let $R'_j = R_n$. Strategy-proofness of f implies $f_j(R'_i, R_{-i}) \leq f_j(R'_i, R'_j, R_{-\{i,j\}})$. By the definition of uniform rule, $g_j(R'_i, R_{-i}) = g_j(R'_i, R'_j, R_{-\{i,j\}})$ since $g_j(R'_i, R_{-i}) < \tau(R_j) < \tau(R'_j)$. Combining all these we get, $g_j(R'_i, R'_j, R_{-\{i,j\}}) < f_j(R'_i, R'_j, R_{-\{i,j\}})$. If $(R'_i, R'_j, R_{-\{i,j\}}) = (R_n, \cdots, R_n)$ the by equal treatment of equals we have a contradiction, otherwise we apply Step 1 to $(R'_i, R'_j, R_{-\{i,j\}})$.

Since N is finite and at every Step we change the preference of a new agent by R_n, eventually it will lead to a contradiction. ■

References

[1] Salvador Barberà, Matthew O Jackson, and Alejandro Neme. Strategy-proof allotment rules. *Games and Economic Behavior*, 18(1):1–21, 1997.

[2] Salvador Barberà, Jordi Masso, and Alejandro Neme. Voting under constraints. *journal of economic theory*, 76(2):298–321, 1997.

[3] J-P Benassy. The economics of market disequilibriu. *New York: Academic Press*, 1982.

[4] Shurojit Chatterji, Remzi Sanver, and Arunava Sen. On domains that admit well-behaved strategy-proof social choice functions. *Journal of Economic Theory*, 148(3):1050–1073, 2013.

[5] Stephen Ching. An alternative characterization of the uniform rule. *Social Choice and Welfare*, 11(2):131–136, 1994.

[6] Yves Sprumont. The division problem with single-peaked preferences: a characterization of the uniform allocation rule. *Econometrica: Journal of the Econometric Society*, pages 509–519, 1991.