Utilitarianism with interpersonally significant norms

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October 25, 2019

Abstract

We characterize utilitarianism with interpersonally significant norms in a multi-profile and purely ordinal framework, i.e. without assuming that utilities have been measured beforehand.

1 Introduction

In a recent paper, Marchant [2019] has shown that it is possible to characterize utilitarianism in a purely ordinal framework, thereby avoiding the pitfalls of cardinal social choice theory and Social Welfare Functionals [Morreau and Weymark, 2016]. Marchant's paper assumes that the agents have Von Neumann-Morgenstern preferences over all lotteries defined on the set of alternatives. A weakness of the model characterized by Marchant is that two agents with exactly the same preference relation necessarily have the same weight in the utilitarian rule although it may be the case that one agent has a very strong preference for x over y while the other has much weaker preferences.

In the present paper, we enrich the setting by assuming the existence of interpersonally significant norms [e.g. Blackorby and Donaldson, 1982]. These norms permit us to make a distinction between the two above-mentioned agents and we characterize the utilitarian rule that scales the utility function of each agent by means of the interpersonally significant norms.

2 Notation and definitions

Blackorby and Donaldson [1982], List [2003], Zuber [2018] use the concept of an interpersonally significant norm: it is the description of a life (or situation) supposed to result in the same level of happiness or well-being for different agents. They use, for instance, a situation in which life is no more worthwhile than death. Blackorby et al. [1999] also use a situation in which life is at an excellent level. Let us respectively call these two norms the neutral and the high norm. The existence and meaningfulness of such interpersonally significant norms will not be discussed in this paper.

Let $\mathbb{N}$ represent the set of all potential agents and $X = \{x, y, z, \ldots\}$ be the set (finite, with $\#X \geq 3$) of alternatives out of which a society $N \subseteq \mathbb{N}$ has to make a choice. We will enlarge this set with two particular alternatives: x_0 (resp. x_1) is an alternative resulting in the neutral (resp. high) norm for all agents in $\mathbb{N}$. Formally, $X' = X \cup \{x_0, x_1\}$.

We define $\Pi = \{p, q, r, \ldots\}$ as the set of all probability distributions on X'. Each such probability distribution is called a lottery. Given the lottery p in Π, the probability that x obtains is denoted by p_x. The lottery such that x obtains with certainty is denoted by $\overline{x}$. It is called a safe lottery. The set of all binary relations on Π is $\mathcal{R} = 2^{\Pi \times \Pi}$. If $R \in \mathcal{R}$, then P and I respectively denote the asymmetric and the symmetric part thereof. A binary relation R on Π is a von Neumann-Morgenstern (VNM) relation [Jensen, 1967] if it satisfies

- weak order: it is transitive, reflexive and complete;
- independence: if $p\ P\ q$, then $\lambda p + (1-\lambda) r\ P\ \lambda q + (1-\lambda) r$ for all $\lambda \in\ ]0,1[$;
- continuity: if $p\ P\ q$ and $q\ P\ r$, then there are $\lambda, \lambda' \in\ ]0,1[$ such that $\lambda p + (1-\lambda) r\ P\ q$ and $q\ P\ \lambda' p + (1-\lambda') r$.

Most other definitions of VNM relations would work equally well. Let $V \subset \mathcal{R}$ be the set of all VNM relations on Π such that $\overline{x}_1$ is strictly preferred to $\overline{x}_0$. We say a binary relation R on X' has an expected utility representation if there exists a mapping $v : X' \to \mathbb{R}$ such that

$$p\ R\ q \iff \sum_{x \in X'} p_x v(x) \geq \sum_{x \in X'} q_x v(x), \text{ for all } p, q \in \Pi. \tag{1}$$

A binary relation has an expected utility representation as in (1) if and only if it is a VNM relation [Jensen, 1967]. The utility function v in (1) is a VNM utility function; it is unique up to a positive affine transformation.

Given a set of agents $N \subset \mathbb{N}$, a profile $\succsim\ = (\succsim_i)_{i \in N}$ is an element of V^N indexed by the elements of N, where $\succsim_i$ is the preference relation of individual i. Let $\mathcal{P}_N$ be the set of all possible profiles given X' and N and $\mathcal{P} = \bigcup_{N \subset \mathbb{N}} \mathcal{P}_N$. We define a VNM Social Choice Correspondence (SCC) as a mapping $f : \mathcal{P} \to 2^X \setminus \emptyset$, that is, a mapping from the set of all possible profiles to the set of all non-empty subsets of X. Notice that the choice set is a subset of X and not of X' or of Π. We want to choose alternatives, i.e. elements of X, even though the preferential information we use is defined on the richer set Π, involving X'.

Let σ_X be a permutation on X and Σ the set of all such permutations. Then σ_Π is a permutation on Π defined by $(\sigma_\Pi(p))_{x_0} = p_{x_0}$, $(\sigma_\Pi(p))_{x_1} = p_{x_1}$ and $(\sigma_\Pi(p))_x = p_{\sigma(x)}$ for all $x \in X$ and $p \in \Pi$. Similarly, σ_V is a permutation on V defined by $\sigma_\Pi(p)\ \sigma_V(R)\ \sigma_\Pi(q)$ iff $p\ R\ q$ for all $p, q \in \Pi$ and all $R \in V$. And $\sigma_{\mathcal{P}}$ is a permutation on $\mathcal{P}$ defined by $\sigma_{\mathcal{P}}((\succsim_i)_{i \in N}) = (\sigma_V(\succsim_i))_{i \in N}$ for all $(\succsim_i)_{i \in N} \in \mathcal{P}$. We will henceforth abuse notation and write σ without subscript for all these permutations.

The aim of this section is to characterize the normalized utilitarian VNM Social Choice Correspondence, defined by

$$f((\succsim_i)_{i \in N}) = \operatorname*{argmax}_{x \in X} \sum_{i \in N} u(\succsim_i, x) \tag{2}$$

where $u : V \times X' \to \mathbb{R}$ is such that

(i) $u(R,\cdot)$ is a VNM utility function representing R and

(ii) $u(R,x_0)=0$ and $u(R,x_1)=1$.

In the sequel, when a utility function satisfies (ii), we will say it is a normalized utility function. Notice that it is possible to have $u(R,x)\notin[0,1]$ because x_0 and x_1 are not necessarily minimal or maximal elements in R. The mapping u in (2) is unique, because of constraints (i) and (ii). The VNM SCC f defined by (2) is also unique. Notice that the uniqueness of u is not essential, contrary the uniqueness of f. Indeed, if u' is such that, for all $R\in V$, $u'(R,\cdot)=\alpha u(R,\cdot)+\beta_R$, and if we define

$$f'((\succsim_i)_{i\in N})=\operatorname*{argmax}_{x\in X}\sum_{i\in N}u'(\succsim_i,x),$$

then $f'((\succsim_i)_{i\in N})=f((\succsim_i)_{i\in N})$. The uniqueness of u in (2) is thus artificial and could be avoided by weakening (ii) as follows:

(ii') $u(R,x_1)-u(R,x_0)=u(S,x_1)-u(S,x_0)$ for all $R,S\in V$.

Yet, since (ii) is so handy, we will keep it.

In the introduction, we mentioned that the utilitarian rule characterized by Marchant [2019] does not permit to give different weights to two agents that have identical preferences over all lotteries defined on X. In (2), there are no weights, but we see that two agents with identical preferences over all lotteries defined on X may have different preferences over all lotteries defined on X'. Their utility functions can therefore be different: they can be scaled in a different way and this amounts to giving them different weights.

3 Axioms and preliminary results

3.1 Standard axioms

In order to characterize the normalized utilitarian VNM SCC, we will use a result by Pivato [2014] extending a result of Myerson [1995], which is itself an extension of a result of Young [1975][1]. Our axioms are therefore very similar to those of Young [1975]. We present them hereunder without much comment, because they have been extensively discussed elsewhere. The first condition says that all alternatives are treated equally.

A 1 *Neutrality. For each profile $\succsim\in\mathcal{P}$ and permutation σ on X,*

$$\sigma(f(\succsim))=f(\sigma(\succsim)).$$

Notice that the permutation is defined on X and not on X' because we want x_0 and x_1 to be treated differently. The second condition says that all agents are treated equally.

[1] The latter is closely linked to [Smith, 1973]

A 2 *Anonymity. For all finite $N \subset \mathbb{N}$, all profiles $\succsim, \succsim' \in \mathcal{P}_N$ and every permutation γ on N such that $\succsim_i = \succsim'_{\gamma(i)}$, for all $i \in N$,*

$$f(\succsim) \;=\; f(\succsim').$$

Young [1975] groups these two conditions under the name 'Symmetry'.

We introduce a new piece of notation before next condition. Let $\succsim = (\succsim_i)_{i \in N}$ and $\succsim' = (\succsim_i)_{i \in M}$ be two profiles with $N \cap M = \emptyset$. Then $\succsim'' = \succsim \circ \succsim'$ is the profile in $\mathcal{P}_{N \cup M}$ defined by

$$\succsim''_i = \begin{cases} \succsim_i & \text{if} \quad i \in N \\ \\ \succsim'_i & \text{if} \quad i \in M. \end{cases}$$

If $f(\succsim)$ is the choice set of an agent group N and $f(\succsim')$ is the choice set of another agent group M disjoint from N, and if $f(\succsim) \cap f(\succsim') \neq \emptyset$, then the group $N \cup M$ should precisely choose the alternatives in $f(\succsim) \cap f(\succsim')$. Formally,

A 3 *Separability. Let $\succsim = (\succsim_i)_{i \in N}$ and $\succsim' = (\succsim'_i)_{i \in M}$ be two profiles with $N \cap M = \emptyset$. If $f(\succsim) \cap f(\succsim') \neq \emptyset$, then $f(\succsim \circ \succsim') = f(\succsim) \cap f(\succsim')$.*

This is what Young [1975] calls Consistency while Myerson [1995] calls it Reinforcement. We call it Separability, like Smith [1973].

Let $\succsim = (\succsim_i)_{i \in N}$ and $\succsim' = (\succsim'_i)_{i \in M}$ be two profiles. We say $\succsim$ and $\succsim'$ are isomorphic if there is a bijection $\mu : N \to M$ such that $\succsim_i = \succsim'_{\mu(i)}$ for all $i \in N$. If $\succsim$ and $\succsim'$ are isomorphic, we can consider $\succsim'$ as a copy of $\succsim$. If $f(\succsim^1)$ is the choice set of a certain group N^1, then given any second group M disjoint from N^1 and with preference profile $\succsim'$, we can replicate the first group (and its preference profile) a sufficient number of times so that it will overwhelm the second group in a combined profile and yield a subset of $f(\succsim^1)$ as choice set. This kind of continuity requirement is our Archimedean condition.

A 4 *Archimedeanness. Let $\{N^j\}_{j \in \mathbb{N}}$ be a collection of disjoint subsets of $\mathbb{N}$, all of size n. Suppose $\{\succsim^j\}_{j \in \mathbb{N}}$ is a collection of isomorphic profiles in $\mathcal{P}_{N^j}$ and $\succsim' \in \mathcal{P}_M$ with $(\bigcup_{j \in \mathbb{N}} N^j) \cap M = \emptyset$. Then there exists $h \in \mathbb{N}$ such that, for every $k > h$,*

$$f(\succsim^1 \circ \ldots \circ \succsim^k \circ \succsim') \subseteq f(\succsim^1).$$

This is exactly Myerson's (1995) Overwhelming Majority. The next condition is a standard Pareto condition [e.g. Donaldson and Weymark, 1988].

A 5 *Weak Pareto. For any profile $\succsim = (\succsim_i)_{i \in N} \in \mathcal{P}$, if there are $x, y \in X$ such that $\overline{x} \succ_i \overline{y}$ for all $i \in N$, then $y \notin f(\succsim)$.*

In our first result, we use a weakening of this condition and a strict version thereof. Let $(R)_i$ denote a profile with a single voter i and a single preference relation R.

A 6 *Non-Triviality. There exists $R \in V$ such that $f((R)_i) \neq X$.*

A 7 *Strict Pareto. For any profile $\succsim = (\succsim_i)_{i \in N} \in \mathcal{P}$, if there are $x, y \in X$ such that $\overline{x} \succsim_i \overline{y}$ for all $i \in N$ and $\overline{x} \succ_j \overline{y}$ for some $j \in N$, then $y \notin f(\succsim)$.*

3.2 Preliminary result

Using the conditions of previous section, we state a preliminary result that can almost be considered as a corollary to a result by Pivato [2014].

Proposition 1 *Let $\#X \geq 3$. A VNM SCC f satisfies Neutrality, Anonymity, Separability, Archimedeanness and Non-Triviality iff there exists $u : V \times X \to \mathbb{R}$ such that*

$$f((\succsim_i)_{i \in N}) = \operatorname*{argmax}_{x \in X} \sum_{i \in N} u(\succsim_i, x), \tag{3}$$

with u neutral, i.e., $u(\sigma(R), x) = u(R, \sigma(x))$, for any $R \in V$, $\sigma \in \Sigma$ and $x \in X$. If f also satisfies Strict Pareto, then $u(R, x) \geq u(R, y) \iff \overline{x}\ R\ \overline{y}$, for all $x, y \in X$ and all $R \in V$.

Let us notice that, without Strict Pareto, the mapping u is not necessarily a VNM utility function and not even a utility function. It can be anything, provided it is neutral and not constant (so as to satisfy Non-Triviality). Yet, with Strict Pareto, the mapping u is a utility function and Proposition 1 can thus be considered as a first characterization of utilitarianism in a very general sense, leaving full freedom for the choice of u as long as $u(R, \cdot)$ is a utility representation of R.

We do not present the proof of Proposition 1 because this proposition is almost identical to Proposition 1 in [Marchant, 2019].

3.3 A recent condition

All axioms presented so far are standard in the social choice literature. Usually, they are imposed on SCCs acting on profiles of preference relations on unstructured sets, but nothing prevents us from imposing them on a SCC acting on profiles of preference relations defined on a structured set (e.g., Π), as we just did. Yet, none of these axioms makes use of the structure of Π; none of them helps us to exploit the potentially cardinal information contained in the VNM preference relations. Our next condition precisely do this.

A 8 *VNM-Comparability. There exists an infinite subset O of $\mathbb{N}$ such that, whenever*

- *$R \in V$ is such that $\overline{x}\ P\ \overline{y}\ P\ \overline{z}$ and $\overline{y}\ I\ \lambda\overline{x} + (1-\lambda)\overline{z}$,*
- *N^1 and N^2 are disjoint subsets of O,*
- *$\sigma \in \Sigma$ is such that $\sigma(x) = y$, $\sigma(z) = x$*
- *$\#N^2/\#N^1 = 1 - \lambda$,*

then

$$x \in f\big((R)_{i \in N^1} \circ (\sigma(R))_{i \in N^2}\big) \iff y \in f\big((R)_{i \in N^1} \circ (\sigma(R))_{i \in N^2}\big).$$

For a detailed presentation of this condition, see [Marchant, 2019].

3.4 Another preliminary result

The following theorem is almost identical to the main result in [Marchant, 2019]. We therefore do not present its proof. The only difference is the addition of x_0 and x_1, but they do not play a significant role in this result.

Theorem 1 *Let $\#X \geq 3$. A VNM SCC satisfies Neutrality, Anonymity, Separability, Archimedeanness, Weak Pareto and VNM-Comparability iff it is an anonymous utilitarian VNM SCC defined by*

$$f((\succsim_i)_{i\in N}) = \underset{x\in X}{\operatorname{argmax}} \sum_{i\in N} u(\succsim_i, x) \tag{4}$$

where $u : V \times X \to \mathbb{R}$ is such that

(i) $u(R,\cdot)$ is the restriction to X of a VNM utility function representing R, for any $R \in V$,

(ii) u is neutral, i.e., $u(\sigma(R), x) = u(R, \sigma(x))$, for any $R \in V$, $\sigma \in \Sigma$ and $x \in X$.

4 New condition and main result

We finally present a condition in which the interpersonally significant norms play a role.

A 9 *Normalized Comparability. There exits an infinite subset O of $\mathbb{N}$ with the following property. If $N = \{i, j\} \subset O$, $\succsim = (\succsim_i, \succsim_j)$, $x, y \in X$ and $\alpha \in [0, 1]$ are such that*

- $(\alpha\overline{x}_1 + (1-\alpha)\overline{y}) \sim_i (\alpha\overline{x}_0 + (1-\alpha)\overline{x})$ *and*
- $(\alpha\overline{x}_1 + (1-\alpha)\overline{x}) \sim_j (\alpha\overline{x}_0 + (1-\alpha)\overline{y})$,

then $x \in f(\succsim) \iff y \in f(\succsim)$.

The rationale for this condition is simple. Suppose we consider that VNM utilities can adequately model strength or intensity of preference. Suppose also that x_0 and x_1 indeed represent states in which all agents are equally happy. Then $(\alpha\overline{x}_1 + (1-\alpha)\overline{y}) \sim_i (\alpha\overline{x}_0 + (1-\alpha)\overline{x})$ implies that agent i prefers x over y with intensity $\alpha/(1-\alpha)$. Similarly, $(\alpha\overline{x}_1 + (1-\alpha)\overline{x}) \sim_j (\alpha\overline{x}_0 + (1-\alpha)\overline{y})$ implies that agent j has the opposite preference (y over x) with the same intensity. Then, if the society contains only agents i and j, because of the symmetric position of x and y in their preferences, it seems very reasonable to choose x if and only if y is also chosen. We are now ready to present our main result.

Theorem 2 *A VNM SCC satisfies Neutrality, Anonymity, Separability, Archimedeanness, Weak Pareto, VNM-Comparability and Normalized Comparability iff it is the normalized utilitarian VNM SCC defined by (2).*

Proof of Theorem 2. By Theorem 1, f is an anonymous utilitarian VNM SCC. We extend the mapping u of Theorem 1 from $V \times X$ to $V \times X'$ by setting, for all $R \in V$, $u(R, x_0)$ and $u(R, x_1)$ in such a way that $u(R, \cdot)$ is a VNM representation of R.

Let us define $u' : V \times X' \to \mathbb{R}$ by $u'(R, \cdot) = u(R, \cdot) - u(R, x_0)$, for all $R \in V$. As a result, $u'(R, x_0) = 0$ for all $R \in V$. Then

$$f'((\succsim_i)_{i \in N}) = \operatorname*{argmax}_{x \in X} \sum_{i \in N} u'(\succsim_i, x)$$

is identical to f.

Choose some $R^* \in V$ such that $\overline{x}^* \ P^* \ \overline{y}^* \ P^* \ \overline{z}^*$ for some $x^*, y^*, z^* \in X$ and $\overline{z}^* \ R^* \ \overline{x}$ for all $x \in X \setminus \{x^*, y^*, z^*\}$. Let us define $u'' : V \times X' \to \mathbb{R}$ by $u'' = u'/u'(R^*, x_1)$. Notice that this automatically sets $u''(R, x_1) = 1$ for all $R \in V$ such that $R = \sigma(R^*)$ for some $\sigma \in \Sigma$. Then

$$f''((\succsim_i)_{i \in N}) = \operatorname*{argmax}_{x \in X} \sum_{i \in N} u''(\succsim_i, x)$$

is identical to f.

Let us define $u''' : V \times X' \to \mathbb{R}$ by

$$u'''(R, \cdot) = \begin{cases} u''(R, \cdot)/u''(R, x_1) & \text{if } \overline{x} \ I \ \overline{y} \text{ for all } x, y \in X \\ u''(R, \cdot) & \text{otherwise,} \end{cases}$$

for all $R \in V$. Notice that this transformation modifies $u(R, \cdot)$ only when R is such that $u(R, x) = u(R, y)$ for all $x, y \in X$. Hence

$$f'''((\succsim_i)_{i \in N}) = \operatorname*{argmax}_{x \in X} \sum_{i \in N} u'''(\succsim_i, x)$$

is identical to f.

To complete the proof, we need to show that $u'''(R, x_1) = 1$ for all $R \in V$. Take any $R \in V$ such that $R \neq \sigma(R^*)$, for all $\sigma \in \Sigma$, and such that $\overline{x} \ P \ \overline{y}$ for some $x, y \in X$. Let $w, z \in X$ be respectively of rank 1 and 2 in R. Formally, $\overline{w} \ P \ \overline{z}$ and, for all $x \in X$, we have $\overline{x} \ I \ \overline{w}$ or $\overline{x} \ I \ \overline{z}$ or $\overline{z} \ P \ \overline{x}$. Let $R' = \sigma(R)$ with $\sigma \in \Sigma$, $\sigma(w) = y^*, \sigma(z) = z^*$ and $\overline{z}^* \ R' \ \overline{x}^*$. There exists $R'' \in V$ and $\alpha, \beta \in \mathbb{R}$ such that $\overline{z}^*, \overline{x}^*, \overline{y}^*$ respectively have rank 1, 2, 3 in R'' and

$$(\alpha \overline{x}_1 + (1-\alpha)\overline{y}^*) \ I'' \ (\alpha \overline{x}_0 + (1-\alpha)\overline{z}^*), \tag{5}$$

$$(\alpha \overline{x}_1 + (1-\alpha)\overline{z}^*) \ I' \ (\alpha \overline{x}_0 + (1-\alpha)\overline{y}^*), \tag{6}$$

$$(\beta \overline{x}_1 + (1-\beta)\overline{x}^*) \ I'' \ (\beta \overline{x}_0 + (1-\beta)\overline{z}^*), \tag{7}$$

$$(\beta \overline{x}_1 + (1-\beta)\overline{z}^*) \ I^* \ (\beta \overline{x}_0 + (1-\beta)\overline{x}^*). \tag{8}$$

Using (5–8), we obtain

$$\begin{aligned} \alpha u'''(R'', x_1) + (1-\alpha)u'''(R'', y^*) &= (1-\alpha)u'''(R'', z^*) \\ \alpha u'''(R', x_1) + (1-\alpha)u'''(R', z^*) &= (1-\alpha)u'''(R', y^*) \\ \beta u'''(R'', x_1) + (1-\beta)u'''(R'', x^*) &= (1-\beta)u'''(R'', z^*) \\ \beta + (1-\beta)u'''(R^*, z^*) &= (1-\beta)u'''(R^*, x^*) \end{aligned}$$

or

$$u'''(R'', y^*) = u'''(R'', z^*) - \frac{\alpha}{1-\alpha}\ u'''(R'', x_1) \tag{9}$$

$$u'''(R', z^*) = u'''(R', y^*) - \frac{\alpha}{1-\alpha}\ u'''(R', x_1) \tag{10}$$

$$u'''(R'', x^*) = u'''(R'', z^*) - \frac{\beta}{1-\beta}\ u'''(R'', x_1) \tag{11}$$

$$u'''(R^*, z^*) = u'''(R^*, x^*) - \frac{\beta}{1-\beta} \tag{12}$$

By Weak Pareto, $y^* \notin f(R', R'')$. By virtue of (4), this implies that at least one of x^*, z^* belongs to $f(R', R'')$ and, by Normalized Comparability, both belong to $f(R', R'')$. Hence

$$u'''(R^*, x^*) + u'''(R'', x^*) = u'''(R^*, z^*) + u'''(R'', z^*). \tag{13}$$

Substituting (11) and (12) in (13) yields $u'''(R'', x_1) = 1$.

By Weak Pareto again, $x^* \notin f(R', R'')$. This implies that at least one of y^*, z^* belongs to $f(R', R'')$ and, by Normalized Comparability, both belong to $f(R', R'')$. Hence

$$u'''(R', y^*) + u'''(R'', y^*) = u'''(R', z^*) + u'''(R'', z^*). \tag{14}$$

Substituting (9) and (10) in (14) yields $u'''(R', x_1) = u'''(R'', x_1) = 1$. Thanks to Neutrality, $u'''(R, x_1) = 1$. □

5 Discussion

Theorem 2 is by no means a justification of normalized utilitarianism. In order to justify normalized utilitarianism, we need to scrutinize all assumptions of Theorem 2 and to check for each one whether it is plausible or desirable. For the standard conditions of Theorem 2 (Neutrality, Anonymity, Separability, Archimedeanness and Weak Pareto), we skip the scrutiny because these conditions have been discussed in many other places. We only discuss VNM-Comparability and Normalized Comparability.

VNM-comparability is a desirable condition if (i) preference intensity for $\overline{x}$ over a mixture $\lambda\overline{x} + (1-\lambda)\overline{z}$ is proportional to $1-\lambda$, (ii) a small unanimous group with a strong preference for x over y can compensate a large unanimous group with a weaker preference for y over x and, more precisely, (iii) preference intensity and group size combine multiplicatively. Normalized-comparability is a desirable condition if assumption (i), as above, holds, (iv) interpersonally significant norms are meaningful and (v) the particular alternatives x_0 and x_1 are meaningful.

(i) Although the mathematical foundations of expected utility do not say a word about preference intensity [Weymark, 2005], it seems intuitively plausible that $(1 - \lambda)$ is somehow related to preference intensity and Abdellaoui et al. [2007] provide some empirical support hereto.

Nevertheless, using mixtures of lotteries is not the only way to try to capture preference intensity; we could also use, for instance, conjoint measurement [Debreu, 1960, Krantz et al., 1971] or difference measurement [Krantz et al., 1971]. They do not necessarily yield the same utilities as techniques based on mixtures of lotteries. It is therefore not clear why we should rely on mixtures of lotteries for measuring preference intensity.

(*ii*) This is probably the least disputable assumption.

(*iii*) Our second assumption (*ii*) says that the strength of a unanimous group (in favouring x against y) is a combination of group size and preference intensity. Assumption (*iii*) goes further and precisely defines the strength of a unanimous group (in favouring x against y) as group size times preference intensity. Why this specific form? We have no convincing argument for this.

(*iv*) As mentioned in Section 2, we will not discuss the existence and meaningfulness of interpersonally significant norms. We just mention a recent contribution about interpersonal comparisons of well-being [Kaminitz, 2018], balancing some arguments in favour of interpersonal comparisons against the well-known objections.

(*v*) The meaningfulness of interpersonally significant norms is not enough for Normalized Comparability. We also need two special alternatives (x_0 and x_1) resulting in a neutral and a high interpersonally significant norm for every agent in $\mathbb{N}$. Do such alternatives make sense? For instance one in which everyone has an excellent life? What if an agent considers that the happiness of his neighbour is intrinsically incompatible with his own happiness?

All the questions we just raised, go much beyond the scope of this paper but nevertheless deserve a thorough examination. Compared to these important questions, the contribution made by our paper is perhaps modest, but necessary: we provided mathematically unambiguous conditions characterizing a particular form of utilitarianism.

6 Logical independence of the conditions

We are not able to prove the logical independence of the conditions in Theorem 2. This result makes use of seven conditions. For each of these conditions, we provide below an example violating that condition (starred hereunder) and as few other conditions (not starred) as possible. Proving the independence of the conditions (or providing alternative characterizations with weaker conditions) is therefore left as an open problem.

For Theorem 1 also, we cannot prove the logical independence of the conditions. Archimedeanness could possibly be implied by the other conditions. Examples 1–5 nevertheless prove the logical independence of the conditions in Proposition 1 (as already shown by Marchant [2019]).

Example 1 (Neutrality* and Normalized Comparability) *Let $x \in X$ and let V^* be a proper subset of V containing all relations $R \in V$ such that $\overline{x}\ P\ \overline{w}$ or $\overline{w}\ P\ \overline{x}$ for all $w \in X$. Define $g : V \times X' \to \mathbb{R}$ so that, for all $R \in V$, $g(R, \cdot)$ is the VNM utility function representing R and such that $g(R, x_0) = 0$ and $g(R, x_1) = 1$. Define $u : V \times X' \to \mathbb{R}$ by*

$$u(R, \cdot) = \begin{cases} g(R, \cdot) & \textit{if } R \in V^* \\ 2g(R, \cdot) & \textit{otherwise} \end{cases}$$

and f by

$$f\big((\succsim_i)_{i \in N}\big) = \operatorname*{argmax}_{x \in X} \sum_{i \in N} u(\succsim_i, x).$$

This VNM SCC obviously violates Neutrality and Normalized Comparability. The reason it satisfies VNM-Comparability is that the relations R and $\sigma(R)$ in the statement of VNM-Comparability both belong to V^* or both to $V \setminus V^*$. The other conditions are clearly satisfied.

Example 2 (Anonymity*) *Let $u : V \times X' \to \mathbb{R}$ be such that, for all $R \in V$, $u(R, \cdot)$ is a normalized VNM utility function representing R. Let O be any proper infinite subset of $\mathbb{N}$; for instance the set of all odd natural numbers. Define*

$$f\big((\succsim_i)_{i \in N}\big) = \operatorname*{argmax}_{x \in X} \Big(\sum_{i \in N \setminus O} 2u(\succsim_i, x) + \sum_{i \in N \cap O} u(\succsim_i, x) \Big).$$

Anonymity is blatantly violated. To understand why f satisfies VNM-Comparability and Normalized Comparability, notice that both conditions only apply to agents in O. In that case, f can be rewritten as

$$f\big((\succsim_i)_{i \in N}\big) = \operatorname*{argmax}_{x \in X} \sum_{i \in N} u(\succsim_i, x),$$

which is the plain normalized utilitarian VNM SCC. The other conditions obviously hold.

Example 3 (Separability* and Normalized Comparability) *Let us say that R and $R' \in V$ are isomorphic if there exists $\sigma \in \Sigma$ such that $\sigma(R) = R'$. Consider the profile $(\succsim_i)_{i \in N}$. Let $(N_1, \ldots, N_q)$ be a partition of N such that, for all $j, l \in \{1, \ldots, q\}$, and all $i \in N_j, k \in N_l$, $\succsim_i$ and $\succsim_k$ are isomorphic iff $j = l$. Define a new profile $\succsim_*$ of weak orders on X where, for each $j \in \{1, \ldots, q\}$, $\#N_j$ voters have identical preferences, induced by $\sum_{i \in N_j} u(\succsim_i, \cdot)$, where $u(R, \cdot)$ is a normalized VNM utility representation of R. Define then $f\big((\succsim_i)_{i \in N}\big)$ as the Copeland Social Choice Correspondence applied to the new profile $\succsim_*$.*

	$u(R_1,\cdot)$	$u(R_2,\cdot)$	$u(R_3,\cdot)$	$u(R_4,\cdot)$	$u(R_5,\cdot)$	$u(R_6,\cdot)$
x	0.8	0	0.9	0.5	0.9	0.3
y	0.9	1	1	0	0.4	1
z	0	0.9	0.6	0.9	1	0
w	1	0.7	0	1	0	0.9

Table 1: VNM representations of $R_1, \ldots, R_6$

This VNM SCC clearly satisfies Neutrality, Anonymity and Weak Pareto. Since the statement of the VNM Comparability condition involves two isomorphic preference relations, it is easy to see that VNM Comparability is satisfied.

Let us show that f violates Normalized Comparability. If $\succsim_i$ and $\succsim_j$ in the statement of Normalized Comparability are isomorphic, then $\succsim_*$ consists of a single relation and the Copeland rule yields the same result as the normalized utilitarian VNM SCC. But, if $\succsim_i$ and $\succsim_j$ are not isomorphic, then $\succsim_*$ consists of two relations and the Copeland rule does not necessarily yield the same result as the normalized utilitarian VNM SCC.

Let us show that f violates Separability. Suppose $X = \{x, y, z, w\}$, $R_1, \ldots, R_6 \in V$ and the normalized VNM representations of $R_1, \ldots, R_6$ are as in Table 1. Notice that no two relations among $R_1, \ldots, R_6$ are isomorphic. Let $N = \{1, \ldots, 6\}$ and $\succsim = (\succsim_i)_{i \in N}$ be defined by $\succsim_1 = \succsim_2 = \succsim_3 = \succsim_4 = R_1$, $\succsim_5 = R_2$ and $\succsim_6 = R_3$. The corresponding profile $\succsim_*$ has 6 weak orders: 4 voters with preferences $wyxz$, 1 voter with $yzwx$ and 1 voter with $yxzw$. The largest Copeland score in this new profile $\succsim_*$ is 3 and corresponds to w. Hence $f((\succsim_i)_{i \in N}) = \{w\}$.

Let $M = \{11, \ldots, 24\}$ and $\succsim' = (\succsim'_i)_{i \in M}$ be defined by $\succsim'_{11} = \succsim'_{12} = \succsim'_{13} = \succsim'_{14} = \succsim'_{15} = R_4$, $\succsim'_{16} = \succsim'_{17} = \succsim'_{18} = \succsim'_{19} = R_5$ and $\succsim'_{20} = \succsim'_{21} = \succsim'_{22} = \succsim'_{23} = \succsim'_{24} = R_6$. The corresponding profile $\succsim'_*$ has 14 weak orders: 5 voters with preferences $wzxy$, 4 voters with $zxyw$ and 5 voters with $ywxz$. The largest Copeland score in this new profile $\succsim'_*$ is 1 and corresponds to both w and z. Hence $f((\succsim'_i)_{i \in N}) = \{w, z\}$.

Let us now consider the profile $\succsim'' = \succsim \circ \succsim'$. The corresponding profile $\succsim''_*$ has 20 voters and simple arithmetic shows that $f(\succsim'') = \{y\}$ while Separability implies $f(\succsim'') = \{w\}$.

We now prove f satisfies Archimedeanness. Suppose the partition corresponding to $\succsim^1$ has q components. The corresponding profile $\succsim^1_*$ has, for each $i \in \{1, \ldots, q\}$, $\#N^1_i$ identical weak orders, induced by the sum of the utilities in N^1_i. If the partition corresponding to $\succsim'$ has q' components, then the partition corresponding to $\succsim^1 \circ \ldots \circ \succsim^k \circ \succsim'$ has then $q + s$ components, with $0 \leq s \leq q'$. When $k \to \infty$, the corresponding profile $(\succsim^1 \circ \ldots \circ \succsim^k \circ \succsim')_*$ has, for each $i \in \{1, \ldots, q\}$, at least $k\#N^1_i$ identical weak orders (the same ones as in $\succsim^1_*$) and, for each $i \in \{q+1, \ldots, s\}$, exactly $\#N'_i$ identical weak orders, induced by the sum of the utilities in N'_i. So, when $k \to \infty$, the Copeland rule applied to $(\succsim^1 \circ \ldots \circ \succsim^k \circ \succsim')_*$ yields a subset of $f(\succsim^1)$.

Example 4 (Archimedeanness*, VNM-Comparability and Normalized Comparability) *Let $u : V \times X' \to \mathbb{R}$ be such that, for all $R \in V$, $u(R, \cdot)$ is a neutral utility func-*

tion (not necessarily VNM) representing R. Define

$$h\big((\succsim_i)_{i\in N}\big) = \underset{x\in X}{\operatorname{argmax}} \sum_{i\in N} u(\succsim_i, x).$$

$$f\big((\succsim_i)_{i\in N}\big) = \underset{x\in h\left((\succsim_i)_{i\in N}\right)}{\operatorname{argmax}} \#\{i \in N : \overline{x} \succsim_i \overline{z} \ \ \textit{for all } z \in X\}.$$

Put differently, this VNM SCC successively applies the argmax to two different criteria: first the sum of the utilities and, then, a criterion based on the number of times an alternative is maximal in individual preferences. This VNM SCC clearly satisfies Neutrality, Anonymity and Weak Pareto.

If an alternative x is selected in $f(\succsim)$, it is maximal in $\succsim$ according to the utilitarian criterion and according to the second criterion. If the same alternative x is selected in $f(\succsim')$, it is also maximal in $\succsim'$ according to both criteria. Since both criteria are additive, x is again maximal in $\succsim \circ \succsim'$ according to both criteria and, hence, Separability holds.

Normalized Comparability is obviously violated, even if $u(R,\cdot)$ is VNM and normalized.

Suppose $X = \{x,y,z\}$, $u(R,x) = 1, u(R,y) = u(R,z) = 0$, $u(R',y) = 1, u(R',x) = 0.5$ and $u(R',z) = 0$. Let $N^1 = \{2,3,4\}, M = \{1\}, \succsim^1 = (\succsim_i^1)_{i\in N^1} = (R,R',R')$ and $\succsim' = (\succsim_i)_{i\in M} = (R)$. Then $h(\succsim^1) = \{x,y\}$ and $f(\succsim^1) = \{y\}$. For any $k > 0$, $h(\succsim^1 \circ \ldots \circ \succsim^k \circ \succsim') = \{x\}$ and $f(\succsim^1 \circ \ldots \circ \succsim^k \circ \succsim') = \{x\}$, thereby violating Archimedeanness.

We finally show that this example also violates VNM-Comparability. Let $X = \{x,y,z\}$. Suppose $u(R,\cdot)$ is a VNM utility function for all $R \in V$, $u(R,x) = 1, u(R,y) = 0.5, u(R,z) = 0$ and $u(R',y) = 1, u(R',z) = 0.5$ and $u(R',x) = 0$. Let $N = \{1,2\}, M = \{3\}, \succsim = ((R)_{i\in N} \circ (R')_{i\in M})$. Then $h(\succsim) = \{x,y\}$ and $f(\succsim) = \{x\}$, thereby violating VNM-Comparability.

Example 5 (Weak Pareto*) *Let $u : V \times X' \to \mathbb{R}$ be such that, for all $R \in V$, $u(R,\cdot)$ is a normalized VNM utility function representing R. Define*

$$f\big((\succsim_i)_{i\in N}\big) = \underset{x\in X}{\operatorname{argmin}} \sum_{i\in N} u(\succsim_i, x).$$

Example 6 (VNM-Comparability* and Normalized Comparability) *Let $g : V \times X' \to \mathbb{R}$ be such that, for all $R \in V$, $g(R,\cdot)$ is a VNM utility function representing R, with g neutral. Define $u : V \times X' \to \mathbb{R}$ by $u(R,x) = \big(g(R,x)\big)^3$ for all $R \in V$ and $x \in X'$. Define*

$$f\big((\succsim_i)_{i\in N}\big) = \underset{x\in X}{\operatorname{argmax}} \sum_{i\in N} u(\succsim_i, x).$$

This VNM SCC violates VNM-Comparability and Normalized Comparability because $u(R,\cdot)$ is the third power of a VNM utility function representing R. It is therefore monotonically but not linearly related to a VNM utility function representing R. It clearly satisfies all other conditions.

Example 7 (Normalized Comparability*) *Let* $u : V \times X' \to \mathbb{R}$ *be such that, for all* $R \in V$*,* $u(R, \cdot)$ *is a VNM utility function representing* R *and* u *is neutral but* $u(R', x_1) - u(R', x_0) \neq u(R'', x_1) - u(R'', x_0)$ *for some* $R', R'' \in V$*. Define*

$$f\big((\succsim_i)_{i\in N}\big) = \operatorname*{argmax}_{x\in X} \sum_{i\in N} u(\succsim_i, x).$$

This VNM SCC clearly violates Normalized Comparability and satisfies all other conditions.

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