Permanent Primary Deficits, Idiosyncratic Long-Run Risk, and Growth

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September 14, 2022

[1]We thank seminar participants at USC, UCLA, and the Federal Reserve Bank of Minneapolis for useful comments on earlier versions of this paper.

Abstract

We propose an economy with perpetual youth and inelastic labor supply that grows endogenously. Consumers are subject to idiosyncratic capital accumulation risk and markets are incomplete. The government makes purchases of consumption goods and transfers in the form of baby bonds and can use a consumption tax to finance its outlays. The wealth distribution is given in closed form. When the intertemporal elasticity of substitution ε is equal to 1, the government can run a permanent primary deficit, up to a finite limit, if the coefficient of relative risk aversion is high enough and the factor share of labor is not too close to 1. This causes $r - g < 0$. If $\varepsilon \neq 1$, then, for an open set of parameters, there is no upper bound on the permanent primary deficits of the government. Large deficits make consumers worse off when $\varepsilon \in (0, 1)$, but they can imply unbounded utility gains when $\varepsilon \in (1, \infty)$. We give a detailed characterization of when and how this happens.

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1 Introduction

What are the effects on growth, inequality, and welfare, of persistent and possibly large primary government deficits?

In this paper we take a detailed look at this question in a model of an endogenously growing economy in which consumers have access to accumulation technologies that are subject to idiosyncratic Brownian shocks, and in which markets are incomplete. Capital can be traded, but there are no markets to hedge the risky returns that come with accumulating capital. So nobody can earn the returns to holding capital without also being exposed to idiosyncratic risk.

The capital accumulation technology is linear. Consumption is produced using a Cobb-Douglas technology that uses capital and labor. The economy can be viewed as a special case of the two-sector economy of Uzawa [1965], but with a capital producing sector that is extremely capital intensive (linear in capital only), and a consumption sector that is relatively labor intensive. As in Uzawa [1965], the production possibility frontier for consumption and new capital is strictly concave, resulting in an endogenously determined price of capital.[1] As in Epstein and Zin [1989], consumers have nested CES preferences, with distinct elasticities of substitution across time and across states of the world.

The government issues nominal liabilities and prices are flexible. The government can purchase consumption goods, make transfers to newborn consumers (baby bonds), and tax consumption. Given stable government policies, aggregates in this economy are always on a balanced growth path. Over time, the distribution of wealth scaled by aggregate wealth also settles down, to a stationary distribution that can be given in closed form. As in Toda [2014], this distribution is a double Pareto distribution. But Toda [2014] used the standard AK structure and did not have a fixed factor. Benhabib and Bisin [2010] developed a closely related model of the distribution of wealth that relies on overlapping generations and perpetual youth. Here, the thickness of the right tail of the wealth distribution depends not only on idiosyncrasies in life spans, but also on the idiosyncratic returns on capital that individual households face. The factor share of labor is an important parameter for determining how thick the tail of the wealth distribution will be.

A special case of the economy we analyze is one in which consumers have a unit elasticity of substitution. We show that it is possible for the government to run permanent

[1] Rebelo [1991] already mentions a deterministic version of this model. Jones and Manuelli [1988] use it to avoid stagnation in an overlapping generations economy. See Galor [1992] for a detailed analysis of an Uzawa economy with overlapping generations. Luttmer [2012] shows that the competitive quality ladder model of Boldrin and Levine [2010] converges to this economy as the ladder steps become small.

primary deficits as long as there is enough idiosyncratic risk (or risk aversion) and the factor share of labor is not too close to 1. In contrast to an economy without labor, it is not possible to make permanent primary deficits possible simply by taking idiosyncratic risk to be large enough. Intuitively, labor income already provides some insurance against the risk associated with capital accumulation. There is a finite upper bound on how large deficits can be relative to household consumption expenditures.

Importantly, the upper bound on primary deficits depends on how the government uses these deficits. For example, suppose the government provides a universal basic income instead of baby bonds. Then, in effect, the government is providing consumers with a lifetime safe source of income. This reduces their demand for safe government securities, which in turn lowers the upper bound on how large deficits can be. In this sense, there is no single notion of "fiscal space." How much the government can borrow depends on what it does with the proceeds.

One consequence of a unit intertemporal elasticity of substitution is that utility is well defined for any growth rate, no matter how large or how small. If the intertemporal elasticity of substitution is below 1, utility is zero if consumption growth is too low. If the elasticity is above 1, utility is unbounded if consumption growth is too fast. Consumption risk acts like a mean reduction in consumption growth, and this has the effect of shifting these thresholds.

For example, when the elasticity of substitution is greater than one, there will be a range of economies, indexed by the productivity of the capital accumulation technology, that implies infinite utility if consumption risk can be shared perfectly, but finite utility when consumption risk is, somehow, uninsurable. In this range, there can be no competitive equilibrium with complete markets, because utility would be infinite. But there can be competitive equilibria in which consumers cannot avoid idiosyncratic risk. We show that in this range of economies, the government can run arbitrarily large primary deficits. For any large but finite level of these deficits, the economy has a well defined equilibrium. Government securities become a large component of wealth and this reduces consumption risk. We prove that utility goes to infinity as government deficits become large.

Similarly, the government can also run very large deficits when the elasticity of substitution is below 1 and the economy has a well defined complete markets equilibrium but no equilibrium without government securities. But for this range of economies, there is always an equilibrium in which unbacked government securities are valued. Utility is positive, even though it would be zero in the absence of any government securities. But now large deficits make things worse: utility converges back to zero when these deficits become large.

The unbounded utility result is intimately related to our perpetual youth assumption. We also consider an economy in which households cannot last beyond a definite age T. In the finite-T economy, we show that it is possible to construct fiscal policies that approximate the utilities in the perpetual youth economy as T becomes large.

Related Literature Going back to Samuelson [1958] and Diamond [1965], there is, of course, a vast literature on economies with real interest rates that are below their growth rates. Blanchard [2019] and the facts of the US economy in recent decades have triggered a renewed interest in the topic. Here we focus on the most closely related work that has appeared relatively recently.

Brunnermeier, Merkel and Sannikov [2022] and Reis [2021] both describe AK economies with infinitely-lived consumers, Brownian uncertainty, and incomplete markets, in which the rates of return on safe government securities is low. But there is no fixed factor of production in their models. In our economy, labor is a fixed factor, and there are overlapping generations of consumers who are not perfectly altruistically linked. Markets are incomplete but there are no other frictions, and permanent primary deficits would not be possible when households are infinitely lived. Also, because of the overlapping generations structure, fiscal policies that hurt growth can never be Pareto improving in our economy. Another important distinction is that we do not restrict attention to consumer preferences with a unit intertemporal elasticity of substitution. This is essential for making welfare improving large deficits a robust possibility.

Incomplete markets also play an important role in the Aiyagari-Bewley-Huggett economy with a neoclassical technology described by Aguiar, Amador and Arellano [2021]. They characterize the types of fiscal policies for the government that can result in Pareto improvements. In their economy, equilibrium allocations can be dynamically inefficient. This is ruled out by our AK-style Uzawa technology.

Kocherlakota [2020] considers an Aiyagari-Bewley-Huggett economy in which consumers face a near-zero probability of a highly adverse outcome. Due to a precautionary savings motive accentuated by this tail risk, there is a strong demand for risk-free government bonds. This makes a large supply of government debt possible and also desirable. Given any level of government debt, it is possible to select the parameters of the adverse tail so that that level of government debt is part of an equilibrium. In our economy, the uninsurable idiosyncratic investment risk faced by households grows without bound over long horizons. If dynastic households can survive with positive probability beyond any given age, then there is, for certain preference and technology parameters, no limit on how much a government can borrow.

Outline We introduce the economy with incomplete markets in Section 2. In Section 3, we describe the stationary complete markets allocations for the underlying economy. Section 4 shows when unbounded deficits are feasible, and when they are desirable. We specialize to a unit elasticity of substitution in Section 5 and provide explicit conditions under which the government can actually run a permanent primary deficit. In Section 6 we prove the utility approximation result for finitely-lived households. Section 7 concludes.

2 The Economy

We only consider balanced growth paths of the economy we are interested in.

2.1 Demographics and Preferences

There is a flow $\delta > 0$ of newborn consumers. At all times, consumers supply L units of labor inelastically. They die randomly at the rate δ. The population is assumed to be in a steady state, so that there is a unit measure of consumers. Given consumption flows $C_{j,t}$ and information generated by a Brownian motion $Z_{j,t}$, the utility process $U_{j,t}$ of a typical consumer j evolves according to

$$\mathrm{d}U_{j,t} = U_{j,t}\left(\mathcal{A}_t U_j \mathrm{d}t + \mathcal{S}_t U_j' \mathrm{d}Z_{j,t}\right)$$

where $\mathcal{A}_t U_j$ and $\mathcal{S}_t U_j$ satisfy

$$(\rho+\delta)U_{j,t}^{1-1/\varepsilon} = (\rho+\delta)C_{j,t}^{1-1/\varepsilon} + \left(1-\frac{1}{\varepsilon}\right)U_{j,t}^{1-1/\varepsilon}\left(\mathcal{A}_t U_j - \frac{1}{2}\xi\|\mathcal{S}_t U_j\|^2\right).$$

This version of Epstein-Zin preferences makes utility homogeneous of degree 1 in consumption. The parameter $\varepsilon \in (0,\infty)$ is the intertemporal elasticity of substitution, and ξ is the coefficient of relative risk aversion. In the standard additively separable case, $\varepsilon = 1/\xi$. Our results apply to this special case, but, following Bansal and Yaron [2004], we are especially interested in scenarios with both ξ significantly above 1 and with $\varepsilon \in (1,\infty)$.

2.2 Idiosyncratic Capital Accumulation

There is a single type of capital in this economy, but consumers can only accumulate this capital subject to idiosyncratic shocks. The capital stock of a consumer j who holds $K_{j,t}$

units of capital, evolves according to

$$\mathrm{d}K_{j,t} = (\mu K_{j,t} - X_{j,t})\,\mathrm{d}t + \varsigma K_{j,t}\mathrm{d}Z_{j,t} + \mathrm{d}I_{j,t},$$

where $X_{j,t} \geq 0$ is a flow of capital used for consumption, $I_{j,t}$ represents cumulative purchases of capital, and $Z_{j,t}$ is a standard Brownian motion that is unrelated across consumers. The parameters μ and $\varsigma^2 > 0$ are common. A central feature of the economy we consider is that there are no financial markets contingent on the $Z_{j,t}$.

2.3 The Aggregate Technology

The technology in the consumption sector is Cobb-Douglas,

$$Y_t = X_t^{1-\beta} L^{\beta},$$

where $L > 0$ is inelastically supplied labor, and $X_t \geq 0$ is the flow of capital used up during the process of producing consumption. Throughout, it is assumed that $\beta \in (0, 1)$.[2]

Because idiosyncratic shocks average out, the aggregate capital stock evolves according to

$$\mathrm{d}K_t = \mu K_t \mathrm{d}t - X_t \mathrm{d}t.$$

The price of capital in units of consumption is q_t. Since X_t depletes capital at a one-for-one rate, this is also the factor price of capital faced by producers of consumption. The factor price of labor is w_t in units of consumption. Profit maximization in the consumption sector then implies

$$\begin{bmatrix} q_t X_t \\ w_t L \end{bmatrix} = \begin{bmatrix} 1-\beta \\ \beta \end{bmatrix} Y_t.$$

A closely related interpretation is that $X_t/\mu \in [0, K_t]$ is capital that is employed to produce consumption rather than new capital. This makes this economy a special case of Uzawa [1965], with a linear technology in the capital accumulation sector. We will refer to this as the Uzawa-AK technology.

[2]Of course, any economy will have to have land, another fixed factor. This is not innocuous, because a claim to labor is not a claim to an infinitely-lived asset, and land could be (Muller and Woodford [1988]). We assume there is an unbounded supply of unimproved land and interpret capital as including improved land.

2.4 Balanced Growth

Suppose $X_t/K_t = x \in (0, \infty)$. Then the aggregate capital stock grows at the rate $\mu - x$, and therefore Y_t grows at the rate

$$g = (1 - \beta)(\mu - x). \tag{1}$$

This relation describes an immediate trade-off between the level and the growth rate of consumption. Since Y_t grows at the rate $(1-\beta)(\mu - x)$, and X_t grows at the rate $\mu - x$, the fact that factor shares are constant implies that $(\mathrm{d}q_t/\mathrm{d}t)/q_t = \mu_q$ is given by $\mu_q = -\beta(\mu - x)$, and therefore $(1-\beta)\mu_q = -\beta g$. The technology of this economy says that a high growth rate must go together with a rapidly declining price of capital.[3] This is the same as saying that the aggregate return on capital is given by $\mu + \mu_q = (1-\beta)\mu + \beta x$. Using (1), yet another way to put this is

$$x = \mu + \mu_q - g. \tag{2}$$

Notice that the dividend-price ratio for the aggregate capital stock is simply $(q_t X_t)/(q_t K_t) = x$. Therefore, in Gordon-growth fashion, $x = \mu + \mu_q - g$ can be interpreted as the effective discount rate for the dividends produced by the aggregate capital stock.

Let r be the risk-free rate in this economy. It will be useful to note that the right-hand side of (2) can be written as the sum of the excess return $\mu + \mu_q - r$ on capital and the effective discount rate $r - g$ for risk-free dividends that grow at the same rate g as Y_t.

2.5 Government

Household consumption at time t is C_t, and wealth is W_t (this will be the present value of current and future consumption of consumers alive at time t). The government consumes G_t. The aggregate resource constraint is $C_t + G_t = Y_t$. The target for government consumption is $G_t = \gamma C_t$. The government also targets a consumption tax $\tau \geq 0$ and a wealth tax ω that raises revenues equal to $T_t = \tau C_t + \omega W_t$. In addition, the government targets aggregate transfers to newborn consumers equal to σY_t. We will only consider steady state equilibria that are consistent with the target parameters γ, τ, ω, and σ.

[3]This also means that GDP and the value of aggregate output in units of consumption grow at different rates.

2.5.1 The Primary Surplus or Deficit

Including the consumption tax, household consumption expenditures are $E_t = (1+\tau)C_t$. Given the fiscal targets γ, τ, ω, and σ, the primary surplus of the government can be written as

$$\mathcal{S}_t = \frac{T_t - G_t - \sigma Y_t}{E_t} = 1 + \frac{\omega}{E_t/W_t} - \frac{(1+\gamma)(1+\sigma)}{1+\tau}. \tag{3}$$

In a steady state, aggregate consumption and wealth grow at the rate g. Write $\mathcal{S}$ for the steady state surplus ratio, and $[C_t, G_t, Y_t, E_t, W_t] = [C, G, Y, E, W]\, e^{gt}$ for the balanced growth path for the consumption sector.

2.5.2 Government Issued Deposits

The government runs a bank that issues deposits. Purchases of consumption goods are paid for and basic income transfers are made by issuing more deposits. Taxes are used to retire deposits. The government also pays interest on these deposits, by issuing more deposits. For simplicity, take the nominal interest rate to be constant at some real number $i \geq 0$. The price of consumption in units of government deposits is P_t.

The supply of government deposits D_t therefore evolves according to

$$\mathrm{d}D_t = iD_t\mathrm{d}t + P_t(G_t + \sigma Y_t - T_t)\mathrm{d}t.$$

Since government deposits are risk-free, it must be that $i = r + (\mathrm{d}P_t/\mathrm{d}t)/P_t$. This implies

$$\mathrm{d}\left(\frac{D_t}{P_tE_t}\right) = (r-g)\left(\frac{D_t}{P_tE_t}\right)\mathrm{d}t - \mathcal{S}_t\mathrm{d}t. \tag{4}$$

In a steady state, the surplus relative to consumption expenditures (3) will be constant, and so $D_t/(P_tE_t)$ must be constant. Since $E_t = Ee^{gt}$, and writing $[D, P] = [D_0, P_0]$, this means that $D_t/P_t = (D/P)e^{gt}$. Since the price level grows at the rate $i - r$, this gives $D_t = De^{(i-(r-g))t}$.

Government policy is never to lend to the public. The equilibrium value of government deposits can therefore be zero or positive, but not negative. The steady state value of $D/(PE)$ must therefore satisfy

$$(r-g) \times \frac{D}{PE} = 1 + \frac{\omega}{E/W} - \frac{(1+\gamma)(1+\sigma)}{1+\tau}, \qquad \frac{D}{PE} \geq 0. \tag{5}$$

That is, the fiscal targets γ, τ, and σ, together with a government policy not to lend to the public, force $r - g > 0$ if these fiscal targets imply a primary surplus, and $r - g < 0$

if these fiscal targets imply a primary deficit. Off an equilibrium path, this requires the government to deviate from its fiscal targets if the trajectory of P_tE_t is such that (4) leads to negative $D_t/(P_tE_t)$.

2.6 Aggregate Wealth and Portfolio Shares

Consumers can pledge assets held at their time of death in exchange for an annuity income, as in Yaari [1965] and Blanchard [1985]. Conditional on survival, their labor incomes w_tL grow at the same rate g as aggregate consumption. At birth, date-t newborn consumers sell their future labor income for $w_tL/(\delta + r - g)$ and buy capital and risk-free securities.[4] In any equilibrium, this present value must be well defined and finite. As long as $\beta \in (0,1)$, this requires $\delta + r - g > 0$. Aggregate household wealth at any point in time can then be defined as

$$W_t = q_tK_t + \frac{w_tL}{\delta + r - g} + \frac{D_t}{P_t}.$$

As already anticipated, the steady state conditions imply that $W_t = We^{gt}$.

Let $\psi = qK/W$ be the steady state portfolio share of capital. Given $X = xK$, $qX = (1-\beta)Y$, and $Y_t/E_t = (1+\gamma)/(1+\tau)$, this implies

$$\psi = \frac{1-\beta}{x}\frac{1+\gamma}{1+\tau}\frac{E}{W}, \tag{6}$$

where x is given by (2). This can be read as saying that wealth held in the form of capital, ψW, is equal to the present value of the capital income flows $(1-\beta)Y_t = \beta Ye^{gt}$, discounted at the effective rate $x = \mu + \mu_q - g$. Using $w_tL = \beta Y_t$ and $Y_t/E_t = (1+\gamma)/(1+\tau)$, the definition of wealth then implies a risk-free portfolio share

$$1 - \psi = \left(\frac{\beta}{\delta + r - g}\frac{1+\gamma}{1+\tau} + \frac{D}{PE}\right)\frac{E}{W}. \tag{7}$$

One could imagine an economy in which consumers follow inelastic decision rules defined simply by taking ψ and E/W as parameters. Then one can view (6) and (7) as market clearing conditions for capital and risk-free assets, respectively. Together with the steady state condition (5) for government deposits, these conditions must be solved for x,

[4] On a given equilibrium path, this is equivalent to the assumption that consumers can only issue real debt, subject to a present value borrowing constraint. In contrast, the government simply issues deposits, and this enables it to effectively run a Ponzi scheme when $r - g < 0$. One possible interpretation is that the private sector faces legal restrictions against such Ponzi schemes (Wallace [1983]).

$r-g$, and $D/(PE)$. The growth rate of the economy then follows from $g=(1-\beta)(\mu-x)$.

2.7 Consumer Decision Rules and Utility

With the economy on a balanced growth path, consumers face constant expected rates of return. Although shocks are idiosyncratic, everyone faces the same return parameters. So we can take E/W and ψ to also represent the decision rules of individual consumers. Conditional on survival, the average growth rate of individual consumption is given by

$$g_y = r+\delta+\psi(\mu+\mu_q-r)-\left(\omega+\frac{E}{W}\right). \tag{8}$$

Including annuity payments, consumers face expected returns $r+\delta$ and $\mu+\mu_q+\delta$, but the wealth tax lowers the rate at which wealth grows by ω. The Epstein-Zin preferences we have adopted imply

$$\frac{E}{W} = \rho+\delta-\left(1-\frac{1}{\varepsilon}\right)\left(g_y-\frac{1}{2}\xi\varsigma^2\psi^2\right) \tag{9}$$

$$\psi = \frac{\mu+\mu_q-r}{\xi\varsigma^2}. \tag{10}$$

Using (10) to eliminate $\mu+\mu_q-r$ from (8) gives $g_y = r+\delta+\xi\varsigma^2\psi^2-(\omega+E/W)$, and plugging this into (9) gives E/W in terms of r and ψ. The decision rules (8)-(10) are well defined if and only if E/W is strictly positive. The resulting utility for a consumer j with wealth $W_{j,t}$ is determined by

$$U_{j,t} = C_{j,t}\left(\frac{E/W}{\rho+\delta}\right)^{-1/(1-1/\varepsilon)}, \qquad C_{j,t}=\frac{E}{W}\frac{W_{j,t}}{1+\tau}. \tag{11}$$

So $U_{j,t}$ is linear in $W_{j,t}$, with a slope that scales with $(E/W)^{1/(1-\varepsilon)}$. Holding fixed $W_{j,t}$, the partial effects on utility of an increase in the risk-free rate r and of an increase in the risk premium $\mu+\mu_q-r$ are both positive.

The expression (9) for E/W emphasizes how the consumption-wealth ratio of a consumer whose consumption process follows a geometric Brownian motion depends on its geometric drift parameter g_y, its diffusion coefficient $\varsigma\psi$, and the risk-aversion coefficient ξ. What matters is the risk-adjusted growth rate $g_y-\frac{1}{2}\xi\varsigma^2\psi^2$. When $\varepsilon\in(0,1)$, the consumption-wealth ratio E/W is positive if and only if this risk-adjusted growth rate is not too low. When $\varepsilon\in(1,\infty)$, it is positive if and only if the risk-adjusted growth rate is not too high.

2.8 Equilibrium

Balanced growth paths for this economy can be constructed by solving the market clearing conditions (6)-(7), taking into account the decision rules (8)-(10), as well as (2) and (5). Note that (2) and (10) immediately imply $x = \xi\varsigma^2\psi + r - g$, and this can be used to eliminate x from (6). That leaves four equilibrium conditions, (5)-(9), that have to be solved for $r - g$, ψ, E/W, and $D/(PE)$.

Observe that $r - g$ and ψ pin down E/W via (8)-(10), and then (7) can be used to infer $D/(PE)$. So $r - g$ and ψ are sufficient to identify a particular equilibrium.

As in AK economies without a fixed factor, the economy will be on a balanced growth path from the start, and following any unforeseen changes in government policy. Our assumption that newborn consumers can sell their labor income at birth means that all consumers alive at a point in time use the same portfolio shares for risky and risk-free securities. Everyone holds shares in a risk-free mutual fund, backed by labor income and government securities, as well as physical capital subject to idiosyncratic shocks. The value of this mutual fund will typically jump following an unforeseen change in government policy.[5]

2.8.1 The Dynamics of Consumption, Wealth and Utility

Conditional on survival, the individual wealth of consumer j evolves according to

$$\mathrm{d}W_{j,t} = W_{j,t}\left(g_y \mathrm{d}t + \psi\varsigma \mathrm{d}Z_{j,t}\right),$$

The wealth distribution is the determined by how g_y differs from the growth rate g of aggregate consumption, and by ψ. Since an unforeseen change in government policy does not redistribute wealth among consumers alive at the time of the change, the distribution of wealth will only adjust to the new policy over time. As pointed out by Gabaix, Lasry, Lyons and Moll [2017], this may take a long time.

2.8.2 Aggregate and Individual Consumption Growth

The distribution of wealth is driven by idiosyncratic capital accumulation shocks, and by the discrepancy between the drift g_y of individual consumption and the aggregate consumption growth rate g. Newborn consumers start with wealth $W_{y,t} = W_y e^{gt}$, where

[5] An equivalent alternative assumption is that consumers borrow against their labor income at the real risk-free rate, and that consumers alive at a point in time have in place a separate perfect risk-sharing arrangement for unforeseen policy changes.

W_y/W satisfies

$$\sigma \times \frac{1+\gamma}{1+\tau}\frac{E}{W} = \delta\left(\frac{W_y}{W} - \frac{\beta}{\delta+r-g}\frac{1+\gamma}{1+\tau}\frac{E}{W}\right). \tag{12}$$

The left-hand side gives aggregate transfers σY relative to aggregate wealth, using the fact that $Y/E = (1+\gamma)/(1+\tau)$. The right-hand side accounts for the fact that these transfers add an instantaneous jump $\sigma Y/\delta$ to the initial wealth of every one of these consumers. Observe that (11) and (12) allow one to relate the utility of newborn consumers to the utility of the average consumer.

As a matter of accounting, the ratio W_y/W pins down the relation between aggregate consumption and the drift of individual consumption,

$$g = g_y - \delta\left(1 - \frac{W_y}{W}\right). \tag{13}$$

The term g_y is the consumption growth rate of surviving consumers. The consumption of consumers who randomly die scales with W, while the consumption of newborn consumers is proportional to W_y. Since $W_y > 0$, the accounting equation (13) implies $g_y < g + \delta$. The consumption growth rate of surviving consumers can, on average, exceed the aggregate growth rate of consumption, but by no more than δ.

It should be emphasized that this accounting relation is already implied by (**??**)-(12) and the equilibrium conditions (2), (5), (6)-(7) and (9)-(10). This is a consequence of Walras' law.

2.8.3 The Stationary Wealth Distribution

By Ito's lemma, we have

$$\mathrm{d}\ln\left(\frac{W_{j,t+a}}{W_{y,t+a}}\right) = \left(g_y - \left(g + \frac{1}{2}\psi^2\varsigma^2\right)\right)\mathrm{d}a + \psi\varsigma\mathrm{d}Z_{j,t+a}.$$

Taking logs reduces the drift g_y by the Ito term $\psi^2\varsigma^2/2$, and de-trending with newborn wealth reduces the drift by g. In a cohort of age a, the distribution of wealth relative to current newborn wealth among surviving consumers is therefore normal with mean $\left(g_y - \left(g + \frac{1}{2}\psi^2\varsigma^2\right)\right)a$ and variance $\psi^2\varsigma^2 a$. Random death implies that the age distribution is exponential with a density $\delta e^{-\delta a}$. Combining these two distributions gives the distribution of wealth relative to newborn wealth. As is well known, this implies a double Pareto distribution.

Proposition 1 *The stationary distribution of wealth relative to newborn wealth,* $u_{j,t} = W_{j,t}/W_{y,t}$, *has a density given by*

$$f(u) = \frac{\min\left\{u^{-(1+\zeta_-)}, u^{-(1+\zeta_+)}\right\}}{\frac{1}{\zeta_-} + \frac{1}{\zeta_+}}, \quad u \in (0, \infty),$$

where

$$\zeta_\pm = -\frac{d}{s^2} \pm \sqrt{\left(\frac{d}{s^2}\right)^2 + \frac{\delta}{s^2/2}}, \quad d = g_y - \left(g + \frac{1}{2}\psi^2\varsigma^2\right), \quad s = \psi\varsigma.$$

This satisfies $\zeta_- < 0 < \zeta_+$ *and* $\zeta_+ > 1$.[6]

The fact that $g_y < g + \delta$ implies that $d + \frac{1}{2}s^2 < \delta$, and therefore $\zeta_+ > 1$. The construction of a balanced growth path therefore guarantees that the distribution of wealth has a finite mean. The closer g_y is to its upper bound $g + \delta$, the closer is the tail index ζ_+ to 1, which is Zipf's law. In turn, this happens when W_y/W is particularly small.

2.9 Ricardian Policy Changes

As in settings in which classical Ricardian results apply, there is, in our incomplete markets economy, a certain arbitrariness to how government transfers to consumers are implemented.

To illustrate, suppose that the government not only makes one-time transfers to newborn consumers but also pays everyone a universal basic income (UBI). Suppose the aggregate UBI transfers are θY_t. The primary surplus of the government is then given by

$$\frac{T_t - G_t - (\sigma + \theta)Y_t}{E_t} = 1 + \frac{\omega}{E_t/W_t} - \frac{(1+\gamma)(1+\sigma+\theta)}{1+\tau},$$

and this appears in the steady state condition (5) for $D/(PE)$. To account for the additional consumer income, the labor share parameter β in (7) and (12) (but not in (6)) must be replaced by $\beta+\theta$. The remaining equilibrium conditions for $r-g$, ψ, E/W, and $D/(PE)$ are unaffected.

Proposition 2 shows that any policy with a UBI component can be transformed into a policy with only baby bonds, with no effect on the consumption allocation, of anyone, at any time.

Proposition 2 *Suppose the equilibrium for a policy* (θ, σ) *is given by* ψ *and* $r - g$*. Then* ψ *and*

[6] Because wealth is de-trended by newborn wealth, this proposition generalizes easily to a setting with Brownian aggregate shocks to the technology for accumulating capital. Our main result for such an economy is that nominal interest rate policy can be used to make the price level mean-reverting.

$r - g$ are also an equilibrium for the policy (θ', σ') defined by

$$\theta' = 0, \quad \sigma' = \sigma + \delta \times \frac{\theta}{\delta + r - g}.$$

Given an unforeseen one-time change in policy from (θ, σ) to (θ', σ'), the price level is also not affected if the government makes an instantaneous transfer of deposits equal to

$$\frac{D' - D}{P} = \frac{\theta Y}{\delta + r - g}$$

to consumers alive at the time of the policy change.

This policy is constructed to leave newborn consumers with the same amount of wealth when the universal basic income is abolished and replaced by baby bonds. And consumers already alive are compensated for the universal basic income they lose as a result of the new policy. This instantaneous transfer causes a jump $D' - D > 0$ in the supply of government deposits. Observe that the definition of σ' implies that the primary surplus of the government changes by the amount

$$-(\Lambda' - \Lambda)E_t = -\sigma' Y_t + (\sigma + \theta)Y_t = (r - g) \times \frac{\theta Y_t}{\delta + r - g} = (r - g) \times \frac{D' - D}{P}.$$

If $r - g > 0$, then this implies $\Lambda' < \Lambda \leq 1$, and an increase in the primary surplus of the government. But if $r - g < 0$, then this implies $1 \leq \Lambda < \Lambda'$, and an increase in the primary deficit of the government. In both cases, this implies an increase in the steady state supply of government deposits. The instantaneous transfer of deposits to consumers already alive exactly matches this steady state increase, without a change in the price level. If consumers already alive were not made good by the government for losing their universal basic income, then there would be a drop in the price level that ensures $(r - g)(D/P' - D/P) = -(\Lambda' - \Lambda)E_t$. That would give these consumers a capital gain that exactly compensates them for losing the universal basic income.

For the government, the flow cost of making instantaneous transfers with a present value $1/(\delta + r - g)$ to a flow δ of agents is $\delta/(\delta + r - g)$. The flow cost of transferring a unit flow of consumption to a unit measure of agents is 1.[7] Therefore, switching from baby bonds to an equivalent UBI could turn a surplus into a deficit if $r - g > 0$, and a deficit into a surplus if $r - g < 0$. In both cases, this would then require the government to lend to the public.

[7]If $r - g > 0$, then the present value of transfers made to all consumers born after a given initial date is $\delta/((r - g)(\delta + r - g))$ in both cases.

As we shall see, there is a range of baby bond policies that cannot be replicated using a universal basic income when the government does not lend to consumers.

3 Complete Markets Economies

To set the stage, it is useful to discuss the properties of complete markets versions of this economy in some detail.

3.1 Infinitely Lived Consumers

Suppose $\delta = 0$ and that markets are complete. So there is a fixed population of consumers who live forever, and these consumers can perfectly share the idiosyncratic risk of their capital accumulation technologies. This implies a representative consumer. Accounting for possible wealth taxes, the standard Euler condition is then

$$g = \varepsilon(r - (\rho + \omega)). \tag{14}$$

Since there can be no risk premium, $\mu + \mu_q - r = 0$, and hence (2) becomes $x = r - g$. Together with $g = (1 - \beta)(\mu - x)$, this yields

$$\beta g = (1 - \beta)(\mu - r). \tag{15}$$

The negative technological relation between the return on capital and the growth rate of the economy becomes a negative relation between the risk-free rate and the growth rate of the economy.

Figure 1 shows the equilibrium (14) and (15). Variation in the productivity μ of the capital accumulation technology results in shifts of (15) along the Euler condition (14), and this leads to co-movement of r and g. On the other hand, variation in ρ produces movements in the Euler condition along the technological restriction (15) on r and g. As long as the factor share of capital is strictly positive, this leads to r and g that move in opposite directions.

In any equilibrium x, and hence $r - g$, must be positive. In Figure 1, the intersection of $g = \varepsilon(r - (\rho + \omega))$ with the diagonal defines the boundary of the region where this is the case. Here, $\varepsilon \in (1, \infty)$, resulting in an upper bound on r and g for which $r - g$ is positive. The economy only has a well defined equilibrium if $\mu - \omega$ is low enough to ensure that the intersection of $\beta g = (1 - \beta)(\mu - \omega - (r - \omega))$ and $g = \varepsilon(r - \omega - \rho)$ is below this upper bound. If $\varepsilon \in (0, 1)$, then the requirement that $r - g > 0$ implies a lower bound on $\mu - \omega$.

Away from these boundaries, an increase in the wealth tax lowers the growth rate of this economy. The technology implies that this raises the level of current consumption at the same time.

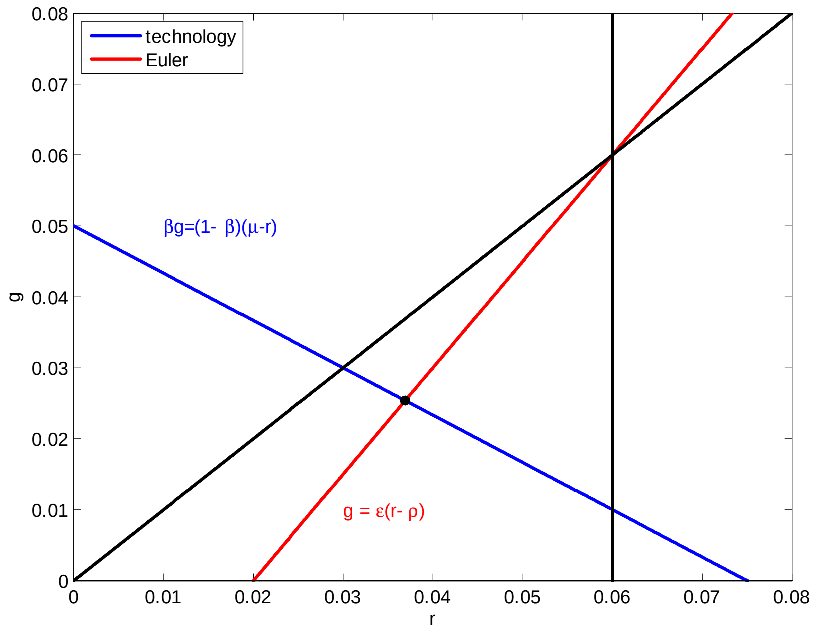

Figure 1 *Equilibrium in an economy with a representative agent.*

3.2 Perpetual Youth

When there is a flow $\delta > 0$ of newborn consumers and consumers die randomly at some rate $\delta > 0$, the economy no longer has a representative agent. We will need to clear markets explicitly. Fiscal policy can have important effects on the growth rate of the economy.

Continuing to assume that markets are complete, the fact that $r - g = x > 0$ implies that the present value of aggregate consumption will be finite. If the wealth tax is zero, this will be enough to guarantee that competitive equilibria are Pareto efficient. The Uzawa-AK technology we are using automatically rules out the dynamic inefficiencies that are central in Samuelson [1958] and Diamond [1965].[8]

3.2.1 The Equilibrium Conditions

The fact that $x = r - g > 0$ in any equilibrium, together with our assumption that the government does not lend to consumers, also implies that the government cannot run

[8] It is easy to show that this is still true when the capital accumulation technologies are also subject to aggregate shocks.

permanent primary deficits. For simplicity, focus on the case $(1+\gamma)(1+\sigma)/(1+\tau) \leq 1$, so that wealth taxes are never necessary to avoid a deficit. Together with (5), the risky and risk-free market clearing conditions (6)-(7) imply

$$1 = \left(\frac{1-\beta}{x} + \frac{\beta}{\delta + x}\right) \frac{1+\gamma}{1+\tau} \frac{E}{W} + \frac{1}{x}\left(\omega + \left(1 - \frac{(1+\gamma)(1+\sigma)}{1+\tau}\right) \frac{E}{W}\right). \tag{16}$$

This is simply a decomposition of the components of consumer wealth, into capital, claims to labor, and government securities. The consumer decision rules (8)-(10) reduce to

$$g = (1-\beta)(\mu - x), \quad g_y = g + \delta + x - \left(\omega + \frac{E}{W}\right), \quad \frac{E}{W} = \rho + \delta - \left(1 - \frac{1}{\varepsilon}\right) g_y. \tag{17}$$

The side conditions are $x > 0$ and $E/W > 0$. Using the fact that $\delta > 0$, $\sigma \geq 0$, and $\omega \geq 0$, it is easy to see from (16) that $x \in (\omega, \omega + E/W)$ in any equilibrium. The second condition in (17) then ensures that $g_y < g + \delta$ in any equilibrium. Note that (17) and $x = r - g$ implies the Euler condition $g_y = \varepsilon(r - (\rho + \omega))$ for individual consumption growth. The Euler condition (14) for aggregate consumption growth no longer applies.

Given a solution to these equilibrium conditions, and an initial capital stock K, the utility U of the average consumer already alive and the utility U_y of the current generation of newborn consumers are

$$U = \frac{(xK)^{1-\beta} L^{\beta}}{1+\gamma} \left(\frac{E/W}{\rho + \delta}\right)^{-1/(1-1/\varepsilon)}, \quad U_y = \frac{g + \delta - g_y}{\delta} \times U. \tag{18}$$

This follows from (11) and the fact that $W_y/W = (g + \delta - g_y)/\delta$ in any steady state. Since aggregate consumption grows at the rate g, the utility of a consumer who will be born at some future date $T \geq 0$ is $U_y e^{gT}$.

3.2.2 Existence of Equilibrium

Solving both (16) and (17) for E/W and eliminating g and g_y gives

$$\begin{aligned} \frac{E}{W} &= \frac{1 - \frac{\omega}{x}}{\left(\frac{1-\beta}{x} + \frac{\beta}{\delta+x}\right)\frac{1+\gamma}{1+\tau} + \frac{1}{x}\left(1 - \frac{(1+\gamma)(1+\sigma)}{1+\tau}\right)}, & (19) \\ \frac{E}{W} &= \varepsilon \times \left(\rho + \delta - \left(1 - \frac{1}{\varepsilon}\right)((1-\beta)(\mu - \omega) + \beta(x - \omega) + \delta)\right). & (20) \end{aligned}$$

By taking derivatives, one can verify that (19) is a positive, increasing, and concave function of $x \in (\omega, \infty)$. Importantly, its slope converges to a limit that is strictly greater than 1 as x becomes large. The equation (20) is a line with slope $(1-\varepsilon)\beta < 1$. Therefore, the only way these two equilibrium conditions can intersect with $E/W > 0$ is for the line to be strictly positive at $x = \omega$. One can verify that an increase in ω raises the equilibrium value of x. This proves our next proposition.

Proposition 3 *Suppose the fiscal targets of the government satisfy $(1+\gamma)(1+\sigma)/(1+\tau) \leq 1$ and that there is a wealth tax $\omega \geq 0$. Then the complete markets economy has an equilibrium if and only if*

$$\left(1 - \frac{1}{\varepsilon}\right) \frac{(1-\beta)(\mu-\omega)+\delta}{\rho+\delta} < 1. \tag{21}$$

The equilibrium is unique and satisfies $x \in (\omega, \omega + E/W)$. The equilibrium is Pareto efficient if and only if $\omega = 0$. An increase in ω increases the level of aggregate consumption but lowers its growth rate.

The bound (21) holds trivially if $\varepsilon = 1$. For $\varepsilon \in (0,1)$, this bound is a lower bound on $(1-\beta)(\mu-\omega)+\delta$, while for $\varepsilon \in (1,\infty)$ it is an upper bound. In any stationary allocation, the deterministic rate at which individual consumption can grow must satisfy $g_y < g + \delta$, and a planner can take $g = (1-\beta)(\mu-x)$ up to $(1-\beta)\mu$ by taking x close to zero. Therefore, evaluated at $\omega = 0$, (21) is simply the bound that $(1-\beta)\mu + \delta$ must satisfy for the utility of every stationary allocation to be positive if $\varepsilon \in (0,1)$, and finite if $\varepsilon \in (1,\infty)$.

More generally, $x > \omega$ implies the rate at which individual consumption can grow in a competitive equilibrium must be less than $(1-\beta)(\mu-\omega)+\delta$. As a consequence, an increase in the wealth tax ω shrinks the set of economies with a well-defined complete markets equilibrium if $\varepsilon \in (0,1)$, and expands it when $\varepsilon \in (1,\infty)$. In particular, when $\varepsilon \in (1,\infty)$, a large enough ω will ensure that (21) holds.[9]

3.2.3 Unbounded Utilities

Suppose $\varepsilon \in (1,\infty)$ and (21) is violated at $\omega = 0$. Then there will be a $\omega_\infty > 0$ so that the economy has a well defined complete markets equilibrium for all $\omega > \omega_\infty$. The proof of Proposition 3 shows that $x - \omega \downarrow 0$ and $E/W \downarrow 0$ as $\omega \downarrow \omega_\infty$. The equilibrium condition (19) means that E/W behaves like $x - \omega$, and the second equation in (17) can be written as $E/W = x - \omega + g + \delta - g_y$, implying that $g + \delta - g_y$ also behaves like $x - \omega$. Since

[9] If $\varepsilon \in (0,1)$, then a large enough negative ω will ensure that the inequality (21) holds, even if it fails at $\omega = 0$. But there can be no equilibrium for any $\omega \leq 0$ if (21) fails at $\omega = 0$.

$1 - 1/(1 - 1/\varepsilon) > 1$, it then follows from (18) that both U and U_y grow without bound as $\omega \downarrow \omega_\infty$.

If $\varepsilon \in (1, \infty)$ and $\omega_\infty > 0$, the economy is so productive that an omniscient central planner can deliver unbounded utilities to everyone. But the economy does not have a complete markets equilibrium if $\omega = 0$. We can only speculate what will happen in such an economy. By setting a wealth tax $\omega > \omega_\infty$, a government can ensure the economy does have a complete markets equilibrium. If a government takes $\omega > \omega_\infty$ close to ω_∞, utilities will be very high. In the presence of any type of uncertainty about the structure of the economy, such a government must face the risk of setting ω too low.

3.2.4 A Ricardian Corollary

Consider some $\omega \geq 0$ for which (21) holds, and fix the solution to (16)-(17) obtained for certain fiscal targets γ, τ, and σ. These fiscal targets only appear in (16). One way to rewrite (16) is

$$\left(1 - \frac{x - \omega}{E/W}\right) \frac{1 + \tau}{1 + \gamma} = \sigma + \frac{\delta\beta}{\delta + x}.$$

This confirms $x \in (\omega, \omega + E/W)$, as argued in Proposition 3. Given the fixed solution for x and E/W, one can choose any $\tau > -1$ and $\sigma \geq 0$ subject to this affine restriction and obtain the same equilibrium. In particular, one could set $\sigma = 0$, possibly requiring consumption subsidies because the implied $\tau > -1$ is negative. But one can also let τ and σ increase without bound. In that case, the first term in (16) must converge to zero. From (6), this means that ψ converges to zero, and so does the portfolio weight of labor income. The condition (16) immediately implies that the limiting value of $(1 + \gamma)(1 + \sigma)/(1 + \tau)$ is $(x - \omega)/(E/W)$, and this is strictly inside $(0, 1)$.

In other words, for every feasible government policy, there is an unbounded range of equivalent policies, implying the same equilibrium allocation of consumption, in which the government uses large consumption taxes to make large transfers to newborn consumers and run a primary surplus. Government securities in such equilibria account for almost all consumer wealth. Since E/W and the trajectory for aggregate consumption are the same across all these policies, and since $E_t = (1 + \tau)C_t$, the construction of these equivalent policies implies that W_t scales with $1 + \tau$. An unforeseen Ricardian increase in τ and σ will cause an upward jump in aggregate wealth, all of which is held by consumers already alive, that compensates everyone for the higher consumption taxes they will have to pay.

3.2.5 The Welfare Properties of Stationary Allocations

Stationary allocations in this economy are characterized by pairs (x, g_y) with $x > 0$ and $g_y < g + \delta = (1-\beta)(\mu - x) + \delta$. The resource constraint on aggregate consumption C and newborn consumption C_y is $gC = (g_y - \delta)C + \delta C_y$. This implies $C_y/C = (g + \delta - g_y)/\delta$, as in (13). The resulting utilities are

$$U = \frac{(xK)^{1-\beta} L^{\beta}}{1+\gamma} \left(1 - \left(1 - \frac{1}{\varepsilon}\right) \frac{g_y}{\rho + \delta}\right)^{-1/(1-1/\varepsilon)}, \qquad U_y = \frac{(1-\beta)(\mu - x) + \delta - g_y}{\delta} \times U.$$

There are pairs (x, g_y) for which $U \in (0, \infty)$ and $U_y/U > 0$ if and only if (21) holds at $\omega = 0$. It is easy to see that U is increasing in x and g_y, and that U_y has a unique maximizer $x = ((1-\beta)\mu + \delta - g_y)/(2-\beta)$ given any g_y that satisfies $(1 - 1/\varepsilon)g_y < \rho + \delta$ and $g_y < (1-\beta)\mu + \delta$. The newborn utility U_y has a unique global maximum if (21) holds at $\omega = 0$ and $\varepsilon < 1 + 1/(1-\beta)$. If ε is larger, then U_y maximized over x is decreasing in g_y. When ε exceeds $1 + 1/(1-\beta)$, the prospect of more rapid consumption growth cannot overcome the implied reduction in the level of consumption at the start of life.

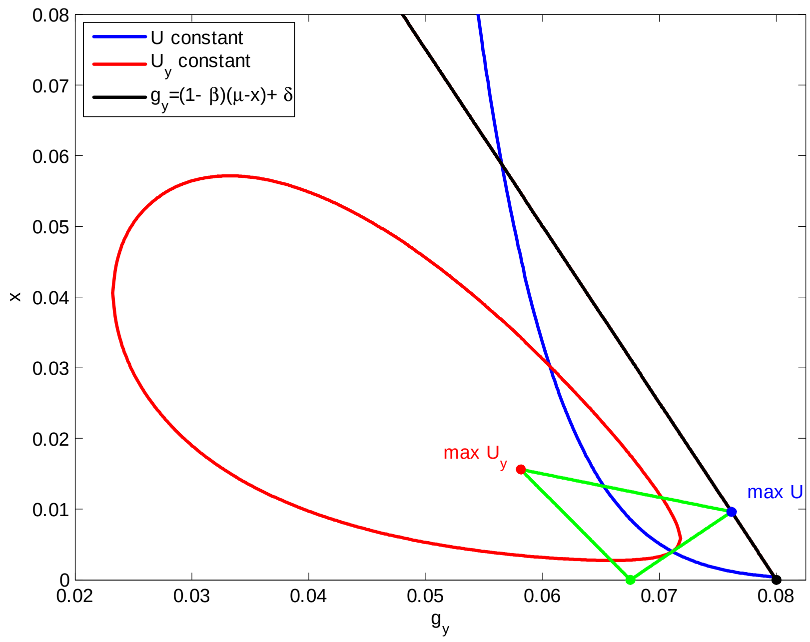

Figure 2 *Stationary allocations with perfect risk sharing*

As long as (21) holds at $\omega = 0$, stationary allocations are Pareto efficient if and only if $g_y = \varepsilon(x + (1-\beta)(\mu - x) - \rho)$ and $g_y < (1-\beta)(\mu - x) + \delta$.[10] Figure 2 shows the line $g_y = (1-\beta)(\mu - x) + \delta$ and the indifference curves of U and U_y for the unique

[10]This is seen most easily from the fact that in any decentralization there will be an Euler condition $g_y = \varepsilon(r - \rho)$, and $r = x + (1-\beta)(\mu - x)$.

Pareto efficient allocation that maximizes U_y. In this example, $\varepsilon \in (1, 1 + 1/(1 - \beta))$, so that U_y has a global maximum, as indicated in Figure 2. Also shown are the allocation that maximizes U, the limiting Pareto efficient allocation that maximizes aggregate growth, and the allocation that maximizes both g and g_y. These four allocations generate a convex quadrilateral. The upward sloping diagonal of this quadrilateral represents the Pareto efficient allocations that are stationary. The two edges of this quadrilateral connected to the allocation that maximizes U_y correspond to the conditions $\partial U_y/\partial g_y = 0$ and $(\partial U_y/\partial x)/(\partial U_y/\partial g_y) = (\partial U/\partial x)/(\partial U/\partial g_y)$. These edges can be interpreted as contract curves for stationary allocations. One can use this to argue that this convex quadrilateral is the set of stationary allocations that are not Pareto dominated by other stationary allocations.

The full range of stationary Pareto efficient allocations can be implemented by setting $\omega = 0$ and varying τ and σ. At one end of this range is the allocation preferred by consumers already alive. It can be found by imposing $E/W = x$ in (20). Given that we restrict attention to non-negative transfers σ, the implementation of this allocation requires that $(1 + \gamma)(1 + \sigma)/(1 + \tau) \downarrow 0$, so that (19) reduces to $E/W = x$. As indicated in Figure 2, this means $g_y \uparrow g + \delta$ and therefore $C_y/C \downarrow 0$. Consumption inequality approaches Zipf's law because every new generation has to start with a very low initial level of consumption if its growth rate g_y conditional on survival is close to the maximal feasible rate $g + \delta$. At the other end of the range of stationary Pareto efficient allocations, approximating a competitive equilibrium with an aggregate growth rate that approaches its technological upper bound $(1 - \beta)\mu$ requires balancing the budget and taking τ and σ to be large. This delivers the allocations preferred by generations that will be born very far into the future.

As evidenced by Figure 2, almost all stationary allocations that are not Pareto dominated by other stationary allocations are not Pareto optimal. In particular, the stationary allocation preferred by the current newborn generation over all other stationary allocations is not a Pareto efficient allocation. That allocation can be approximated by setting the wealth tax equal to the desired x and letting consumption taxes become large while balancing the budget.

4 Incomplete Markets Economies

In a complete markets economy, a government that does not lend to the private sector cannot run a permanent primary deficit. When markets are incomplete, this is no longer true.

4.1 A Summary of the Equilibrium Conditions

Relative to aggregate wealth, (5) says that the value of government securities outstanding must be non-negative,

$$(r-g) \times \frac{D/P}{W} = \omega + \left(1 - \frac{(1+\gamma)(1+\sigma)}{1+\tau}\right)\frac{E}{W}, \qquad \frac{D/P}{W} \geq 0. \tag{22}$$

The risky and risk-free market clearing conditions (6)-(7) are

$$\psi = \frac{1-\beta}{x}\frac{1+\gamma}{1+\tau}\frac{E}{W}, \tag{23}$$

$$1-\psi = \frac{\beta}{\delta+r-g}\frac{1+\gamma}{1+\tau}\frac{E}{W} + \frac{D/P}{W}. \tag{24}$$

Both x and g are functions of ψ and $r-g$,

$$x = \xi\varsigma^2\psi + r - g, \qquad g = (1-\beta)(\mu - x). \tag{25}$$

The consumer decision rules (8)-(9) then become

$$g_y = g + \delta + x - \xi\varsigma^2\psi(1-\psi) - \left(\omega + \frac{E}{W}\right), \tag{26}$$

$$\frac{E}{W} = \rho + \delta - \left(1 - \frac{1}{\varepsilon}\right)\left(g_y - \frac{1}{2}\xi\varsigma^2\psi^2\right). \tag{27}$$

The side conditions are $\psi \in (0,1)$, $x > 0$, $E/W > 0$, and $r - g \in (-\delta, \infty)$.

Given a solution to these equilibrium conditions, and an initial capital stock K, the utility U of consumers already alive and U_y of newborn consumers are again given by (18). But E/W includes now the risk adjustment $-\frac{1}{2}\xi\varsigma^2\psi^2$ and $g + \delta - g_y$ includes a term $-\xi\varsigma^2\psi^2$ that reflects expected returns on wealth in excess of r.

4.1.1 Solving this System

These equilibrium conditions can be reduced to two equations in $\psi = (\mu + \mu_q - r)/(\xi\varsigma^2)$ and $r-g$ that can be interpreted as risky and risk-free market clearing conditions. First, using (25)-(27) to eliminate g_y and g gives x and E/W as functions of ψ and $r-g$. Then, using (25)-(27) to eliminate x and E/W from (23) gives a risky market clearing condition that is a linear equation in $r-g$ given ψ. And using (22) and (25)-(27) to eliminate $(D/P)/W$ and E/W from (24) gives a risk-free market clearing condition that is a quadratic equation in ψ given $r - g \neq 0$.

4.2 Primary Surplus Policies

In the complete markets economy, we know that there is an unbounded range of fiscal policies that all implement the same allocation. And the government can make the portfolio share of capital ψ arbitrarily small by using large consumption taxes to make transfers and back its outstanding securities. In the incomplete markets economy, this gives rise to the following approximation result.

Proposition 4 *Suppose $\omega \geq 0$ and the economy satisfies the condition (21) for the existence of a complete markets equilibrium when the fiscal parameters satisfy $(1+\gamma)(1+\sigma)/(1+\tau) = \Lambda \in (0,1]$. Then the resulting complete markets utilities can be approximated in the incomplete markets economy by taking τ and σ to be large, subject to $(1+\gamma)(1+\sigma)/(1+\tau) \to \Lambda$.*

The precise proof is in the appendix. The basic intuition is that the government can use large τ and σ to make $\psi > 0$ small in a desired complete markets allocation. But then the ψ and $r - g$ that form part of a complete markets equilibrium will also approximately solve the conditions for an incomplete markets equilibrium, because the terms $\xi\varsigma^2\psi$, $\xi\varsigma^2\psi(1-\psi)$, and $\frac{1}{2}\xi\varsigma^2\psi^2$ in (25)-(27) will be small. Given a large-τ and large-σ complete markets solution for ψ and $r - g$, the continuity of (25)-(27) implies that the resulting E/W must be close to the complete markets solution. One can then use (23) to back out a τ, and the combination of (22) and (24) to back out σ, to make this an equilibrium in the incomplete markets economy.[11]

4.2.1 Unbounded Utilities Again

If $\varepsilon \in (0,1)$, then a violation of (21) at $\omega = 0$ means that μ is so low that there are no stationary allocations that deliver positive utility, not even when these allocations are risk-free. This certainly rules out the existence of competitive equilibria in an economy with incomplete markets.

Violations of (21) at $\omega = 0$ are more interesting when $\varepsilon \in (1,\infty)$. We have already shown, for $\varepsilon \in (1,\infty)$, that the complete markets economy has a unique equilibrium as long as $\omega > \omega_\infty$, where $\omega_\infty > 0$ is the value of ω at which (21) holds with equality. And utilities are unbounded as $\omega > \omega_\infty$ approaches ω_∞. In combination with Proposition 4, this means that the a government can deliver unbounded utilities also in the incomplete markets economy by setting $\omega > \omega_\infty$ close to ω_∞, imposing large consumption taxes,

[11] In the complete markets economy, Ricardian increases in τ and σ only reduce $\psi \in (0,1)$, with no effect on anyone's utility. Here, increasing τ and σ not only reduces idiosyncratic risk but also redistributes wealth across the generations. We will return to this in the special case of $\varepsilon = 1$.

and using these taxes to both make large transfers to newborn consumers and back its outstanding securities.

4.3 Balanced Budget Policies

Unlike in the complete markets economy, the incomplete markets economy may have a pure bubble equilibrium—an equilibrium in which government securities trade at a strictly positive price even if government budgets are balanced at all times. Here we restrict attention to economies in which both $\omega = 0$ and $(1+\gamma)(1+\sigma)/(1+\tau) = 1$.

The conditions for a pure bubble equilibrium can be obtained from (23)-(27) by setting $r - g = 0$ and replacing the risk-free market clearing condition (24) with the inequality $1 - \psi \geq (\beta/\delta)(E/W)/(1+\sigma)$. This ensures that the value of government securities is non-negative. Using the risky market clearing condition (23) to eliminate $(E/W)/(1+\sigma)$ from this inequality, and noting that $x = \xi\varsigma^2\psi$, gives

$$\psi + \frac{\xi\varsigma^2}{\delta}\frac{\beta}{1-\beta} \times \psi^2 \leq 1. \tag{28}$$

This inequality will be satisfied for all $\psi \in (0,1)$ close enough to zero. The price of government securities will be strictly positive if and only if (28) is strict. To attempt to construct an equilibrium, one can pick a $\psi \in (0,1)$ that satisfies (28) and then use the remaining equilibrium conditions to search for balanced budget parameters $\tau > -1$ and $\sigma \geq 0$ that make this ψ an equilibrium.

The resulting equations for τ and σ are linear, but $\sigma \geq 0$ together with the balanced budget requirement forces $\tau \geq \gamma$. From (23), this is equivalent to $E/W \geq \psi x/(1-\beta)$. Using the decision rule (26)-(27) to eliminate E/W from this inequality, and then (25) to eliminate x and g from the result, gives

$$\frac{\xi\varsigma^2}{\rho+\delta}\left(\frac{1}{\varepsilon}\frac{\psi^2}{1-\beta} + \left(1-\frac{1}{\varepsilon}\right)\left(\beta\psi - \frac{1}{2}\left(1-(1-\psi)^2\right)\right)\right) \leq 1 - \left(1-\frac{1}{\varepsilon}\right)\frac{(1-\beta)\mu+\delta}{\rho+\delta}. \tag{29}$$

The left-hand side of this inequality is convex in ψ, equal to zero at $\psi = 0$, with a positive slope at $\psi = 0$ if $\varepsilon \in (0,1)$, and a negative slope if $\varepsilon \in (1,\infty)$. As a result, (29) holds for all ψ close enough to zero if the right-hand side is strictly positive, and only then if $\varepsilon \in (0,1)$.

Given some $\psi \in (0,1)$ that satisfies (28) and (29), the equilibrium conditions (25)-(27) pin down x and E/W, and then (23) can be used to back out $(1+\gamma)/(1+\tau)$ and $1+\sigma = (1+\tau)/(1+\gamma)$. By construction, this delivers a balanced budget equilibrium for the resulting fiscal targets. One can verify that the implied σ will be large when $\psi > 0$

is close to zero. If the right-hand side of (29) is sufficiently large and positive, then (29) will hold strictly for all $\psi \in (0, 1)$. In that case pure bubble equilibria are possibly only for strictly positive σ. This proves the following proposition.

Proposition 5 *Fix $\omega = 0$ and $\gamma \geq 0$. If $\varepsilon \in (0, 1)$, then the condition (21) is necessary and sufficient for the economy to have pure bubble equilibria for all large enough $\tau \geq 0$ and $\sigma \geq 0$ that satisfy $(1 + \gamma)(1 + \sigma)/(1 + \tau) = 1$. If $\varepsilon \in (1, \infty)$, then the condition (21) is sufficient but not necessary.*

In other words, if the economy has stationary allocations that produce positive utility, and the resulting utilities are bounded above, then there exist balanced budget policies for which the economy has a pure bubble equilibrium. The positive utility condition is obviously necessary if $\varepsilon \in (0, 1)$. But if $\varepsilon \in (1, \infty)$, then a stationary allocation subject to idiosyncratic risk may have finite utility even if the corresponding risk-free allocation generates infinite utility. As a result, there will be a range of economies that do have a pure bubble equilibrium, even though, in the absence of a wealth tax, these economies are too productive to have a well defined complete markets equilibrium.

The following proposition gives the range of economies for which the incomplete markets economy has a balanced budget equilibrium when there are no bubble securities. The proof is in the appendix.

Proposition 6 *Fix $\omega = 0$ and $\gamma \geq 0$. Suppose $(1 + \gamma)(1 + \sigma)/(1 + \tau) = 1$ and there is no bubble security. If $\varepsilon \in (0, 1)$ then the bound*

$$\left(1 - \frac{1}{\varepsilon}\right) \frac{(1 - \beta)\mu + \delta}{\rho + \delta} < 1 + \left(1 - \frac{1}{\varepsilon}\right) \frac{\delta}{\rho + \delta} \times \begin{cases} \frac{1}{2}\frac{\xi\varsigma^2}{\delta} & \text{if } \frac{\xi\varsigma^2}{\delta} < 1 \\ 1 - \frac{1}{2}\frac{1}{\xi\varsigma^2/\delta} & \text{if } \frac{\xi\varsigma^2}{\delta} > 1 \end{cases} \tag{30}$$

is necessary and sufficient for the incomplete markets economy to have an equilibrium. If $\varepsilon \in (1, \infty)$, then this bound is sufficient for existence, and necessary if $(1 - \beta)^2\xi\varsigma^2/\delta < 1$, or if $(1 - \beta)^2\xi\varsigma^2/\delta \geq 1$ and $\varepsilon \in (1, \varepsilon_\infty)$ for some $\varepsilon_\infty \in (1, \infty)$ that depends on σ.

If $\varepsilon \in (0, 1)$, then the combination of Propositions 3 and 6 says that there is a range of economies with a complete markets equilibrium that do not have a no-bubble incomplete markets equilibrium because idiosyncratic risk is too high. But then Proposition 5 restores the existence of equilibrium for these economies if bubble securities are allowed. If $\varepsilon \in (1, \infty)$, then the Propositions 3, 5, and 6 imply that the range of economies with no-bubble and bubble equilibria is larger than the range of economies with a well defined complete markets equilibrium at $\omega = 0$.

4.4 Primary Deficit Policies

Given incomplete markets, there are economies in which there is no bound on how large primary deficits can be.

4.4.1 The Possibility of Unbounded Deficits

To illustrate, Figure 3 shows an example of the risky and risk-free market clearing conditions in an economy with $\varepsilon > 1$. The figure also shows the line $0 = \xi\varsigma^2\psi + r - g$, and a curve that gives the large-σ limit of the risk-free market clearing condition.[12] The risky market clearing condition does not depend on σ. As indicated by Figure 3, increasing σ shrinks the risk-free market clearing condition, viewed as a mapping $r - g \mapsto \psi$, towards its large-σ limit. And that limit continues to intersect the risky market clearing condition. In this example, there is no bound on how large transfers can be.

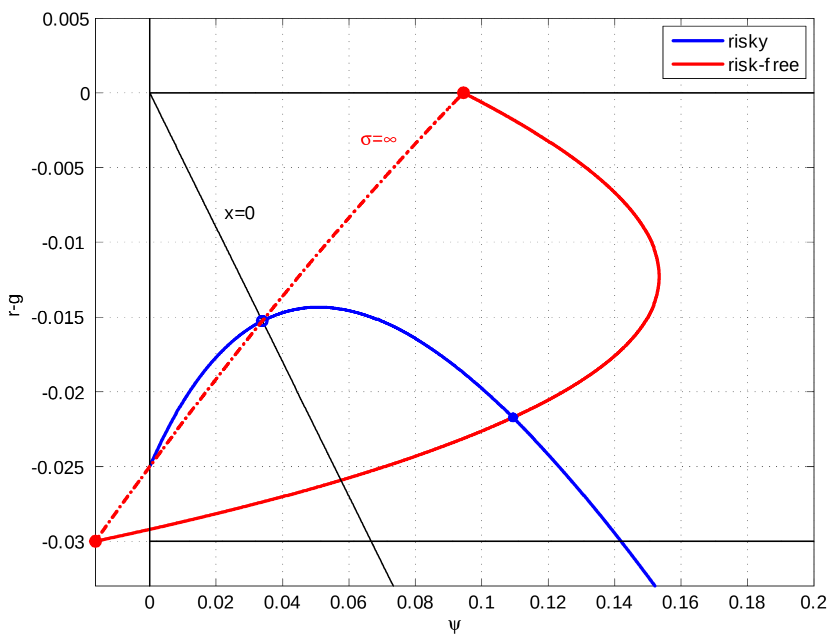

Figure 3 *The equilibrium conditions for $\sigma \in (0, \infty)$ and $\sigma = \infty$.*

To understand why unbounded deficits are a possibility, focus on $\omega = 0$ so that large $\sigma > 0$ automatically imply large primary deficits and hence require $r - g \in (-\delta, 0)$.

Consider a sequence of σ that become large, and take a sequence of $\psi \in (0, 1)$, $r - g \in (-\delta, 0)$, and $E/W > 0$ that satisfy (24). Along such a sequence, (24) implies that $E/W > 0$

[12] The risk-free market clearing condition implied by (24) and (25)-(27) is a quadratic in ψ. It has another branch that is not shown. That branch does not intersect the risky market clearing condition. In Section 5 we provide a more detailed characterization of the risk-free market clearing condition in the $\varepsilon = 1$ special case.

must converge to zero. The fact that the portfolio share of government securities has to be bounded above by 1 means that when primary deficits become large as a share of consumer expenditures, consumer wealth must also become large relative to expenditures—something that is a possibility only if $\varepsilon \neq 1$. Furthermore, the risky market clearing condition (23) implies that, for a converging sequence of deficit equilibria, with $E/W > 0$ converging to zero, $x = \xi\varsigma^2\psi + r - g$ must also converge to zero. If not, then ψ would have to converge to zero, and a strictly positive limit for $x = \xi\varsigma^2\psi + r - g$ together with a zero limit for ψ would imply a strictly positive limit for $r - g$. This would violate the side condition $r - g \in (-\delta, 0)$ that must hold when the government runs a primary deficit. In sum, if there is a converging sequence of deficit equilibria, then it must have the property that E/W and x converge to zero.

Suppose that it is indeed possible to construct equilibria for all large σ, and suppose that ψ_∞ and $(r-g)_\infty$ are large-σ limits of equilibrium values for ψ and $r - g$. Then the argument just given says that $(r-g)_\infty = -\xi\varsigma^2\psi_\infty$, and that ψ_∞ must solve the quadratic equation

$$\rho + \delta = \left(1 - \frac{1}{\varepsilon}\right)\left((1-\beta)\mu + \delta - \omega - \left(1 - (1-\psi_\infty)^2\right) \times \frac{1}{2}\xi\varsigma^2\right).$$

This equation follows from imposing $E/W = 0$ and $x = 0$ in (25)-(27). The factor multiplying $1 - 1/\varepsilon$ on the right-hand side is simply the limiting value of the risk-adjusted consumption growth rate $g_y - \frac{1}{2}\xi\varsigma^2\psi^2$. Since $\rho + \delta > 0$, this risk-adjusted growth rate will have to be negative if $\varepsilon \in (0,1)$, and positive if $\varepsilon \in (1,\infty)$. In any case, the only solution for ψ_∞ that could possibly be in $(0,1)$ is

$$\psi_\infty = 1 - \sqrt{1 - \frac{1}{\xi\varsigma^2/2}\left((1-\beta)\mu + \delta - \omega - \frac{\rho+\delta}{1-1/\varepsilon}\right)}. \tag{31}$$

If this leads to $\psi_\infty \in (0,1)$ and $(r-g)_\infty \in (-\delta, 0)$, then (24) automatically implies that the government does indeed run a primary deficit in the limit.

But the formula (31) can only lead to a positive ψ_∞ if μ is large enough to ensure that $(1-\beta)\mu + \delta - \omega > (\rho+\delta)/(1-1/\varepsilon)$. And μ cannot be too large either, since ψ_∞ will reach 1 for a large enough μ and then become complex with any further increases in μ. Moreover, if $\xi\varsigma^2 > \delta$, then the requirement that $(r-g)_\infty/\delta = -(\xi\varsigma^2/\delta)\psi_\infty > -1$ forces $\psi_\infty < 1$. This further lowers the upper bound on μ. Given any $\varepsilon \in (0,1) \cup (1,\infty)$, these considerations define a non-empty interval of μ for which it is possible to construct $\psi_\infty \in (0,1)$ and $(r-g)_\infty \in (-\delta, 0)$ that can be interpreted as large-σ limits of equilibria.

Proposition 7 *Fix some wealth tax $\omega \geq 0$. The economy has an equilibrium for all large enough σ if and only if*

$$0 < \frac{(1-\beta)\mu + \delta - \omega}{\rho + \delta} - \frac{1}{1 - 1/\varepsilon} < \frac{\delta}{\rho + \delta} \times \begin{cases} \frac{1}{2}\frac{\xi\varsigma^2}{\delta} & \text{if } \frac{\xi\varsigma^2}{\delta} < 1, \\ 1 - \frac{1}{2}\frac{1}{\xi\varsigma^2/\delta} & \text{if } \frac{\xi\varsigma^2}{\delta} > 1. \end{cases} \tag{32}$$

As σ grows without bound, the equilibrium values of ψ and $r - g$ converge to the $\psi_\infty \in (0, 1)$ defined in (31) and $(r - g)_\infty = -\xi\varsigma^2\psi_\infty \in (-\delta, 0)$. And x and E/W converge to zero at the same rate, giving rise to zero utility for everyone if $\varepsilon \in (0, 1)$, and unbounded utility if $\varepsilon \in (1, \infty)$. The growth rate of aggregate consumption goes to its technological upper bound $g = (1 - \beta)\mu$.

The complete proof is in the appendix.

To understand the utility implications described in Proposition 7, first note that $C = (xK)^{1-\beta}L^{1-\beta}/(1 + \gamma)$ goes to zero because x goes to zero.[13] If $\varepsilon \in (0, 1)$ then (18) implies that U/C goes to zero as E/W goes to zero, and so utility must go to zero. Arbitrarily large transfers are the worst thing a government can do. But if $\varepsilon \in (1, \infty)$, then (18) implies that U/C goes to infinity as E/W goes to zero. Furthermore, the risky market clearing condition (23) implies that E/W and x converge to zero at the same rate. The fact that $1 - \beta - 1/(1 - 1/\varepsilon) > 0$ therefore ensures that U/C goes to infinity fast enough to overcome the fact that C goes to zero. In this case, the government can increase U without bound by setting σ large enough. Because $\psi_\infty \in (0, 1)$, consumption remains risky, unlike what happens when a government backs its securities with large consumption taxes. Nevertheless, the risk-adjusted individual consumption growth rate $g_y - \frac{1}{2}\xi\varsigma^2\psi^2$ increases by just enough to make utility explode when $\varepsilon \in (1, \infty)$. Given that wealth taxes are non-negative, (26) together with $\psi_\infty \in (0, 1)$ ensures $g_y < g + \delta$ in the limit. So U_y/U has a limit in $(0, \infty)$, and therefore U_y inherits the large-σ limits of U. Everyone will benefit from large transfers.

[13] From (23), aggregate wealth $W = (1 - \beta)K^{1-\beta}L^{\beta}/(\psi x^{\beta})$ goes to infinity. Also, (24) implies that the portfolio share of claims to labor income converges to zero. When deficits are large, most of consumer wealth is invested in physical capital and government securities. A newborn consumer who somehow fails to receive baby bonds would be in dire shape.

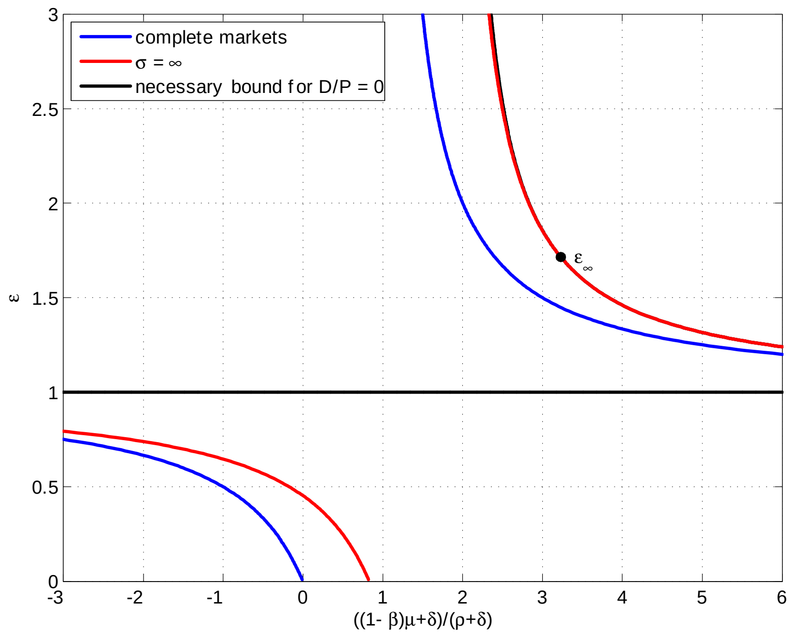

Figure 4 *The feasible region for* $\sigma \to \infty$.

Figure 4 pulls together the main implications of Propositions 4-7. It shows the $\omega = 0$ version of (21) together with (30) and the $\omega = 0$ version of (32). Note that (21) and (30) define a lower bound on μ when $\varepsilon \in (0, 1)$ and an upper bound when $\varepsilon \in (1, \infty)$. The upper bound in (32) coincides with the bound in (30) when both hold with equality.

4.4.2 Harmful Large Deficits

If $\varepsilon \in (0, 1)$, then Figure 4 says that there is range of relatively low μ for which the economy has a complete markets equilibrium but no incomplete markets equilibrium if the governrment runs a balanced budget and there is no bubble asset. This range of μ coincides with the range in which there is no upper bound on the transfers σ that the government can make to newborn consumers. But economies in this range also have a bubble equilibrium with positive utility, and large transfers make everyone worse off.

4.4.3 Large Deficits and Unbounded Utility Gains

If $\varepsilon \in (1, \infty)$, then the economy has a complete markets equilibrium as long as μ is not too high. For μ above this range, but not too much, utility is unbounded if the government makes large enough transfers to newborn consumers. For every economy in which this is the case, the economy also has balanced budget equilibrium, without a bubble asset, and possibly with a bubble asset as well. But these equilibria are Pareto dominated by the permanent deficit equilibria that are guaranteed to exist when transfers to newborn consumers are high enough. As shown by (32), an increase in ω moves the range of μ for which equilibrium utilities are unbounded to the right. There is, in this economy, no limit to how large wealth taxes can be. Therefore, every economy that is too productive for a complete markets equilibrium to exist will have an equilibrium if the government sets the wealth tax in a certain range and makes large transfers to newborn consumers.

4.4.4 Pareto Improvements from Small Increases in Transfers

Although Proposition 7 only describes what happens for large enough transfers to newborn consumers, one can also construct robust examples of Pareto improvements that arise from small deficit financed increases in σ. Figure 5 gives an example.

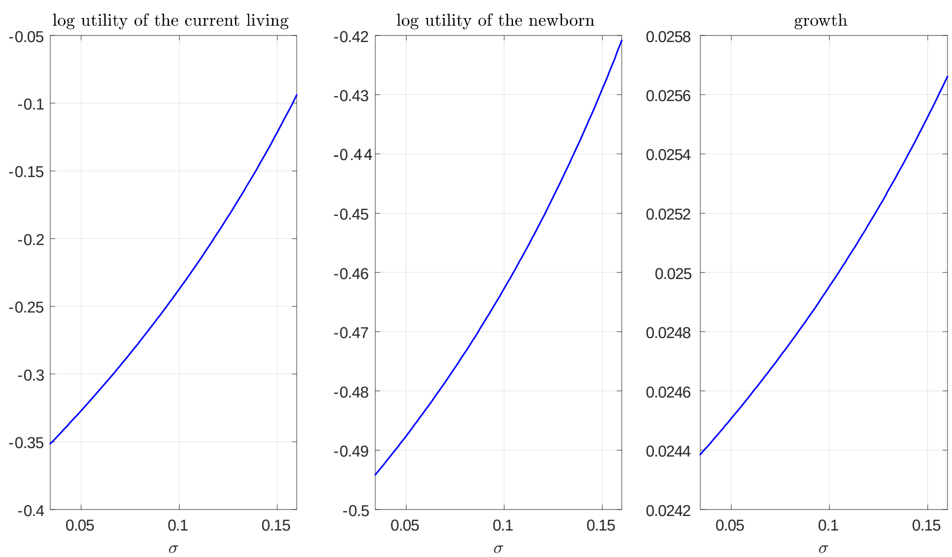

Figure 5 *Pareto improvements from increases in σ*

In this example, $\varepsilon = 2$ and μ is in the region where unbounded welfare improvements emerge when σ is taken to be very large.[14] As in the large-σ limit displayed in Figure 3, the equilibria shown in Figure 5 are the only balanced growth equilibria. The lowest σ in Figure 5 satisfies $(1+\gamma)(1+\sigma)/(1+\tau) = 1$. In this example, therefore, Pareto improvements happen immediately as the government begins to use a primary deficit to fund transfers to newborn consumers. This happens when the economy is sufficiently productive.

[14]More precisely, $\rho = 0.005$, $\delta = 0.03$, $\xi = 7.5$, $\varsigma = 0.25$, $\beta = 0.6$, and $\mu = 0.12$. The fiscal parameters satisfy $(1 + \gamma)/(1 + \tau) = 0.967$.

5 The Special Case $\varepsilon = 1$

The equilibrium conditions simplify greatly in the special case $\varepsilon = 1$, allowing us to characterize equilibria in much more detail. We show that large deficits are not possible when $\varepsilon = 1$ and give explicit constraints on the fiscal parameters that are consistent with equilibrium.

5.1 The Equilibrium Conditions

The equilibrium conditions (22)-(27) depend on ε only via the equilibrium condition (27) for E/W. The assumption $\varepsilon = 1$ implies the very convenient simplification $E/W = \rho + \delta$.[15] In turn, this means that the surplus ratio $\mathcal{S}_t$ defined in (3) is now a parameter, given by

$$\mathcal{S} = \frac{\rho + \delta + \omega}{\rho + \delta} - \frac{(1+\gamma)(1+\sigma)}{1+\tau}.$$

We restrict attention to wealth taxes that satisfy $\rho + \delta + \omega > 0$, so that primary surpluses are a possibility. Suppose the fiscal targets imply $\mathcal{S} \neq 0$, so that (22) gives $D/(PE) = \mathcal{S}/(r-g) \geq 0$. Taking into account that (25) implies $x = \xi\varsigma^2\psi + r - g$, the market clearing conditions for risky and risk-free assets (23)-(24) then reduce to

$$\frac{\psi}{\rho+\delta} = \frac{1-\beta}{\xi\varsigma^2\psi + r - g}\frac{1+\gamma}{1+\tau}, \tag{33}$$

$$\frac{1-\psi}{\rho+\delta} = \frac{\beta}{\delta + r - g}\frac{1+\gamma}{1+\tau} + \frac{\mathcal{S}}{r-g}. \tag{34}$$

The side conditions are $\psi \in (0,1)$ together with $r - g > 0$ if $\mathcal{S} > 0$ and $r - g \in (-\delta, 0)$ if $\mathcal{S} < 0$. If $\mathcal{S} = 0$ and $r - g = 0$, then the term $\mathcal{S}/(r-g)$ on the right-hand side of (34) must be replaced by $D/(PE) \geq 0$. These are now two equilibrium conditions to be solved for ψ and $r - g$. As before, x, g, and g_y follow from (25)-(26).

Adding up (33)-(34) and using $\xi\varsigma^2\psi > 0$, $\delta > 0$ and $\sigma \geq 0$ shows that $r - g \in (0, \rho+\delta+\omega)$ if $\mathcal{S} > 0$. So we can infer that

$$r - g \in (-\delta, \rho + \delta + \omega)$$

in any equilibrium.

[15] In fact, one can interpret everything we say about the feasibility of alternative fiscal targets as applying to an economy in which E/W is simply an exogenously specified decision rule.

The Special Role of the Wealth Tax The definition of $\mathcal{S}$ shows that there is an unbounded range of wealth taxes $\omega > -(\rho + \delta)$ and transfers $\sigma \geq 0$ of baby bonds that lead to the same surplus ratio $\mathcal{S}$. The equilibrium values of ψ, $r - g$, x, and g therefore only depend on ω and σ via $\mathcal{S}$. In particular, an increase in the wealth tax does not hurt aggregate growth as long as it is accompanied by the appropriate increase in transfers to newborn consumers. But (26) shows that the individual consumption growth rate g_y does depend separately, and negatively, on ω. Alternative policies for ω and σ can be used to target g_y without affecting the aggregate consumption growth rate.

5.2 The Three Primary Surplus Scenarios

Figure 6 shows the equilibrium conditions (33)-(34) for $\omega = 0$, a common $(1 + \gamma)/(1 + \tau) < 1$, and with $\sigma \geq 0$ selected to illustrate each of the three possible primary surplus scenarios. Observe that changing σ shifts (34) but not (33).

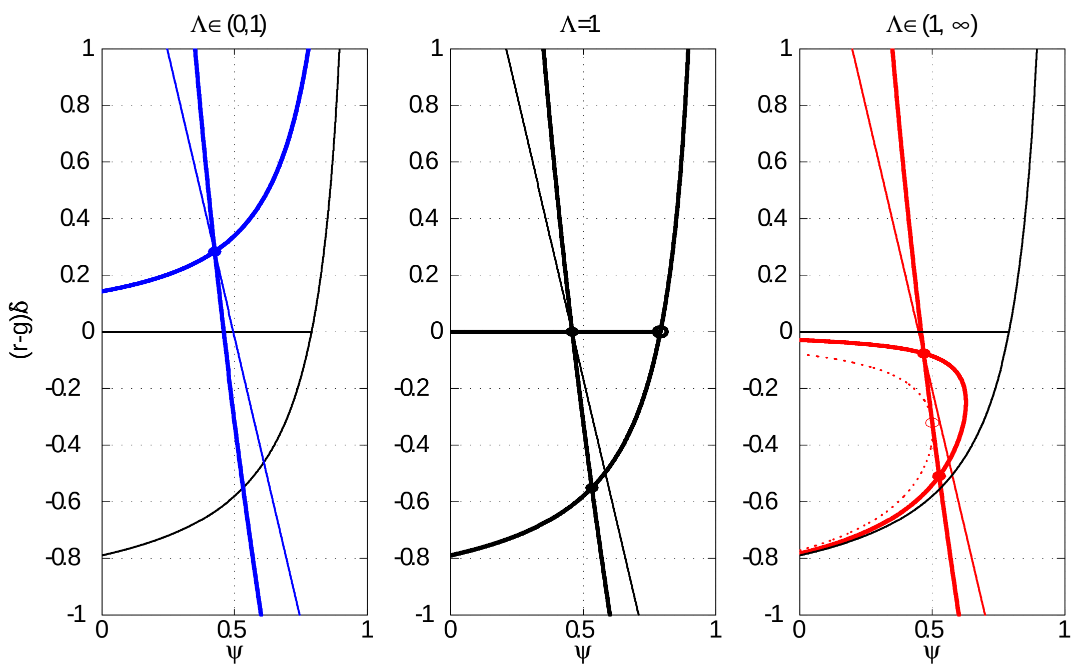

Figure 6 *Equilibria for the three primary surplus scenarios.*

The thick downward-sloping curve is the equilibrium condition (33), which is the same curve in each of the three panels. This equilibrium condition is a hyperbola, with a vertical asymptote at $\psi = 0$, and a large-ψ asymptote $-\xi\varsigma^2\psi$. The upward-sloping curve in the $\mathcal{S} = 0$ panel represents (34) for $r - g \neq 0$, extended by continuity to $r - g = 0$. Because $D/(PE)$ can be any non-negative number when budgets are balanced, the $\mathcal{S} = 0$ version of the equilibrium condition (34) also includes the positive horizontal axis up to the point where (34) crosses the horizontal axis. For reference, the $\mathcal{S} = 0$ version of (34) is also shown in the background of the $\mathcal{S} > 0$ and $\mathcal{S} < 0$ panels. Letting σ adjust so that $\mathcal{S} \downarrow 0$ causes the $\mathcal{S} > 0$ version of (34) to converge to the $r - g \geq 0$ segment of the $\mathcal{S} = 0$ version of (34). Similarly, letting $\mathcal{S} \uparrow 0$ causes the $\mathcal{S} < 0$ version of (34) to converge to the $r - g \leq 0$ segment of the $\mathcal{S} = 0$ version of (34).

Given ψ and $r - g$ that solve (33)-(34), the equilibrium growth rate g follows from $x = \xi\varsigma^2\psi + r - g$ and $g = (1 - \beta)(\mu - x)$, and then the risk-free rate is $r = g + r - g$. The thin downward sloping lines in Figure 6 represent lines with a constant value of $x = \xi\varsigma^2\psi + r - g$. Outcomes along these lines imply the level of consumption and the same growth rate as in the corresponding equilibrium. It is important to note that, where they cross, the curve $\psi \mapsto r - g$ implied by (33) must always be steeper than the line $r - g = x - \xi\varsigma^2\psi$. As Figure 6 shows, the economy with a surplus grows more slowly than the economy with a balanced budget, for both of the two possible equilibria. The effect on growth of increasing σ further depends on which of the two equilibria the economy is in.

5.3 Conditions for Existence

As $\mathcal{S}$ decreases from $\mathcal{S} > 0$ to $\mathcal{S} < 0$, the domain for $r - g$ switches from $(0, \infty)$ to $(-\delta, 0)$. In between, at $\mathcal{S} = 0$, the risk-free market clearing condition (34) is a correspondence. We discuss these regimes in sequence.

5.3.1 Primary Surpluses

If $\mathcal{S} > 0$, then the right-hand side of (34) is large for $r - g$ close to zero, strictly decreasing in $r - g > 0$, and converging to zero as $r - g$ becomes large. As a result, (34) defines an increasing function that maps $\psi \in (0, 1)$ into $r - g \in (0, \infty)$. Together with the properties of (33) this proves the first part of the following proposition.

Proposition 8 *The economy has a unique steady state equilibrium for any combination of fiscal targets that satisfies $\mathcal{S} > 0$. An increase in σ that preserves $\mathcal{S} > 0$ increases the growth rate of the economy.*

The positive effect on growth of increased transfers comes from the fact that the line $r - g = x - \xi\varsigma^2\psi$ is not as steep as the equilibrium condition (33).

5.3.2 Balanced Budgets

As illustrated by the second panel in Figure 6, the $\mathcal{S} = 0$ economy may well have two equilibria: a no-bubble equilibrium with $r - g < 0$ and $D/(PE) = 0$, and a bubble equilibrium with $r - g = 0$ and $D/(PE) > 0$. There is always a no-bubble equilibrium, determined by the intersection of (33) and the $\mathcal{S} = 0$ version of (34). It is easy to verify that this no-bubble equilibrium is unique.

To construct a possible bubble equilibrium, set $r - g = 0$, and use (33) to infer that

$$\psi = \sqrt{\frac{1-\beta}{\xi\varsigma^2/\delta}\frac{\rho+\delta}{\delta}\frac{1+\gamma}{1+\tau}}.$$

Plugging this into (34) gives

$$\frac{D}{PE} = \frac{1}{\rho+\delta}\left(1 - \left(\beta \times \frac{\rho+\delta}{\delta}\frac{1+\gamma}{1+\tau} + \sqrt{\frac{1-\beta}{\xi\varsigma^2/\delta}\frac{\rho+\delta}{\delta}\frac{1+\gamma}{1+\tau}}\right)\right). \tag{35}$$

The $\mathcal{S} = 0$ economy has a pure bubble equilibrium if and only if the $D/(PE)$ implied by (35) is positive. The no-bubble equilibrium is the only equilibrium when (33) crosses the horizontal axis to the right of (34). In such a scenario, $r - g$ is strictly positive. The bubble equilibrium emerges if (33) crosses the horizontal axis to the left of (34). In Figure 6, the value $D/(PE)$ of the bubble is measured by the distance on the horizontal axis between the points where (33) and (34) cross the horizontal axis (marked by a solid dot and a circle, respectively). The properties of the balanced budget economy can be summarized as follows.

Proposition 9 *When fiscal targets imply balanced budgets, the economy has a unique steady state equilibrium without a bubble. This no-bubble equilibrium has $r - g < 0$ if and only if the right-hand side of (35) is positive. The economy then also has a unique steady state equilibrium with a strictly positive bubble. When $\omega = 0$ and $\sigma = 0$, the requirement that (35) is positive is equivalent to*

$$\beta < \frac{\delta}{\rho+\delta}\left(1 - \frac{\delta/2}{\xi\varsigma^2} - \sqrt{\left(\frac{\delta/2}{\xi\varsigma^2}\right)^2 + \frac{\rho}{\xi\varsigma^2}}\right). \tag{36}$$

More generally, an economy with $\mathcal{S} = 0$ has a pure bubble equilibrium if and only if $(1+\gamma)/(1+\tau)$

satisfies

$$\beta \times \frac{1+\gamma}{1+\tau} < \frac{\delta}{\rho+\delta}\left(-\frac{1}{2}\sqrt{\frac{\delta}{\xi\varsigma^2}\frac{1-\beta}{\beta}} + \sqrt{\left(\frac{1}{2}\sqrt{\frac{\delta}{\xi\varsigma^2}\frac{1-\beta}{\beta}}\right)^2 + 1}\right)^2. \tag{37}$$

The economy grows faster in the no-bubble equilibrium than in the bubble equilibrium.

In the special case given by $\omega = 0$ and $\sigma = 0$, the upper bound (36) on the labor share parameter β is decreasing in $\xi\varsigma^2$ and positive if and only if $\xi\varsigma^2 > \rho + \delta$. There must be enough idiosyncratic risk. But no amount of idiosyncratic risk makes a bubble possible if $\beta > \delta/(\rho+\delta)$. In contrast, (37) says that for any $\xi\varsigma^2 > 0$, there will be a bubble equilibrium as long as consumption taxes are high enough. This is already clear from the fact that the expression for $D/(PE)$ given in (35) is positive when τ is large enough. Lowering $(1+\gamma)/(1+\tau)$ shifts the risky market clearing condition (33) to the left and the no-bubble version of the risk-free market clearing condition (34) to the right. The result is a lower $r-g$, and once $r-g$ becomes negative, this makes a bubble possible.

The size of the bubble is maximized by taking τ to infinity, which, for $r-g \neq 0$, makes (33) converge to the vertical axis, and (34) to the vertical line $\psi = 1$. This is a situation in which, because budgets are balanced, high consumption taxes are mostly used to make large transfers to newborn consumers.[16] Imposing $r-g = 0$ in (33) shows that, in the bubble equilibrium, ψ and thus x both converge to zero as τ and σ become large. The growth rate of the economy converges to its technological upper bound, and consumer exposure to idiosyncratic risk disappears.

5.3.3 Primary Deficits

When $\mathcal{S} < 0$, the risk-free market clearing condition (34) generates a hump-shaped mapping $r-g \mapsto \psi$, as shown in the $\mathcal{S} < 0$ panel of Figure 6. This is reminiscent of the risk-free market clearing condition obtained for $\varepsilon \in (1,\infty)$ in Figure 3.

To see why this mapping is hump-shaped, observe that, as usual, the present value of labor earnings is decreasing in the effective discount rate $\delta + r - g \in (0,\delta)$. But with $\mathcal{S} < 0$ and $r-g \in (-\delta, 0)$, $\mathcal{S}/(r-g)$ is a steady state value, not a present value. And it is increasing in $r-g \in (-\delta,0)$, not decreasing. Higher interest rates imply a larger value of the steady state supply of government securities. Therefore, a low $r-g \in (-\delta,0)$ gives

[16] This is very familiar from what happens in an exchange economy with two-period lived consumers and overlapping generations. A transfer to the young financed by a consumption tax can make the interest rate at which the young are willing to save negative.

rise to a high value of private risk-free assets and a low steady state value of government securities, and the reverse is true for a high $r - g \in (-\delta, 0)$.

We can use this to construct a first bound on how large primary deficits can be.

Weak Upper Bounds Since $\psi \geq 0$, one implication of (34) is that

$$\frac{\beta}{\delta + r - g} \frac{1+\gamma}{1+\tau} + \frac{\mathcal{S}}{r-g} \leq \frac{1}{\rho + \delta}.$$

Since the left-hand side is U-shaped in $r - g \in (-\delta, 0)$, with vertical asymptotes at $-\delta$ and 0, it will have a minimum for some $r - g \in (-\delta, 0)$. The envelope theorem implies that this minimum increases as the deficit ratio $-\mathcal{S} > 0$ increases, holding fixed $(1+\gamma)/(1+\tau)$. Since $r - g > -\delta$, this minimum must become large as $-\mathcal{S}$ becomes large. Given some $(1+\gamma)/(1+\tau)$, this says that the government faces an upper bound on its deficit ratio $-\mathcal{S}$.

This upper bound on primary deficits relies on preferences only via the decision rule $E/W = \rho + \delta$. It is an easy bound to calculate, but not tight. An even easier bound follows from the fact that (34) together with $\psi \geq 0$ and $-(r-g) < \delta$ implies $-\mathcal{S} < \delta/(\rho + \delta)$. By Proposition 11 below, this turns out to be an accurate bound when there is a lot of idiosyncratic risk and consumption taxes are high relative to government purchases.

Tight Upper Bounds The following proposition gives a tight upper bound on $\mathcal{S}$, as well as a tight lower bound on τ.

Proposition 10 *Fix ω and suppose the economy with $\mathcal{S} = 0$ has an equilibrium with a strictly positive bubble. Then the economy also has equilibria for the same $(1+\gamma)/(1+\tau)$ and all $\sigma' \in (\sigma, \infty)$ up to some finite upper bound, and for the same $(1+\gamma)(1+\sigma)$ and all $\tau' \in (-1, \tau)$ above some positive lower bound. These bounds can be found by requiring (33) and (34) to be tangent.*

Holding $(1+\gamma)/(1+\tau)$ fixed means that (33) does not change. Taking $\sigma' > \sigma$ causes (34) to shift to the left, and for large enough σ' this removes all equilibria. Holding fixed $(1+\gamma)(1+\sigma)$, a reduction in τ moves (33) to the right, and (34) to the left. The lowest possible value for the consumption tax is reached when the two equilibria merge. The tangency results follow because the mappings $r - g \mapsto \psi$ implied by (33) and (34) are, respectively, convex and concave. The $\mathcal{S} < 0$ panel of Figure 6 gives an illustration.

Joint Upper Bounds Proposition 10 gives a bound on one tax parameter while holding the other fixed. The next proposition gives an explicit bound on the surplus ratio $\mathcal{S}$ that applies when fiscal targets can be varied jointly.

Proposition 11 *For any surplus ratio that satisfies*

$$0 < \mathcal{S} + \frac{\delta}{\rho+\delta} \times \begin{cases} \frac{1}{2} \times \frac{\xi\varsigma^2/2}{\delta} & \text{if} \quad \frac{1}{2}\xi\varsigma^2 < \delta \\ 1 - \frac{1}{2} \times \frac{\delta}{\xi\varsigma^2/2} & \text{if} \quad \frac{1}{2}\xi\varsigma^2 > \delta \end{cases}$$

there are fiscal targets for which the economy has an equilibrium. As $-\mathcal{S}$ approaches this upper bound, the required consumption taxes and transfers to newborn consumers become large, and the growth rate approaches its maximal feasible rate.

When there is a large amount of idiosyncratic risk, this says that the upper bound on $-\mathcal{S}$ is approximately $\delta/(\rho+\delta)$, implying primary deficits almost as large as aggregate consumption expenditures when ρ is very close to zero.

To prove this proposition, use (33) to eliminate $(1+\gamma)/(1+\tau)$ from the first term in (34). The result is an equation that maps ψ and $r-g$ into $\mathcal{S}$. Varying $\psi \in (0,1)$ and $r-g \in (-\delta, 0)$ subject to $\xi\varsigma^2\psi + r - g > 0$ then gives the feasible range for $\mathcal{S}$. Subject to these constraints, the supremum of $-\mathcal{S}$ is approached by letting $x = \xi\varsigma^2\psi + r - g \downarrow 0$ and $r - g \downarrow -\min\{\delta, \xi\varsigma^2/2\}$. The fact that $x \downarrow 0$ means that aggregate growth is maximal. The fact that large consumption taxes will be required in such a limit is immediate from (33). A detailed version of this backsolving argument is in the appendix.

5.3.4 Primary Deficits in the UBI Economy

The UBI version of (33) is the same as in the baby bonds economy. In the UBI version of (34), σ must be replaced by θ, and β by $\beta+\theta$. Define

$$\mathcal{S}_\omega = 1 - \frac{\rho+\delta}{\rho+\delta+\omega} \frac{(1+\gamma)(1+\theta)}{1+\tau}$$

and note that $\mathcal{S}_0 = \mathcal{S}$. The UBI version of (33)-(34) can then be written as

$$\begin{aligned} \frac{\psi}{\rho+\delta+\omega} &= \frac{1-\frac{\beta+\theta}{1+\theta}}{\xi\varsigma^2\psi + r - g} \times (1-\mathcal{S}_\omega), \\ \frac{1-\psi}{\rho+\delta+\omega} &= \frac{\frac{\beta+\theta}{1+\theta}}{\delta + r - g} \times (1-\mathcal{S}_\omega) + \frac{\mathcal{S}_\omega}{r-g}. \end{aligned}$$

This is of the exact same form as (33)-(34) with $\omega = 0$ and $\sigma = 0$. The upper bound (36) therefore applies to $(\beta+\theta)/(1+\theta) \in (\beta, 1)$ rather than to β itself. This immediately implies that there can only be a bubble equilibrium if θ is not too large. Certainly, $(\beta+\theta)/(1+\theta) < \delta/(\rho+\delta+\omega)$ is necessary, and this is sufficient only if $\xi\varsigma^2/\delta$ is large. This is in sharp contrast to the fact that a large baby bonds parameter σ can ensure the existence of a bubble equilibrium. Furthermore, taking the UBI parameter θ to be positive lowers the maximal size of the steady state primary deficit that is consistent with equilibrium.

Proposition 12 *If transfers are in the form of a universal basic income, then, holding fixed ω, the largest possible steady state primary deficit is attained by setting the universal basic income to zero.*

The resulting upper bound on $-\mathcal{S}$ is the same as the bound already described for the baby bonds economy with $\sigma = 0$. The risk-free income generated by a UBI competes with the government's ability to sell risk-free securities, while transfers to newborn consumers, to some extent, generate their own demand for risk-free securities. An increase in the UBI tightens the upper bound on primary deficits faced by the government. If primary deficits are close to their upper bound, a non-trivial increase in the UBI must therefore be accompanied by some combination of higher consumption taxes and lower government purchases that exceeds the extent to which the higher UBI would raise the deficit.

"Fiscal Space" The combination of Propositions 10, 11 and 12 highlights the fact that how large primary deficits can be depends very much on how these deficits are used. There is no sense in which there is a single notion of "fiscal space" that the government can use for whatever purpose it chooses.

5.4 Welfare Consequences

A government with access to consumption and wealth taxes, and able to make transfers to newborn consumers, can approximate all stationary allocations that are not Pareto dominated by other stationary allocations.

5.4.1 Implementable Allocations

Consider the equilibrium conditions (33)-(34). Take some $\psi \in (0, 1)$ and $r - g \in (-\delta, \rho + \delta + \omega)$, and use the risky market clearing condition (33) to back out the underlying $(1 + \gamma)/(1 + \tau)$. Then plug this $(1+\gamma)/(1+\tau)$ into the first term of the risk-free market clearing condition (34), and use that same equation to infer $\mathcal{S}$. The pair ψ and $r-g$ is an equilibrium if and only if (i) $x = \xi\varsigma^2\psi + r - g$ is positive and (ii) the implied value of government securities (the second term in (34)) is non-negative. These constraints define the set of ψ and $r - g$ that can be implemented by a government that does not lend to the public, using time-invariant taxes on consumption and wealth, and transfers in the form of baby bonds.

5.4.2 Stationary Utilities

The utilities for the incomplete markets economy can be obtained by taking the $\varepsilon \to 1$ limit in (9)-(11) and using $C = (xK)^{1-\beta}L^{\beta}/(1+\gamma)$. For the average consumer already alive, and for the current newborn generation, this yields

$$U = \frac{(xK)^{1-\beta}L^{\beta}}{1+\gamma} \exp\left(\frac{g_y - \frac{1}{2}\xi\varsigma^2\psi^2}{\rho+\delta}\right), \quad U_y = \frac{W_y}{W} \times U, \tag{38}$$

where g_y and W_y/W are given by

$$g_y = g + \delta + x - \xi\varsigma^2(1-\psi)\psi - (\rho+\delta+\omega), \quad \frac{W_y}{W} = \frac{g+\delta-g_y}{\delta}. \tag{39}$$

The first equation in (39) is a convenient expression for the growth rate of consumption chosen by consumers with $\varepsilon = 1$—observe that $x = \xi\varsigma^2\psi + r - g$ means that $g + \delta + x - \xi\varsigma^2(1-\psi)\psi$ is just the expected return $r + \delta + \psi(\mu + \mu_q - r)$. The second equation in (39) is the steady state accounting relation (13).

A consumer with wealth W_j has utility $U_j = (W_j/W)U$. Because all consumers alive at the time of an unforeseen change in government policy use the same portfolio weights, such changes will have no effect on the W_j/W. But newborn consumers start life with an unbalanced portfolio, and this means that their relative wealth position is affected by policy.

5.4.3 Approximating Complete Markets Allocations

The quadrilateral in Figure 2, describing stationary allocations not dominated by other stationary allocations, becomes a parallelogram when $\varepsilon = 1$. A version of Proposition 4 applies. Take some $x \in (0, \rho+\delta)$ and consider fiscal targets that satisfy $\mathcal{S} = x/(\rho+\delta)$. Holding fixed ω, one can let both τ and σ become large without changing the surplus ratio $\mathcal{S}$. Doing this makes the right-hand side of the risky market clearing condition (33) go to zero at $\psi = 0$ and $r - g = x$. The same is true for the first term on the right-hand side of the risk-free market clearing condition. This means that equilibrium values for ψ and $r - g$ converge to 0 and x, respectively, as τ and σ become large, holding fixed $\mathcal{S}$. The growth rate of individual consumption converges to $g_y = x + g + \delta - (\rho + \delta + \omega)$, and individual consumption becomes risk-free because ψ converges to zero. The allocation converges to a risk-free stationary allocation, and one can vary $\omega \in (-(\rho+\delta), \rho+\delta)$ to attain every stationary allocation that is not dominated by another stationary allocation. In particular, by letting $\omega = 0$ one can approximate all the stationary allocations that are also Pareto efficient.

5.4.4 Varying Transfers Only

Consider the effects of varying σ while holding ω and $(1+\gamma)/(1+\tau)$ fixed. We already know from Proposition 7 and Figure 5 that increases in σ can lead to Pareto improvements when $\varepsilon \in (1, \infty)$ and the economy is sufficiently productive. Figure 7 shows what happens when $\varepsilon = 1$: the utility of consumers already alive and the growth rate of the economy always move in opposite directions. In other words, even without considering U_y, varying transfers cannot be used to create Pareto improvements. Any attempt to improve the steady state utility of consumers already alive comes at the cost of lower growth, and that will hurt future generations.

To see why this is true, begin by observe that varying σ implies changes in the surplus ratio $\mathcal{S}$ that lead to shifts in the risk-free market clearing condition (34) along a fixed risky market clearing condition (33).

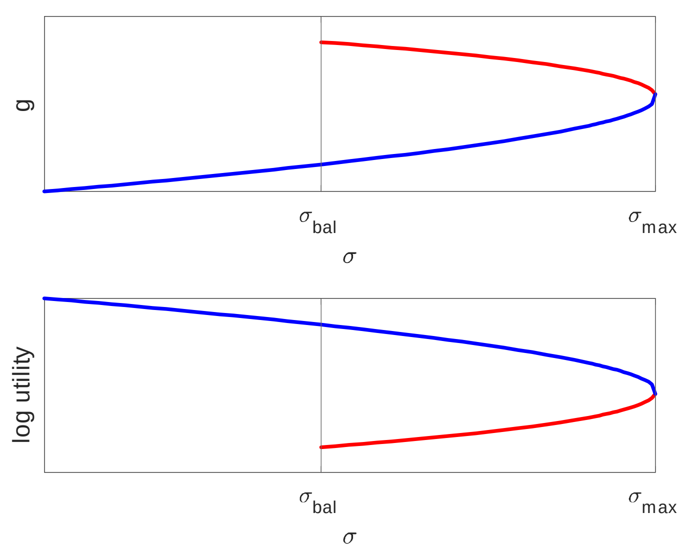

Figure 7 *The welfare consequences of varying σ*

Starting with some $\mathcal{S} > 0$, increasing σ lowers $\mathcal{S}$ and shifts the risk-free market clearing condition (34) down towards the $\mathcal{S} = 0$ version of that condition. As can be seen using Figure 6, this lowers $r - g$, increases ψ, and lowers x. The fact that $g + x = (1-\beta)\mu + \beta x$ is increasing in x and $-(1-\psi)\psi - \frac{1}{2}\psi^2$ is decreasing in $\psi \in (0,1)$ implies that this lowers U. But $g = (1-\beta)(\mu - x)$ increases with the reduction in x, and so newborn consumers sufficiently far into the future will gain.

Next, suppose (37) holds, so that the economy will have both a no-bubble equilibrium and a bubble equilibrium when $\mathcal{S}$ reaches zero. Consider the equilibria with $r-g<0$ close to zero that emerge when $\mathcal{S}$ becomes negative as a result of further increases in σ. For these equilibria, as (34) shrinks towards the vertical axis in Figure 6, the increases in σ further lower $r-g$, increase ψ, and lower x. The result is further reductions in U and increases in g. This continues until $-\mathcal{S}$ reaches its upper bound. At $\mathcal{S}=0$, the no-bubble equilibrium also has a lower $r-g$, a larger ψ, and a lower x than the bubble equilibrium. This implies a worse outcome for U and a better outcome for g in the no-bubble equilibrium than in the bubble equilibrium. It is easy to verify that the same conflict of interest arises for the $\mathcal{S}<0$ equilibria with $r-g<0$ close to $-\delta$.

This proves the following proposition and explains the example shown in Figure 7.

Proposition 13 *Holding fixed consumption and wealth taxes, increases in transfers to newborn consumers lead to lower and more risky individual consumption growth, a lower level of aggregate consumption, and faster aggregate consumption growth if $r-g$ is positive or negative and relatively close to zero. The opposite happens when $r-g$ is negative and relatively close to $-\delta$. This leads to a conflict of interest between consumers already alive and consumers who will be born sufficiently far into the future.*

5.4.5 No Wealth Taxes

The government can use large consumption taxes and transfers to newborn consumers to approximate the stationary allocations that are Pareto efficient. But there will then be many types of consumers who are worse off than they would be with more limited government interventions.

The absence of a wealth tax now implies that g_y is tied down by ψ and x. Policies that reduce ψ also have consequences for g_y. It is easy to see from (38)-(39) that $\partial U/\partial\psi<0$ and $\partial U/\partial x>0$. A higher ψ implies more risk and reduces g_y, and a higher x implies both a higher level of consumption and higher rates of return, because slow growth implies a low rate at which capital depreciates. Since any equilibrium must have $\psi\in(0,1)$ and $r-g\in(-\delta,\rho+\delta)$, this means that consumers already alive want to be as close as possible to $\psi=0$ and $x=\rho+\delta$.

Combining (39) with $g=(1-\beta)(\mu-x)$ shows that $\partial(W_y/W)/\partial\psi>0$ if and only if $\psi\in(0,1/2)$, and $\partial(W_y/W)/\partial x<0$. So the effects of $\psi\in(0,1/2)$ and $x>0$ on W_y/W are the opposite of those on U, essentially because feasibility requires that faster individual consumption growth relative to aggregate growth must come at the expense of lower newborn wealth relative to aggregate wealth. For low ψ and high x, these effects will be large enough to create disagreement between consumers already alive and the current generation of newborn consumers about the desirability of reducing ψ or increasing x. In particular, at $\psi = 0$, the utility U_y is hump-shaped in $x \in (0, \rho + \delta)$ and $\partial \ln(U_y)/\partial\psi \propto x/(\rho + \delta - x)$ is positive.

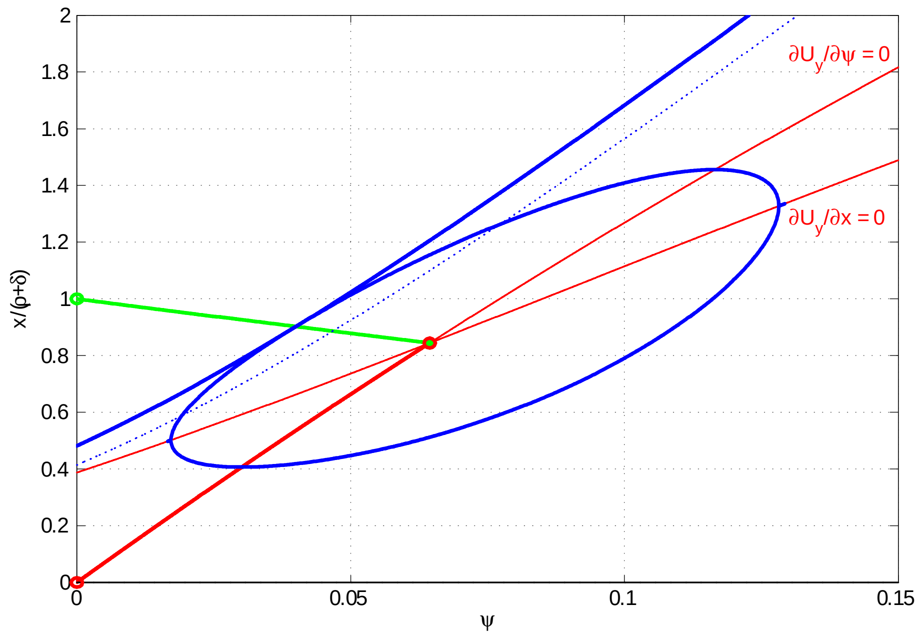

Figure 8 *Indifference curves and the efficient region*

Figure 8 shows indifference curves for U_y and U at an allocation where U_y is maximal given a lower bound on U—on the contract curve for U_y and U. The dotted curve is a second indifference curve for U, at a lower level of utility. The other two upward sloping curves are the lower boundaries of the regions where $\partial U_y/\partial\psi \geq 0$ and $\partial U_y/\partial x \leq 0$. Note that the lower of the two indifference curves for U intersects the indifference curve for U_y twice, once below the contract curve for U and U_y, and once above it. At the upper intersection, it would be possible to increase both U and U_y by lowering both ψ and x. Moreover, the resulting increase in the growth rate of the economy would also benefit all future generations. So the upper intersection is inefficient. At the lower intersection, it would be possible to increase both U and U_y by increasing both ψ and x. But this lowers the growth rate of the economy and therefore hurts newborn consumers who will be born far enough into the future. So this intersection represents a constrained efficient allocation.

The bold segment of the curve $\partial U_y/\partial\psi = 0$ in Figure 8 can be viewed as another contract curve, for the current newborn consumers and consumers who will be born far into the future. The latter only care about growth, and so maximizing U_y subject to a lower bound on $g = (1-\beta)(\mu - x)$ requires $\partial U_y/\partial\psi = 0$ and $\partial U_y/\partial x \geq 0$. In fact, every point on this contract curve maximizes $\ln(U_y) + gT$ for some $T \in (0, \infty)$. Similarly, the vertical axis between $(0, 0)$ and $(0, 1)$ is the contract curve between consumers already alive and consumers who will be born very far into the future. Taking into account everyone in the economy, the constrained efficient allocations are those in between these contract curves.

A full proof of the following proposition is in the appendix.[17]

Proposition 14 *An equilibrium allocation (ψ,x) is constrained efficient if and only if*

$$\frac{\partial U_y}{\partial \psi} \geq 0 \quad \textit{and} \quad -\frac{\partial U/\partial x}{\partial U/\partial \psi} \geq -\frac{\partial U_y/\partial x}{\partial U_y/\partial \psi}. \tag{40}$$

The two inequalities in (40) can be written as, respectively,

$$(1-\psi)\psi \times \frac{\xi\varsigma^2}{\rho+\delta} + \frac{\psi}{1-\psi} \leq \frac{x}{\rho+\delta} \leq \frac{(1-\beta)(1-2\psi)}{(1-\beta)(1-\psi)+\beta\psi}. \tag{41}$$

In turn, (41) implies that $\psi \in (0, 1/2)$, $r - g > 0$, and $x < \rho + \delta$ are all necessary for constrained efficiency.

The necessary condition $\psi \in (0, 1/2)$ for efficiency means that W_y/W is increasing in ψ. Since $\partial U/\partial \psi < 0$, this is necessary for $\partial U_y/\partial \psi \geq 0$. The fact that $r - g > 0$ is necessary for efficiency means that permanent primary deficits can never be constrained efficient in this economy. But unlike in an economy with complete markets, the condition $r - g > 0$ is not sufficient for efficiency.[18] The proposition shows that the $x < \rho + \delta$ property of Pareto efficient equilibria generalizes.

In Figure 8, their preferred equilibrium has transfers that are about 8% of the output of aggregate consumption goods and total government outlays about 30% of tax revenues. Cutting taxes to balance the budget would increase both ψ and x in a way that makes everyone worse off.

[17] In the example of Figure 4, all outcomes in the region bounded by the three contract curves, minus the vertical axis, are indeed implementable. But for low $\xi\varsigma^2 > 0$ and high $\beta \in (0, 1)$, it is possible for a subset of allocations in this region to violate the constraint that the government does not lend to the public. This shrinks but does not empty the set of efficient allocations that are implementable. All allocations with $x \in (0, \rho + \delta)$ and ψ close enough to zero are implementable.

[18] A sharper lower bound on $\frac{r-g}{\rho+\delta}$ implied by (41) is $\frac{r-g}{\rho+\delta} > \frac{\psi}{1-\psi}\frac{\psi}{(1-\beta)(1-\psi)+\beta\psi}$. This lower bound can be inferred from the labor share in the consumption sector and the portfolio share of risky capital. It is increasing in ψ, but of second order for ψ close to zero.

6 Finitely Lived Consumers

The assumption that consumers die randomly at some rate $\delta > 0$ plays an absolutely critical role in generating the possibility of unbounded utilities. This perpetual youth assumption is clearly a bad assumption for individual consumers. But the consumers in this economy can also be viewed as dynastically linked individuals who care about their descendants (Weil [1989]). The rate δ can then be interpreted as the rate at which altruistic links break down. If $\varepsilon \in (1, \infty)$ and the conditions of Proposition 7 apply, then potential dynastic utilities are unbounded.

Consider the other extreme: consumers who live finite lives and who do not care about their descendants. Specifically, suppose consumers die randomly at the rate δ, and for certain when they reach the age $T > 0$. The flow of new births is $\delta/(1 - e^{-\delta T})$, which implies a unit measure of consumers in the steady state. There is no bequest motive, and so consumers will choose to spend all their wealth by the time they reach age T.

In this setting we show that it is possible to construct fiscal policies so that the equilibrium utilities in the finite-T economy approximate their $T = \infty$ counterparts.

6.1 Decision Rules and Aggregation

For consumers faced with constant rates of return, the Epstein-Zin preferences we have used all along again give rise to the portfolio choice $\psi = (\mu + \mu_q - r)/(\xi \varsigma^2)$. But the optimal consumption-wealth ratio does depend on age.[19] It is of the form

$$\phi_a = \frac{\phi_\infty}{1 - e^{-\phi_\infty (T-a)}}, \quad a \in [0, T]. \tag{42}$$

Observe that ϕ_a is increasing in age and that $\phi_a \to \infty$ as a approaches T from below. This is how consumers end up spending all their wealth as they approach their terminal age T.

The dependence on age of these consumption-wealth ratios means that an equilibrium must, in general, depend on the distribution of wealth across different age cohorts of consumers alive at a given date. Here we will describe only how steady state equilibria are determined. It will no longer be the case that an unforeseen change in government policy immediately puts the economy in a new aggregate steady state. But it is possible for the government to augment an unforeseen change in fiscal targets with one-time age-specific proportional taxes on wealth and transfers of wealth to immediately implement

[19] See Schroder and Skiadas [1999] for the solution to the finite-horizon Epstein-Zin version of a Merton problem.

the new steady state distribution of wealth across age cohorts.[20]

Given a risk-free rate equal to $r = g + x - \xi\varsigma^2\psi$, the parameter ϕ_∞ of the decision rule (42) and the resulting individual consumption growth rate g_y are determined by two conditions that are completely analogous to the conditions (26)-(27) for the $T = \infty$ economy,

$$
\begin{aligned}
g_y &= g + \delta + x - \xi\varsigma^2\psi(1-\psi) - (\omega + \phi_\infty)\,, \qquad (43)\\
\phi_\infty &= \rho + \delta - \left(1 - \frac{1}{\varepsilon}\right)\left(g_y - \frac{1}{2}\xi\varsigma^2\psi^2\right). \qquad (44)
\end{aligned}
$$

Given the decision rules ψ and ϕ_a, individual consumer wealth is no longer a geometric Brownian motion. Its drift decreases with age, and very rapidly as a approaches T. But all consumers alive at a given point in time face the same expected returns and the same uncertainty. Because of this, the consumption $C_{j,t}$ of consumer j alive at time t again follows $\mathrm{d}C_{j,t} = C_{j,t}\left(g_y\mathrm{d}t + \psi\varsigma\mathrm{d}Z_{j,t}\right)$ conditional on survival. At time t, aggregate consumption of the cohort born at date $t-a$ is then $C_{y,t}e^{-(g+\delta-g_y)a}$. In a steady state, $[C_t, C_{y,t}] = [C, C_y]e^{gt}$, and accounting for births and deaths shows that $C_y/C = ((1-e^{-\delta T})/\delta)/((1-e^{-(g+\delta-g_y)T})/(g+\delta-g_y))$.

Wealth at time t of a consumer j born at $t-a$ can be inferred from $C_{j,t}/\phi_a$. This can be used to calculate aggregate steady state wealth and infer the aggregate consumption-wealth ratio. This yields

$$
\frac{E}{W} = \phi_\infty\left(1 - e^{-\phi_\infty T}\left(\frac{1-e^{-(g+\delta-g_y)T}}{g+\delta-g_y}\right)^{-1}\frac{1-e^{-(g+\delta-(\phi_\infty+g_y))T}}{g+\delta-(\phi_\infty+g_y)}\right)^{-1}. \qquad (45)
$$

Even though ϕ_∞ could be negative in a finite-T economy, the E/W implied by (45) is strictly positive by construction—it is a ratio of positive aggregate consumption and positive aggregate wealth. If $\phi_\infty > 0$ and $g_y < g + \delta$, as would be the case in the $T = \infty$ economy, then E/W converges to ϕ_∞ as T becomes large.

6.2 The Finite-T Equilibrium Conditions

The risky market clearing condition (23) for the $T = \infty$ economy still applies here, for finite T. But the risk-free market clearing condition (24) changes because the aggregate present value of the labor income of consumers alive at a given point in time has to ac-

[20] To emphasize: the distribution of wealth within an age cohort still does not matter for determining the equilibrium.

count for their ages. A straightforward calculation gives

$$1-\psi=\frac{\beta}{\delta+r-g}\left(1-\frac{1-e^{-(r-g)T}}{r-g}\frac{\delta e^{-\delta T}}{1-e^{-\delta T}}\right)\frac{1+\gamma}{1+\tau}\frac{E}{W}+\frac{D/P}{W}. \tag{46}$$

The conditions for a balanced growth path can then be obtained from (22)-(27) by replacing (24) with (46), replacing (26)-(27) with (43)-(44), and adding the new condition (45). The additional variable is the parameter ϕ_∞ of the age-dependent consumption-wealth ratio ϕ_a. Its sign is unrestricted because ϕ_a is automatically positive, as is the aggregate consumption-wealth ratio (45). As before, x has to be positive, and $r-g$ has to have the sign of the primary surplus. There is no requirement that $\delta+r-g$ is positive, because the present value of anyone's labor income is automatically finite for any $r-g$.

6.3 Large-T Convergence

Given an equilibrium in the $T=\infty$ economy, it is rather straightforward pick fiscal targets for a large but finite T economy that generate an equilibrium close to that of the $T=\infty$ economy. Fix some $\psi\in(0,1)$ and $r-g>-\delta$ that characterize an equilibrium in the $T=\infty$ economy, given some fiscal targets τ and σ. This implies an $x>0$, an aggregate consumption-wealth ratio $E/W>0$, as well as g_y and g that satisfy $g_y<g+\delta$. To construct fiscal targets and an equilibrium for the finite-T economy, define $\phi_\infty=E/W$ and take ψ, $r-g$, x, g and g_y for the finite-T economy to be same as in the $T=\infty$ economy. By construction, this means that (43)-(44) holds. Furthermore, (45) implies that one can take T large enough so that E/W is arbitrarily close to ϕ_∞ in the finite-T economy. One can then use the risky market clearing condition (23) to construct a τ_T for the finite-T economy. Since E/W converges to its $T=\infty$ counterpart ϕ_∞, this τ_T converges to τ. The risk-free market clearing condition (46) can then be used to back out a σ_T. It will also converge to σ because the aggregate present value of labor income in the finite-T economy converges to its $T=\infty$ counterpart.

Proposition 15 *Fix some $\omega\geq 0$ and fiscal targets $\tau>-1$ and $\sigma>0$. Suppose the $T=\infty$ economy has a well-defined equilibrium balanced growth path characterized by some ψ and $r-g$. For all T large enough, it is possible to find $\tau_T>-1$ and $\sigma_T>0$ so that these ψ and $r-g$ are also part of an equilibrium balanced growth path in the finite-T economy. The resulting sequence of fiscal targets satisfies $(\tau_T,\sigma_T)\to(\tau,\sigma)$.*

For simplicity, the case $\sigma=0$ is ruled out in this proposition to avoid complications that could arise from our assumption that transfers to newborn consumers have to be non-negative. With that caveat, this proposition applies to all $T = \infty$ economies that have an equilibrium, including the ones for which there is no upper bound on utility.

In the finite-T economy, and the utility at time t of a consumer j born at time $t - a$ can be written as

$$U_{j,a} = C_t \times \frac{g + \delta - g_y}{\delta} \frac{1 - e^{-\delta T}}{1 - e^{-(g+\delta-g_y)T}} \left(\frac{\phi_a}{\rho + \delta} \right)^{-1/(1-1/\varepsilon)} M_{j,a} e^{(g_y - g)a}, \tag{47}$$

where ϕ_a is defined in (42) and $M_{j,a}$ is an individual-specific positive Brownian martingale with diffusion coefficient $\varsigma\psi M_{j,a}$ and initial value $M_{j,0} = 1$. For a newborn consumer at time t, this reduces to $C_t(C_y/C)(\phi_0/(\rho + \delta))^{-1/(1-1/\varepsilon)}$. Aggregate consumption is $C_t = (xK_t)^{1-\beta} L^\beta/(1 + \gamma)$. For the finite-$T$ equilibria constructed in the proof of Proposition 15, the x, g, and g_y, as well as the trajectory of K_t and the $\{M_{j,a}\}_{a\in[0,T]}$ are identical to what they are in the corresponding $T = \infty$ economy. For any age interval $[0, A] \subset [0, T)$, it then follows that the date-t utilities $\{U_{j,a}\}_{a\in[0,A]}$ converge to the corresponding utilities for the $T = \infty$ economy. In this sense, finite-T utilities also converge to their $T = \infty$ counterparts.

As already noted, it is possible for the finite-T economy to immediately switch to a new balanced growth path following an unforeseen change in fiscal targets, provided that such a change is accompanied by age-dependent taxes and transfers that put the distribution of wealth across age cohorts into its new steady state. Such a redistribution of wealth can also be implemented in the $T = \infty$ economy, and then (47) can be used to evaluate the welfare consequences for both $T < \infty$ and $T = \infty$. But for the $T = \infty$ economy, this results in a policy experiment that differs from the changes in τ and σ only that we have considered throughout. We leave the transitional dynamics in a finite-T economy of changes in τ and σ only to future work.

6.4 A Quantitative Example

When an economy is sufficiently productive, we know from Proposition 7 that large Pareto improvements will result from large transfers to newborn consumers, combined, if necessary, with a positive wealth tax. Large consumption taxes are not needed. Although it is not necessarily the case that the effect on utilities of increasing σ is monotone, it is certainly possible to construct robust examples in which even small increases in these transfers are Pareto improving.

Here we add to this an example showing that increases in transfers to newborn consumers can be Pareto improving even in an economy in which consumers are finitely lived and do not care about their descendants. In the example, we consider unforeseen increases in σ that are accompanied by one-time age-dependent proportional wealth taxes and transfers at the time a new policy is implemented, in such a way that the distribution of wealth across a cohorts immediately jumps to its new steady state. In a steady state, the aggregate consumption at time t of consumers born at date $t-a$ is $C_{y,t}e^{-(g+\delta-g_y)a}$, and the size of this cohort is $e^{-\delta a}/(1-e^{-\delta T})$, implying that the stationary distribution of per-capita consumption across cohorts of ages $a \in [0,T]$ has a density that scales with $e^{-(g-g_y)a}$. A new policy implies a new steady state value for $g-g_y$ and ϕ_a, and therefore for the distribution of wealth across cohorts as well. Age-dependent wealth taxes and transfers leave the within-cohort wealth distributions unaffected. The overall distribution of wealth will be in its new steady state only after T units of time, when the last cohort that lived through the unforeseen policy change leaves the scene.

In such a setting, the effects of an unforeseen one-time increase in σ are implied by the balanced growth conditions for the finite-T economy and (47). For individual consumers, the $M_{j,a}$ are unaffected, while the other factors in (47) jump upon the arrival of the new policy. Figure 9 displays a scenario in which increases in σ can lead to Pareto improvements. Over a bounded range of σ, the entire curve $\{U_{j,a}\}_{a\in[0,A]}$ shifts up with increases in σ, and the growth rate g increases as well. The increase in g ensures that C_s/C_t increases for all $s>t$, and hence that all future cohorts of consumers also gain from the increased transfers to newborn consumers.

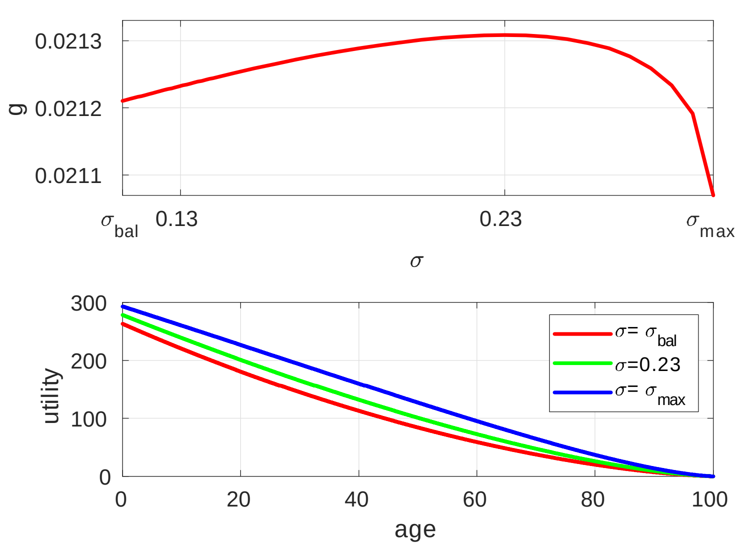

Figure 9 *Pareto improvements for $T = 100$.*

In this example, $T = 100$ and $\delta = 0.005$, resulting in an average age of about 46 years, and an average life span of almost 79 years. The intertemporal elasticity of substitution is large, $\varepsilon = 3$, and the economy is productive enough that its $T = \infty$ counterpart does not have an equilibrium when the wealth tax is zero.[21]

7 Conclusion

How much governments can borrow depends very much on how they plan to use the proceeds. If they use the proceeds to fund "baby bonds" —that is, make transfers to newborn consumers— then there may not be a bound on government borrowing in a sufficiently productive economy with in which consumers are subject to uninsurable indiosyncratic investment risk. If their preferences imply an intertemporal elasticity of substitution greater than 1, then unbounded Pareto improvements may be possible.

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