On the core of an economy with arbitrary consumption sets and asymmetric information

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Abstract

This paper analyses the properties of (strong) core allocations in a two-period asymmetric information economy that also involves both negligible and non-negligible agents as well as an infinite dimensional commodity space. Within this setup, we allow the consumption set of each agent to be an arbitrary subset of the commodity space that may not have any lower bound. Our first result deals with robustness of the the core and the strong core allocations with respect to the restrictions imposed on the size of the blocking coalitions in an economy with only non-negligible agents. As an application of this result, we formulate a characterization of the Aubin core in terms of the set of Aubin non-dominated allocations in a finite economy. The second result is a generalization of the first result to an economy that allow simultaneous presence of negligible as well as non-negligible agents with the consideration of Aubin coalitions. Finally, we show that (strong) core allocations are coalitional fair in the sense that no coalition of negligible agents could redistribute among its members the net trade of any other coalition containing all non-negligible agents in a way which could assign a preferred bundle to each of its members, and vice versa.

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1 Introduction

The core of an economy is a solution concept which acknowledges the fact that coalitions of agents may corporate to improve their own welfare. In other words, for any allocation not belonging to the core, there is a coalition whose members achieve better commodity bundles that the non-core allocation by redistributing their initial endowments with themselves. In a classical exchange economy with a continuum of agents, the core coincides with set of competitive allocations, refer to Aumann (1965). However, the equivalence theorem fails to hold, in general, if there are some non-negligible market participants in addition to the negligible ones (see Schitovitz (1973)). The purpose of the paper is to study the core allocations in the exchange economy embodying large number of agents some which are non-negligible. Note that the the market participants become non-negligible due to the following two reasons: (i) first reason is some agents may endowed with an *exceptional* initial endowments, because their initial ownership of commodities are sufficiently large with respect to the total market endowment. This is typical in monopolistic or, more generally, in oligopolistic markets; and (ii) the second reason is, while the initial endowment is spread over of continuum of negligible agents, some of them may join forces and decide a act a single agent in the form of cartels, syndicates, or similar institutions.

The economic activity is taken into account uncertainty, where agents subscribe contracts at time $\tau = 0$ (ex-ante) that are contingent upon the realized state of nature at time $\tau = 1$ (ex-post), in a way so that their expected payoff is maximized. In this paper, we consider an infinite dimensional commodity space, as it arises naturally due to several reasons: modeling allocations over an infinite time horizon, economies with commodity differentiation, among others. We refer to Mas-colell and Zame (1991) for more details. Our primary focus is an ordered Banach space having non-empty positive interior. One major issues arises while dealing with main results is that Lyapunov's convexity theorem fails to hold in the exact form. The consumption set for each agent in each state to assumed to an arbitrary subset of the commodity space, which may not have any lower bound. Thus, not only the private information restricts the trade of individuals in the ex-ante stage the structure of the consumption sets prevent us to apply strong monotonicity condition at certain bundles.

In the above setup, we study the veto power of arbitrary sized coalitions for non-core allocations and the coalitional fairness of the core allocations. This significantly extends the scope of the theory, incorporating much larger of models as it involves the four aspects together: negligible as well as non-negligible agents, infinite dimensional commodity spaces, uncertainty with asymmetric information, and arbitrary consumption sets not necessarily having lower bounds.

Extensions of the Schmeidler-Vind theorems: For an atomless economy with restricted consumption sets and asymmetric information, we investigate the size of the blocking coalition for a non-core allocation in our setup. This type of investigations goes back to the seminal contributions of Schmeidler (1972) and Vind (1972) in a framework with the positive cone of the Euclidean space as the consumption sets of agents and without uncertainty. More precisely, Schmeidler (1972) showed that if a feasible allocation is not the core of the economy then it can be blocked by coalitions of small measures. Thus, the core (in particular, the set of competitive allocations) can be implemented only through the formation of small coalitions. Schmeidler's idea of blocking mechanism was further extended by Vind (1972) by showing that for any feasible allocation outside the core of an economy then for any measure ε less than the measure of the grand coalition there is a coalition S whose measure is exactly ε such that the non-core allocation is blocked by S. One of the implications of this theorem is normative in the following sense: as an arbitrary large size of coalitions are entitled to block each non-core allocation, the core can be seen as a solution supported by an arbitrary large majority of agents. Later, these results were extended to several frameworks by Bhowmik and Cao (2012), Bhowmik and Cao (2013), Bhowmik (2015), Bhowmik and Graziano (2015), Evren and Hüsseinov (2008), Hervés-Beloso et al. (2000), Hervés-Beloso et al. (2005), Graziano and Romaniello (2012), Pesce (2010) and Pesce (2014) among others. Recently, Bhowmik and Graziano (2020) have extended this result in a setting where agents' consumption sets are arbitrary subsets (without any lower bound restrictions) of a finite dimensional space and ex ante trades are defined in terms of some general restrictions. Such restrictions include two different scenarios: asymmetric information economies and asset market economies. In the present paper, we generalizes the above result of Bhowmik and Graziano (2020) to an economy with an infinite dimensional commodity space but only considering asymmetric information scenario. Not only that, our paper also extends all of the above results in the following direction.

- First, we consider an ex-ante strong core allocation, which is a feasible allocation that cannot be weakly blocked by a non-null coalition. By definition, the ex-ante strong core allocation is an ex-ante core allocation, but the converse fails to hold, in general. Nevertheless, by adopting continuity and strong monotonicity of preferences, one can readily verify that two core notions are the same in a classical economy without uncertainty with asymmetric information in which the positive cone is the consumption set of each agent. However, in a model that involves either uncertainty with asymmetric information or arbitrary consumption sets, such a conclusion cannot be immediately drawn in the presence of continuity and strong monotonicity of preferences. The present paper deals with this issue and formulates a set of sufficient condition that ensures the equivalence of two core notions in our framework. As a consequence, our Vind's theorem is also valid for the ex ante strong core in a framework of an infinite dimensional commodity space.

- We show that the core of a mixed economy coincides with the core of an atomless economy derived by splitting each atom into a continuum of small agents, and vice versa. In view of this result and Vind's theorem for atomless economies, we can generalizes Vind's theorem to a mixed economy by considering generalized coalitions. It is worthwhile to pointing out that this result is the first generalization of Vind's theorem in an economy with asymmetric information and a finite dimensional commodity space, where the feasibility is defined as exact and the consumption set for each agent is an arbitrary subset of the commodity space.

Coalitional fairness of ex-core allocations: Next, we investigate the coalitional fairness of core allocations, which is property of equity, introduced by Gabszewicz in his seminal paper Gabszewicz (1975), in which bundle comparisons are allowed between coalitions of agents according to the concept of coalitional envy.[1] According to Gabszewicz (1975), an allocation is coalitionally unfair if a coalition is treated under the allocation in a discriminatory way by the market. More generally, an allocation is coalitionally fair if no coalition could benefit from achieving the net trade of some other coalition, which means under coalitional fairness, no coalition could redistribute among its members the net trade of any other coalition in a way which could assign a preferred bundle to each of its members. It is well-known that a core allocation is not necessarily coalitionally fair in a mixed economy, refer to Gabszewicz (1975). Thus, we restricts ourselves to coalitions containing either no large agents or all of them, and show that any core allocation is coalitionally fair in the sense that no coalition of small agents envies the net trade of a disjoint coalition comprised of all large agents or vice versa. Therefore, large agents, despite of their privileged initial position, can not enforce a core allocation because this would render the allocation unfair towards some coalition of small agents, and vice versa. Related research in this direction either focuses on a finite dimensional commodity space or an infinite dimensional commodity space with the positive cone as the consumption set of each agent, see Bhowmik (2015), Bhowmik and Graziano (2020), and Gabszewicz (1975). Thus, our result generalizes the above results to centain extent.

[1]See also Schmeidler and Vind (1972) and Varian (1974).

2 Description of the model

We consider a standard pure exchange economy with uncertainty and asymmetric information. We assume that the economic activity takes place over two periods $\tau = 0, 1$. The exogenous uncertainty is described by a measurable space $(\Omega, \mathscr{F})$, where Ω is a finite set denoting all possible states of nature at time $\tau = 1$ and the σ-algebra $\mathscr{F}$ denotes all events. At time $\tau = 0$ (ex-ante stage) there is uncertainty about the state of nature that will be realized at time $\tau = 1$ (ex-post stage). At the ex-ante stage, agents arrange contract on redistribution of their initial endowments. At $\tau = 1$, agents carry out previously made agreements, and consumption takes place[2].

Economic agents: The space of **economic agents** is described by a complete probability space $(T, \mathscr{T}, \mu)$, where T represents the set of agents, the σ-algebra $\mathscr{T}$ represents the collections of allowable coalitions whose economic weights on the market are given by μ. A **non-null coalition** of $\mathscr{E}$ is a member of $\mathscr{T}$ whose economic weight on the market is positive. Since $\mu(T) < \infty$, the set T of agents can be decomposed in the disjoint union of an atomless sector T_0 of non-influential (small or negligible) agents and the set T_1 of influential (large or non-negligible) agents, which is the union of at most countable family $\{A_1, A_2, \cdots\}$ of atoms of μ. Abusing notation, we also denote by T_1 the collection $\{A_1, A_2, \cdots\}$. Thus, the space of agents not only allow us to investigate in a unified manner the markets that are competitive and the markets that are not, but also deal with the simultaneous action of influential and non-influential agents. This general representation permits to cover simultaneously the case of an economy with a finite set of agents (when T_0 is empty and T_1 is finite), the case of an atomless economy (when T_1 is empty), the case of mixed markets in which an ocean of negligible agents coexists with few influential agents (when both T_0 and T_1 have positive measure). Moving from this representation, we can also identify two relevant subfamilies from $\mathscr{T}$ by defining

$$\mathscr{T}_0 := \{S \in \mathscr{T} : S \subseteq T_0\} \text{ and } \mathscr{T}_1 := \{S \in \mathscr{T} : T_1 \subseteq S\}.$$

Thus, $\mathscr{T}_0$ is a subfamily of $\mathscr{T}$ containing no atoms whereas $\mathscr{T}_1$ is a subfamily of $\mathscr{T}$ containing all atoms. Finally, we denote by

$$\mathscr{T}_2 := \mathscr{T}_0 \cup \mathscr{T}_1 = \{S \in \mathscr{T} : S \in \mathscr{T}_0 \text{ or } S \in \mathscr{T}_1\}$$

the subfamily of $\mathscr{T}$ formed by coalitions containing either no atoms or all atoms.

[2]For simplicity, we assume that there are no endowments and thus no consumption at $\tau = 0$. Hence, agents are only concerned with allocating their second period ($\tau = 1$) endowments.

Commodity Spaces: The **commodity space** in our model is an ordered separable Banach space with the interior of the positive cone is non-empty. We denote by $\mathbb{Y}$ the commodity space of our economy whereas the notation $\mathbb{Y}_+$ is employed to denote the positive cone of $\mathbb{Y}$. Let $\mathbb{Y}_{++}$ be the interior of $\mathbb{Y}_+$.

Defining an economy: We introduce a mixed economy with uncertainity and asymmetric information, and an ordered separable Banach space whose positive cone has non-empty interior as the commodity space.

Definition 2.1. An **economy** is defined as $\mathscr{E} := \{(X_t, \mathscr{F}_t, u_t, e(t,\cdot), \mathbb{P}_t) : t \in T\}$ with the following specifications:

(A) $X_t : \Omega \rightrightarrows \mathbb{Y}$ denotes the (state-contingent) consumption set of agent $t \in T$[3];

(B) $\mathscr{F}_t$ is the σ-algebra generated by a measurable partition $\mathscr{P}_t$ of Ω (i.e. $\mathscr{P}_t \subseteq \mathscr{F}$) denoting the private information of agent t;

(C) $u_t : \Omega \times \mathbb{Y} \to \mathbb{R}$ is the state-dependent utility function of agent t;

(D) $e(t,\cdot) : \Omega \to \mathbb{Y}$ is the random initial endowment of agent t;

(E) $\mathbb{P}_t : \Omega \to [0,1]$ is the prior of agent t.

Available Information and Expected Utilities: The family of all paritions of Ω is denoted by $\mathfrak{P}$. Since Ω is finite, $\mathfrak{P}$ has only finitely many different elements: $\mathscr{P}_1, \cdots, \mathscr{P}_n$. We assume that $T_i := \{t \in T : \mathscr{P}_t = \mathscr{P}_i\}$ is $\mathscr{T}$-measurable for all $1 \leq i \leq n$. For every $1 \leq i \leq n$, define $\mathscr{G}_i$ to be the set of all functions $\varphi : \Omega \to \mathbb{Y}$ such that φ is $\mathscr{P}_i$-measurable.[4] For any $x : \Omega \to \mathbb{Y}$, define the **ex-ante expected utility** of agent t by the usual formula

$$V_t(x) = \sum_{\omega \in \Omega} u_t(\omega, x(\omega))\mathbb{P}_t(\omega).$$

We now state our main assumptions to be used throughout the paper.

Assumption 2.2. Consider an ecoonomy $\mathscr{E}$ as defined in Definition 2.1.

($\mathbf{A}_1$) For all $(t,\omega) \in T \times \Omega$, $X_t(\omega)$ is a closed convex cone.

($\mathbf{A}_2$) The correspondence $\mathbf{\Theta} : T \times \Omega \rightrightarrows \mathbb{Y}$, defined by $\mathbf{\Theta}(t,\omega) := X_t(\omega)$, is such that $\mathbf{\Theta}(\cdot,\omega)$ is $\mathscr{T}$-measurable for all $\omega \in \Omega$.

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[3]Notice that we do not impose non-negative constraints on consumption sets. Thus, short sales are allowed.

[4]By $\mathscr{P}_i$-measurability, we mean the measurability with respect to the σ-algebra generated by $\mathscr{P}_i$.

($\mathbf{A}_3$) The mapping $e(\cdot,\omega) : T \to \mathbb{Y}$ is $\mathscr{T}$-measurable for all $\omega \in \Omega$ and $e(t,\omega)$ is an interior point of $X_t(\omega)$ for all $\omega \in \Omega$.

($\mathbf{A}_4$) The mapping $\varphi : T \to [0,1]^\Omega$, defined by $\varphi(t) = \mathbb{P}_t$, is $\mathscr{T}$-measurable.

($\mathbf{A}_5$) For all $(t,\omega) \in T_1 \times \Omega$, $u_t(\omega,\cdot)$ is concave.

($\mathbf{A}_6$) For all $(t,\omega) \in T \times \Omega$, $u_t(\omega,\cdot)$ is continuous and for all $x \in \mathbb{R}^\ell$, $t \mapsto u_t(\omega,x)$ is $\mathscr{T}$-measurable.

($\mathbf{A}_7$) For all $(t,\omega) \in T \times \Omega$, $u_t(\omega,y) > u_t(\omega,x)$ for all $x,y \in X_t(\omega)$ with $y \geq x$ and $x \neq y$.

($\mathbf{A}_8$) For all $(t,\omega) \in T \times \Omega$, $x \in \mathscr{A}_t$ and $\varepsilon > 0$, there is an $y \in \bigcap\{\varepsilon\mathscr{G}_i : i \in \mathbb{K}\} \cap \mathbb{B}(0,\varepsilon)^\Omega$ such that $x + y \in X_t$ and $u_t(\omega, x(\omega) + y(\omega)) > u_t(\omega, x(\omega))$[5].

($\mathbf{A}'_8$) For all $(t,\omega) \in T \times \Omega$, $x \in \mathscr{A}_t$ and $\varepsilon > 0$, there is an $y \in \bigcap\{\varepsilon\mathscr{G}_i : i \in \mathbb{K}\} \cap \mathbb{B}(0,\varepsilon)^\Omega$ such that $x + y \in \mathrm{int}X_t$ and $u_t(\omega, x(\omega) + y(\omega)) > u_t(\omega, x(\omega))$.

Remark 2.3. The assumptions in ($\mathbf{A}_1$)-($\mathbf{A}_7$) are standard in the literature of general equilibrium in economies with asymmetric information and/ or restricted consumption sets.

3 Extensions of the Schmeidler-Vind theorems

Our aim in this section is to introduce the (strong) core allocations in an economy with a mixed measure space of agents and provide characterizations by means of the size of the coalitions in the sense of Schmeidler (1972) and Vind (1972) in an economy containing only a continuum of negligible agents or negligible as well as non-negligible agents.

3.1 Defining Core Allocations

In this subsection, we first introduce the classical **ex-ante (strong) core** for a two period economy with uncertainty, where it is assumed implicitly that the trade takes place at time $\tau = 0$ and that contracts are binding: they are carried out after the resolution of uncertainty and there is no possibility of their renegotiation. Moreover, the consumption of each agent is compatible with her private information. Secondly,

[5] $\mathbb{B}(0,\varepsilon)$ denotes the ball centered at 0 and radius ε in $\mathbb{R}^\ell$.

we deal with the relationship between the two different notions of core allocations. We start introducing the concept of an allocation, which is a specification of the amount of commodities assigned to each agent.

Definition 3.1. An **allocation** in $\mathscr{E}$ is a Bochner integrable function $f : T \times \Omega \to \mathbb{Y}$ such that

(i) $f(t, \omega) \in X_t(\omega)$ for all $(t, \omega) \in T \times \Omega$; and

(ii) $f(t, \cdot) \in \mathscr{G}_i$ for all $(t, \omega) \in T_i \times \Omega$ and all $1 \leq i \leq n$.

It is said to be **feasible** if $\int_T f(\cdot, \omega) d\mu = \int_T e(\cdot, \omega) d\mu$ for all $\omega \in \Omega$. We assume that e is an allocation.

Our first notion of core aims to study the blocking mechanism under the assumptions that a coalition deviates from a proposed allocation if it's members guarantee a *stictly better* commodity bundle for themselves by the redistribution.

Definition 3.2. A feasible allocation f is **ex-ante blocked by a non-null coalition** S if there is an allocation g such that $V_t(g(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on S, and

$$\int_S g(\cdot, \omega) d\mu = \int_S e(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$. The **ex-ante core** of $\mathscr{E}$, denoted by $\mathscr{C}(\mathscr{E})$, is the set of feasible allocations that are not ex ante blocked by any non-null coalition.

The next formalisation of core differs from the earlier one in the sense that agents within a blocking coalition are not worse-off by the re-distribution whereas members of a sub-coalition are strctly better-off.

Definition 3.3. A feasible allocation f is **ex-ante weakly blocked by a non-null coalition** S if there is a sub-coalition R of S and an allocation g such that

(i) $V_t(g(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on R;

(ii) $V_t(g(t, \cdot)) \geq V_t(f(t, \cdot))$ μ-a.e. on S; and

(iii) $\int_S g(\cdot, \omega) d\mu = \int_S e(\cdot, \omega) d\mu$ for all $\omega \in \Omega$.

The **ex-ante strong core** of $\mathscr{E}$, denoted by $\mathscr{C}^s(\mathscr{E})$, is the set of feasible allocations that are not ex ante weakly blocked by any non-null coalition.

Recognized that any ex ante strong core allocation is also an ex ante core allocation. For the converse, we additionally assume in our next result that if an allocation f is ex ante weakly blocked by a coalition S via some allocation g and if B is a sub-coalition of S in which members of B strictly prefer g to f then the information available to both coalitions are the same. The basic intuition is that members belonging to R_i, where $R := S \setminus B$, can be allocated $\mathscr{P}_i$-measurable consumption bundles which give higher utilities by reducing the utility level of the members of B_i under g, due to continuity and strong monotonicity. However, such an argument cannot be done easily in the presence of arbitrary consumption sets. In what follows, we establish this result in an economy with either no atom or finitely many atoms by applying Lemma 5.1 and Lemma 5.2 in Appendix. To this end, we define $\mathbb{I}_S := \{i : \mu(S_i) > 0\}$ for any non-null coalition S, where $S_i := S \cap T_i$ for all $1 \leq i \leq n$.

Proposition 3.4. *Suppose that f is weakly blocked by a coalition S via some $\mathscr{G}$-assignemnt g satisfying $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on some sub-coalition B of S satisfying $\mathbb{I}_B = \mathbb{I}_S$. Further, assume that either of the following two conditions is satisfied:*

(1) $T = T_0$; *and*

(2) $B = S$ *and* T_1 *has finitely many atoms.*

Then there are coalitions E, R and an $\mathscr{G}$-assignment y such that

(i) $R \subseteq E \subseteq S$, $\mathbb{I}_R = \mathbb{I}_E = \mathbb{I}_S$ *and* $T_1 \subseteq R$;[6]

(ii) f *is blocked by* E *via* y;

(iii) $g(t,\cdot) + z \in X_t$ *for all* $z \in \mathbb{B}(0,\eta)^\Omega$ *and* μ*-a.e. on* R; *and*

(iv) $V_t(g(t,\cdot) + z) > V_t(f(t,\cdot))$ *for all* $z \in \mathbb{B}(0,\eta)^\Omega$ *and* μ*-a.e. on* R.

Proof. The proof of the proposition is relegated to Appendix. □

3.2 The Size of Blocking Coalitions in a continuum economy

In this subsection, we address the issues related to the size of a bolcking coalition, extending the corresponding results of Schmeidler (1972) and Vind (1972) to the case of a continuum economy with arbitrary consumption sets and private information.

[6] If $T_1 = \emptyset$ then $T_1 \subseteq R$ is automatically satisfied.

Extending the Schmeidler Theorem: The insight of Schmeidler theorem was that, in a continuum economy, if a feasible non-core allocation is blocked by some coalition S then it can also be blocked by a coalition of any given measure less than that of S. The immediate implication of this theorem includes the fact that the core (and thus, the set of competitive allocations) can be implemented by the formation of small coalitions only. In what follows, we extend this result to our framework. This definitely extend the corresponding result of Bhowmik and Graziano (2019) to a certain extent. It is worthwhile to pointing out that the techniques adopted in the proof of Bhowmik and Graziano (2019) are not appropriate in our setup of infinitely many commodities. Thus, in order to obtain the Schmeidler theorem in our framework, we first establish the following proposition. This proposition can be considered as an extension of the Lyapunov convexity theorem.

Proposition 3.5. *Let $\mathscr{E}$ be a continuum economy and let the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ *be satisfied. Suppose that ψ, f and g are allocations such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on some non-null coalition S and $0 < \delta < 1$. Assume further that $g(t,\omega)$ is an interior point of $X_t(\omega)$ for all $(t,\omega) \in R \times \Omega$ for some sub-coalition R of S satisfying $\mathbb{I}_R = \mathbb{I}_S$. Then there are an $\eta_0 > 0$, two non-null coalitions B and C, and an allocation φ such that*

(i) *$C \subseteq B \subseteq S$, $\mathbb{I}_C = \mathbb{I}_B = \mathbb{I}_S$ and $\mu(B) = \delta\mu(S)$;*

(ii) *$\varphi(t,\omega) + z(\omega) \in X_t(\omega)$ for all $z(\omega) \in \mathbb{B}(0,\eta_0)$ and $(t,\omega) \in C \times \Omega$;*

(iii) *$V_t(\varphi(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on C;*

(iv) *$V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on $B \setminus C$; and*

(v) *$\int_B (\varphi(\cdot,\omega) - \psi(\cdot,\omega))d\mu = \delta \int_S (g(\cdot,\omega) - \psi(\cdot,\omega))d\mu$ for all $\omega \in \Omega$.*

Proof. The proof of the proposition is relegated to Appendix. □

The following theorem is an immediate implication of above proposition, which extends Schmeidler's (1972) theorem to our framework.

Theorem 3.6. *Consider a continuum economy $\mathscr{E}$ and assume that the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ *are satisfied. Let f be a feasible allocation $\mathscr{E}$ blocked by some coalition S. Then for any $\varepsilon \in (0,\mu(S))$, there is a coalition R such that $\mu(R) = \varepsilon$ and f is blocked by R.*

Proof. The proof of the theorem is relegated to Appendix. □

Extending the Vind Theorem: The Vind (1972) states that, in a continuum economy, if a feasible allocation is not in the core of the economy then there is a blocking coalition of any given measure less than the measure of the grand coalition. Thus, the core allocations (and thus, the competitive allocations) can also be characterized by coalitions of arbitrary large sizes. We now intended to show a similar result in our framework. To this end, we first the establish the following result, which claims that if an allocation is blocked by a coalition S via some allocation g then there is another allocation h in which everbody is better off than what she gets under f. This Proposition extends the corresponding results in Bhowmik and Cao (2013), Hervés-Beloso and Moreno-García (2008), and Vind (1972).

Proposition 3.7. *Suppose that f and g are two allocations such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on some non-null coalition S with $g(t,\omega)$ being an interior point of $X_t(\omega)$ for all $(t,\omega) \in R \times \Omega$ for some sub-coalition R of S satisfying $\mathbb{I}_R = \mathbb{I}_S$, and $0 < \delta < 1$. Then there exist some allocation h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S, $h(t,\cdot)$ is an interior point of X_t for all $t \in G$ for some sub-coalition G of S with $\mathbb{I}_G = \mathbb{I}_S$, and*

$$\int_S h(\cdot,\omega)d\mu = \int_S (\delta g(\cdot,\omega) + (1-\delta)f(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$.

Proof. The proof of the proposition is relegated to Appendix. □

Corollary 3.8. *Consider now a mixed economy where all large agents have continuous and quasi-concave utility functions. For any large agent A and $x, y \in X_A$, if $V_A(y) > V_A(x)$ and $0 < \delta < 1$ then, by Lemma 5.26 of Aliprantis and Border (2006), we have $V_A(\delta y + (1-\delta)x) > V_A(x)$. In view of this, the conclusion of Proposition 3.7 can be obtained in a mixed model.*

Next, we formulate a version of Vind's (1972) theorem on the blocking of an arbitrary coalition.

Theorem 3.9. *Consider a continuum economy $\mathscr{E}$ in which the assumptions $(\mathbf{A}_1)$-$(\mathbf{A}'_8)$ are satisfied. Let f be a feasible allocation such that $f \notin \mathscr{C}(\mathscr{E})$. Then for any $\varepsilon \in (0, \mu(T))$, there is some coalition R such that $\mu(R) = \varepsilon$ and f is blocked by R.*

Proof. The proof of the theorem is relegated to Appendix. □

Remark 3.10. We now complete the proof by replacing the assumption $[\mathbf{A}'_8]$ with $\mathbb{I}_{T\setminus S} \subseteq \mathbb{I}_S$ and $[\mathbf{A}_8]$. Let $D := T \setminus S$. For each $i \in \mathbb{I}_D$, there is some $F_i \in \mathscr{T}_{D_i}$ such that

$\mu(F_i) = \delta\mu(D_i)$ and

$$b_i(\omega) := \delta \int_{D_i} (g(\cdot,\omega) - e(\cdot,\omega))\, d\mu - \int_{F_i} (g(\cdot,\omega) - e(\cdot,\omega))\, d\mu \in \mathbb{B}\left(0, \eta\delta\mu(C_i)\right).$$

Define $z : C \times \Omega \to \mathbb{Y}$ by letting $z(t,\omega) := \frac{b_i(\omega)}{\delta\mu(C_i)}$ if $(t,\omega) \in R_i \times \Omega$ and $i \in \mathbb{I}_D$; and $z(t,\omega) := 0$, otherwise. Let $\tilde{g} : T \times \Omega \to \mathbb{Y}$ be an allocation such that

$$\tilde{g}(t,\omega) := \begin{cases} g(t,\omega) - z(t,\omega), & \text{if } (t,\omega) \in C \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

By Proposition 3.7, there exist some $\mathscr{G}$-assignment h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S, and

$$\int_S h(\cdot,\omega) d\mu = \int_S (\delta\tilde{g}(\cdot,\omega) + (1-\delta) f(\cdot,\omega)) d\mu$$

for all $\omega \in \Omega$. We define an assignment $y : T \times \Omega \to \mathbb{Y}$ defined by

$$y(t,\omega) := \begin{cases} \psi(t,\omega), & \text{if } (t,\omega) \in F \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

It can be raedily verified that f is blocked by the coalition $E := F \cup S$ via y. $\square$

As an application of Theorem 3.9, we have the following result.

Theorem 3.11. *The Aubin core of $\mathscr{E}^{\text{finite}}$ coincides with the set of feasible allocations that are Aubin non-dominated in $\mathscr{E}^{\text{finite}}$.*

Proof. The proof of the theorem is relegated to Appendix. $\square$

3.3 The Size of Blocking Coalitions in a mixed economy

In this subsection, we first associate $\mathscr{E}$ with an atomless ecomomy $\widetilde{\mathscr{E}}$ and study the connection between the ex-ante core allocations of these two economies. This extends the result of Greenberg and Shitovitz (1986) and some of its follow up papers as mentioned in Section 1.

Given the economy $\mathscr{E}$, the economy $\widetilde{\mathscr{E}}$ is obtained by *splitting* each large agent into a continuum of small agents whose characteristics are the same as that of large agent. Therefore, the space of agents of $\widetilde{\mathscr{E}}$, denoted by $(\widetilde{T}, \widetilde{\mathscr{T}}, \widetilde{\mu})$, satisfies the following: (i) $\widetilde{T}_0 = T_0$ and $\widetilde{\mu}(\widetilde{T}_1) = \mu(T_1)$, where $\widetilde{T}_1 := T \setminus T_0$; (ii) $\widetilde{\mathscr{T}}$ and $\widetilde{\mu}$ are obtained by the direct sum of $\mathscr{T}$ and μ restricted to T_0 and the Lebesgue atomless measure space over $\widetilde{T}_1$; and (iii) each atom A_i one-to-one corresponds to a Lebesgue measurable subset $\widetilde{A_i}$ of $\widetilde{T}_1$ such that $\mu(A_i) = \widetilde{\mu}(\widetilde{A_i})$, where $\{\widetilde{A_i} : i \geq 1\}$ can be expressed as the disjoint union of the intervals $\{\widetilde{A_i} : i \geq 1\}$ given by $\widetilde{A_1} := [\mu(T_0), \mu(T_0) + \mu(A_1))$, and

$$\widetilde{A_i} := \left[\mu(T_0) + \mu\left(\bigcup_{j=1}^{i-1} A_j \right),\ \mu(T_0) + \mu\left(\bigcup_{j=1}^{i} A_j \right) \right),$$

for all $i \geq 2$. Furthermore, the space of states of nature and the commodity space of $\widetilde{\mathscr{E}}$ are the same as those of $\mathscr{E}$. Finally, the characteristics $(\widetilde{X}_t \widetilde{\mathscr{F}_t}, \widetilde{u}_t, \widetilde{e}(t, \cdot), \widetilde{\mathbb{P}}_t)$ of each agent $t \in \widetilde{T}$ in $\widetilde{\mathscr{E}}$ are defined as follows:

$$\widetilde{X}_t := \begin{cases} X_t, & \text{if } t \in T_0; \\ X_{A_i}, & \text{if } t \in \widetilde{A}_i, \end{cases}$$

$$\widetilde{\mathscr{F}_t} := \begin{cases} \mathscr{F}_t, & \text{if } t \in T_0; \\ \mathscr{F}_{A_i}, & \text{if } t \in \widetilde{A}_i, \end{cases}$$

$$\widetilde{u}_t := \begin{cases} u_t, & \text{if } t \in T_0; \\ u_{A_i}, & \text{if } t \in \widetilde{A}_i, \end{cases}$$

$$\widetilde{e}(t, \cdot) := \begin{cases} e(t, \cdot), & \text{if } t \in T_0; \\ e(A_i, \cdot), & \text{if } t \in \widetilde{A}_i, \end{cases}$$

and

$$\widetilde{\mathbb{P}}_t := \begin{cases} \mathbb{P}_t, & \text{if } t \in T_0; \\ \mathbb{P}_{A_i}, & \text{if } t \in \widetilde{A}_i. \end{cases}$$

We now introduce some notations for the rest of the section. To an allocation f in $\mathscr{E}$, we associate an allocation $\widetilde{f} := \Xi[f]$ in $\widetilde{\mathscr{E}}$, defined by

$$\widetilde{f}(t) := \begin{cases} f(t), & \text{if } t \in T_0; \\ f(A_i), & \text{if } t \in \widetilde{A}_i. \end{cases}$$

Reciprocally, for each allocation $\widetilde{f}$ in $\widetilde{\mathscr{E}}$, we define an allocation $f := \Phi[\widetilde{f}]$ in $\mathscr{E}$ such that

$$f(t) := \begin{cases} \widetilde{f}(t), & \text{if } t \in T_0; \\ \frac{1}{\mu(A_i)} \int_{A_i} \widetilde{f} d\widetilde{\mu}, & \text{if } t = A_i. \end{cases}$$

Recognized that if f is a feasible allocation in $\mathscr{E}$ then $\Xi[f]$ is a feasible allocation in $\widetilde{\mathscr{E}}$. Similarly, for each feasible allocation $\widetilde{f}$ in $\widetilde{\mathscr{E}}$, the allocation $\Phi[\widetilde{f}]$ is feasible in $\mathscr{E}$.

We show that an allocation is in the ex ante core of a mixed economy assigns indifferent consumption plans to all large agents. This is due to the fact that all agents have the same characteristics.

Proposition 3.12. *Let the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ *be satisfied for a mixed economy* $\mathscr{E}$*. Let* R *be a coalition in* $\mathscr{T}_1$[7] *having the same characteristics. If* f *is in the ex ante core of* $\mathscr{E}$ *then* $V_t(f(t,\cdot)) = V_t(\mathbf{x}_f)$ μ*-a.e. on* R*, where*

$$\mathbf{x}_f(\omega) := \frac{1}{\mu(R)} \int_R f(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$*.*

Proof. The proof of the proposition is relegated to Appendix. □

Remark 3.13. If $\mathbb{Y}$ is finite dimensional then one can dispense with the assumption $\mathbf{A}'_8$. In fact, the assumption $\mathbf{A}'_8$ help us to apply Proposition 3.5 in the proof of Proposition 3.12. In the case of finite dimension, we can just use $\mathbf{A}_8$ and apply Lyapunov convexity theorem instead of Proposition 3.5.

Lemma 3.14. *Let* $\mathscr{E}$ *be a continuum economy and let the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$ *be satisfied. Suppose that* f *and* g *are allocations such that* $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ*-a.e. on some coalition* S *and* $g(t,\omega)$ *is an interior point of* $X_t(\omega)$ *for all* $(t,\omega) \in R \times \Omega$ *for some sub-coalition* R *of* S *satisfying* $\mathbb{I}_R = \mathbb{I}_S$*. Assume further that* $\mu(S \cap H) \geq \alpha > 0$ *for some coalition* H *of* $\mathscr{E}$*. Then there are a coalition* B *and an allocation* h *such that* f *is blocked by* B *via* h *and* $\mu(B \cap H) = \alpha$*.*

Proof. The proof of the lemma is relegated to Appendix. □

Theorem 3.15. *Let* $\mathscr{E}$ *be a mixed economy satisfying the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$*. If* $\widetilde{f} \in \mathscr{C}(\widetilde{\mathscr{E}})$ *then* $f \in \mathscr{C}(\mathscr{E})$*.*

Proof. The proof of theorem is relegated to Appendix. □

Theorem 3.16. *Let* $\mathscr{E}$ *be a mixed economy satisfying the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8)$*. Suppose also that* $R \in \mathscr{T}_1$ *is a coalition having the same characteristics. Then* $f \in \mathscr{C}(\mathscr{E}) \Rightarrow \widetilde{f} \in \mathscr{C}(\widetilde{\mathscr{E}})$ *if either of the following two conditions are true:*

(i) $(\mathbf{A}'_8)$ *is satisfied and* $R = T_1$ *has at least two elements; and*

(ii) T_1 *has exactly one element and* $\mu(R \setminus T_1) > 0$*.*

[7] If T_1 is empty then R contains only negligible agents.

Proof. The proof of the theorem is relegated to Appendix. □

In view of above results and Theorem 3.9, one can readily derive the following result as in Bhowmik and Graziano (2015).

Theorem 3.17. *Consider a mixed economy $\mathscr{E}$ in which the assumptions* $(\mathbf{A}_1)$-$(\mathbf{A}_8')$ *are satisfied. Let f be a feasible allocation such that $f \notin \mathscr{C}(\mathscr{E})$. Then for any $\varepsilon \in (0, \mu(T))$, there is some coalition R such that $\mu^{\mathrm{A}}(R) = \varepsilon$ and f is blocked by R.*

4 Coalitional Fairness of Core Allocations

In this section, we study the coalitional fairness of the ex ante core allocations. This means that the stability of an allocation under the coalitional improvement guarantees that it is also equitable in the sense that no coalition *envies* the net trade of any other disjoint coalition. The concept of a *coalitionally fair allocation* was first proposed by Gabszewicz in his seminal paper Gabszewicz (1975) for an exchange economy, where an allocation is said to be **coalitionally fair** if no coalition can redistribute among its members the net trade of any other coalition, in such a way that each of them is better-off. It is worthwhile to pointing out that competitive equilibrium allocations are coalitionally fair, which also belongs to the core of the economy. For a classical economy with an atomless measure space of agents, Aumman's core-equivalence theorem guarantees that the set of coalitionally fair allocations coincides with the core of the economy. However, it is unclear to us whether such a result hold true whenever the consumption sets are not bounded from below and restrictions imposed on ex ante trade. On the other hand, in a classical mixed economy, it is well known (refer to Shitovitz (????????????)) that the core-equivalence theorem does not hold in general, and one should expect a kind of *exploitation* of small agents by large agents. Fortunately, Bhowmik and Graziano (?????) obtained a partial result on the coalitional fairness of the core allocations in the sense that no coalition of small agents envies the net trade of a disjoint coalition comprised of all large agents and vice versa in a framework similar to us but only for finitely many commodities. This result extends Theorem 2 in Gabszewicz (1975), who established the result in a classical deterministic economy with finitely many commodities.

The first notion fairness requires that no coalition of small agents envies the net trade of a disjoint coalition comprised of all large agents.

Definition 4.1. A feasible allocation f is called $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E})$*-fair* if there do not exist two disjoint elements $S \in \mathscr{T}_0$, $E \in \mathscr{T}_1$ and an $\mathscr{G}$-assignment g such that μ-a.e. on S and for each $\omega \in \Omega$:

(i) $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$; and

(ii) $$\int_S (g(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_E (f(\cdot,\omega) - e(\cdot,\omega))d\mu.$$

In what follows, we show that any allocation in the ex-ante core is coalitionally fair in a way that no coalition of small agents can redistribute among its members the net trade of any other coalition containing all large agents, in such a way that each of them is better-off.

Theorem 4.2. *Let* $(\mathbf{A}_1)$-$(\mathbf{A}_?)$ *be satisfied. Then any allocation in the ex-ante core of* $\mathscr{E}$ *is* $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E})$*-fair.*

Proof. The proof of the theorem is relegated to Appendix. □

In the next notion fairness, the role of coalitions are opposite, i.e., no coalition containing all large agents envies the net trade of a disjoint coalition of small agents.

Definition 4.3. A feasible allocation f is called $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$*-fair* if there do not exist two disjoint elements $S \in \mathscr{T}_1$, $E \in \mathscr{T}_0$ and an $\mathscr{G}$-assignment g such that μ-a.e. on S and for each $\omega \in \Omega$:

(i) $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$; and

(ii) $$\int_S (g(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_E (f(\cdot,\omega) - e(\cdot,\omega))d\mu.$$

To prove that any ex-ante core allocation is $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$-fair we establish the following lemma.

Lemma 4.4. *Assume that f and h are two allocations such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on some coalition S. Then there exist $0 < \lambda, \eta < 1$, a sub-coalition R of S and an allocation such that*

(i) *$y(t,\cdot) + z \in X_t$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on R;*

(ii) *$V_t(y(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on S;*

(iii) *$V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on $S \setminus R$; and*

(iv) *$\int_S (y - e)d\mu = (1-\lambda)\int_S (h - e)d\mu$.*

Proof. The proof of theorem is relegated to Appendix. □

The following theorem demonstrates that any allocation in the ex-ante core is coalitionally fair in a way that no coalition comprised of all large agents can redistribute among its members the net trade of any other coalition containing only small agents, in such a way that each of them is better-off.

Theorem 4.5. *Let* $(\mathbf{A}_1)$-$(\mathbf{A}_?)$ *be satisfied. Then any allocation in the ex-ante core is* $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$*-fair.*

Proof. The proof the theorem is relegated to Appendix. □

5 Appendix

Lemma 5.1. *Suppose that* f *and* g *are two* $\mathscr{G}$*-allocations such that* $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ*-a.e. on some coalition* S *with* $g(t,\omega)$ *being an interior point of* $X_t(\omega)$ *for all* $(t,\omega) \in S \times \Omega$*. Then for any* $0 < \varepsilon < \mu(S)$*, there are some* $\eta > 0$ *and a sub-coalition* R *of* S *such that*

(i) $\mu(R) > \mu(S) - \varepsilon$;

(ii) $g(t,\omega) + z(\omega) \in X_t(\omega)$ *for all* $z(\omega) \in \mathbb{B}(0,\eta)$ *and* $(t,\omega) \in R \times \Omega$*; and*

(iii) $V_t(g(t,\cdot) + z) > V_t(f(t,\cdot))$ *for all* $z \in \mathbb{B}(0,\eta)^\Omega$ *and* μ*-a.e. on* R*.*

Proof. Define a correspondence $\mathbf{\Upsilon} : S \rightrightarrows \mathbb{R}_+$ by letting

$$\mathbf{\Upsilon}(t) := \left\{\eta \in (0,\infty) : g(t,\cdot) + z \in X_t \text{ and } V_t(g(t,\cdot) + z) > V_t(f(t,\cdot)) \text{ for all } z \in \mathbb{B}(0,\eta)^\Omega\right\}.$$

By the continuity of preferences and the fact that $g(t,\omega)$ is an interior point of $X_t(\omega)$ for all $(t,\omega) \in S \times \Omega$, we have $\mathbf{\Upsilon}(t) \neq \emptyset$ μ-a.e. on S. As $\mathbf{\Upsilon}(t)$ is bounded from above, the function $\varphi : T \to \mathbb{R}_+$, defined by $\varphi(t) = \sup \mathbf{\Upsilon}(t)$, is well-defined. We show that φ is $\mathscr{T}_S$-measurable. To this end, first note that the function $\psi : S \times \mathbb{Y}^\Omega \to \mathbb{R}$, defined by $\psi(t,z) := V_t(g(t,\cdot) + z) - V_t(f(t,\cdot))$, is a Carathéodory function, and thus, it is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Y}^\Omega)$-measurable. Define a correspondence $\mathbf{G} : S \rightrightarrows \mathbb{Y}^\Omega$ by letting

$$\mathbf{G}(t) := \left\{z \in \mathbb{Y}^\Omega : \psi(t,z) > 0\right\}.$$

It follows that $\mathbf{G}$ is non-empty valued and has $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Y}^\Omega)$-measurable graph. Consider a correspondence $\mathbf{H} : S \rightrightarrows \mathbb{Y}^\Omega$ defined by

$$\mathbf{H}(t) := \left\{z \in \mathbb{Y}^\Omega : g(t,\omega) + z(\omega) \in X_t(\omega) \text{ for all } \omega \in \Omega\right\}.$$

Due to the closeness of $X_t(\omega)$, $\mathbf{H}(t)$ can be equivalently expressed as

$$\mathbf{H}(t) = \left\{ z \in \mathbb{Y}^\Omega : \mathrm{dist}(g(t,\omega) + z(\omega), X_t(\omega)) = 0 \text{ for all } \omega \in \Omega \right\}.$$

In view of the fact that $0 \in \mathbf{H}(t)$, we have $\mathbf{H}(t) \neq \emptyset$ for all μ-a.e. on S. Moreover, $\mathrm{Gr}_{\mathbf{H}}$ is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Y}^\Omega)$-measurable as $\mathrm{Gr}_{\mathbf{H}} = y^{-1}(\{0\})$, where $y : S \times \mathbb{Y}^\Omega \to \mathbb{R}$, defined by $y(t,\omega) := \mathrm{dist}(g(t,\omega) + z(\omega), X_t(\omega))$, is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Y}^\Omega)$-measurable. Finally, define a correspondence $\mathbf{\Phi} : S \rightrightarrows \mathbb{Y}^\Omega$ such that $\mathbf{\Phi}(t) := \mathbf{G}(t) \cap \mathbf{H}(t)$ for all $t \in S$. As $0 \in \mathbf{\Phi}(t)$, we have $\mathbf{\Phi}(t) \neq \emptyset$ for all $t \in S$. Moreover, $\mathrm{Gr}_{\mathbf{\Phi}}$ is $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{R}^\Omega)$-measurable. Analogously, the correspondnce $\mathbf{\Theta}_\eta : S \rightrightarrows \mathbb{Y}^\Omega$, defined by $\mathbf{\Theta}_\eta(t) := \mathbb{B}(0,\eta)^\Omega$, has $\mathscr{T}_S \otimes \mathscr{B}(\mathbb{Y}^\Omega)$-measurable graph, for all $\eta > 0$. Thus,

$$\mathbf{\Upsilon}(t) = \{\eta \in \mathbb{R}_+ : \mathbf{\Theta}_\eta(t) \subseteq \mathbf{\Phi}(t)\} = \{\eta \in \mathbb{R}_+ : \mathbf{\Lambda}_\eta(t) = \emptyset\},$$

where $\mathbf{\Lambda}_\eta : S \rightrightarrows \mathbb{Y}^\Omega$, defined as $\mathbf{\Lambda}_\eta(t) := \mathbf{\Theta}_\eta(t) \cap (\mathbb{Y}^\Omega \setminus \mathbf{\Phi}(t))$, has $\mathscr{T}_S$-measurable graph. Finally, the $\mathscr{T}_S$-measurability of φ follows from the fact that for each $\alpha > 0$, we have

$$\{t \in S : \varphi(t) < \alpha\} = \bigcup_{\eta \in \mathbb{Q} \cap (0,\alpha)} \mathrm{Proj}_S \mathbf{\Lambda}_\eta.$$

For each $\eta \in \mathbb{Q} \cap (0,1)$, define $B_\eta := \{t \in S : \varphi(t) \geq \eta\}$. Thus, $\{B_\eta : \eta \in \mathbb{Q} \cap (0,1)\}$ is family of $\mathscr{T}_S$-measurable sets such that $B_\eta \subseteq B_{\eta'}$ if and only if $\eta \geq \eta'$ and $S \sim \bigcup\{B_\eta : \eta \in \mathbb{Q} \cap (0,1)\}$[8]. Let $\varepsilon \in (0, \mu(S))$. Then there is some $\eta_0 \in \mathbb{Q} \cap (0,1)$ such that $\mu(B_{\eta_0}) > \mu(S) - \varepsilon$. Set $R := B_{\eta_0}$ and note that, for $t \in R$, as $\varphi(t) \geq \eta_0$, we have $\mathbb{B}(0,\eta_0)^\Omega \subseteq \mathbf{\Phi}(t)$. This completes the proof. □

The following lemma on the convexity of vector measure is an application of the infinite dimensional version of the Lyapunov convexity theorem (refer to Uhl (???????)), whose proof can be found in Bhowmik and Cao (2013) and Evren and Hüsenov (2008).

Lemma 5.2. *Consider a continuum economy and assume that $f \in L_1\left(\mu, \mathbb{Y}^\Omega\right)$. Suppose also that S, R are two coalitions of $\mathscr{E}$ such that $\mu(S \cap R) > 0$. Then,*

$$H := \mathrm{cl}\left\{ \left(\mu(B \cap R), \int_B f d\mu \right) : B \in \mathscr{T}_S \right\}$$

is a convex subset of $\mathbb{R} \times \mathbb{Y}^\Omega$. Moreover, for any $0 < \delta < 1$, there is a sequence $\{G_n\}_{n \geq 1} \subseteq \mathscr{T}_S$ such that $\mu(G_n \cap R) = \delta\mu(S \cap R)$ for all $n \geq 1$ and

$$\lim_{n \to \infty} \int_{G_n} f(\cdot,\omega) d\mu = \delta \int_S f(\cdot,\omega) d\mu$$

for all $\omega \in \Omega$.

[8] $C \sim D$ means $\mu(C \Delta D) = 0$, where $C \Delta D = (C \setminus D) \cup (D \setminus C)$.

Proof of Proposition 3.4: It is given that

(i) $V_t(g(t,\cdot)) \geq V_t(f(t,\cdot))$ μ-a.e. on S;

(ii) $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on B and $\mathbb{I}_B = \mathbb{I}_S$; and

(iii) $\int_S g(\cdot,\omega)d\mu = \int_S e(\cdot,\omega)d\mu$ for all $\omega \in \Omega$.

Define $\varphi : T \times \Omega \times (0,1) \to \mathbb{Y}$ by letting $\varphi(t,\omega,\lambda) := \lambda g(t,\omega) + (1-\lambda)e(t,\omega)$. By Lemma 5.26 of Aliprantis and Border (2005), we conclude that $\varphi(t,\omega,\lambda)$ is an interior point of $X_t(\omega)$ for all $(t,\omega,\lambda) \in T \times \Omega \times (0,1)$. Let $\boldsymbol{\Upsilon} : B \rightrightarrows (0,1)$ be a correspondence such that

$$\boldsymbol{\Upsilon}(t) := \{\lambda \in (0,1) : V_t(\varphi(t,\cdot,\lambda)) > V_t(f(t,\cdot))\}.$$

By the continuity of preference, we have $\boldsymbol{\Upsilon}(t) \neq \emptyset$ μ-a.e. on B. Also, $\mathrm{Gr}_{\boldsymbol{\Upsilon}}$ is $\mathscr{T}_B \otimes \mathscr{B}((0,1))$-measurable as $\mathrm{Gr}_{\boldsymbol{\Upsilon}} = \psi^{-1}(\{0\})$, where $\psi : B \times (0,1) \to \mathbb{R}$, defined by $\psi(t,\lambda) := V_t(\varphi(t,\cdot,\lambda)) - V_t(f(t,\cdot))$, is $\mathscr{T}_B \otimes \mathscr{B}((0,1))$-measurable. Therefore, $\boldsymbol{\Upsilon}$ has $\mathscr{T}_B$-measurable selection γ. For each $\lambda \in (0,1) \cap \mathbb{Q}$, define $B_\lambda := \{t \in B : \gamma(t) \geq \lambda\}$. Thus, $\{B_\lambda : \lambda \in \mathbb{Q} \cap (0,1)\}$ is family of $\mathscr{T}_B$-measurable sets such that $B_\lambda \subseteq B_{\lambda'}$ if and only if $\lambda \leq \lambda'$ and $B = \bigcup\{B_\lambda : \lambda \in \mathbb{Q} \cap (0,1)\}$. Let $\varepsilon > 0$ be such that

$$\varepsilon < \min\{\mu(B_i) : i \in \mathbb{I}_B\} \text{ if } T_1 = \emptyset,$$

and

$$\varepsilon < \min\{\mu(B_i) : i \in \mathbb{I}_B\} \text{ and } \varepsilon < \min\{\mu(A_i) : 1 \leq i \leq n\} \text{ if } T_1 \neq \emptyset,$$

where $\{A_1, \cdots, A_n\}$ is the collection of all atoms in T_1. Therefore, there is some $\lambda_0 \in (0,1) \cap \mathbb{Q}$ such that $\mu(B_{\lambda_0}) > \mu(B) - \varepsilon$. Hence, $\mathbb{I}_{B_{\lambda_0}} = \mathbb{I}_B = \mathbb{I}_S$ and $T_1 \subseteq B_{\lambda_0}$. Similar to above, by Lemma 5.1, there are some $\eta > 0$ and a sub-coalition R of B_{λ_0} such that

(a) $\mathbb{I}_R = \mathbb{I}_{B_{\lambda_0}}$ and $T_1 \subseteq R$;

(b) $\varphi(t,\omega,\lambda_0) + z(\omega) \in X_t(\omega)$ for all $z(\omega) \in \mathbb{B}(0,\eta)$ and $(t,\omega) \in R \times \Omega$; and

(c) $V_t(\varphi(t,\cdot,\lambda_0) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on R.

Define $F := S \setminus R$ and note that $\mathbb{I}_F \subseteq \mathbb{I}_S$. Pick an $i \in \mathbb{I}_F$. By Lemma 5.2, there exists a sequence $\{G_n\}_{n\geq 1} \subseteq \mathscr{T}_{F_i}$ such that $\mu(G_n) = \lambda_0 \mu(F_i)$ and for all $\omega \in \Omega$,

$$\lim_{n\to\infty} \int_{G_n} (g(\cdot,\omega) - e(\cdot,\omega))d\mu = \lambda_0 \int_{F_i} (g(\cdot,\omega) - e(\cdot,\omega))\, d\mu.$$

The function $\xi_n : \Omega \to \mathbb{Y}$, defined by

$$\xi_n(\omega) = \lambda_0 \int_{F_i} (g(\cdot,\omega) - e(\cdot,\omega))\, d\mu - \int_{G_n} (g(\cdot,\omega) - e(\cdot,\omega))\, d\mu,$$

satisfies $\xi_n \in \mathscr{G}_i$ for all $n \geq 1$ and $\{\|\xi_n(\omega)\| : n \geq 1\}$ converges to 0 for all $\omega \in \Omega$. Choose an $n_i \geq 1$ such that $\xi_{n_i}(\omega) \in \mathbb{B}\left(0, \frac{\eta\mu(R_i)}{2|\mathbb{I}_F|}\right)$, for all $\omega \in \Omega$. Define $z : R \times \Omega \to \mathbb{Y}$ by letting $z(t,\omega) := \frac{\xi_{n_i}(\omega)}{\mu(R_i)}$ if $(t,\omega) \in R_i \times \Omega$ and $i \in \mathbb{I}_F$; and $z(t,\omega) := 0$, otherwise. Let

$$\mathbb{D} := \bigcap \left\{ \frac{\eta\mu(R)}{2} \mathscr{G}_i : 1 \leq i \leq n \right\} \cap \mathbb{B}\left(0, \frac{\eta\mu(R)}{2}\right)^{\Omega}.$$

Define $C := \bigcup \{G_{n_i} : i \in \mathbb{I}_F\}$. As before, one can find an allocation $\xi : C \times \Omega \to \mathbb{Y}$ such that $\xi(t,\cdot) \in \mathbb{D}$ and $V_t(g(t,\cdot) + \xi(t,\cdot)) > V_t(g(t,\cdot))$ μ-a.e. on C. Define

$$\zeta := \frac{1}{\mu(C)} \int_C \xi d\mu.$$

By Lemma 5 in Shitovitz (1973), one has $\zeta \in \mathbb{D}$, which further implies $\alpha := \zeta\mu(C) \in \mathbb{D}$. Consequently,

$$\beta := \frac{\alpha}{\mu(R)} \in \bigcap \left\{ \frac{\eta}{2} \mathscr{G}_i : 1 \leq i \leq n \right\} \cap \mathbb{B}\left(0, \frac{\eta}{2}\right)^{\Omega}.$$

Finally, we define an assignment $y : T \times \Omega \to \mathbb{Y}$ defined by[9]

$$y(t,\omega) := \begin{cases} \varphi(t,\omega,\lambda_0) - \beta(\omega) + z(t,\omega), & \text{if } (t,\omega) \in R \times \Omega; \\ g(t,\omega) + \xi(t,\omega), & \text{if } (t,\omega) \in C \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

Recognized that y is an $\mathscr{G}$-allocation with $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on $E := C \cup R$. It can be readily verified that

$$\int_E (y(\cdot,\omega) - e(\cdot,\omega)) d\mu = \lambda_0 \int_S (g(\cdot,\omega) - e(\cdot,\omega)) d\mu = 0.$$

For each $t \in R_i$, define

$$\eta_i := \min \{\eta - \operatorname{dist}(0, -\beta(\omega) + z(t,\omega)) : \omega \in \Omega\}.^{[10]}$$

Let $\eta_0 := \min\{\eta_i : i \in \mathbb{I}_R\}$. As a consequence, we have $y(t,\cdot) + \mathbb{B}(0,\eta_0)^{\Omega} \subseteq X_t$ and $V_t(y(t,\cdot) + z) > V_t(f(t,\cdot))$ μ-a.e. on R. This completes the proof. □

Proof of Proposition 3.5: Let $\varepsilon > 0$ be such that $\varepsilon < \min\{\mu(R_i) : i \in \mathbb{I}_R\}$. By Lemma 5.1, one can find an $\eta > 0$ and a sub-coalition C of R such that

[9] $\xi(t,\omega)$ denotes the ω^{th}-coordinate of $\xi(t)$.

[10] Note that $z(\cdot,\omega)$ is constant on R_i.

(i) $\mu(C) > \mu(R) - \varepsilon$;

(ii) $g(t,\omega) + z(\omega) \in X_t(\omega)$ for all $z(\omega) \in \mathbb{B}(0,\eta)$ and $(t,\omega) \in C \times \Omega$; and

(iii) $V_t(g(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^{\Omega}$ and μ-a.e. on C.

It follows from (i) that $\mathbb{I}_C = \mathbb{I}_R = \mathbb{I}_S$ and $\mathbb{I}_{S\setminus C} \subseteq \mathbb{I}_S$. Pick an $i \in \mathbb{I}_C$. By Lemma 5.2, there exists a sequence $\{G_n\}_{n\geq 1} \subseteq \mathscr{T}_{C_i}$ such that $\mu(G_n) = \delta\mu(C_i)$ and for all $\omega \in \Omega$,

$$\lim_{n\to\infty} \int_{G_n} (g(\cdot,\omega) - \psi(\cdot,\omega))d\mu = \delta \int_{C_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu.$$

The function $\xi_n : \Omega \to \mathbb{Y}$, defined by

$$\xi_n(\omega) = \delta \int_{C_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu - \int_{G_n} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu,$$

satisfies $\xi_n \in \mathscr{G}_i$ for all $n \geq 1$ and $\{\|\xi_n(\omega)\| : n \geq 1\}$ converges to 0 for all $\omega \in \Omega$. Choose an $n_i \geq 1$ such that $\xi_{n_i}(\omega) \in \mathbb{B}\left(0, \frac{\eta\delta\mu(C_i)}{2}\right)$ for all $i \in \mathbb{I}_S$ and $\omega \in \Omega$. Similarly, for each $i \in \mathbb{I}_D$ (where $D := S \setminus C$), there is some $F_i \in \mathscr{T}_{D_i}$ such that $\mu(F_i) = \delta\mu(D_i)$ and

$$b_i(\omega) := \delta \int_{D_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu - \int_{F_i} (g(\cdot,\omega) - \psi(\cdot,\omega))\, d\mu \in \mathbb{B}\left(0, \frac{\eta\delta\mu(C_i)}{2}\right).$$

For each $\omega \in \Omega$, define $z_i(\omega) := b_i(\omega)$ if $i \in \mathbb{I}_D$; and $z_i(\omega) := 0$, if $i \in \mathbb{I}_S \setminus \mathbb{I}_D$. Analogously, define

$$K_i := \begin{cases} G_{n_i} \cup F_i, & \text{if } i \in \mathbb{I}_D; \\ G_{n_i}, & \text{if } i \in \mathbb{I}_S \setminus \mathbb{I}_D, \end{cases}$$

and

$$S_i := \begin{cases} C_i \cup D_i, & \text{if } i \in \mathbb{I}_D; \\ C_i, & \text{if } i \in \mathbb{I}_S \setminus \mathbb{I}_D. \end{cases}$$

For each $i \in \mathbb{I}_S$, define a function $\varphi^i : K_i \times \Omega \to \mathbb{Y}$ such that

$$\varphi^i(t,\omega) = \begin{cases} g(t,\omega) + \frac{1}{\delta\mu(C_i)}(\xi_{n_i}(\omega) + z_i(\omega)), & \text{if } (t,\omega) \in G_{n_i} \times \Omega; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

It follows that $\varphi^i(t,\cdot) - e(t,\cdot)$ is $\mathscr{F}_i$-measurable. Furthermore, in the light of (ii) and (iii), we have $\varphi^i(t,\omega) \in X_t(\omega)$ for all $(t,\omega) \in K_i \times \Omega$ and $V_t(\varphi^i(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on K_i. Lastly, note that

$$\int_{K_i} (\varphi^i(\cdot,\omega) - \psi(\cdot,\omega))d\mu = \delta \int_{S_i} (g(\cdot,\omega) - \psi(\cdot,\omega))d\mu$$

for all $\omega \in \Omega$. Let $B := \bigcup\{K_i : i \in \mathbb{I}_S\}$ and

$$\eta_0 := \min\left\{\eta - \operatorname{dist}\left(0, \frac{1}{\delta\mu(C_i)}(\xi_{n_i}(\omega) + z_i(\omega))\right) : i \in \mathbb{I}_R \text{ and } \omega \in \Omega\right\}.$$

Thus, the function $\varphi : T \times \Omega \to \mathbb{Y}$, defined by $\varphi(t,\omega) := \varphi^i(t,\omega)$ for all $(t,\omega) \in T_i \times \Omega$, satisfies the requied properties for the above choices of B, C and η_0. $\square$

Proof of Theorem 3.6: It follows from the fact that $f \notin \mathscr{C}(\mathscr{E})$, there exists a coalition S and an assignment g such that f is blocked by S via g. In view of Proposition 3.4, we can assume that $g(t,\omega)$ is an interior point of $X_t(\omega)$, for all $(t,\omega) \in B \times \Omega$ for some sub-coalition B of S satisfying $\mathbb{I}_B = \mathbb{I}_S$. Pick an $\varepsilon \in (0, \mu(S))$. Let $\delta \in (0,1)$ be such that $\varepsilon = \delta\mu(S)$. By Proposition 3.5 that there are a coalition R and an $\mathscr{G}$-assignment φ such that

(i) $\mu(R) = \delta\mu(S)$;

(ii) $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on R; and

(iii) $\int_R(\varphi(\cdot,\omega) - e(\cdot,\omega))d\mu = \delta \int_S(g(\cdot,\omega) - e(\cdot,\omega))d\mu$ for all $\omega \in \Omega$.

Consequently, $\mu(R) = \varepsilon$ and $\int_R(\varphi(\cdot,\omega) - e(\cdot,\omega))d\mu = 0$ for all $\omega \in \Omega$. This means that f is blocked by R. $\square$

Proof of Proposition 3.7: Let $0 < \delta < 1$. Thus, there are an $\eta_0 > 0$, two coalitions B and C, and an $\mathscr{G}$-assignment φ such that

(i) $C \subseteq B \subseteq S$, $\mathbb{I}_C = \mathbb{I}_B = \mathbb{I}_S$ and $\mu(B) = \delta\mu(S)$;

(ii) $\varphi(t,\omega) + z(\omega) \in X_t(\omega)$ for all $z(\omega) \in \mathbb{B}(0,\eta_0)$ and $(t,\omega) \in C \times \Omega$;

(iii) $V_t(\varphi(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in X_t$ and μ-a.e. on C;

(iii) $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on $B \setminus C$; and

(iv) $\int_B(\varphi(\cdot,\omega) - f(\cdot,\omega))d\mu = \delta \int_S(g(\cdot,\omega) - f(\cdot,\omega))d\mu$ for all $\omega \in \Omega$.

Define

$$\mathbb{D} := \bigcap\{\eta_0\mu(C)\mathscr{G}_i : 1 \le i \le n\} \cap \mathbb{B}\left(0, \eta_0\mu(C)\right)^{\Omega}.$$

As in Lemma 5.1, the correspondence $\mathbf{F} : S \setminus B \rightrightarrows \mathbb{D}$, defined by

$$\mathbf{F}(t) := \{z \in \mathbb{D} : f(t,\cdot) + z \in X_t \text{ and } V_t(f(t,\cdot) + z) > V_t(f(t,\cdot))\},$$

has a $\mathscr{T}_{S\setminus B} \otimes \mathscr{B}(\mathbb{D})$-measurable graph. By our stated assumptions, $\mathbf{F}(t) \neq \emptyset$ for all $t \in S \setminus B$. By the Aumann-Saint-Beuve measurable selection theorem, there is a $\mathscr{T}_{S\setminus B}$-measurable selection ξ of $\mathbf{F}$. Define

$$\zeta := \frac{1}{\mu(S \setminus B)} \int_{S\setminus B} \xi d\mu.$$

By Lemma 5 in Shitovitz (1973), one has $\zeta \in \mathbb{D}$. So, $\varepsilon := \zeta\mu(S \setminus B) \in \mathbb{D}$ and

$$\gamma := \frac{\varepsilon}{\mu(C)} \in \bigcap \{\eta_0 \mathscr{G}_i : 1 \leq i \leq n\} \cap \mathbb{B}\left(0, \eta_0\right)^{\Omega}.$$

Let $h : S \times \Omega \to \mathbb{Y}$ be an assignment such that[11]

$$h(t, \omega) := \begin{cases} \varphi(t, \omega) - \gamma, & \text{if } (t, \omega) \in C \times \Omega; \\ f(t, \omega) + \xi(t, \omega), & \text{if } (t, \omega) \in (S \setminus B) \times \Omega; \\ \varphi(t, \omega), & \text{otherwise.} \end{cases}$$

It is evident that h is an $\mathscr{G}$-assignment and $V_t(h(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on S. It can be easily verified that

$$\int_S h(\cdot, \omega) d\mu = \int_S (\delta g(\cdot, \omega) + (1 - \delta) f(\cdot, \omega)) d\mu$$

for all $\omega \in \Omega$. This completes the proof. □

Proof of Theorem 3.9: Since f is a non-core allocation, there exists a coalition S and an allocation g such that f is blocked by S via g. Then for each $\varepsilon \in (0, \mu(S))$, by Theorem 3.6, there is a coalition R and an allocation φ such that $\mu(R) = \varepsilon$ and f is blocked by R via φ. If $\mu(S) = \mu(T)$, then there is nothing more to verify. Thus, we assume that $\mu(S) < \mu(T)$ and choose an $\varepsilon \in (\mu(S), \mu(T))$. Define

$$\delta := 1 - \frac{\varepsilon - \mu(S)}{\mu(T \setminus S)}.$$

By Lemma 5.1, one can find an $\eta_0 > 0$ and a sub-coalition C of B such that

(A) $\mathbb{I}_C = \mathbb{I}_B$;

(B) $g(t, \omega) + z(\omega) \in X_t(\omega)$ for all $z(\omega) \in \mathbb{B}(0, \eta_0)$ and $(t, \omega) \in C \times \Omega$; and

(C) $V_t(g(t, \cdot) + z) > V_t(f(t, \cdot))$ for all $z \in \mathbb{B}(0, \eta_0)^{\Omega}$ and μ-a.e. on C.

[11] $\xi(t, \omega)$ denotes the ω^{th}-coordinate of $\xi(t)$.

Define

$$\mathbb{D} := \bigcap \{\eta_0 \delta \mu(C) \mathscr{G}_i : 1 \leq i \leq n\} \cap \mathbb{B}\left(0, \eta_0 \delta \mu(C)\right)^{\Omega}.$$

As in Lemma 5.1, the correspondence $\mathbf{F} : T \setminus S \rightrightarrows \mathbb{D}$, defined by

$$\mathbf{F}(t) := \{z \in \mathbb{D} : f(t, \cdot) + z \in \mathrm{int} X_t \text{ and } V_t(f(t, \cdot) + z) > V_t(f(t, \cdot))\},$$

is non-empty valued and has $\mathscr{T}_{T \setminus S} \otimes \mathscr{B}(\mathbb{D})$-measurable graph, which further implies the existence of a $\mathscr{T}_{T \setminus S}$-measurable selection ξ of $\mathbf{F}$. Define

$$\zeta := \frac{1}{\mu(T \setminus S)} \int_{T \setminus S} \xi d\mu.$$

By Lemma 5 in Shitovitz (1973), one has $\zeta \in \mathbb{D}$. So, $\varepsilon := \zeta \mu(T \setminus S) \in \mathbb{D}$ and

$$\gamma := \frac{\varepsilon}{\delta \mu(C)} \in \bigcap \{\eta_0 \mathscr{G}_i : 1 \leq i \leq n\} \cap \mathbb{B}\left(0, \eta_0\right)^{\Omega}.$$

In view of Proposition 3.5 there exist a coalition F and an $\mathscr{G}$-assignment ψ such that

(a) $\mu(F) = (1 - \delta)\mu(T \setminus S)$;

(b) $V_t(\psi(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on F; and

(c) $\int_F (\psi(\cdot, \omega) - e(\cdot, \omega)) d\mu = (1 - \delta) \int_{T \setminus S} (f(\cdot, \omega) + \xi(\cdot, \omega) - e(\cdot, \omega)) d\mu$ for all $\omega \in \Omega$.

Let $\widetilde{g} : T \times \Omega \to \mathbb{Y}$ be an allocation such that

$$\widetilde{g}(t, \omega) := \begin{cases} g(t, \omega) - \gamma(\omega), & \text{if } (t, \omega) \in C \times \Omega; \\ g(t, \omega), & \text{otherwise.} \end{cases}$$

By Proposition 3.7, there exist some $\mathscr{G}$-assignment h such that $V_t(h(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on S, and

$$\int_S h(\cdot, \omega) d\mu = \int_S (\delta \widetilde{g}(\cdot, \omega) + (1 - \delta) f(\cdot, \omega)) d\mu$$

for all $\omega \in \Omega$. We define an assignment $y : T \times \Omega \to \mathbb{Y}$ defined by

$$y(t, \omega) := \begin{cases} \psi(t, \omega), & \text{if } (t, \omega) \in F \times \Omega; \\ h(t, \omega), & \text{otherwise.} \end{cases}$$

Recognized that y is an $\mathscr{G}$-allocation with $V_t(y(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on $E := F \cup S$. It can be readily verified that $\mu(E) = \varepsilon$ and

$$\int_E (y(\cdot, \omega) - e(\cdot, \omega)) d\mu = (1 - \delta) \int_T (f(\cdot, \omega) - e(\cdot, \omega)) d\mu = 0.$$

This completes the proof. $\square$

Proof of Proposition 3.12: Denoting by X_R, $\mathscr{F}_R$, V_R, and $e_R(\cdot)$ the common values of X_t, $\mathscr{F}_t$, V_t, and $e(t,\cdot)$, respectively. Suppose, on contrary, that $V_R(\mathbf{x}_f) > V_R(f(t,\cdot))$ for all $t \in B$ for some sub-coalition B of R. Then there are an $\lambda \in (0,1)$ and a sub-coalition D of B such that

$$V_R(\lambda \mathbf{x}_f + (1-\lambda)e_R) > V_R(f(t,\cdot))$$

for all $t \in D$. By Lemma 5.26 of Aliprantis and Border (2005), we conclude that $\lambda \mathbf{x}_f + (1-\lambda)e_R$ is an interior point of X_R. It follows that there are an $\eta > 0$ and a sub-coalition E of D such that

$$V_{T_1}(\lambda \mathbf{x}_f + (1-\lambda)e_{T_1} - z) > V_{T_1}(f(t,\cdot))$$

for all $z \in \mathbb{B}(0,\eta)^\Omega$ and $t \in E$. Let $\delta \in (0,1]$ be such that $\mu(E) = \delta\mu(R)$. Define

$$\mathbb{D} := \bigcap \{\eta\mu(E)\mathscr{G}_i : 1 \le i \le n\} \cap \mathbb{B}\left(0, \eta\mu(E)\right)^\Omega.$$

As before, one can find an allocation $\xi : T_0 \times \Omega \to \mathbb{Y}$ such that

(i) $\xi(t,\cdot) \in \mathbb{D}$ μ-a.e. on T_0;

(ii) $f(t,\cdot) + \xi(t,\cdot) \in \mathrm{int}X_t$ μ-a.e. on T_0; and

(iii) $V_t(f(t,\cdot) + \xi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on T_0.

By Proposition 3.5, there exists an element $C \in \mathscr{T}_{T\setminus R}$ and an $\mathscr{G}$-assignment φ such that

(A) $\mu(C) = \delta\mu(T \setminus R)$;

(B) $V_t(\varphi(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on C; and

(C) $\displaystyle\int_C (\varphi - e)d\mu = \lambda\delta \int_{T\setminus R} (f + \xi - e)d\mu.$

Define

$$\zeta := \frac{1}{\mu(C)} \int_C \xi d\mu.$$

By Lemma 5 in Shitovitz (1973), one has $\zeta \in \mathbb{D}$, which further implies $\alpha := \zeta\mu(C) \in \mathbb{D}$. Consequently,

$$\gamma := \frac{\alpha}{\mu(E)} \in \bigcap \{\eta\mathscr{G}_i : 1 \le i \le n\} \cap \mathbb{B}\left(0, \eta\right)^\Omega.$$

Finally, we define an assignment $y : T \times \Omega \to \mathbb{Y}$ defined by

$$y(t, \omega) := \begin{cases} \lambda \mathbf{x}_f(\omega) + (1-\lambda) e_R(\omega) - \gamma(\omega), & \text{if } (t,\omega) \in C \times \Omega; \\ \varphi(t,\omega), & \text{otherwise.} \end{cases}$$

It can be readily verified that $S := C \cup E$ blocks f via y. This is a contradiction. Hence, $V_R(f(t,\cdot)) \geq V_R(\mathbf{x}_f)$ μ-a.e. on R. Let B be a sub-coalition of R such that$V_R(f(t,\cdot)) > V_R(\mathbf{x}_f)$ for all $t \in B$. By applying Jensen's inequality, one obtains

$$V_R\left(\frac{1}{\mu(B)} \int_B f d\mu\right) > V_R(\mathbf{x}_f)$$

and

$$V_R\left(\frac{1}{\mu(R \setminus B)} \int_{R \setminus B} f d\mu\right) \geq V_R(\mathbf{x}_f).$$

Let $\lambda = \frac{\mu(B)}{\mu(R)}$. By Lemma 5.28 in Aliprantis and Border (2005), one has

$$\begin{aligned} V_R(\mathbf{x}_f) &= V_R\left(\frac{\lambda}{\mu(B)} \int_B f d\mu + \frac{1-\lambda}{\mu(R \setminus B)} \int_{R \setminus B} f d\mu\right) \\ &> V_R(\mathbf{x}_f), \end{aligned}$$

which is a contradiction. Therefore, $V_R(f(t,\cdot)) = V_R(\mathbf{x}_f)$ μ-a.e. on R. $\square$

Proof of Lemma 3.14: Let $\delta \in (0,1]$ be a number such that $\alpha = \delta\mu(S \cap H)$. Pick an $i \in \mathbb{I}_S$. By Lemma 3.6 in Bhowmik and Cao (2013), there is a sequence $\{C_n^i :\geq 1\} \subseteq \mathscr{T}_{R^i}$ such that $\mu(C_n^i \cap H) = \delta\mu(R^i \cap H)$ for all $n \geq 1$ and $\{x_n^i : n \geq 1\}$ converges to 0 in norm-topology, where

$$x_n^i := \delta \int_{S^i} (g-e) d\mu - \int_{C_n^i} (g-e) d\mu,$$

for all $n \geq 1$. Let $F := S \setminus R$. Analogusly, there is a sequence $\{D_n^i :\geq 1\} \subseteq \mathscr{T}_{F_i}$ such that $\mu(D_n^i \cap H) = \delta\mu(F^i \cap H)$ for all $n \geq 1$ and $\{y_n^i : n \geq 1\}$ converges to 0 in norm-topology, where

$$y_n^i := \delta \int_{F^i} (g-e) d\mu - \int_{D_n^i} (g-e) d\mu,$$

for all $n \geq 1$. By Lemma 5.1, there exists an $\eta > 0$ such that $g(t,\cdot) + z \in X_t$ and $V_t(g(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $t \in R$. Let $n_0 \geq 1$ be such that

$$\frac{1}{\mu(R_i)} \left(x_{n_0}^i + y_{n_0}^i\right) \in \mathbb{B}(0,\eta)^\Omega.$$

Let $G_i := C^i_{n_0} \cup D^i_{n_0}$ for each $i \in \mathbb{I}_S$ and define $B := \bigcup\{G_i : i \in \mathbb{I}_S\}$. Define an allocation $h : T \times \Omega \to \mathbb{Y}$ such that

$$h(t,\omega) := \begin{cases} g(t,\omega) + \frac{1}{\mu(R_i)}\left(x^i_{n_0} + y^i_{n_0}\right), & \text{if } (t,\omega) \in C^i_{n_0} \times \Omega \text{ and } i \in \mathbb{I}_S; \\ g(t,\omega), & \text{otherwise.} \end{cases}$$

It can be easily verified that f is blocked by B via h and $\mu(B \cap H) = \alpha$. □

Proof of Theorem 3.15: Let $\widetilde{f} \in \mathscr{C}(\widetilde{\mathscr{E}})$. Suppose by the way of contradiction that $f \notin \mathscr{C}(\mathscr{E})$. Consequently, there are a coalition S and an $\mathscr{G}$-allocation g such that $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S and

$$\int_S g(\cdot,\omega)d\mu = \int_S e(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$. Define

$$\widetilde{S} := (S \cap T_0) \cup \bigcup\{\widetilde{A}_i : A_i \in S\}.$$

Pick an $i \geq 1$. For a continuum economy $\widetilde{\mathscr{E}}$, by taking $R = \widetilde{A_i}$ in Proposition 3.12, we have $V_t(f(A_i,\cdot)) = V_t(\widetilde{f}(t,\cdot))$ μ-a.e. on $\widetilde{A_i}$. Hence, $\widetilde{f}$ is blocked by $\widetilde{S}$ via $\widetilde{g} := \Xi(g)$, which leads to a contradiction. □

Proof of Theorem 3.16: First, we define $\mathbf{x}_f : \Omega \to \mathbb{Y}$ by letting

$$\mathbf{x}_f(\omega) := \frac{1}{\mu(R)}\int_R f(\cdot,\omega)d\mu$$

for all $\omega \in \Omega$. Thus, consider a feasible allocation $f^A : T \times \Omega \to \mathbb{Y}$ such that

$$f^A(t,\omega) := \begin{cases} f(t,\omega), & \text{if } (t,\omega) \in (T \setminus R) \times \Omega; \\ \mathbf{x}_f(\omega), & \text{otherwise.} \end{cases}$$

In view of Proposition 3.12, we have $V_t(f(t,\cdot)) = V_t(f^A(t,\cdot))$ μ-a.e. on R. Suppose, by the way of contradiction, that $\widetilde{f} \notin \mathscr{C}(\widetilde{\mathscr{E}})$. Recognize that $\widetilde{f^A} = \widetilde{f}^A$ and $V_t(\widetilde{f}(t,\cdot)) = V_t(\widetilde{f^A}(t,\cdot))$ μ-a.e. on R. Thus, $\widetilde{f^A}$ is not in the core of $\widetilde{\mathscr{E}}$.

Case 1. $R = T_1$ and $|T_1| \geq 2$. Choose an element $A_0 \in T_1$ and let $\mu(A_0) = \varepsilon > 0$. By Theorem 3.9, $\widetilde{f^A}$ is blocked by a coalition $\widetilde{B}$ of $\widetilde{\mathscr{E}}$ with $\widetilde{\mu}(\widetilde{B}) = \widetilde{\mu}(T_0) + \varepsilon$, which gives $\widetilde{\mu}(\widetilde{B} \cap \widetilde{T}_1) \geq \varepsilon$. Therefore, in the light of Lemma 3.14, there exist a coalition $\widetilde{E}$ and an assignment $\widetilde{y}$ such that $\widetilde{f^A}$ will be blocked by $\widetilde{E}$ via $\widetilde{y}$ and $\widetilde{\mu}(\widetilde{E} \cap \widetilde{T}_1) = \varepsilon$. Define a coalition S of $\mathscr{E}$ such that $S := (\widetilde{E} \cap T_0) \cup A_0$, and define a function $y : T \times \Omega \to \mathbb{Y}$ by

$$y(t,\omega) = \begin{cases} \widetilde{y}(t,\omega), & \text{if } (t,\omega) \in (T \setminus A_0) \times \Omega; \\ \frac{1}{\varepsilon}\int_{\widetilde{E} \cap \widetilde{T}_1} \widetilde{y}(\cdot,\omega)d\widetilde{\mu}, & \text{otherwise.} \end{cases}$$

Recognized that y is an $\mathscr{G}$-assignment such that $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S and $\int_S y(\cdot,\omega)d\mu = \int_S e(\cdot,\omega)d\mu$ for all $\omega \in \Omega$. This is a contradiction.

Case 2. $\mu(R \setminus T_1) > 0$. Define $C := R \cap T_0$. Since $\widetilde{f^A}$ is not in the core of $\widetilde{\mathscr{E}}$, by Proposition 3.4, we conclude that there are a coalition $\widetilde{B}$ and an assignment $\widetilde{y}$ such that $\widetilde{f^A}$ will be blocked by $\widetilde{B}$ via $\widetilde{y}$ and $\widetilde{y}(t,\cdot)$ is an interior point of X_t for all $t \in \widetilde{G}$ for some sub-coalition $\widetilde{G}$ of $\widetilde{B}$ satisfying $\mathbb{I}_{\widetilde{B}} = \mathbb{I}_{\widetilde{G}}$. If $\widetilde{B} \subseteq T_0$, there is noting more to verify. Thereofore, we assume that $\widetilde{\mu}(\widetilde{B} \cap \widetilde{T}_1) > 0$. Let $\varepsilon := \widetilde{\mu}(\widetilde{B} \cap \widetilde{T}_1)$. Define a function $\widetilde{y^A} : \widetilde{T} \times \Omega \to \mathbb{Y}$ by

$$\widetilde{y^A}(t,\omega) = \begin{cases} \frac{1}{\varepsilon}\int_{\widetilde{B}\cap\widetilde{T}_1} \widetilde{y}(\cdot,\omega)d\widetilde{\mu}, & \text{if } (t,\omega) \in (\widetilde{B} \cap \widetilde{T}_1) \times \Omega; \\ \widetilde{y}(t,\omega), & \text{otherwise.} \end{cases}$$

As $\mathbb{I}_{\widetilde{B}} = \mathbb{I}_{\widetilde{G}}$, one of the following must hold: $\widetilde{\mu}(\widetilde{G} \cap \widetilde{T}_1) > 0$ or $\mu(\widetilde{G} \cap R) > 0$. It follows that $V_{T_1}(\widetilde{y^A}(t,\cdot)) > V_{T_1}(\widetilde{f^A}(t,\cdot))$ μ-a.e. on $\widetilde{B}$ and

$$\int_{\widetilde{B}\cap T_0} (\widetilde{y^A} - e)d\widetilde{\mu} + \varepsilon(\widetilde{y^A} - e_{T_1}) = 0. \tag{5.1}$$

If $\widetilde{\mu}(C) \geq \varepsilon$ then we choose a coalition $\widehat{R} \subseteq C$ such that $\widetilde{\mu}(\widehat{R}) = \varepsilon$. Consequently, by Equation (5.1), we have

$$\int_{\widetilde{B}\cap T_0} (\widetilde{y^A} - e)d\widetilde{\mu} + \widetilde{\mu}(\widehat{R})(\widetilde{y^A} - e_{T_1}) = 0.$$

If $\widetilde{\mu}(C) < \varepsilon$ then first choose an $\alpha \in (0,1)$ such that $\widetilde{\mu}(C) = \alpha\varepsilon$. By Proposition 3.5, there are two coalitions $\widehat{K}$ and $\widehat{D}$ and an $\mathscr{G}$-assignment φ such that $\widehat{K} \subseteq \widehat{D} \subseteq \widetilde{B} \cap T_0$ with $\mathbb{I}_{\widehat{K}} = \mathbb{I}_{\widehat{D}} = \mathbb{I}_{\widetilde{B}\cap T_0}$; $\varphi(t,\omega)$ is an interior point of $X_t(\omega)$ for all $(t,\omega) \in \widehat{K} \times \Omega$; and

$$\int_{\widehat{D}} (\varphi - e)d\widetilde{\mu} = \alpha \int_{\widetilde{B}\cap T_0} (\widetilde{y^A} - e)d\widetilde{\mu}.$$

In view of Equation (5.1), we have $\mathbb{I}_{\widehat{K}\cup C} = \mathbb{I}_{\widehat{D}\cup C} = \mathbb{I}_{\widetilde{B}}$ and

$$\int_{\widehat{D}} (\varphi - e)d\widetilde{\mu} + \widetilde{\mu}(C)(\widetilde{y^A} - e_{T_1}) = 0.$$

Hence, in either of these cases, there are coalitions D, K, R and $\mathscr{G}$-allocation ξ such that $K \subseteq D \subseteq \widetilde{B} \cap T_0$ and $N \subseteq C$ such that $\mathbb{I}_{K\cup N} = \mathbb{I}_{D\cup N} = \mathbb{I}_{\widetilde{B}}$ and

$$\int_D (\xi - e)d\widetilde{\mu} + \widetilde{\mu}(N)(\widetilde{y^A} - e_{T_1}) = 0.$$

If $\widetilde{\mu}(D\cap N)=0$ then $D\cup N$ blocks the allocation $\widetilde{f^A}$ via ζ, where the $\mathscr{G}$-allocation ζ is defined by

$$\zeta(t,\omega)=\begin{cases}\xi(t,\omega), & \text{if } (t,\omega)\in D\times\Omega;\\ \widetilde{y^A}(t,\omega), & \text{otherwise.}\end{cases}$$

If $\widetilde{\mu}(D\cap N)>0$ the define $E:=(D\setminus N)\cup(N\setminus D)$ and $G:=D\cap N$. Recognized that $\zeta(t,\omega)$ is an interior point of $X_t(\omega)$ for all $t\in H$ for some sub-coalition H of $K\cup N$ satisfying $\mathbb{I}_H=\mathbb{I}_{(K\cup N)\cap E}$. By Proposition 3.5, there is some coalition $F\subseteq E$ and an $\mathscr{G}$-allocation h such that

$$\int_F (h-e)d\widetilde{\mu}=\frac{1}{2}\int_E(\zeta-e)d\widetilde{\mu}.$$

By Proposition 3.7, there exist an $\mathscr{G}$-allocation ι and a sub-coalition V of G such that $\mathbb{I}_V=\mathbb{I}_G$, $V_t(\iota(t,\cdot))>V_t(f(t,\cdot))$; and

$$\int_G(\iota-e)d\mu=\frac{1}{2}\int_G(\xi-e)d\mu+\frac{1}{2}\int_G(\widetilde{y^A}-e)d\mu.$$

Then $S:=F\cup G$ blocks the allocation $\widetilde{f^A}$ via ψ, where the $\mathscr{G}$-allocation ψ is defined by

$$\psi(t,\omega)=\begin{cases}h(t,\omega), & \text{if } (t,\omega)\in F\times\Omega;\\ \iota(t,\omega), & \text{otherwise.}\end{cases}$$

This contradicts with the fact that f is in the ex-ante core of $\mathscr{E}$. $\square$

Proof of Theorem 3.11: By definition, any allocation in the Aubin core of $\mathscr{E}^{\text{finite}}$ is also Aubin non-dominated. To show the other direction, let x be a feasible allocation that is Aubin non-dominated in $\mathscr{E}^{\text{finite}}$. Assume by the way of contradiction that x is not in the Aubin core of $\mathscr{E}^{\text{finite}}$. Then there are some Aubin coalition γ and an allocation y such that $V_t(y_i)>V_t(x_i)$ for all $i\in\text{supp}(\gamma)$ and $\sum_{i\in N}\gamma_i y_i=\sum_{i\in N}\gamma_i e_i$. Let $f:=\Xi(x)$, $g:=\Xi(y)$, and define $R_i\subseteq I_i$ such that $\mu(S_i)=\gamma_i$ for all $i\in\text{supp}(\gamma)$. Thus, the coalition $R:=\bigcup\{R_i: i\in\text{supp}(\gamma)\}$ blocks the allocation f via g. This means that f is not in the core of $\mathscr{E}^{\text{cont}}$. By Theorem 3.9, there are some coalition S and an allocation h such that f is blocked by S via h and $\mu(S)>1-\frac{1}{n}$. Recognized that $\delta_i:=\mu(S\cap I_i)>0$ for all $i\in N$, and define $z_i:=\frac{1}{\delta_i}\int_{S\cap T_i}hd\mu$ for all $i\in N$. It follows that $V_i(z_i)>V_i(x_i)$ for all $i\in N$, and $\sum_{i\in N}\delta_i z_i=\sum_{i\in N}\delta_i e_i$. This means that x is Aubin dominated in $\mathscr{E}^{\text{finite}}$, which is a contradiction. $\square$

Proof of Theorem 4.2: Let f be in the ex-ante core of $\mathscr{E}$. Assume by the way of contradiction that f is not $\mathscr{C}_{(\mathscr{T}_0,\mathscr{T}_1)}(\mathscr{E})$-fair. This means that there exist two disjoint elements $S\in\mathscr{T}_0$, $E\in\mathscr{T}_1$ and an $\mathscr{G}$-assignment g such that μ-a.e. on S and for each $\omega\in\Omega$:

(i) $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$; and

(ii) $\displaystyle\int_S (g(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_E (f(\cdot,\omega) - e(\cdot,\omega))d\mu.$

By Proposition 3.4, there are an $\eta_0 > 0$, two coalitions B and R, and an $\mathscr{G}$-assignment $\widetilde{g}$ such that

(i) $R \subseteq B \subseteq S$ and $\mathbb{I}_R = \mathbb{I}_B = \mathbb{I}_S$;

(ii) $\widetilde{g}(t,\cdot) + z \in X_t$ for all $z \in \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on R;

(iii) $V_t(\widetilde{g}(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on R;

(iv) $V_t(\widetilde{g}(t,\cdot)) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta_0)^\Omega$ and μ-a.e. on $B \setminus R$; and

(v) $\int_B (\widetilde{g}(\cdot,\omega) - e(\cdot,\omega))d\mu = \frac{1}{2}\int_S (g(\cdot,\omega) - e(\cdot,\omega))d\mu$ for all $\omega \in \Omega$.

Define $G := S \setminus B$ and

$$\mathbb{D}_i := \bigcap \left\{ \frac{\eta_0 \mu(R_i)}{3} \mathscr{G}_i : 1 \leq i \leq n \right\} \cap \mathbb{B}\left(0, \frac{\eta_0 \mu(R_i)}{3}\right)^\Omega.$$

for each $i \in \mathbb{I}_G$. Pick an $i \in \mathbb{I}_G$. Then there is some $C_i \in \mathscr{T}_{G_i}$ such that

$$b_i := \frac{1}{3}\int_{G_i} (f-e)\, d\mu - \int_{C_i} (f-e)\, d\mu \in \mathbb{B}\left(0, \frac{\eta_0 \mu(R_i)}{3}\right)^\Omega.$$

As in Lemma 5.1, the correspondence $\mathbf{F}_i : C_i \rightrightarrows \mathbb{D}_i$, defined by

$$\mathbf{F}_i(t) := \{ z \in \mathbb{D}_i : f(t,\cdot) + z \in X_t \text{ and } V_t(f(t,\cdot) + z) > V_t(f(t,\cdot)) \},$$

is non-empty valued and has $\mathscr{T}_{C_i} \otimes \mathscr{B}(\mathbb{D}_i)$-measurable graph, which further implies the existence of a $\mathscr{T}_{C_i}$-measurable selection ξ_i of $\mathbf{F}_i$. Define

$$\zeta_i := \frac{1}{\mu(C_i)} \int_{C_i} \xi d\mu.$$

By Lemma 5 in Shitovitz (1973), one has $\zeta_i \in \mathbb{D}_i$. So, $\varepsilon_i := \zeta_i \mu(C_i) \in \mathbb{D}_i$ and

$$\gamma_i := \frac{\varepsilon_i}{\mu(R_i)} \in \bigcap \left\{ \frac{\eta_0}{3} \mathscr{G}_i : 1 \leq i \leq n \right\} \cap \mathbb{B}\left(0, \frac{\eta_0}{3}\right)^\Omega.$$

Let $z_i : S_i \times \Omega \to \mathbb{Y}$ be a function define by $z_i(t,\omega) := \frac{1}{\mu(R_i)}(b_i(\omega) - \gamma_i(\omega))$ if $(t,\omega) \in R_i \times \Omega$ and $i \in \mathbb{I}_G$; and $z_i(t,\omega) := 0$, if $i \in \mathbb{I}_S \setminus \mathbb{I}_G$. Thus,

$$z_i(t,\omega) \in \bigcap \left\{ \frac{\eta_0}{3} \mathscr{G}_i : 1 \leq i \leq n \right\} \cap \mathbb{B}\left(0, \frac{\eta_0}{3}\right)^\Omega$$

for all $(t, \omega) \in S_i \times \Omega$. Consider an $\mathscr{G}$-assignment $\varphi : T \times \Omega \to \mathbb{Y}$ defined by

$$\varphi(t, \omega) = \begin{cases} \widetilde{g}(t, \omega) + z_i(t, \omega), & \text{if } (t, \omega) \in R_i \times \Omega \text{ and } i \in \mathbb{I}_S; \\ \widetilde{g}(t, \omega), & \text{otherwise.} \end{cases}$$

Firstly, note that $V_t(\varphi(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on B. Furthermore, by (ii) and (iii), we have

$$\varphi(t, \omega) + \mathbb{B}\left(0, \frac{\eta_0}{3}\right) \subseteq X_t(\omega)$$

for all $(t, \omega) \in R \times \Omega$.

Case 1. $\mu(S \cup E) = \mu(T)$. By Proposition 3.7, there is an $\mathscr{G}$-assignment h such that $V_t(h(t, \cdot)) > V_t(f(t, \cdot))$ μ-a.e. on B, and

$$\int_B (h - e) d\mu = \frac{2}{3} \int_B (\varphi - e) d\mu + \frac{1}{3} \int_B (f - e) d\mu.$$

Define

$$C := \bigcup \{C_i : i \in \mathbb{I}_G\} \text{ and } K := B \cup C.$$

Let $\psi : T \times \Omega \to \mathbb{Y}$ be an $\mathscr{G}$-assignment such that

$$\psi(t, \omega) = \begin{cases} f(t, \omega) + \xi(t, \omega), & \text{if } (t, \omega) \in C \times \Omega; \\ h(t, \omega), & \text{otherwise.} \end{cases}$$

It can be readily verified that

$$\int_K (\psi - e) d\mu = \frac{1}{3} \int_{S \cup E} (f - e) d\mu = 0,$$

which contradicts with the fact that f is in the ex-ante core of $\mathscr{E}$.

Case 2. $\mu(S \cup E) < \mu(T)$. Define $Q := T \setminus (S \cup E)$. Applying an argument similar to that in the proof of Theorem 3.9, one can show that there exists an element

$$c \in \bigcap \left\{ \frac{\eta_0}{3} \mathscr{G}_i : 1 \leq i \leq n \right\} \cap \mathbb{B}\left(0, \frac{\eta_0}{3}\right)^{\Omega}$$

such that $c = \displaystyle\int_Q \xi d\mu$, where

(i) $\xi(t) \in \bigcap \left\{ \frac{\eta_0}{3} \mathscr{G}_i : 1 \leq i \leq n \right\} \cap \mathbb{B}\left(0, \frac{\eta_0}{3}\right)^{\Omega}$;

(ii) $f(t, \cdot) + \xi(t) \in \text{int} G_t$; and

(iii) $V_t(f(t,\cdot)+\xi(t)) > V_t(f(t,\cdot))$

μ-a.e. Q. By applying Proposition 3.5, one obtains a coalition D and an $\mathscr{G}$-assignment κ such that $V_t(\kappa(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on D and

$$\int_D (\kappa - e)d\mu = \frac{1}{3}\int_Q (f+\xi-e)d\mu$$

for all $\omega \in \Omega$. Let $\widetilde{\varphi} : T \times \Omega \to \mathbb{Y}$ be an allocation such that

$$\widetilde{\varphi}(t,\omega) := \begin{cases} \varphi(t,\omega) - \frac{1}{\mu(R)}c(\omega), & \text{if } (t,\omega) \in R \times \Omega; \\ \varphi(t,\omega), & \text{otherwise.} \end{cases}$$

By Proposition 3.7, there is an $\mathscr{G}$-assignment h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S, and

$$\int_B (h-e)d\mu = \frac{2}{3}\int_B (\widetilde{\varphi}-e)d\mu + \frac{1}{3}\int_B (f-e)d\mu.$$

Let

$$C := D \cup \bigcup \{C_i : i \in \mathbb{I}_G\} \text{ and } K := B \cup C$$

Let $\psi : T \times \Omega \to \mathbb{Y}$ be an $\mathscr{G}$-assignment such that

$$\psi(t,\omega) = \begin{cases} f(t,\omega) + \xi(t,\omega), & \text{if } (t,\omega) \in C \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

Therefore, as in **Case 1**, ψ is blocked by K via ψ. This is a contradiction. $\square$

Proof of Lemma 4.4: For any $i \in \mathbb{I}_S$ and $r \geq 1$, define

$$S_i^r := \left\{ t \in S_i : e(t,\omega) + \mathbb{B}\left(0, \frac{1}{r}\right) \subseteq X_t(\omega) \text{ for all } \omega \in \Omega \right\}.$$

It follows that $\{S_i^r : r \geq 1\}$ is an increasing sequnce of $\mathscr{T}$-measurable sets and $\lim_{r\to\infty} \mu(S_i \setminus S_i^r) = 0$ for all $i \in \mathbb{I}_S$. Pick an interger r_0 such that $\mu(S_i^{r_0}) > \frac{2\mu(S_i)}{3}$ for all $i \in \mathbb{I}_S$. Let $\{\eta_m : m \geq 1\} \subseteq (0,1)$ be a sequence of real numbers converging to 0, and $b \in \mathbb{Y}_{++}$ be such that $b \in \mathbb{B}\left(0, \frac{1}{3r_0}\right)$. Consider a function $h^m : S \times \Omega \to \mathbb{Y}$ defined by

$$h^m(t,\omega) := (1-\eta_m)h(t,\omega) + \eta_m(e(t,\omega) - 2b).$$

Put,

$$B^m := \left\{ t \in S : V_t(h^k(t,\cdot)) > V_t(f(t,\cdot)) \text{ for all } k \geq m \right\}.$$

By ($\mathbf{A}_{??}$) and ($\mathbf{A}_{??}$), the mapping $\xi^k : S \to \mathbb{R}$, defined by

$$\xi^k(t) := V_t(h^k(t,\cdot)) - V_t(f(t,\cdot)),$$

is $\mathscr{T}$-measurable and so is B^m. It is obvious that $\{B^m : m \geq 1\}$ is increasing and $S \sim \bigcup\{B^m : m \geq 1\}$. Define $c := \min\{\mu(S_i) : i \in \mathbb{I}_S\}$, and choose some $\upsilon > 0$ such that

$$\frac{2}{c}\int_R (h-e)d\mu \in \mathbb{B}\left(0, \frac{1}{3r_0}\right)^\Omega$$

for any $R \in \mathscr{T}$ with $R \subseteq S$ and $\mu(R) < \upsilon$. Let $m_0 \geq 1$ be an integer such that $\mu(S \setminus B^{m_0}) < \min\{\upsilon, \frac{d}{4}\}$, where $d := \min\{\mu(S_i^{r_0}) : i \in \mathbb{I}_S\}$. In view of this, we get

$$\mu(S_i^{r_0} \cap B^{m_0}) > \frac{3\mu(S_i^{r_0})}{4} > \frac{\mu(S_i)}{2} \geq \frac{c}{2}$$

and

$$\frac{2}{c}\int_{S_i \setminus B^{m_0}} (h-e)d\mu \in \mathbb{B}\left(0, \frac{1}{3r_0}\right)^\Omega$$

for all $i \in \mathbb{I}_S$, which further implies

$$\frac{1}{\mu(S_i \cap B^{m_0})}\int_{S_i \setminus B^{m_0}} (h-e)d\mu \in \mathbb{B}\left(0, \frac{1}{3r_0}\right)^\Omega.$$

for all $i \in \mathbb{I}_S$. We are ready to choose $\lambda := \eta_{m_0}$. By Lemma 5 in Shitovitz (1973), one has

$$\frac{1}{\mu(S_i \setminus B^{m_0})}\int_{S_i \setminus B^{m_0}} (h-e)d\mu \in \mathscr{G}_i$$

for all $i \in \mathbb{I}_S$. Recognized that $\mu(S_i \setminus B^{m_0}) < \mu(S_i^{r_0} \cap B^{m_0})$ for each $i \in \mathbb{I}_S$. Convexity of $\mathscr{G}_i$ and $0 \in \mathscr{G}_i$ further yield that

$$\frac{1}{\mu(S_i^{r_0} \cap B^{m_0})}\int_{S_i \setminus B^{m_0}} (h-e)d\mu \in \mathscr{G}_i \cap \mathbb{B}\left(0, \frac{1}{3r_0}\right)^\Omega$$

for all $i \in \mathbb{I}_S$. Thus,

$$x_i := \frac{1}{\mu(S_i^{r_0} \cap B^{m_0})}\int_{S_i \setminus B^{m_0}} (h-e)d\mu$$

satisfies $x_i \in \mathscr{G}_i \cap \mathbb{B}\left(0, \frac{1}{r_0}\right)^\Omega$ for all $i \in \mathbb{I}_S$, and thus, by the definition of $S_i^{r_0}$, we have $e(t,\omega) - x_i(\omega) \in X_t(\omega)$ for all $(t,\omega) \in S_i^{r_0} \times \Omega$ and $i \in \mathbb{I}_S$. For all $i \in \mathbb{I}_S$, consider an assignment $g_i : S_i \times \Omega \to \mathbb{Y}$ defined by

$$g_i(t,\omega) := \begin{cases} (1-\lambda)h(t,\omega) + \lambda(e(t,\omega) - x_i(\omega)), & \text{if } (t,\omega) \in (S_i^{r_0} \cap B^{m_0}) \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

It is obvious that $g_i(t,\omega) \in X_t(\omega)$ for all $(t,\omega) \in S_i \times \Omega$ and $g_i(t,\cdot) - e(t,\cdot) \in \mathscr{G}_i$ for all $t \in S_i$. As $g_i(t,\omega) \gg h^{m_0}(t,\omega)$ for all $t \in S_i^{r_0} \cap B^{m_0}$, we have $V_t(g_i(t,\cdot)) > V_t(h^{m_0}(t,\cdot)) > V_t(f(t,\cdot))$ for all $t \in S_i^{r_0} \cap B^{m_0}$. Therefore, $V_t(g_i(t,\cdot)) > V_t(f(t,\cdot))$ for all $t \in S_i$, and

$$\int_{S_i} (g_i - e)d\mu = (1-\lambda) \int_{S_i} (h-e)d\mu.$$

for all $i \in I$. Thus, the assignment $y : T \times \Omega \to \mathbb{Y}$, defined by

$$y(t,\omega) := \begin{cases} g_i(t,\omega), & \text{if } (t,\omega) \in S_i \times \Omega, i \in I; \\ h(t,\omega), & \text{otherwise,} \end{cases}$$

satisfies the required condition. □

Proof of Theorem 4.5: Let f be not in the ex-ante core of $\mathscr{E}$. Suppose, on contrary, that it is not $\mathscr{C}_{(\mathscr{T}_1,\mathscr{T}_0)}(\mathscr{E})$-fair, which means that there exist two disjoint elements $S \in \mathscr{T}_1$, $E \in \mathscr{T}_0$ and an $\mathscr{G}$-assignment g such that μ-a.e. on S and for each $\omega \in \Omega$:

(i) $V_t(g(t,\cdot)) > V_t(f(t,\cdot))$; and

(ii) $\displaystyle\int_S (g(\cdot,\omega) - e(\cdot,\omega))d\mu = \int_E (f(\cdot,\omega) - e(\cdot,\omega))d\mu.$

By Lemma 3.5, there exist $0 < \lambda, \eta < 1$, a sub-coalition R of S with $\mathbb{I}_R = \mathbb{I}_S$ and an $\mathscr{G}$-assignment y such that

(iii) $y(t,\cdot) + z \in X_t$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on R;

(iv) $V_t(y(t,\cdot) + z) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on R;

(v) $V_t(y(t,\cdot)) > V_t(f(t,\cdot))$ for all $z \in \mathbb{B}(0,\eta)^\Omega$ and μ-a.e. on $S \setminus R$; and

(vi) $\displaystyle\int_S (y-e)d\mu = (1-\lambda)\int_S (g-e)d\mu.$

By combining (ii) and (vi), we have

$$\int_S (y-e)d\mu = (1-\lambda)\int_E (f-e)d\mu.$$

This implies that

$$\int_S (y-e)d\mu + \lambda \int_E (f-e)d\mu + \int_{T \setminus E} (f-e)d\mu = 0.$$

As a consequence, we have

$$\frac{1}{2}\int_S (y-e)d\mu + \frac{1}{2}\int_S (f-e)d\mu + \frac{\lambda}{2}\int_E (f-e)d\mu + \frac{1}{2}\int_{T\setminus(S\cup E)} (f-e)d\mu = 0.$$

Applying an argument similar to that in the proof of Theorem 4.2, one can show that there exists a function $\xi : E\times\Omega \to \mathbb{Y}$ such that

(i) $\xi(t,\cdot) \in \bigcap\left\{\frac{\eta\mu(R)}{2}\mathscr{G}_i : 1\le i\le n\right\} \cap \mathbb{B}\left(0, \frac{\eta\mu(R)}{2}\right)^{\Omega}$;

(ii) $f(t,\cdot)+\xi(t,\cdot) \in \mathrm{int}X_t$; and

(iii) $V_t(f(t,\cdot)+\xi(t,\cdot)) > V_t(f(t,\cdot))$

for all $t\in E$. Again, by Lemma 3.5, there exist a sub-coalition B of S with $\mathbb{I}_B = \mathbb{I}_E$ and an $\mathscr{G}$-assignment φ such that

$$\int_B (\varphi - e)d\mu = \frac{\lambda}{2}\int_E (f+\xi-e)d\mu.$$

Define $c := \int_E \xi d\mu$ and note that

$$c \in \bigcap\left\{\frac{\eta\mu(R)}{2}\mathscr{G}_i : 1\le i\le n\right\} \cap \mathbb{B}\left(0, \frac{\eta\mu(R)}{2}\right)^{\Omega}.$$

It follows that

$$\gamma := \frac{c}{\mu(R)} \in \bigcap\left\{\frac{\eta}{2}\mathscr{G}_i : 1\le i\le n\right\} \cap \mathbb{B}\left(0, \frac{\eta}{2}\right)^{\Omega}.$$

Consider an $\mathscr{G}$-assignment $\widetilde{y} : T\times\Omega \to \mathbb{Y}$ defined by

$$\widetilde{y}(t,\omega) = \begin{cases} y(t,\omega)-\gamma(\omega), & \text{if } (t,\omega)\in R\times\Omega; \\ y(t,\omega), & \text{otherwise.} \end{cases}$$

It is obvious that $V_t(\widetilde{y}(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on S. Furthermore, by (ii) and (iii), we have

$$\widetilde{y}(t,\omega) + \mathbb{B}\left(0, \frac{\eta}{2}\right) \subseteq X_t(\omega)$$

for all $(t,\omega)\in R\times\Omega$.

Case 1. $\mu(S\cup E) = \mu(T)$. By Proposition 3.7, there exists an $\mathscr{G}$-assignment h such that $V_t(h(t,\cdot)) > V_t(f(t,\cdot))$ μ-a.e. on B, and

$$\int_S (h-e)d\mu = \frac{1}{2}\int_S (\widetilde{y}-e)d\mu + \frac{1}{2}\int_S (f-e)d\mu.$$

Let $\psi : T \times \Omega \to \mathbb{Y}$ be an $\mathscr{G}$-assignment such that

$$\psi(t,\omega) = \begin{cases} \varphi(t,\omega), & \text{if } (t,\omega) \in B \times \Omega; \\ h(t,\omega), & \text{otherwise.} \end{cases}$$

It can be readily verified that $K := B \cup S$ blocks f via ψ, which leads to a contradiction.

Case 2. $\mu(S \cup E) < \mu(T)$. As in the proof of **Case 2** of Theorem 4.2, one can derive a contradiction. □

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