Specialized Allocation of Public Goods with Incomplete Information

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Abstract

This study examines allocation of public goods in network with incomplete information. We consider specialized equilibrium where agents either make the entire contribution to provide a particular good or free rides. We assume that the agents connected to others via a social network receive some information regarding the chances of whether neighbours specialize or not. This information is termed as signals. We further assume that agents can observe the signals received by their neighbours. Each agent is categorized as persuaded or non-persauded based on whether they follow their neighbours' signals or their own signals. While an agent can observe its number of neighbours, it cannot observe whether the neighbours are persuaded ot not. On the basis of this incomplete information, agents decide whether to contribute or not. We see that allocation of goods depends on the information received as well as on the network structure at equilibrium. We also take examples of some specific signal structures and analyse the the network structure which brings about allocation as well as efficient allocation of goods to each agent. Lastly, a comparison of the welfare is made when agents cannot observe the signals of the neighbours directly but can estimate them from a uniformly distributed function.

Keywords: Social Network, Public goods, Information

JEL Classification Codes: C72, C44, D85

1 Introduction

This paper takes a look at allocation of public goods in network modelled by Bramoulli and Kranton (2007) [1] in a different light. We look only into specialized equilibrium of the model. Instead of simultaneously choosing the level of contribution in the game to maximize payoff, agents acts upon signals recived by an independent authority. This signal gives the probability of a neighbour specializing at equilibrium. Each agent receives a private signal. Agents can observe its own signal and can also observe the signals received by their neighbours. An agent contributes only if it estimates that none of the neighbours make any contribution towards the provision of goods. This estimation is done differently by the agents based on the category they belong to. An agent can only follow its private signal (known as non-persuaded agent) or it can only follows the neighbourhood signal (known as persuaded agents). An agent is aware of its own category but not the category of its neighbours. Hence, informaion about the network structure is incomplete.

The paper proceedes as follows: The first section deals with the model specification which is followed by conditions for Bayesian Equilibrium of the model. The relation of network structure with allocation as well as efficient allocation of goods is also mentioned in this section. The next section deals with signal structures that make allocation of goods possible in equilibrium. After that two special cases are taken: when all the agents of the model are only of persuaded type or non-persuaded type and equilibirum conditions for allocation is studied. Lastly, we make a comparison of the welfare of this model when communication is restricted between the agents so that agents cannot observe the private signals received by its neighbours.

2 The Model

- There are n agents $N = \{1, \ldots, n\}$ connected by a network. Agent $i's$ neighbourhood is denoted by N_i. Every agent can see the number of neighbours they are connected to but not the entire network. The number of neighbours an agent is connected to is known as its degree and is denoted by d_i. We assume that there are no isolated nodes in the network.
- Similar to Bramouli and Kranton (2007) model [1], agent receives their

benefit from own and neighbour's effort according to a twice differentiable strictly concave benefit function $b(e)$. There is also a cost associated with effort which is denoted by c. The total effort e^* required for the good to be allocated is given by $b'(e^*) = c$, which is the utility maximizing effort. Therefore, we can say that atleast e^* effort is required for the good to be allocated. We assign the value of $b(e^*) = 1$. We further assume that $0 \leq e^*, c \leq 1$, so that $0 \leq ce^* \leq 1$.

- Agents can chose efforts e_i from the set $\{0, e*\}$. That is we assume that agents can either specialize or free ride. In this regard, we are only considering the specilaized euilibrium that has been mentioned in [1]. Therefore, the cost of effort will either be 0 or ce^* depending on whether or not the agent specialize.

- The total effort given by agent i's neighbours are denoted by $\bar{e}_i$. Therefore, the total payoff that an agent recieves is equal to 1 if the good is allocated (i.e $e_i + \bar{e}_i \geq e^*$) and equal to zero if the good is not allocated (i.e $e_i + \bar{e}_i < e^*$).

- Each agent receives a signal s_i from a central agency which gives a probability value that none of the neighbours would specialize in the game i.e $s_i = Pr(\bar{e}_i = 0)$. The set of signal is denoted by S. Further, we also assume that signals are informative which means that if $s_i < s_j$, then $Pr(\bar{e}_i = 0) < Pr(\bar{e}_j = 0)$.

- Every agents receives a signal and since the neighbourhood is visible to the agents, neighbours' signals are also visible. However, there are some agents that follow only the private signal that they receive and others who follow only their nighbourhood signals. Let us call these types as non-persuaded and persuaded agents as in agents who are either persuaded by their neighbours or not. This categorization is similar to the one carried out in the paper [2]. We denote the set of persuaded agents as P and set of non-persuaded agents as P'.

- We further assume that even though agents can see their neighbour's signals, they cannot observe their neighbour's type. Therefore the network structure that consists of degree and types of agents is not complete information.

3 Equilibrium

In this section, we talk about the game that the agents play and the type of Equilibrium that the model achieves. Since, information about specialization and the network is not perfect, we get a Bayesian Nash Equilibrium.

Definition 1 *A strategy profile σ is pure strategy Bayesian Nash Equilibrium of this game if for each agent i, σ_i maximizes the expected payoff of the agent given strategies of other agents.*

Probability that good is allocated is given by $Pr(e_i + \bar{e}_i \geq e^*)$ and the probability that the good is not allocated is given by $Pr(e_i + \bar{e}_i < e^*)$.

When an agent specialize, the good is allocated to the agent. However, when the agent free rides, the good is allocated only if atleast one of its neighbour specializes. Probability that atleast one of the neighbours specialize is given by $Pr(\bar{e}_i \geq e^*)$ and the probability none of the neighbour specialize is given by $Pr(\bar{e}_i < e^*)$. Since, agents can only play specialisation or play zero; $Pr(\bar{e}_i < e^*) = Pr(\bar{e}_i = 0)$.

Each agent would try to maximize their utility. The expected utility of agent i is given by: Probability that a good is allocated X $(1 - ce_i)$ + Probability that a good is not allocated $(-ce_i)$. e_i can take the value 0 or e^*.

Therefore, expected utility of an agent playing effort equal to e^*:

$$U_{e^*} = 1 - ce^*$$

Since, the agent specialize, the good will be allocated to the agent and the cost incurred by it is equal to ce^*. The expected utility of an agent playing effort equal to 0 is given by:

$$U_0 = Pr(\bar{e}_i \geq e^*)$$

Therefore, an agent i plays e^*, when

$$\begin{aligned} & 1 - ce^* \geq Pr(\bar{e}_i \geq e^*) \\ \implies & Pr(\bar{e}_i < e^*) \geq ce^* \\ \implies & Pr(\bar{e}_i = 0) \geq ce^* \end{aligned}$$

3.1 Allocation of goods in equilibrium

Each agent receives a private signal $s_i = Pr(\bar{e}_i = 0)$ which is visible oly to the neighbours. Since the values of c and e^* as known to all, agents can observe whether the signals received are greater or less than ce^*. The total effort required for the allocation of the good is $e = e_i + \bar{e}_i$. Agents do not know the value of $\bar{e}_i$ but can get an idea from the signal recieved. Recall from earlier that the agent specialize when $Pr(\bar{e}_i = 0) \geq ce^*$. Therefore, for an non-persuaded agent, this condition becomes:

$$s_i \geq ce^*$$

So, non-persuaded agents at equilibrium free rides when $0 \leq s_i < ce^*$ and specialize when $ce^* \leq s_i \leq 1$.

Let $b_i = \max_{j \epsilon N_i} s_j$. Therefore b_i denotes the maximum value of neihbourhood signal for an agent i. Therefore for a persuauded agent, $Pr(\bar{e}_i = 0)$ can take the value 1 or zero when $b_i \geq ce^*$ or $b_i < ce^*$ respectively. A persuauded agent free rides when $b_i \geq ce^*$ because it beleives that atleast one of the neighbours specialize in this case; and it specializes when $b_i < ce^*$ beacuse it belibves that none of the neihbours specialize in that case. These equilibrium profile leads to our first proposition regarding the existence of Bayesian Nash Equilibrium of the model.

Proposition 1 *For any signal structure $s_i \in S$, Bayesian Nash Equibrium of the model would exist as long as $0 \leq s_i \leq 1$.*

The proof of this is quite straight forward. $0 \leq s_i \leq 1$ makes $Pr(\bar{e}_i)$ a properly defined distribution. Therefore, as long $(0 \leq s_i \leq 1)$, we can see that Bayesian Nash Equilibrium will always exists for the game.

One of the result of Bramoulli and Kranton (2007) [1] paper was that the set of specialized agents form a maximal independent set in equilibrium. However, unlike the paper of Bramoulli and Kranton, we see here the set of specialized agents do not form a maximal independent set. In fact equilibrium is very trivially attained in this model. So, instead we focus on the provision of good to all agents. However, before moving forward, we provide the following definitions.

Definition 2 *A dominating set D in a network is such that every vertex not in D is adjacent to atleast one member of D.*

Proposition 2 *Under a signal structure, an equilibrium of the game will allocate goods to every agent if and only if the set of specialist is a dominating set.*

$$
\begin{aligned}
& Pr(e_i + \bar{e}_i \geq e^*) = 1 \\
\Longrightarrow\ & Pr(e_i + \bar{e}_i \geq e^* | e_i = 0).Pr(e_i = 0) + Pr(e_i + \bar{e}_i \geq e^* | e_i = e^*).Pr(e_i = e^*) = 1 \\
\Longrightarrow\ & Pr(e_i + \bar{e}_i \geq e^* | e_i = 0).Pr(e_i = 0) + 1.Pr(e_i = e^*) = 1 \\
\Longrightarrow\ & Pr(e_i + \bar{e}_i \geq e^* | e_i = 0).Pr(e_i = 0) - Pr(e_i = 0) = 0 \\
\Longrightarrow\ & Pr(e_i = 0).[1 - Pr(e_i + \bar{e}_i \geq e^* | e_i = 0)] = 0 \\
\Longrightarrow\ & Pr(e_i = 0).[1 - Pr(\bar{e}_i \geq e^* | e_i = 0)] = 0
\end{aligned}
$$

Note: $Pr(e_i + \bar{e}_i \geq e^* | e_i = e^*) = 1$ since $e_i = e^*$ ensures that the good is allocated.

Therefore, either of the conditions $Pr(e_i = 0) = 0$ i.e $Pr(e_i = e^*) = 1$ or $Pr(\bar{e}_i \geq e^* | e_i = 0) = 1$ needs to be satisfied for allocation of goods to all agents. Let D be the set of specialized agents. We need to prove that set D needs to be dominating in order to provide goods to every agent. From the condition of allocation of goods, we see that either an agents needs to specialize (i.e belong to set D) or else if it free rides, it should have at least one neighbours who specilaize (i.e the agent should be connected to atleast one agent in set D). Therefore, under provision of goods, D must be a dominating set.

Now we come to the other part of the proof. Let the dominating set denoted by D be the set of specilaized agents. We know that for agent i who is in the dominating set, $e_i = e^*$ i.e $Pr(e_i = e^*) = 1$. Any agent i who is not in dominating set, $e_i = 0$ must be connected to atleast one agent from D i.e must have atleast one neighbour who specilaizes, which means $Pr(\bar{e}_i \geq e^* | e_i = 0) = 1$. This proofs that when a set of specialized agents forms a dominating set, provision of goods is ensured. In the model of [1], equilibrium would automatically allocate goods to every agent. However, this is not true in this model. Allocation of goods would depend on the signal structure as well as the network structure which we pursue further.

3.2 Efficient allocation of goods

This can be treated as welfare. Among the equilibrium which allocates the good to every agent, we consider the efficient allocations where our cost is minimized. Efficient allocation of goods takes place when there is welfare maximization in equilibrium. For this, firstly we give the defination of welfare and then find the conditions under which welfare is maximized in equilibrium.

Definition 3 *A dominating set D in a network is minimal dominating set when D is not a proper subset of any other dominating set*

The expected welfare of the model is as follows:

$$\sum_{i=1}^{n}[(1-ce_i)Pr(e_i+\bar{e}_i \geq e^*) + (-ce_i)Pr(e_i+\bar{e}_i < e^*)]$$

$$\implies \sum_{i=1}^{n}[Pr(e_i+\bar{e}_i \geq e^*) - ce_i]$$

The maximum value of $Pr(e_i + \bar{e}_i \geq e^*)$ for every agent is 1 which occurs when every individual is allocated the good. Since $ce^* \leq 1$, the welfare is maximized when $Pr(e_i + \bar{e}_i \geq e^*) = 1$. Therefore, in order to maximize welfare the cost needs to be minimized. If every individual can access the good, then the benefit that every individual receives is 1. For the entire network, the total benefit would be equal to n. The cost incurred by the agent would be equal to ce^* when the agent specialize and zero when they free ride. Welfare of the model when good is provided to every agent is $n - ce^*$(number of agents specialising at equilibrium). Therefore to minimize the cost, we need to minimize the number of agents specialising at equilibrium such that goods are allocated to every individual. **Given any network structure under any signal structure, welfare maximization takes place when the set of specilaized agents form a minimal dominating set.**

4 Bayesian game in a dyad

Consider two agents with index i and j be called by their respective index. They are connected via a dyad as per the three following cases. We assume that agents do not know before hand whether others are persuaded or not:

- Let $s_i, s_j > ce^*$. There are three scenarios that can happen here:

  1. if $i, j \in P$, then $e_i = e_j = 0$.
  2. if $i, j \in P'$, then $e_i = e_j = e^*$.
  3. if $i \in P$ and $j \in P'$, then $e_i = 0$ (because i thinks that j will play e^*) and $e_j = e^*$.

  In this case, equilibrium is achieved in all three scenarios (1,2,3). Allocation of goods happen in scenario 2 and 3.

- Let $s_i, s_j < ce^*$. There are three scenarios that can happen here:

  1. if $i, j \in P$, then $e_i = e_j = e^*$.
  2. if $i, j \in P'$, then $e_i = e_j = 0$.
  3. if $i \in P$ and $j \in P'$, then $e_i = e^*$ and $e_j = 0$.

  In this case, equilibrium is achieved in all three scenarios (1,2,3). Allocation of goods happen in scenario 1 and 3.

- Let $s_i > ce^*$ and $s_j < ce^*$. There are three scenarios that can happen here:

  1. if $i, j \in P$, then $e_i = e^*$ and $e_j = 0$.
  2. if $i, j \in P'$, then $e_i = e^*$ and $e_j = 0$.
  3. This scenario can be divided into two parts:
     - if $i \in P$ and $j \in P'$, then $e_i = e^*$ and $e_j = 0$.
     - if $i \in P'$ and $j \in P$, then $e_i = e^*$ and $e_j = 0$.

  In this case, equilibrium is achieved in all three scenarios (1,2,3). Allocation of goods happens in all three scenarios. We also see that in this particular case, the categorization of agents in persauded and non-persuaded is not required to reach equilibrium.

5 Signal Structure and Allocation

In this section we see how allocation of goods is possible under some conditions. However, before proceeding further, we need to define the following with respect to any network.

Definition 4 *A set of vertices D in a graph is an independent set if no two vertices in the D are linked.*

Definition 5 *A maximal independent set D in a network is defined as an independent set such that it is not a proper subset of any other independent set.*

As mentioned before, a central agency is responsible for sending signals to the individuals. We assume that the network structure is visible to this agency but the position of P and P' is not. The signals are a function of network structure. On this basis, we have the following propositions.

Proposition 3 *For any network with no isolated nodes, when the set of agents receiving signals greater than ce^* forms a maximal independent set, then the equilibrium allocates goods to every agent.*

Let the set of agents receiving signals greater than ce^* be denoted by $\mathcal{D}$. Any maximal independent set is also a dominating as well as an independent set. If we show that agents in D specilaizes and agents in $N - D$ free rides, by Proposition 2, it implies allocation of goods to every agent.

Since D is independent, for any agent i in D (i.e any agent receiving $s_i > ce^*$), $N_i \cap D = \emptyset$. This means that the maximum signal from its neighbour is less than ce^*. Therefore for any agent $i \in D$ in the population, $Pr(s_i > ce^*, b_i > ce^*) = 0$. Since, $Pr(s_i > e^*) = Pr(s_i > ce^*, b_i < ce^*) + Pr(s_i > ce^*, b_i > ce^*)$ this further implies,

$$Pr(s_i > e^*) = Pr(s_i > ce^*, b_i < ce^*) \tag{1}$$

Further, since D is dominant, any agent i in $N - D$ i.e any agent i with $s_i < ce^*$ must be linked to atleast one agent in D i.e an agent with $s_i > ce^*$. It means for agent i, b_i has to be greater than ce^*. This implies that for any agent $Pr(s_i < ce^*, b_i < ce^*) = 0$.

Since, $Pr(s_i < ce^*) = Pr(s_i < ce^*, b_i < ce^*) + Pr(s_i < ce^*, b_i > ce^*)$ this further implies,

$$Pr(s_i < e^*) = Pr(s_i < ce^*, b_i > ce^*) \tag{2}$$

Similarly, using $Pr(b_i < ce^*) = Pr(s_i < ce^*, b_i < ce^*) + Pr(s_i > ce^*, b_i < ce^*)$ and the fact that D is dominant (which implies $Pr(s_i < ce^*, b_i < ce^*) = 0$), we get

$$Pr(b_i < e^*) = Pr(s_i > ce^*, b_i < ce^*) \tag{3}$$

Using Equation 1, for any agent $i \in D$, we need to prove that it specializes i.e., $Pr(e_i = e^*|i \in D) = 1$. Therefore,

$$\begin{aligned} &Pr(e_i = e^*|i \in D) \\ =&Pr(e_i = e^*|s_i > ce^*, b_i < ce^*) \quad \text{[By using Equation 1]} \end{aligned}$$

An agent i specializes if it belongs to P and $s_i > ce^*$ or if i belongs to set P' and $b_i < ce^*$. Therefore the above expression becomes

$$\begin{aligned} &Pr(i \in P) + Pr(i \in P') \\ &= 1 \end{aligned}$$

Therefore, an agent which belongs to set D specializes. This proves one part of the proposition.

Using relation 2, for any agent in the set $N - D$, we get $Pr(s_i < e^*) = Pr(s_i < ce^*, b_i > ce^*) = 1$. Next, we need to prove that $Pr(e_i = 0|i \in N - D) = 1$ to show that agents receiving signals less than ce^* free rides.

$$\begin{aligned} &Pr(e_i = 0|i \in N - D) \\ =&Pr(e_i = 0|s_i < ce^*, b_i > ce^*) \quad \text{[By using Equation 2]} \end{aligned}$$

An agent i belonging to set P free rides because private signal $s_i < ce^*$ induces $i \in P$ to play 0. Similarly, an agent i belonging to set P' free rides because $b_i > 0$ induces $i \in P'$ to play 0. Therefore the above expression becomes,

$$Pr(i \in P) + Pr(i \in P')$$
$$= 1$$

This implies, an agent which belongs to set $N - D$ free rides. This completes the proof.
Since $N - D$ is dominated by D, it can be said that for any agent in $N - D$, $Pr(\bar{e_i} \geq e^*|e_i = 0) = 1$. This satisfies the condition of the allocation of goods i.e either $Pr(e_i = e^*) = 1$ or $Pr(\bar{e_i} \geq e^*|e_i = 0) = 1$.

Since maximal independent set is also a minimal dominating set, it implies that equilibrium in any network under this signal structure maximizes the welfare as well.

Here the set of specialized agents form a maximal independent set. This is similar to the result in [1]. Since, there exists a maximal independent set for every network, therefore, there exists atleast one signal structure which ensures allocation of goods to every agent. However, this is not an if and only if scenario like mentioned in [1]. The following example demonstrates this.

Example

Consider a star network. It has two different maximal independent set : one which consists of only the centre and other with only the peripheries. Without any loss of generality, let us assume that the centre receives signal greater than ce^* and all the peripheries receive signal less than ce^*. This satisfies the condition of Proposition 3.
For the centre i, $s_i > ce^*$ and $b_i < ce^*$. So,

$$Pr(e_i = e^*) = Pr(i \in P) + Pr(i \in P') = 1$$

For any periphery j, $s_i < ce^*$ and $b_i > ce^*$. So,

$$Pr(e_i = 0) = Pr(i \in P) + Pr(i \in P') = 1$$

So, we see that taking a star as an example that good is allocated to every agent when condition of Proposition 3 is satisfied.
However, note that this is not an if and only if condition. Allocation can occur in the same network but under some other signal structure. For example, when all agent of the same network is of type P', then even under the signal structure where every agent receives signal greater than ce^*, goods are allocated to everyone. Even so this is the only signal specification where allocation is guaranteed under any given network structure. In fact, we prove the statement next.

Proposition 4 *When the agents receiving signal greater than ce^* is not a maximal independent set, then there exist a network structure where allocation do not take place.*

Let D be the set of agents receiving sinal greater than ce^*. Since D is not maximal independent then D is either not independent or not dominant. In this proof, we take these two conditions and show that there exist atleast one network structure where allocation does not take place.

- **Case 1 :** When D is not independent, then there exist atleast two agents in D who are connected. Keeping every aspects of the network same, we just allow two agents in D to be linked. Let those agents be i and j such that $\{i, j\}$ belong to set P. Since $\{i, j\} \in D$, $s_i, s_j > ce^*$ and i, j are connected, then $b_i, b_j > ce^*$. Therefore, both agents i and j free rides and allocation does not take place.

- **Case 2 :** Assume D is not dominant. Keeping every aspects of the network same, we just allow one agent in N - D who is not connected to any agent in D. Let that agent be i and belong to P'. For agent i, $s_i < ce^*$ and $b_i < ce^*$ since it is not connected to any agent in D. Therefore, agent i free rides and allocation does not take place.

Combining Proposition 3 and Proposition 4, we can say that given any network structure, allocation of goods is guaranteed if and only if agents receiveing signals greater than ce^* forms a maximal independent set. This signal structure further leads to welfare maximising equilibrium.

Next we look at another signal structure such that if there is a common neighbour between two agents i, j then $s_i = s_j$. We look at the equilibrium profile of the model with this signal structure. However, before that we proof two lemmas.

Lemma 1 *For a network with no isolated nodes receiving signals s_i such that $0 \leq s_i \leq ce^*$, all agents belonging to set P specializes and all agents belonging to set P' free rides.*

Since all agents receives signals $0 \leq s_i \leq ce^*$, for every agent $s_i < ce^*$ and $b_i < ce^*$ which implies $Pr(s_i > ce^*, b_i < ce^*) = Pr(s_i > ce^*, b_i > ce^*) = Pr(s_i < ce^*, b_i > ce^*) = 0$.
Therefore, for any agent i,

$$Pr(s_i < ce^*, b_i < ce^*) = 1 \tag{4}$$

In order to prove this lemma, we need to show that $Pr(e_i = e^*|i \in P) = 1$ and $Pr(e_i = 0|i \in P') = 1$. For any agent in P, we have:

$$\begin{aligned} & Pr(e_i = e^*|i \in P) & \\ =& Pr(b_i < ce^*) & \text{using equilibrium condition} \\ =& Pr(s_i < ce^*, b_i < ce^*) & \text{since } Pr(s_i > ce^*, b_i < ce^*) = 0 \\ =& 1 & \text{using Equation 4} \end{aligned}$$

Similarly, for any agent in P', we have:

$$\begin{aligned} & Pr(e_i = 0|i \in P') & \\ =& Pr(s_i < ce^*) & \text{using equilibrium condition} \\ =& Pr(s_i < ce^*, b_i < ce^*) & \text{since } Pr(s_i < ce^*, b_i > ce^*) = 0 \\ =& 1 & \text{using Equation 4} \end{aligned}$$

Therefore, all the agents belonging to P specializes when $0 \leq s_i \leq ce^*$ and agents belonging to P' free rides.

Lemma 2 *For a network with no isolated nodes receiving signals s_i such that $ce^* \leq s_i \leq 1$, all agents belonging to set P' specializes and all agents belonging to set P free rides.*

Since all agents receive signals $ce^* \leq s_i \leq 1$, for every agent $s_i > ce^*$ and $b_i > ce^*$ which implies $Pr(s_i < ce^*, b_i < ce^*) = Pr(s_i < ce^*, b_i > ce^*) =$

$Pr(s_i > ce^*, b_i < ce^*) = 0$.
Therefore, for any agent i,

$$Pr(s_i > ce^*, b_i > ce^*) = 1 \tag{5}$$

In order to prove this lemma, we need to show that $Pr(e_i = e^*|i \in P') = 1$ and $Pr(e_i = 0|i \in P) = 1$. For any agent in P', we have:

$$\begin{aligned}
&Pr(e_i = e^*|i \in P') & \\
=&Pr(s_i > ce^*) & \text{using equilibrium condition} \\
=&Pr(s_i > ce^*, b_i > ce^*) & \text{since } Pr(s_i > ce^*, b_i < ce^*) = 0 \\
=&1 & \text{using Equation 5}
\end{aligned}$$

Similarly, for any agent in P, we have:

$$\begin{aligned}
&Pr(e_i = 0|i \in P) & \\
=&Pr(b_i > ce^*) & \text{using equilibrium condition} \\
=&Pr(s_i > ce^*, b_i > ce^*) & \text{since } Pr(s_i < ce^*, b_i > ce^*) = 0 \\
=&1 & \text{using Equation 5}
\end{aligned}$$

Therefore, all the agents belonging to P' specializes when $0 \leq s_i \leq ce^*$ and agents belonging to P free rides.

Proposition 5 *When the signal structure of the model is such that all neighbours of any agent recieves the same signal, at equilibrium a network with no isolated nodes consists only of one or more of the following components:*

- *a component where every agent belonging to set P specializes, whereas every agent of P' free rides.*
- *a component where every agent belonging to set P' specializes, whereas every agent of P free rides.*
- *a component where the goods is allocated to every agent.*

When the signal structure is as mentioned in the proposition, there are only three categories of components possible in the network. To understand this, let us take an agent i in the network which receives a signal less than ce^*. There are two possibilities now :

1. All $j \in N_i$ receive signals less than ce^*. Since $i \in N_j$, all agents in N_j also receive the same signal as i which will be less than ce^*. In this case, all agents connected to i receives signal less than ce^*. This implies the entire componenent which includes agent i would receive signal less tha ce^*.

2. All $j \in N_i$ receive signals greater than ce^*. Since $i \in N_j$, all agents in N_j receive the same signal as i which will be less than ce^*. Therefore in this case, if an agents receives signal less than ce^*, all its neighbours receive signal greater than ce^* and all its neighbours of neighbours receive signal less than ce^* and so on. This continues for all agents connected to agent i. Therefore, all agents receiving signal greater than ce^* forms an independent as well as dominant set in this component. This implies, all agents receiving signal greater than ce^* forms a maximal independent set. It is also to note that agents receiving signal less than ce^* also forms a maximal independent set. However for the purpose of our proof, we require only the agents receiving signals greater than ce^* to be a maximal independent set.

Similarly, if we take an agent who receives signal greater than ce^* and repeat the process, we get either a component where all agents receive signal greater than ce^* or a component where set of agents receiving signal greater than ce^* forms a maximal independent set. Now we can categorize all possible components of the network as follows :

- Category 1 : All agents in the component receive signals less than ce^*.
- Category 2 : All agents in the component receive signals greater than ce^*.
- Category 3 : All agents receiving signal greater than ce^* in the component forms a maximal independent set.

Any agent in the model will fall under any one of the categories. Using Lemma 1 we know, when all agents of a component receives signals less than ce^* (component of Category 1), the agents of type P would specialize and agents of type P' would free ride. Similarly, using Lemma 2, when all agents of a component receive signals greater than ce^* (component of Category 2), the agents of type P' would specialize and agents of type P would free ride. Finally in the component of Category 3, agents recieving signals greater than

ce^* forms a maximal independent set. Using Proposition 3 we can say that goods are allocated to all the agents in this component.

Example

Once again consider a star network. When all the periphery receive the same signals, according to Proposition 5, there are only three possibilites at equilibrium. One being all the agents of the star receive signals greater ce^*. In which case agents belonging to set P' specializes. In the second case, all agents receives signals less than ce^* in which case only the agents belonging to set P specialize. In both the cases, allocation of goods is not ensured. However, the last case where agents receiving signals greater than ce^* forms a maximal independent set, allocation of goods is ensured in equilibrium.

6 Population with only one type of agent

So far we have assumed that the agent is not aware of the entire network structure i.e., the position of persuaded or non-persuaded agents.In this section, we take two special cases of the network :

- Where all agents in the network belongs to set P.
- Where all agents in the network belongs to set P'.

Proposition 6 *When all the agents in a network with no isolated nodes are of persuaded type, allocation in equilibrium is possible if and only if set of agents receiving signals greater than ce^* forms an independent set.*

Since this is an if and if condition, we have to first prove that allocation implies that set of agents receiving signals greater than ce^* forms an independent set. Then we need to prove when set of agents receiving signals greater than ce^* forms an independent set, it implies allocation.

Lets assume that set of agents receiving signals greater than ce^* not be an independent set. Therefore there exist atleast one agent i such that $s_i > ce^*$ and who is linked to atleast another agent with signal greater than ce^*. This means for this agent i, $b_i > ce^*$. This induces this agent to free ride since it is of type P. However, since agent i itself has signal greater than ce^*, for any neighbour of i, say j, the condition $b_j > ce^*$ holds true. This induces all its neighbour to free ride. Therefore allocation of goods for all agents is not possible. This proofs that allocation of goods implies that set of agents receiving signals greater than ce^* form an independent set. Now, we proceed to the next part of the proof i.e when set of agents receiving signals greater than ce^* forms an independent set, it implies allocation of goods to all agents.

Since, any agent i such that $s_i > ce^*$ are independent, $Pr(s_i > e^*, b_i > ce^*) = 0$. Since,

$$\begin{aligned}
&Pr(s_i > ce^*) = Pr(s_i > ce^*, b_i < ce^*) + Pr(s_i > ce^*, b_i > ce^* \text{and} \\
&Pr(b_i < ce^*) = Pr(s_i < ce^*, b_i < ce^*) + Pr(s_i > ce^*, b_i < ce^*)
\end{aligned}$$

Using the independence condition, we get

$$Pr(s_i > e^*) = Pr(s_i > ce^*, b_i < ce^*) \tag{6}$$

In order to prove allocation, we need to prove that the set of specialized agents forms a dominating set. Let the set of specialized agents be denoted by D. Since all agents are of type P, an agent specializes if and only if $b_i < ce^*$, i.e.

$$\begin{aligned}
&Pr(e_i = e^*) = 1 \\
\implies &Pr(b_i < ce^*) = 1 \\
\implies &Pr(s_i < ce^*, b_i < ce^*) + Pr(s_i > ce^*, b_i < ce^*) = 1
\end{aligned}$$

This means that for an agent to specialize,

$$Pr(s_i < ce^*, b_i < ce^*) + Pr(s_i > ce^*, b_i < ce^*) = 1 \tag{7}$$

We note that the probability values take one or zero, which means that an agent $i \in D$ either satisfies the conditions $s_i < ce^*, b_i < ce^*$ or $s_i > ce^*, b_i < ce^*$. This further implies that an agent in $N - D$ must satisfy the condition $s_i < ce^*, b_i > ce^*$. This is because independence requires $Pr(s_i > ce^*, b_i > ce^*) = 0$.

For allocation, set D needs to be dominating, which means that any agent $j \in N - D$ should be connected to atleast one agent in D.

Now, each agent j in $N - D$ has $s_j < ce^*$, which means it should be linked to atleast one agent with $s_i > ce^*$ to prove dominance. Using Equation 6, we get, for all j in $N - D$, $\exists (k \in N_j)$ such that:

$$\begin{aligned}
&\exists (k \in N_j) \text{ such that } Pr(s_k > ce^*) = 1 \\
&\text{Using independence condition,} \\
\implies &\exists (k \in N_j) \text{ such that } Pr(s_k > ce^*, b_k < ce^*) = 1 \\
&\text{From condition 7, we get} \\
\implies &(\exists k \in N_j) \text{ such that } Pr(k \in D) = 1
\end{aligned}$$

Therefore, the set of specilaized agents is a dominating set. Thus proving that allocation of good takes place when the set of agents with $s_i > ce^*$ is an independent set.

Proposition 7 *When all the agents in a network with no isolated nodes are of non-persuaded type, allocation in equilibrium is possible if and only if set of agents receiving signals greater than ce^* form a dominant set.*

Since this is an if and if condition, there are two parts of the proof. First we prove that under the given conditions, dominance implies allocation. Then we prove that allocation implies dominance.

1. Let the set of agents with signals greater than ce^* be called D. For any agent $i \in D$, we have the following :

$$\begin{aligned}
&Pr(s_i > ce^* | i \in P') = 1 \\
\implies &Pr(e_i = e^*) = 1
\end{aligned}$$

   This shows all agents in D specialize. Since D is a dominating set, this further implies that allocation takes place in this equilibrium.

2. Let the set of specialized agents be D which means set of free riders is $N - D$. We assume allocation which implies that D is a dominating set. We just need prove that all agents in D receive signals greater than ce^* and all agents in $N - D$ receive signals less than ce^*. For any agent $j \in N - D$, we have the following :

$$Pr(e_j = 0 | j \in P') = 1$$

$$\text{Using equilibirum condition,}$$
$$\implies Pr(s_j < ce^*) = 1$$

This means all agents in $N - D$ receive signals less than ce^*. Since D is a dominating set; all $j \in N - D$ must be connected to atleast one agent in D i.e.,

$$\exists (k \in N_j) \text{ such that } Pr(k \in D) = 1$$

$$\text{Since all agents in D specialize}$$
$$\implies \exists (k \in N_j) \text{ such that } Pr(e_k = e^* | k \in D) = 1$$

$$\text{Using equilibirum condition,}$$
$$\implies (\exists k \in N_j) \text{ such that } Pr(s_k > ce^* | k \in D) = 1$$

This shows that all the agents in set D receive signals greater than ce^*.

7 No communication with neighbours

In this section, we assume that each agent receives a private signal $s_i = Pr(\bar{e}_i = 0)$ which is not visible to the neighbours. In this sense, we restrict communication between neighbours. We find the equilibrium condition of this model and compare it with our previous model.

For a non-persuaded individual this condition is still the same i.e:

$$s_i \geq ce^*$$

Now, coming to the neighbours' signals and how it affects the action played by an individual. Even though agents do not know the signals of the neighbours, we can formulate the probability values of the neighbourhood effort.$Pr(b_i < ce^*)$ gives the probability that no neighbour plays e^*. Whereas, $1 - Pr(b_i < ce^*) = Pr(b_i > ce^*)$ gives the probability that atleast one of the neighbours plays e^*. For a persuaded individual this condition becomes:

$$Pr(b_i > ce^*) \geq ce^*$$

Therefore, non-persuaded agents at equilibrium free rides when $0 \leq s_i < ce^*$ and specialize when $ce^* \leq s_i \leq 1$. However, persuaded agents at equilibrium free rides when $0 \leq Pr(b_i > ce^*) < ce^*$ and specialize when $ce^* \leq Pr(b_i > ce^*) \leq 1$.

We no longer assume the probability values to be zero or one but depends on the distribution of the neighbourhood signals. We assume that the neighbour's signals are distributed uniformly within the range of 0 to 1. We know $Pr(b_i \geq ce^*) = 1 - Pr(b_i < ce^*)$. When the maximum value of signals of neighbours is less then ce^*, it means that all the values of signals of neighbours are less than ce^*.

Since each signal is assumed to be distibuted uniformly from the range of 0 to 1, $Pr(b_i < ce^*) = ce^*$. So, for an agent i of degree d_i, with maximum neighbour signal b_i,

$$Pr(b_i < ce^*) = (ce^*)^{d_i}$$

Agent i of non-persuaded type plays e^* when:

$$\begin{aligned}
&Pr(b_i \geq ce^*) \geq ce^* \\
\implies &1 - (ce^*)^{d_i} \geq ce^* \\
\implies &ln(1 - ce^*) \geq d.ln(ce^*) \\
\implies &d \leq \frac{ln(1 - ce^*)}{ln(ce^*)}
\end{aligned}$$

Let us call $d^* = \frac{ln(1-ce^*)}{ln(ce^*)}$. Therefore agent i specialize when $d_i \leq d^*$ and free rides otherwise. It is assumed that agents can see the neighours.

The condition for allocation of goods and welfare maximization remains the same as previous model i.e., good is allocated to every agent when the set of specialized agents form a dominating set and welfare is maximized at equilibrium when the set of specialized agents form a minimal dominating set. We have the following propositions which compares the model where communication is allowed with the model where no communication takes place.

Proposition 8 *When the signals' range is from 0 to ce^* in a network with no isolated nodes, if allocation takes place in equilibrium with no communication, then allocation also takes place in equilibrium with communication. This further implies that welfare of the model is always greater when communication is not allowed with the neighbours.*

1. **Equilibrium with communication :** When every agent receives signal from 0 to ce^*, using Lemma 1, we can say that all agents belonging to P specialize and all agents belong to P' free rides. This means under allocation, we require P or a subset of P to be a dominating set.

2. **Equilibrium with no communication :** When every agent receives signal from 0 to ce^*, using equilibrium condition, we can say that all agents belonging to P but with degree less than d^* specialize and all other agents free rides. This means the set of specialized agents (say D) is a subset of P. Under allocation, D is a dominating set.

If there is allocation with no communication, a subset of P dominates in equilibrium. This ensures allocation in the equilibrium with no communication as well since $D \subseteq P$. Next, we compare the welfare in both scenarios.

1. **Welfare with communication :** The welfare of the model is equal to $n - ce^*\tilde{P}$, where $\tilde{P}$ is the number of elements in P.

2. **Welfare with no communication :** The welfare of the model is equal to $n - ce^*\tilde{D}$ where $\tilde{D}$ is the number of elements in D.

Since, $D \subseteq P$, we can say that welfare with no communication is always greater than welfare with communication.

Proposition 9 *When the signals' range is from ce^* to 1 in a network with no isolated nodes, if allocation takes place in equilibrium with communication, then allocation also takes place in equilibrium with no communication.*

This further implies that welfare of the model is always greater when communication is allowed with the neighbours.

1. **Equilibrium with communication :** When every agent receives signal from ce^* to 1, using Lemma 2, we can say that all agents belonging to P' specialize and all agents belong to P free rides. This means under allocation, P' is a dominating set.

2. **Equilibrium with no communication :** When every agent receives signal from ce^* to 1, using equilibrium condition, we can say that all agents belonging to P but with degree greater than d^* free rides and all other agents specialize. This means P' is a subset of the set of specialized agents (say D). Under allocation, we require D or a subset of D to be a dominating set.

If there is allocation with communication, P' dominates in equilibrium. This ensures allocation in the equilibrium with no communication as well since $P' \subseteq D$. Next, we compare the welfare in both scenarios.

1. **Welfare with communication :** The welfare of the model is equal to $n - ce^*\tilde{P'}$, where $\tilde{P'}$ is the number of elements in P'.

2. **Welfare with no communication :** The welfare of the model is equal to $n - ce^*\tilde{D}$ where $\tilde{D}$ is the number of elements in D.

Since, $P' \subseteq D$, we can say that welfare with communication is always greater than welfare with no communication.

8 Conclusion

This paper is a modification over the Bramoulli and Kranton (2007) model. The main result of the model is that at specialized equilibrium, the set of agents specialising forms a maximal independent set. This equilibrium also ensures allocation of goods to every agent. Our model introduces additional information in the form of signals which gives the probability that no neighbour makes any effort.

We find that as long as the value of signals comes from a deterministic probabilit distribution, equilibrium would exist in the model under any signal structure and network structure. However, allocation of goods to every agent depends on the network structure. Particularly, when the set of agents receiving signals greater than ce^* forms a maximal independent set, then under any network structure, allocation of goods take place at equilibrium. In line with Bramoulli and Kranton (2007), the set of specialised agents in this case also forms a maximal independent set. However, unlike Bramoulli and Kranton (2007), this is not the only possible case for allocation of goods.

Next, we take two particular cases of the model. One with the entire population being persuaded type and another with all agents being non-persuaded type. We see when all the agents in a network with no isolated nodes are of persuaded type, allocation in equilibrium is possible if and only if set of agents receiving signals greater than ce^* forms an independent set; and when they are of non-persuaded type, allocation in equilibrium is possible if and only if set of agents receiving signals greater than ce^* form a dominant set. Therefore, when it is known that all agents are of a particular type, network structure can help determine whether allocation takes place or not.

Lastly, we analyse the case when communication between the neighbours are restricted, i.e agents do not get to know the neighbours' private signals. We assume that agents believe that neighbour's signals follow an uniform distribution from the range of 0 to 1. In this case, we go on to find the equilibrium profile of the model and compare with the equilibrium of the case where communication between the neighbours is allowed. We find that welfare of the model is always greater when communication is allowed with the neighbours when private signals range from ce^* to 1 and always less when private signals range from 0 to ce^*.

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