Endogenous Transfers and Inequality in a Network of Contesting Agents

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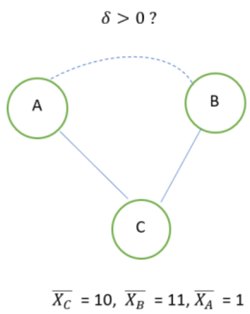
1. Introduction:

In human history conflicts are often multi-front in nature. Conflicting parties are often engaged in multiple conflicts with different parties. A few examples of conflict structures where conflict parties have to deal with multiple rivals in different conflicts simultaneously include conflicts between a centre and its periphery; for instance, an empire fighting its enemies, conflicts among several rivals of similar power; for instance, the so - called peer-polity interactions and conflicts among ideologically different communities, where members of different communities perceive each other as enemies. In all the above cases the outcome of conflict is shaped by the underlying structure of the interdependent conflictive relations. And hence conflicts in a network of conflicting agents, receiving sufficient attention from economists and social scientists at large, comes as no surprise. This paper tries to borrow from two different strands of conflict economics literature (i) the strand concerning the conflict of agents in fixed networks as in Franke and Ozturk (2015) and (ii) the other strand is the one dealing with endogenous alliance formation. Skaperdas (1998), Tan and Wang (1997) and Esteban and Sakovics (2003) have shown that in three-player models, parties have no incentive to form a two-player alliance against the third unless the formation of the alliance generates synergies which enhance the winning probability of the alliance. In this paper we introduce a much weaker notion of alliance, a priori rational transfers.

In this paper we don the hat of a policy maker. The next natural question which one might ask is what is (are) the policy variable(s) and objective(s) and what are the exogenous features of the structure. Before we try to address those questions let us imagine a simple two player contest. In any two - player contest, any unilateral transfer of fighting resources is clearly not rational. Why would I strengthen my opponent and weaken myself at the same time through an unilateral resource transfer? But this question does not have an obvious and trivial answer when we have more than two players engaged in a network of contests. Let us elucidate with the help of an example. If A and B are both fighting C (as depicted in the figure below) is it trivial that A does not have an incentive to provide B with an unilateral transfer? (We assume all contests are characterized by the Tullock (1980) contest success function). Not really. An unilateral transfer from A to B will make B more endowed which in turn will make C allocate more resources in fighting C which in turn reduces C's resource allocation in his conflict with A. Therefore from A's perspective there are two opposite effects. The transfer reduces his strength but at the same time reduces C's allocation in the contest with A. And it is not obvious a priori which effect dominates and when.



Let us consider a transfer of 1 unit of resource from B to A in the case depicted in the figure. The rent being contested over has a valuation of '1' in both the contests. Without the transfer the equilibrium payoffs to A and B are given by π_A = 0.196 and π_B = 0.6507. With the transfer π_A = 0.294 and π_B = 0.657. Thus we see that it is rational for B to make a transfer of 1 unit of resource to A. In fact the transfer leads to an improvement for both A and B. We call such a transfer a "rational transfer". This paper tries to characterize the possibility of such voluntary redistributions (via rational transfers) with in a subset of agents in a network of contests. Ofcourse that will depend on the conflict resource endowments of the agents and the nature of the network. So we try to inspect if a conflict-resource endowment vector is susceptible to such rational transfers. This paper also throws light on the post – conflict inequality or distributional implications of the pre - conflict – resource distribution in different network structures. We also inspect whether the inequality-minmizing distribution can be endogenously achieved through rational transfers and the transfer – susceptibility / stability of the inequality pre- conflict-resource distribution in different networks. The last section of the paper delves into the relationship between pre-conflict resource distribution and the resulting post-conflict distribution. The pre – conflict resources might be perceived as arms & ammunitions, and the post – conflict resource is that what the fight is happening over (for example land or ethnic dominance). So does an equitable distribution of arms a great way of ensuring post – conflict inequality. We show that it depends on the structure of the network. Finally the last section investigates the trade - off between conflict minimization and inequality for different network structures.

2. Model

Let S be the set of agents, who are engaged in a network of conflicts. This is represented by a weighted graph G = <V,E,W>, where V, the set of vertices denote the set of agents and the edge $e_{ij} \in E$ denotes that agent 'i' is contesting with agent j. W: E -> R denotes the weights associated with the edges. W$(e_{ij}) = w_{ij}$ i.e the weight on the edge connecting agent i & j signifies the common valuation of the prize which the two agents are fighting over. Of course W$(e_{ij}) = 0$ if $e_{ij} \notin E$. Also each agent i $\in S$ has conflict resource endowment $\overline{X_\iota}$.

Definitions:

Transfer – susceptible endowment vector: Given a graph G, a conflict resource endowment vector $\{\overline{X_\iota}\}_{i \in S}$ is called transfer susceptible if any unilateral transfer to an agent by another leads to a pareto improvement in payoffs for both.

Transfer – resistant endowment vector: Given a graph G, a conflict resource endowment vector $\{\overline{X_\iota}\}_{i \in S}$ is called transfer resistant if no unilateral transfer between any pair of agents is rational.

Remark: We can possibly imagine a pure exchange economy as an analogue of this model. The "allocation" in an exchange economy is the conflict resource endowment vector in this case. The transfer-resistant conflict endowment vector is the counterpart for a pareto efficient allocation in an exchange economy. In a pure exchange economy we are interested in determining whether an allocation is pareto efficient or not, given the set of utility functions of the agents. In this case we are interested in the transfer susceptibility or transfer resistance of a conflict endowment vector given a network of contests.

Types of Networks under consideration:

In the conflict network literature, analysing general networks is still an open problem. So following Franke and Ozturk (2015), we focus our attention on two broad classes of networks namely, regular and complete bipartite. In the latter category we have especially focussed on star- shaped and linear networks. These categories as mentioned in Franke and Ozturk (2015) are "distinct with respect to their grade of asymmetry which endogenously induces heterogeneity on the agents (depending on their respective location in the network)". But before we move on to the next section let us briefly recall the properties of the above-mentioned networks.

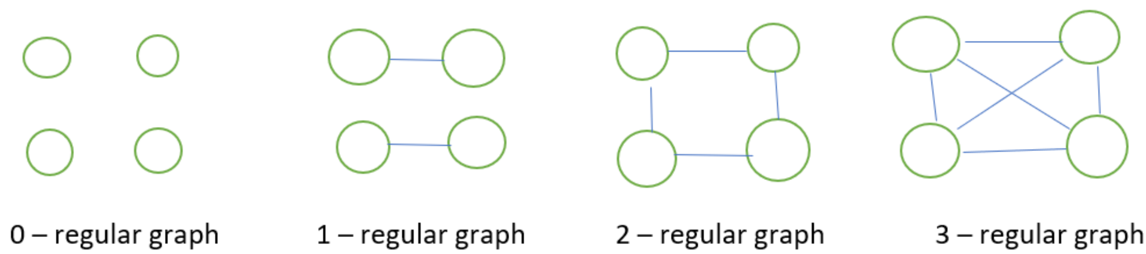
Bipartite Graph: A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets U and V such that every edge of the graph connects a vertex in U with a vertex in V.

Observation 1: The linear network (graph) and the star – shaped network qualify as bipartite graphs.

Proof: Consider any linear network with a finite number (say n) of vertices. We can certainly number the vertices from 1 to n with the left most vertex numbered 1. Now the set of all vertices of the network / graph can be divided into two disjoint sets U and V with U containing the odd – numbered vertices and V containing the even numbered vertices. And every edge in the graph connects a vertex in U to a vertex in V. Hence a linear graph is bipartite. For a star – shaped network the argument is trivial. We can simply put the central vertex in U and the peripheral vertices in V and thereby every edge connects the only vertex in U with the vertices in V, and thereby making the star-shaped graph bipartite.

Regular Graph: A *regular graph* is a graph where every vertex has the same number of neighbours i.e every vertex has the same degree.

A regular graph with vertices of degree k is called a *k – regular graph*. Thus in a k-regular graph every vertex has k neighbours. The figure below shows all possible regular graphs with four vertices.



Connected Graph: A graph is said to connected if there exists a path between every pair of vertices.

Hence amongst the four – vertex regular graphs the 2- regular and 3 – regular graphs are connected, with the 2 – regular graph, which is a polygon (in this case a square) is the simplest (lowest degree) connected regular graph.

Observation 2: All 1 – regular graphs and polygons are necessarily bipartite.

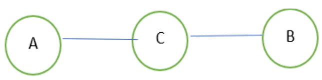
Hence polygons with any finite number (n) of vertices are 2-regular, connected and bipartite.

3. Transfer – resistance & Transfer susceptibility of allocations

This section deals with inspecting the transfer-resistance and transfer – susceptibility of allocation of pre-conflict resources (i.e arms and ammunitions), in case of different networks. As mentioned in the previous section we will focus on bipartite networks (especially linear and star-shaped) and regular networks.

Proposition 1: For a n – vertex linear network (with n>2) of equal edge-weights, there necessarily exists a conflict resource endowment vector which is transfer-susceptible.

Proof: We will argue with the help of mathematical induction. Consider the following linear network (n=3), where the weights of the edges is = 1 (without loss of generality) i.e the common valuation of the prizes in both the contests are equal to 1.



Each player j ∈ $\{A, B, C\}$ is endowed with $\overline{X_J}$ amount of resources. Without loss of generality $\overline{X_B} > \overline{X_A}$. C decides to allocate his resources in the two conflicts. Let us say that the resource allocated in the contest against k ∈ $\{A, B\}$ is given by x_{kc}. Thus C's optimization problem is given by: $maximize\ \pi_C = \frac{x_{Bc}}{x_{Bc}+\overline{X_B}} + \frac{x_{Ac}}{x_{Ac}+\overline{X_A}}$ where $x_{Bc} + x_{Ac} = \overline{X_C}$

In equilibrium $\pi_B{}^* = \frac{\overline{X_B}+\sqrt{\overline{X_A}\ \overline{X_B}}}{\overline{X_A}+\overline{X_B}+\overline{X_C}}$ and $\pi_A{}^* = \frac{\overline{X_A}+\sqrt{\overline{X_A}\ \overline{X_B}}}{\overline{X_A}+\overline{X_B}+\overline{X_C}}$. Now if there is a transfer of resource of amount δ from B to A, then how will the equilibrium payoffs change?

Lemma 1:

(i) *If $\overline{X_B} < (3 + 2\sqrt{2})\, \overline{X_A}$ then no unilateral transfer from B to A is rational for B*

(ii) *If $\overline{X_B} > (3 + 2\sqrt{2})\, \overline{X_A}$ then $\exists\ \delta^* = \frac{\overline{X_B}-(3+2\sqrt{2})\overline{X_A}}{3+2\sqrt{2}} < \overline{X_B} \ni for\ any\ \delta \in (0, \delta^*)$ there is a pareto improvement in the equilibrium payoffs of both A and B.*

Proof of Lemma 1: Let us say that if there is a transfer of δ from B to A then the equilibrium payoffs of B and A are given by $\pi_B{}^*(\delta)\ and\ \pi_A{}^*(\delta)$ respectively.

Clearly $\pi_B{}^*(\delta) = \frac{\overline{X_B}-\delta+\sqrt{(\overline{X_A}+\delta)(\overline{X_B}-\delta)}}{\overline{X_A}+\overline{X_B}+\overline{X_C}}$. Hence $\pi_B{}^*(\delta) - \pi_B{}^* = \frac{\sqrt{(\overline{X_A}+\delta)(\overline{X_B}-\delta)}-\sqrt{\overline{X_A}\ \overline{X_B}}-\delta}{\overline{X_A}+\overline{X_B}+\overline{X_C}}$

Define f(δ) = $\sqrt{(\overline{X_A} + \delta)(\overline{X_B} - \delta)} - \sqrt{\overline{X_A}\ \overline{X_B}} - \delta$

f(0) = 0, and f($\overline{X_B}$) < 0. $f'(\delta) = \frac{(\overline{X_B}-\delta)-(\overline{X_A}+\delta)}{2\sqrt{(\overline{X_A}+\delta)(\overline{X_B}-\delta)}} - 1$. Thus $\lim_{\delta\to 0} f'(\delta) > 0$. Also $f'(\delta) = 0$ at

$\delta = \frac{\overline{X_B}-(3+2\sqrt{2})\overline{X_A}}{3+2\sqrt{2}} = \delta^*$. Thus f($\delta$) is necessarily positive for all $\delta < \delta^*$.

$\pi_A{}^*(\delta) = \frac{\overline{X_A}+\delta+\sqrt{(\overline{X_A}+\delta)(\overline{X_B}-\delta)}}{\overline{X_A}+\overline{X_B}+\overline{X_C}}$. And $\pi_A{}^*(\delta) - \pi_A{}^* = \frac{\sqrt{(\overline{X_A}+\delta)(\overline{X_B}-\delta)}-\sqrt{\overline{X_A}\ \overline{X_B}}+\delta}{\overline{X_A}+\overline{X_B}+\overline{X_C}}$

Define g(δ) = $\sqrt{(\overline{X_A} + \delta)(\overline{X_B} - \delta)} - \sqrt{\overline{X_A}\ \overline{X_B}} + \delta$ = f(δ) + 2δ

g(0) = 0 and g($\overline{X_B}$) > 0. Also $\lim_{\delta\to 0} g'(\delta) = \lim_{\delta\to 0} f'(\delta)$ + 2 > 0.

$g'(\delta) = \frac{(\overline{X_B}-\delta)-(\overline{X_A}+\delta)}{2\sqrt{(\overline{X_A}+\delta)(\overline{X_B}-\delta)}} + 1$. And $g'(\delta) = 0\ at\ \delta = \frac{\overline{X_B}-(3-2\sqrt{2})\overline{X_A}}{3-2\sqrt{2}} = \delta^{**} > \delta^*$. Thus we can infer that g(δ) > 0 for any $\delta \in (0, \delta^{**})$. Thus for any $\delta < \min\{\delta^*, \delta^{**}\} = \delta^*$, $\pi_B{}^*(\delta) > \pi_B{}^*$

and $\pi_A{}^*(\delta) > \pi_A{}^*$, which implies that a transfer of δ from B to A leads to a pareto improvement for both A and B.

Thus we can infer that for all endowment vectors $\bar{X} \subseteq R_+^3$ satisfying the condition in Lemma 1 (ii), $\exists\ \delta > 0$ which when unilaterally transferred by B to A leads to a pareto improvement in their equilibrium payoffs i.e endowment vectors satisfying the condition(s) in Lemma 1(ii) are *transfer-susceptible.*

Proof of Proposition: Let's assume that for an equal edge-weight linear network with n-1 agents there exists a *transfer-susceptible* conflict – resource endowment vector given by $\bar{X} = (\overline{X_1}, \overline{X_2}, \overline{X_3}, \overline{X_4}, \ldots\ldots\ldots.. \overline{X_{n-1}})$ where it is rational for agent i to make an unilateral transfer to agent j where i,j ∈ $\{2,3, \ldots.. n-1\}$.

Fig i

Fig ii

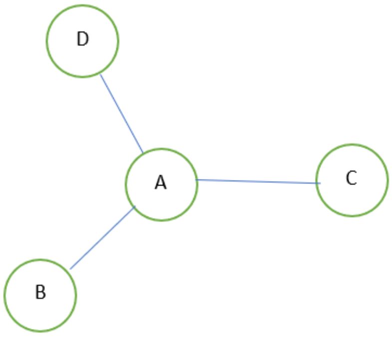
And let's say given $\bar{X} = (\overline{X_1}, \overline{X_2}, \overline{X_3}, \overline{X_4}, \ldots\ldots\ldots.. \overline{X_{n-1}})$ the equilibrium conflict allocations are given by $\{x^*_{i,i-1}, x^*_{i,i+1}\}_{i=2}^{n-2}$ where $x^*_{i,i-1} + x^*_{i,i+1} = \overline{X_i}$. Now imagine an equal edge-weight linear network with n agents and the conflict resource endowment vector is given by $\tilde{X} = (\overline{X_1}, \overline{X_2}, \overline{X_3}, \overline{X_4}, \ldots\ldots\ldots.. \overline{X_{n-1}}, \overline{X_n})$. Now for any conflict allocation vector $\{x_{n-1,n-2}, x_{n-1,n}\}$ where $x_{n-1,n-2} + x_{n-1,n} = \overline{X_{n-1}}$, payoff of agent A_{n-1} is given by

$\pi_{n-1} = \frac{x_{n-1,n}}{x_{n-1,n} + \overline{X_n}} + \frac{\overline{X_{n-1}} - x_{n-1,n}}{\overline{X_{n-1}} - x_{n-1,n} + x_{n-2,n-1}}$. Thus $\lim_{x_{n-1,n}\to 0} \frac{\partial \pi_{n-1}}{\partial x_{n-1,n}} = \frac{1}{\overline{X_n}} - \frac{x_{n-2,n-1}}{(\overline{X_{n-1}} + x_{n-2,n-1})^2}$.

Hence $\lim_{x_{n-1,n}\to 0} \frac{\partial \pi_{n-1}}{\partial x_{n-1,n}} < 0$ if $\overline{X_n}$ is large enough. Thus if $\overline{X_n}$ is large enough we are back to same equilibrium which we had in the previous case. And since the endowments of the first 'n-1' agents are the same in $\tilde{X}$ and $\bar{X}$, we can safely conclude that $\tilde{X}$ is *transfer – susceptible*.

Proposition 2: For a n – vertex star-shaped network of equal edge-weights, there necessarily exists a resource allocation vector where an unilateral transfer is rational.

Proof: We will argue with the help of mathematical induction. Consider the following network (n=4), where the weights of the edges is = 1 (without loss of generality).



The conflict resource endowment vector is given by $\bar{X} = (\overline{X_A}\,, \overline{X_B}, \overline{X_C}, \overline{X_D}\,)$. Payoff of A is given by $\pi_A = \frac{x_{AB}}{x_{AB}+\overline{X_B}} + \frac{x_{AC}}{x_{AC}+\overline{X_C}} + \frac{x_{AD}}{x_{AD}+\overline{X_D}}$ where $x_{AB} + x_{AC} + x_{AD} = \overline{X_A}$. In equilibrium: $\pi_B{}^* = \frac{\overline{X_B} + \sqrt{\overline{X_C}\ \overline{X_B}} + \sqrt{\overline{X_D}\ \overline{X_B}}}{\overline{X_A} + \overline{X_B} + \overline{X_C} + \overline{X_D}}$. Now let's say there is a unilateral transfer δ from B to D, and then the equilibrium payoff of B is given by $\pi_B{}^*(\delta)$.

Now $\pi_B{}^* - \pi_B{}^*(\delta) = \frac{\delta + \sqrt{\overline{X_C}\ \overline{X_B}} - \sqrt{\overline{X_C}\ .(\overline{X_B} - \delta)} + \sqrt{\overline{X_D}\ \overline{X_B}} - \sqrt{(\overline{X_D}+\delta)(\overline{X_B}-\delta)}}{\overline{X_A} + \overline{X_B} + \overline{X_C} + \overline{X_D}}$ = f(δ)

f(0) = 0 & f($\overline{X_B}$) > 0. Also $f'(\delta) = \frac{1 + \sqrt{\frac{\overline{X_C}}{(\overline{X_B} - \delta)}} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{\sqrt{(\overline{X_D}+\delta)(\overline{X_B}-\delta)}} \cdot (\overline{X_B} - \overline{X_D} + 2(\delta))}{\overline{X_A} + \overline{X_B} + \overline{X_C} + \overline{X_D}}$

Thus $\lim_{\delta \to 0} f'(\delta) = \frac{1 + \sqrt{\frac{\overline{X_C}}{\overline{X_B}}} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{\sqrt{(\overline{X_D})(\overline{X_B})}} \cdot (\overline{X_B} - \overline{X_D})}{\overline{X_A} + \overline{X_B} + \overline{X_C} + \overline{X_D}}$

$= \frac{\frac{1}{2}}{\overline{X_A} + \overline{X_B} + \overline{X_C} + \overline{X_D}} \left[\sqrt{\frac{\overline{X_C}}{\overline{X_B}}} + \sqrt{\frac{\overline{X_D}}{\overline{X_B}}} - \sqrt{\frac{\overline{X_B}}{\overline{X_D}}} + 2 \right] = \frac{\frac{1}{2}}{\overline{X_A} + \overline{X_B} + \overline{X_C} + \overline{X_D}} \left[\sqrt{\frac{\overline{X_C}}{\overline{X_B}}} - \left\{ \sqrt{\frac{\overline{X_B}}{\overline{X_D}}} - \sqrt{\frac{\overline{X_D}}{\overline{X_B}}} - 2 \right\} \right]$

It clearly follows that $\lim_{\delta \to 0} f'(\delta) < 0$ if

$$\overline{X_B} > (3 + 2\sqrt{2})\,\overline{X_D} \quad \& \quad \overline{X_C} < \overline{X_B} \left[\sqrt{\frac{\overline{X_B}}{\overline{X_D}}} - \sqrt{\frac{\overline{X_D}}{\overline{X_D}}} - 2 \right]^2 \qquad \text{(C1)}$$

Also $\pi_D{}^*(\delta) - \pi_D{}^* = \frac{\delta + \sqrt{\overline{X_C}\,(\,\overline{X_D} + \delta)} - \sqrt{\overline{X_C}\ \overline{X_D}} - \sqrt{\overline{X_D}\ \overline{X_B}} + \sqrt{(\overline{X_D}+\delta)(\overline{X_B}-\delta)}}{\overline{X_A} + \overline{X_B} + \overline{X_C} + \overline{X_D}}$ = g(δ)

Clearly g(0) = 0 & $g'(\delta) > 0$ if C1 holds. Thus we can infer that for all endowment vectors $\bar{X} \subseteq R^4_{++}$ satisfying C1, $\exists\, \delta > 0$ which when unilaterally transferred by B to D leads to improvement in their equilibrium payoffs of both B and D i.e endowment vectors satisfying C1 are *transfer-susceptible.*

Let's assume that for an equal edge-weight star with n-1 agents (with agent 1 is at the 'centre', without loss of generality) there exists a transfer-susceptible conflict – resource endowment vector given by $\bar{X} = (\overline{X_1}, \overline{X_2}, \overline{X_3}, \overline{X_4}, \dots \dots \dots .. \overline{X_{n-1}})$ where it is rational for agent i to make an unilateral transfer to agent j where i,j ∈ $\{2,3, \dots .. n-1\}$.

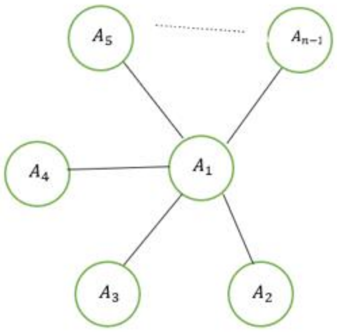


Fig1

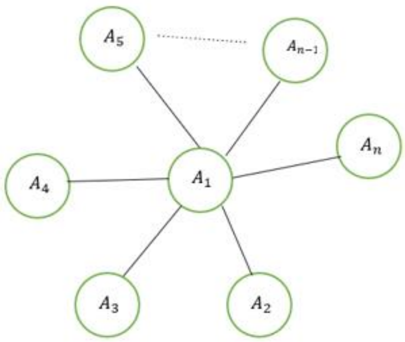
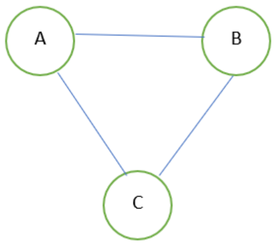


Fig2

Let us say that the equilibrium allocations to conflict by agent 1 is given by $x_1^* = (x_{1,2}^*$ $, x_{1,3}^*, x_{1,4}^*, \dots . x_{1,n-1}^*)$. Now imagine an equal edge-weight star with n agents again with 1 at the centre (without loss of generality) and the conflict resource endowment vector is given by $\tilde{X}$ = $(\overline{X_1}, \overline{X_2}, \overline{X_3}, \overline{X_4}, \dots \dots \dots .. \overline{X_{n-1}}, \overline{X_n})$. Now given any resource allocation vector $x_1 = (x_{1,2}, x_{1,3}, \dots .. x_{1,n})$ chosen by agent 1, his payoff is given by $\pi_1 = \sum_{j=2}^{n} \frac{x_{1,j}}{x_{1,j} + \overline{X_j}}$, where $\sum_{j=2}^{n} x_{1,j} = \overline{X_1}$. Therefore $\lim_{x_{1,n} \to 0} \frac{\partial \pi_1}{\partial x_{1,n}} = \frac{1}{\overline{X_n}} - \sum_{j=2}^{n-1} \frac{\overline{X_j}}{(x_{1,j} + \overline{X_j})^2} < 0$ if $\overline{X_n}$ is large enough. Thus $x_{1,n} > 0$ does not constitute an equilibrium. Therefore in this case the equilibrium allocation vector of agent 1 is given by $(x_{1,2}^*, x_{1,3}^*, x_{1,4}^*, \dots . x_{1,n-1}^*, 0)$, which in turn implies that it will be rational for agent i to make an unilateral transfer to agent j where i,j ∈ $\{2,3, .. n-1\}$. Thus we have proven that $\tilde{X}$ is *transfer-susceptible*.

Proposition 3: A polygon having all edges (but one) of equal weights and one edge with a "high enough" weight, ensures the existence of a transfer-susceptible resource endowment vector.



Consider the following allocation $(\overline{X_A}, \overline{X_B}, \overline{X_C}) =$ (R,R,0). Also let's say the weights of the edges of the graph are given as follows: w(A-B) = 1, w(A-C) = 1 & w(B-C) = w.

In this scenario the equilibrium payoffs are given by $\pi_A{}^*$ = 3/2, $\pi_B{}^*$ = w + ½, $\pi_C{}^*$ = 0.

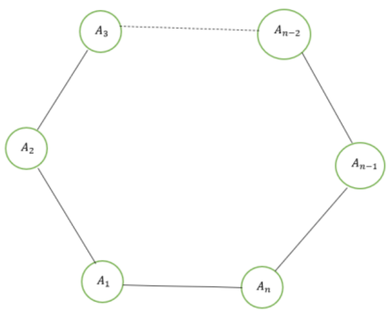
Now if A makes an unilateral transfer of δ to C, the resource allocation vector will be given by (R - δ,R, δ).

Now $\pi_B = \frac{x_{BA}}{x_{BA}+x_{AB}} . 1 + \frac{R- x_{BA}}{R- x_{BA}+x_{CB}}$. w & $\pi_C = \frac{x_{CA}}{x_{CA}+x_{AC}} . 1 + \frac{R- x_{CA}}{R- x_{CA}+x_{BC}}$. w

$\frac{\partial \pi_B}{\partial x_{BA}} = \frac{x_{AB}}{(x_{BA}+x_{AB})^2} - \frac{x_{CB}}{(R- x_{BA}+x_{CB})^2}$. w & $\frac{\partial \pi_C}{\partial x_{CA}} = \frac{x_{AC}}{(x_{CA}+x_{AC})^2} - \frac{x_{BC}}{(R- x_{CA}+x_{BC})^2}$. w

Thus $\exists\, \ddot{w} > 0 \ni \forall\, w > \ddot{w}$, $\lim_{x_{BA}\to 0} \frac{\partial \pi_B}{\partial x_{BA}} < 0$ & $\lim_{x_{CA}\to 0} \frac{\partial \pi_C}{\partial x_{CA}} < 0$. Thus in equilibrium $\pi_A{}^*$ = 2 > 3/2 if $w > \ddot{w}$. Also $\pi_C{}^* > 0$. Thus we see that an unilateral transfer from A to C in the above network leads to a pareto improvement for A and C.

Now consider any polygon with n-sides and no diagonals.



Let's say w$(A_i - A_{i+1})$ = 1 $\forall i = 1,2 \ldots . n-2$ & w$(A_{n-1} - A_n)$ = w. Given this graph consider the conflict resource endowment vector (R,R,R….R,0). Clearly in equilibrium

$\pi_{n-1}{}^{*}$ = 0. Given this, following a similar argument as above $\exists\ \breve{w} > 0 \ni\ \forall\ w > \breve{w}$, an unilateral transfer from A_{n-2} to A_n will lead to a pareto improvement for both agents, which in turn makes the conflict resource endowment vector (R,R,R….R,0) *transfer – susceptible.* Thus we see that for a polygon (with vertices representing the contesting agents) with all equal weighted & one high-weighted edge, there must exist a transfer-susceptible resource endowment vector.

Corollary: If any graph G contains a equal weight star or a linear network or a one-high - weight polygon as a subgraph, there will exist a transfer-susceptible conflict-resource endowment.

Proposition 4: For any 1-regular network with n agents, any allocation is transfer resistant.

Proof: The proof is trivial. Any 1 – regular graph is simply a set of disjoint pairs of vertices. So every agent has exactly one opponent i.e we have a bunch of independent two player contests. Hence the result follows.

4. Allocations, Transfers & Inequality

The Herfindahl-Hirschman Index (HHI) is a commonly accepted measure of wealth concentration and inequality. It is calculated by squaring the share of each agent competing in a system and then summing the resulting numbers. Thus for a system with n agents, HHI = $\sum_{j=1}^{n} s_j^2 = \frac{1}{W^2}\sum_{j=1}^{n} \pi_j^2 \equiv \sum_{j=1}^{n} \pi_j^2$. So a higher HHI would mean higher wealth concentration. Thus a low HHI is desirable.

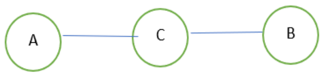
Now we ask three important questions.

i) Given an allocation, is a rational transfer "HHI – improving" (decreases HHI) or "HHI – exacerbating" (increases HHI)?

ii) Given a network & a fixed amount of total-advertising budget, is the HHI – minimizing allocation transfer – susceptible or transfer-resistant?

iii) Does there exist an allocation, given which, a rational transfer will result in the HHI-minimizing allocation?

In this section we'll try to answer these questions for the simplest network we know of, the three - node - linear network with equal weights (=1).



Without loss of generality $\overline{X_B} \geq \overline{X_A}$. Let's also assume that the total amount of resources is fixed i.e $\overline{X_B} + \overline{X_A} + \overline{X_C}$ = T.

Given the network the HHI = $\pi_A^2 + \pi_B^2 + \pi_C^2 = \pi_A^2 + \pi_B^2 + (2 - \pi_B - \pi_A)^2$

Proposition 5: For a three - node - linear network with equal weights (=1)

i. *A rational transfer can be "HHI exacerbating". There exist allocations which are susceptible to HHI – exacerbating transfers.*
ii. *The HHI minimizing allocation is "transfer – resistant"*
iii. *There does not exist a "transfer – susceptible" allocation, where a rational transfer will lead to the HHI minimizing allocation.*

Proof: Let us say after a transfer of δ from B to A increases the payoff of B & A by Δ_1 & Δ_2 respectively i.e $\widetilde{\pi_B} = \pi_B + \Delta_1$ and $\widetilde{\pi_A} = \pi_A + \Delta_2$ where $\widetilde{\pi_j}$ denote the payoff after the transfer has happened.

Hence $\Delta HHI = [\widetilde{\pi_A}^2 + \widetilde{\pi_B}^2 + (2 - \widetilde{\pi_B} - \widetilde{\pi_A})^2]$ - $[\pi_A^2 + \pi_B^2 + (2 - \pi_B - \pi_A)^2]$

$= \Delta_1 [2\pi_B + \pi_A - 1] + \Delta_2 [2\pi_A + \pi_B - 1] + {\Delta_1}^2 + {\Delta_2}^2 + \Delta_1 . \Delta_2$

By Proposition 1 in equilibrium

$${\pi_B}^* = \frac{\overline{X_B} + \sqrt{\overline{X_A}\ \overline{X_B}}}{\overline{X_A} + \overline{X_B} + \overline{X_C}} \text{ and } {\pi_A}^* = \frac{\overline{X_A} + \sqrt{\overline{X_A}\ \overline{X_B}}}{\overline{X_A} + \overline{X_B} + \overline{X_C}}$$

$$\therefore \Delta HHI = \Delta_1 \left[\frac{(2\sqrt{\overline{X_B}} + \sqrt{\overline{X_A}})(\sqrt{\overline{X_A}} + \sqrt{\overline{X_B}})}{\overline{X_B} + \overline{X_A} + \overline{X_C}} - 1\right] + \Delta_2 \left[\frac{(2\sqrt{\overline{X_B}} + \sqrt{\overline{X_A}})(\sqrt{\overline{X_A}} + \sqrt{\overline{X_B}})}{\overline{X_B} + \overline{X_A} + \overline{X_C}} - 1\right] + {\Delta_1}^2 + {\Delta_2}^2 + \Delta_1 . \Delta_2$$

If $(\sqrt{\overline{X_A}} + \sqrt{\overline{X_B}})$. min $\{2\sqrt{\overline{X_B}} + \sqrt{\overline{X_A}}\ , 2\sqrt{\overline{X_A}} + \sqrt{\overline{X_B}}\ \} > \overline{X_B} + \overline{X_A} + \overline{X_C}$, then

$\Delta HHI > 0$. Since $\overline{X_B} \geq \overline{X_A}$, min $\{2\sqrt{\overline{X_B}} + \sqrt{\overline{X_A}}\ , 2\sqrt{\overline{X_A}} + \sqrt{\overline{X_B}}\ \} = 2\sqrt{\overline{X_A}} + \sqrt{\overline{X_B}}$

$\therefore (\sqrt{\overline{X_A}} + \sqrt{\overline{X_B}})$ min $\{2\sqrt{\overline{X_B}} + \sqrt{\overline{X_A}}\ , 2\sqrt{\overline{X_A}} + \sqrt{\overline{X_B}}\ \} > \overline{X_B} + \overline{X_A} + \overline{X_C}$

implies $\overline{X_A} + 3\sqrt{\overline{X_A}\ \overline{X_B}} > \overline{X_C}$.

Moreover by Proposition 1 $\overline{X_B} > (3 + 2\sqrt{2}\,)\ \overline{X_A}$ ensures transfer – susceptibility of the allocation. Hence if $\overline{X_A} + 3\sqrt{\overline{X_A}\ \overline{X_B}} > \overline{X_C}$ & $\overline{X_B} > (3 + 2\sqrt{2}\,)\ \overline{X_A}$ *ensures that the allocation* $(\overline{X_A}\ , \overline{X_B}\ , \overline{X_C}\ )$ susceptible to a HHI – exacerbating rational transfer.

In equilibrium

HHI = $\frac{1}{T^2}[(\sqrt{\overline{X_A}} + \sqrt{\overline{X_B}})^2(\overline{X_A} + \overline{X_B}) + (\sqrt{\overline{X_A}} + \sqrt{\overline{X_B}})\sqrt{\overline{X_A}\ \overline{X_B}} - 2\,T\,(\sqrt{\overline{X_A}} + \sqrt{\overline{X_B}})^2$
]

Thus $\arg min_{\overline{X_A},\ \overline{X_B}}$ HHI $= (\overline{X_A}^*\ , \overline{X_B}^*) = (x^*, x^*)$. The HHI minimizing allocation is such that A & B have the same amount of resources.

Now as we saw in Lemma 1, if there is a transfer of δ from B to A then the equilibrium payoffs of B and A are given by $\pi_B{}^*(\delta)$ *and* $\pi_A{}^*(\delta)$ respectively.

Clearly $\pi_B{}^*(\delta) = \frac{\overline{X_B} - \delta + \sqrt{(\overline{X_A} + \delta)(\overline{X_B} - \delta)}}{\overline{X_A} + \overline{X_B} + \overline{X_C}}$. Hence $\pi_B{}^*(\delta) - \pi_B{}^* =$
$\frac{\sqrt{(\overline{X_A} + \delta)(\overline{X_B} - \delta)} - \sqrt{\overline{X_A}\ \overline{X_B}} - \delta}{\overline{X_A} + \overline{X_B} + \overline{X_C}}$

Define f$(\delta) = \sqrt{(\overline{X_A} + \delta)(\overline{X_B} - \delta)} - \sqrt{\overline{X_A}\ \overline{X_B}} - \delta$

f(0) = 0, and f$(\overline{X_B}) < 0$. $f'(\delta) = \frac{(\overline{X_B} - \delta) - (\overline{X_A} + \delta)}{2\sqrt{(\overline{X_A} + \delta)(\overline{X_B} - \delta)}} - 1$. Thus $\lim_{\delta \to 0} f'(\delta) > 0$

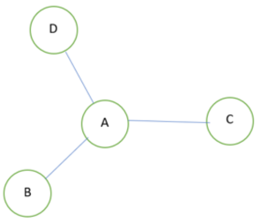
Also $f'(\frac{\overline{X_B} - \overline{X_A}}{2}) = -1 < 0$. Thus the optimal transfer for B, given by $\delta^{optimal} \in (0, \frac{\overline{X_B} - \overline{X_A}}{2})$.
Hence after transfer $\overline{X_B} \neq \overline{X_A}$.

Hence it's clearly proved that for any transfer-susceptible allocation, the rational transfer will never lead to the HHI minimizing allocation.

Also by Lemma 1 since in the HHI minimizing allocation $\overline{X_B}^* = \overline{X_A}^* < (3 + 2\sqrt{2}\,)\ \overline{X_A}^*$ no rational transfer is possible.

5. Pre – conflict & Post – conflict Inequality

In this section we try to investigate the relationship between inequality in the pre-conflict arms distribution to that of the post – conflict inequality in the distribution of the resources which were being fought over. An off-hand conclusion would be that the two should go hand in hand. Higher the inequality in distribution of arms & ammunitions, higher should be the post – conflict inequality. But that is not necessarily the case because there is another factor at play, the network structure. We will aim to drive home this point by using two special kinds of bipartite graphs, namely star-shaped and linear. Let us consider four agent networks as shown below.



The four agents are A, B, C and D. All the edges have equal weight i.e the valuation of the prize being fought over by any two agents is the same (=1 without loss of generality). Now let us say agent j is endowed with X_j amount of conflict resource (i.e the pre-conflict arms & ammunitions). So the pre-conflict resource share of agent j is given by $s_j = \frac{X_j}{\sum_{i \in \{A,B,C,D\}} X_i}$. And therefore the *pre-conflict HHI* = $\sum_{j \in \{A,B,C,D\}} {s_j}^2$. Now given a network and a pre-conflict allocation $\{X_j\}_{j \in \{A,B,C,D\}}$ let the equilibrium payoffs of agents j is given by $\pi_j{}^*$. Then the

post-conflict HHI = $\sum_{j \in \{A,B,C,D\}} \left(\pi_j{}^*\right)^2$. Now we proceed to ask the next natural question, which is whether a pre-conflict HHI minimizing allocation will induce a post-conflict HHI minimizing payoff. We formalize the idea with the help of the following definitions.

- *Feasible Allocation*: Given that the total amount of pre-conflict resource (i.e arms & ammunitions) is = $\bar{X}$, an allocation $\{X_j\}_{j \in \{A,B,C,D\}}$ is said to *feasible* if $\sum_{j \in \{A,B,C,D\}} X_j = \bar{X}$
- *Post conflict HHI minimizing allocation:* A feasible allocation $\{X_j\}_{j \in \{A,B,C,D\}}$ is said to be *the post conflict HHI minimizing allocation* if

  $$\{X_j\}_{j \in \{A,B,C,D\}} = \underset{\{X_j\}_{j \in \{A,B,C,D\}}}{\arg\max} \; post - conflict\ HHI$$

- *Pre conflict HHI minimizing allocation:* A feasible allocation $\{X_j\}_{j \in \{A,B,C,D\}}$ is said to be the *post conflict HHI minimizing allocation* if

  $$\{X_j\}_{j \in \{A,B,C,D\}} = \underset{\{X_j\}_{j \in \{A,B,C,D\}}}{\arg\max} \; pre - conflict\ HHI$$

- *Pre-conflict & Post-conflict Equivalence:* Any given network is said to satisfy the pre-conflict & post-conflict equivalence condition if the *Post - conflict HHI minimizing allocation* is the same as the *Post - conflict HHI minimizing allocation.*

Proposition 6:

i. A four - agent star-shaped network with equal weights (=1) satisfies the *Pre-conflict & Post-conflict Equivalence* condition
ii. A four - agent linear network with equal weights (=1) does not satisfy the *Pre-conflict & Post-conflict Equivalence* condition.
iii. In a four - agent linear network with equal weights (=1) the *Post – conflict HHI minimizing allocation*, allocates higher resources to the agents with lower degree i.e the terminal nodes.
iv. A regular graph need not satisfy the *Pre-conflict & Post-conflict Equivalence* condition, but a regular connected graph necessarily does.

Proof: Let us say the total pre-conflict resources (i.e arms & ammunitions) in the system is $\bar{X}$ *and* consider a feasible allocation $\{X_j\}_{j \in \{A,B,C,D\}}$ i.e $\sum_{j \in \{A,B,C,D\}} X_j = \bar{X}$. Consider the four-agent star-shaped network with A as the central node. Now consider the feasible allocation

$X_j = \frac{\bar{X}}{4} \forall j \in \{A, B, C, D\}$. Now A is engaged in three simultaneous conflicts with B, C and D. Let us say that the resource allocated by A for the conflict with i is given by $X_{A,i}$. Therefore A's optimization problem is given by:

Maximize $\sum_{i \in \{B,C,D\}} \frac{X_{A,i}}{X_{A,i}+\frac{\bar{X}}{4}}$ such that $\sum_{i \in \{B,C,D\}} X_{A,i} = \frac{\bar{X}}{4}$ ……… (A1)

The solution to (A1) is given by $X_{A,i}{}^* = \frac{\bar{X}}{12} \forall i \in \{B, C, D\}$. Hence $\pi_j{}^* = ¾ \forall j \in \{A, B, C, D\}$.

Thus in this case the pre-conflict HHI minimizer (i.e $X_j = \frac{\bar{X}}{4} \forall j \in \{A, B, C, D\}$) leads to post conflict HHI minimization since the resulting $\pi_j{}^* = ¾ \forall j \in \{A, B, C, D\}$. Thus we infer that the four agent star – shaped network with equal edge weights (=1) satisfies the *Pre-conflict & Post-conflict Equivalence* condition.

Now consider the four – agent linear network with equal edge-weights (=1) as shown in the figure above. Consider the following feasible pre – conflict allocation: $X_A = X_D = \frac{9\bar{X}}{32}$ and $X_B = X_C = \frac{7\bar{X}}{32}$. Let us say the resources invested by player j fighting player i is given by $X_{j,i}$. A and D are faced with only one opponent namely B and C respectively. Therefore

$X_{A,B} = \frac{9\bar{X}}{32}$ and $X_{D,C} = \frac{9\bar{X}}{32}$. B and C are faced with two simultaneous conflicts.

B's optimization problem is given by:

$$max_{X_{B,A},X_{B,C}} \quad \frac{X_{B,A}}{X_{B,A}+ X_{A,B}} + \frac{X_{B,C}}{X_{B,C}+ X_{C,B}} \text{ such that } X_{B,A} + X_{B,C} = \frac{7\bar{X}}{32}$$

C's optimization problem is given by:

$$max_{X_{C,B},X_{C,D}} \quad \frac{X_{C,B}}{X_{C,B}+ X_{B,C}} + \frac{X_{C,D}}{X_{C,D}+ X_{D,C}} \text{ such that } X_{C.B} + X_{C,D} = \frac{7\bar{X}}{32}$$

The equilibrium payoffs are given by $\pi_j{}^* = ¾ \forall j \in \{A, B, C, D\}$. Thus we infer that the pre – conflict allocation ($X_A = X_D = \frac{9\bar{X}}{32}$ and $X_B = X_C = \frac{7\bar{X}}{32}$) is the post – conflict HHI minimizing allocation. Thus in this case the pre-conflict HHI minimizing allocation ($X_j = \frac{\bar{X}}{4} \forall j \in \{A, B, C, D\}$) is not the post-conflict HHI minimizer. Thus we conclude that the four-agent linear network with equal edge-weight does not satisfy the *Pre-conflict & Post-conflict Equivalence* condition.

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