# Equilibrium in Asymmetric Auctions* § ¶ ‖

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**Abstract**

We consider first-price auctions with independent and private valuations (IPVFPA) with asymmetric valuation distributions as well as asymmetric supports. We first show the existence of equilibrium in these auctions through a perturbation approach, thereby establishing that the limit of Bayesian Nash equilibria (BNE) of such perturbed auctions is indeed the Bayesian Nash equilibrium (BNE) of the limit IPVFPA with different supports. We then characterize this BNE and show that the $\varepsilon$-equilibrium of the IPVFPAs with different supports is a BNE of an IPVFPA with common support. We then provide a discussion of solution methods for computing the $\varepsilon$-BNE and go on to show some numerical examples of our results.

## 1 Introduction

Consider a government auctioning off the rights to construct an airport terminal; or consider a government auctioning off the rights to construct a high-speed rail corridor. Both large and small firms might participate in such auctions. Symmetric auctions abstract away from these real-life considerations. In the same spirit, it is implausible that any asymmetry is only restricted to the distribution functions of valuations, and does not extend to the support of such distributions. Indeed, there is no reason to believe that in reality, the players' distributions should have common support. Hence, auctions with common supports of valuations are an abstraction of a more realistic scenario wherein there may be asymmetry in both the distribution functions themselves and the supports of such distribution functions. In light of these arguments, the existence and characterization of the Bayesian Nash Equilibrium (BNE) in auctions with asymmetric supports assumes quite a bit of importance in Auction theory literature.

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In this paper, we study a first-price auction with independent and private valuations (IPVFPA) with asymmetric supports and no reserve price. We establish the existence of a BNE and characterize the BNE in IPVFPA with asymmetric supports using a perturbations approach. As a natural result of this approach, we show that the $\varepsilon$-BNE of an IPVFPA with asymmetric supports is the BNE of an IPVFPA with common support. In other words, any IPVFPA with asymmetric supports can be perturbed to get an IPVFPA with common supports, and the BNE of the perturbed game is the $\varepsilon$-BNE of the original game. This allows us to compute the $\varepsilon$-BNE of IPVFPA with asymmetric supports. We then discuss the need for and implement specialized BVP solvers to compute such $\varepsilon$-BNE. Thus, we address three questions in IPVFPA with asymmetric supports: existence, characterization, and implementation. In this regard, our paper bridges an important gap in the application of Auction theory to real-life scenarios.

The first-price sealed-bid auctions have been studied extensively in the literature, beginning with Vickrey's seminal paper (Vickrey, 1961). A cornerstone of this analysis is the existence of a Bayesian Nash equilibrium (BNE) in which players' bidding functions are monotone in their types. Existence can be readily established for the case where valuations are smoothly, independently, and identically distributed, as is done by Milgrom and Weber, 1982. Outside of that restrictive setting, the task becomes more difficult, or as Vickrey, 1961 describes it, "the mathematics of a complete treatment becomes intractable."

Nonetheless, progress has been made in the scenarios where bidders' valuations are independent but not identically distributed. Lebrun, 1999 proves the existence of a BNE in a setting with asymmetric valuation distributions with a common interval of support. To accomplish this task, he characterizes the inverse bidding functions associated with the BNE using a system of ordinary differential equations (ODE) with separable boundary conditions. One of these boundary conditions specifies $\eta$, which is the bid each player makes at the highest possible valuation. In equations, this means that for each player $i$, $\beta_i(d) = \eta$, $\alpha_i(\eta) = d$ where $[c, d]$ is the support of player $i$'s valuation distribution, $\beta_i : [c, d] \to [c, \eta]$ is player $i$'s bidding function and $\alpha_i : [c, \eta] \to [c, d]$ is player $i$'s inverse bidding function. Lebrun, 1999 thus reduces the problem of solving for the BNE in IPVFPA with common support to solving a system of first order two-point Boundary Value Problem (BVP).

Lebrun, 2006 uses a sliding approach to show the existence of equilibrium in IPVFPA with asymmetric supports and reserve price. However, this characterization technique suffers from some major issues. Consider an auction when there are $3$ players, and their distributions of valuations are $F_1 = F_2 = U[0, 10]$, $F_3 = U[0, 8]$. Lebrun, 2006's characterization would compute some $\eta$ and then specify that:

- $\alpha_1(\eta) = \alpha_2(\eta) = \alpha_3(\eta) = 10;$
- $\alpha_3$ is an affine function on $[\beta_3(8), \eta]$.

Notice that in the interval $[0, \beta_3(8)]$, we would have a system of ODE involving all the $3$ players, and on the interval $[\beta_3(8), \eta]$, we would have an ODE system involving only players $1$ and $2$. We obviously cannot solve the whole system in one shot. Indeed, we need $\beta_3(8)$, $\alpha_1(\beta_3(8))$ and $\alpha_2(\beta_3(8))$ to break down the system into two separate systems of ODE and numerically solve them. However, there is no possible way of computing even $\beta_3(8)$. Further, we have no way of computing $\alpha_1(\beta_3(8))$ and $\alpha_2(\beta_3(8))$ either. In other words: we want to solve an ODE system, but we have no idea about the interval over which we have to solve it, and we have no clue about the boundary values we need to solve it. It suffices to say that we cannot numerically solve this problem.

Consequently, the whole approach of breaking down the ODE system into two or more separate systems completely breaks down, and we are at a dead end with no possibility of making any further inroads, and the whole exercise of computation of $\{\beta_i\}_{i=1,2,3}$ goes out of the window. This situation arises because the characterization of the BNE no longer remains a two-point BVP, but becomes an incompletely specified or an ill-posed Differential Algebraic Equations (DAE) problem. We emphasize that this scenario poses absolutely no problem in the existence of an analytic solution, but it wreaks havoc while computing a numerical solution using a numerical ODE solver. Thus, while Lebrun, 2006's existence proof is a seminal contribution, his approach does not allow for any progress in computation.

In light of the significant problems in the computation of BNE in auctions with asymmetric supports, it seems logical to investigate the computation of BNE in auctions with common support and use the BNE of auctions with common supports to approximate the BNE of auctions with asymmetric supports. Naturally, this leads us to the notion of $\varepsilon$-BNE. To that end, we use a perturbation approach to "approximate" an IPVFPA with asymmetric supports with an IPVFPA with common supports. We refer to such approximations as perturbations. However, the literature is silent on the question of the relation between the BNE of the perturbed games and the BNE of the original auction.

It is a well-known result that games that are "close" might have equilibria that are not "close". Therefore, there is no *a priori* reason to expect that the BNE of the perturbed games should give us any idea about the equilibria of the original auction. Our first main contribution in this paper is that we show that as the perturbed games converge to the limit auction[1], the equilibria of the perturbed games converge to the equilibrium of the original auction. This result states that the *limit of the equilibria is the equilibrium of the limit game.* The entire computational exercise stands upon this result. Subsequently, we use this result to show that the BNE of the perturbed game is an $\varepsilon$-BNE of the original auction. Thus, the perturbation approach gives us a natural way to relate the $\varepsilon$-BNE of an IPVFPA with asymmetric supports with the BNE of IPVFPA with common supports.

Once the result relating the $\varepsilon$-BNE of an IPVFPA with asymmetric supports with the BNE of IPVFPA with common supports is at hand, we focus on the computation aspect of BNE in auctions with common supports. We first present a discussion on numerical solvers that should be used for these two-point BVPs, and then we delve into details of some examples. This discussion is important because of the prevalence of two-point BVPs in Economic theory in general, and in Auction theory in particular. This discussion and examples illustrate that numerically solving an ODE system with

[1]We formally define this convergence in Section 2

boundary conditions is far from a straightforward task, and demonstrate the need to use specialized BVP solvers. We use Julia to simulate our numerical examples. We also state the reasons for choosing Julia and mention other options available for solving such kinds of problems.

The paper proceeds as follows. We first give a gist of the relevant literature. We then introduce our model in section 2. We show the existence and characterization of the BNE in Section 3. We subsequently present a discussion of numerical boundary value solvers and some examples in Section 4. Section 5 concludes. Proofs of the claims and the lemmas are included in the Appendix.

## 1.1 Literature

### 1.1.1 Literature on Auctions

The existence and characterization of a monotone BNE in symmetric auctions have been studied quite extensively. Milgrom and Weber, 1982 is the classical reference. Reny, 1999 and Barelli and Meneghel, 2013 establish a condition called Better Reply Security which guarantees the existence of equilibrium in pure strategies for discontinuous games. These results can be used to argue the existence of BNE in IPVFPA.

Seminal work on the existence of a monotone BNE in First Price Auctions is due to Reny and Zamir, 2004. They establish a more general existence result for a setting with interdependent valuations and affiliated signals. Their proof uses a sequence of games with finite bidding sets and does not characterize the equilibrium directly. This is an existence result, but not a characterization result, and hence, is silent on the issue of computation.

Lebrun, 1999 and Lebrun, 2006 are references for characterization of BNE in IPVFPA with symmetric and asymmetric supports of distributions respectively. Lebrun, 2006 comes closest to our paper in terms of characterization of BNE, but, as discussed before, this characterization suffers from some major issues.

### 1.1.2 Literature on ODE and other mathematical tools

Birkhoff and Rota, 1969 remains the classical reference for the theory of ODE. Soetaert et al., 2012 contains a detailed exposition on Explicit Runge Kutta methods and Newton's methods. These methods are well established in ODE literature. Ascher et al., 1995 presents a detailed exposition on the issues arising in numerically solving the BVPs. Cash and Singhal, 1982 first proposed the Mono Implicit Runge Kutta (MIRK) methods for solving stiff two-point BVPs. Subsequently Cash and Wright, 1991 and Cash, 1996 expounded upon the work of Cash and Singhal, 1982.

Barvínek et al., 1991 proposed an important result on the convergence of a sequence of inverse functions. This result adds to the existing results on the convergence of a sequence of functions.

# 2 The Model

We consider a sealed bid auction $\Gamma$ in which a single indivisible item is up for sale to $n$ bidders, where $2 \leq n < \infty$. Bids are submitted simultaneously, with the highest bidder receiving the item and paying a price equal to their bid. If multiple bidders tie for the highest bid, a fair tie-breaking rule is used. There is no reserve price.

We denote by $N := \{1, 2, \ldots, n\}$ the set of bidders. Each bidder $i \in N$ learns their private valuation $v_i$, which is distributed according to $F_i$, prior to bidding. The game $\Gamma$ and all its features are commonly known.

We follow assumptions of Lebrun, 2006:

**A.1** For each $i \in N$, the support of $F_i$ is the interval $[c_i, d_i]$.

**A.2** The pdf of player $i$ is locally bounded away from $0$ in the interval $(c_i, d_i]$.

We note that Lebrun, 1999 implicitly assumes that for each $i \in N$, $f_i$ must be differentiable in the interval $(c_i, d_i)$. Thus, the assumption A.2 strengthens to:

**A.S** $f_i$ is locally bounded away from $0$ and is continuous and differentiable in the interval $(c_i, d_i]$ for each player $i$.

We also assume that:

**A.3** $\bigcap\limits_{i \in N} (c_i, d_i) \neq \emptyset$.

We define $[c, d] := \bigcup\limits_{i \in N} [c_i, d_i]$. We define the extended pdf $g_i$, by

$$g_i(x) := \begin{cases} f_i(x) & x \in [c_i, d_i] \\ 0 & x \in [c, c_i) \cup (d_i, d] \end{cases}$$

As $f_i$ is locally bounded away from $0$ in $(c_i, d_i]$, we can see that $g_i$ is only piece-wise continuous and piece-wise differentiable on $[c, d]$. In fact, $g_i(x)$ is discontinuous at $c_i$ and $d_i$ (if $c_i \neq c$ and $d_i \neq d$). Consequently, we consider a sequence of continuous and differentiable functions $\{g_i^k\}_{k \geq 1}$ with domain $[c, d]$ that converge point-wise to $g_i$ and satisfies $\int_{[c,d]} g_i^k(x)dx = 1$. There are many such sequences and we need not specify one. However, for ease of exposition, we choose such a sequence so that on the interval $[c_i, d_i]$, $|g_i^k - f_i| = \frac{1}{k}$.

Let $\Gamma^k$ denote an auction which is identical to $\Gamma$ in setup, but with valuations of players now distributed according to density functions $\{g_i^k\}_{i=1}^n$ and distribution functions $\{G_i^k\}_{i=1}^n$. We call $\Gamma^k$ a perturbed game. Let $\{\beta_i^k\}_{i=1}^n$ be the BNE of the perturbed game $\Gamma^k$. Notice that as $k \to \infty$, $g_i^k \to g_i$ and $G_i^k \to G_i$. This is what we mean by writing $\Gamma^k \to \Gamma$ or by writing $\Gamma^k$ converges to $\Gamma$. We denote the respective inverse bid functions by $\{\alpha_i^k\}_{i=1}^n$. Define $\phi_{li}^k(v) := \alpha_l^k(\beta_i^k(v))$. Notice that $\{\{\beta_i^k\}_{i=1}^n, \{\phi_{li}^k\}_{l \neq i}\}_{k \geq 1}$ is a sequence of continuous monotonic functions. Therefore, by Helly's selection theorem, it admits a point-wise convergent subsequence. By abuse of notation, we call this subsequence $\{\{\beta_i^k\}_{i=1}^n\}_{k\geq 1}, \{\phi_{li}^k\}_{l\neq i}\}_{k\geq 1}$. Let $\langle\{\beta_i\}_{i=1}^n, \{\phi_{li}\}_{l\neq i}\rangle$ be the point-wise limit of this subsequence, i.e. $\{\{\beta_i^k\}_{i=1}^n\}_{k\geq 1}, \{\phi_{li}^k\}_{l\neq i}\}_{k\geq 1} \to \langle\{\beta_i\}_{i=1}^n, \{\phi_{li}\}_{l\neq i}\rangle$. We also define following notation:

$$
\begin{aligned}
X_{il}(\beta_i, \beta_l, v, \varepsilon) &:= \{v_l \geq c : \beta_l(v) < \beta_i(v) + \varepsilon\} \\
W_{il}(\beta_i, \beta_l, v) &:= \{v_l \geq c : \beta_l(v) = \beta_i(v)\} \\
Z_{il}(\beta_i, \beta_l, v) &:= \{v_l \geq c : \beta_l(v) \leq \beta_i(v)\} \\
v_1^l &:= \sup(X_{il}(\beta_i, \beta_l, v, 0)) \\
v_2^l &:= \sup(\liminf_k (X_{il}(\beta_i^k, \beta_l^k, v, 0))) \\
v_3^l &:= \sup(\limsup_k (X_{il}(\beta_i^k, \beta_l^k, v, 0))) \\
v_4^l &:= \sup(W_{il}(\beta_i, \beta_l, v))
\end{aligned}
$$

# 3 Existence and Characterization

We have two main goals in this section. First, we show that a BNE exists in the limit game $\Gamma$. To that end, we establish that the limit of the equilibria is the equilibrium of the limit game. We formalize this result in Theorem 1. We then proceed to characterize the limit equilibrium, i.e., we show the continuity and piece-wise differentiability of the equilibrium bidding strategies. Theorem 2 formalizes the continuity result, and Theorem 3 and Lemma 6 formalize the piece-wise differentiability results. Subsequently, we show that the equilibrium strategy profile $\{\beta_i^k\}_{i=1}^n$ forms an $\varepsilon$-equilibrium of the limit game $\Gamma$. This result is formalized in Theorem 4. We provide proofs of the theorems in the main text and state the claims and the proofs of lemmas in the Appendix.

**Lemma 1.** *For each player $i$, if the strategy profile $\{\beta_i\}_{i=1}^n$ allows the player $i$ to win with a positive probability, then probability of the player $i$ winning with a tie under the strategy profile $\{\beta_i\}_{i=1}^n$ is $0$.*

Lemma 1 is an analogue of the famous *no tie in the winning bid* condition. Here, we have not yet established that $\{\beta_i\}_{i=1}^n$ is indeed the equilibrium strategy profile, therefore, we need to show that no tie holds for $\{\beta_i\}_{i=1}^n$ without appealing to any kind of better reply security or any of the other standard Auction theory arguments. The crux of the proof is to establish that $v_1^l = v_2^l = v_3^l = v_4^l \; \forall \; l \in N \setminus \{i\}; \forall \; i$[2]. Lemma 1, and the arguments used in the proof of Lemma 1 allow us to state the following corollary.

**Corollary 1.** *For each player $i$, the payoff of the player $i$ in the limit game under the limiting strategies is equal to the limit of the equilibrium payoffs.*

Having established Lemma 1 and Corollary 1, proving the existence of the BNE in the limit game is fairly straightforward.

[2]Please see the appendix for the proof.

**Theorem 1.** $\{\beta_i\}_{i=1}^n$ *is the BNE of the limit game.*

*Proof.* Suppose not, and some player $i$ has an incentive to deviate from $\beta_i$ to some strategy $\gamma$[3]. Then, for a $G_i$-non-negligible set of valuations $v$ of player $i$:

$$(v-\gamma(v))\prod_{l\neq 1} F_l\{v_l \geq c : \beta_l(v_l) < \gamma(v)\} > (v-\beta_i(v))\prod_{l\neq 1} F_l\{v_l \geq c : \beta_l(v_l) < \beta_i(v)\}$$

From Lemma 1 and associated claims[4], for any such valuation $v$, $\exists\, K_v$ such that $\beta_i^k$ is not a best response to $\{\beta_j^k\}_{j\neq i}$ for any $k > K_v$. This is a contradiction. □

**Note:** Here and hereafter, the notation $\{f^k\}_{k\geq 1} \rightrightarrows f$ denotes that the sequence of functions $\{f^k\}_{k\geq 1}$ converges uniformly to the function $f$.

**Note:** Here and hereafter, the moniker "almost everywhere" is supposed to mean "almost everywhere" with respect to the Lebesgue measure on $\mathbb{R}$.

Now we want to establish regularity conditions on the equilibrium bidding functions. The following lemma underlies our continuity result in Theorem 2, and we will show piece-wise differentiability in Theorem 3.

**Lemma 2.** $\{\beta_i\}_{i=1}^n$ *is a continuous function almost everywhere. If $\beta_i$ is discontinuous, then there exist a lower semi-continuous function and an upper semi-continuous functions which disagree with $\beta_i$ only on the points of discontinuities. These functions are also equilibrium strategies of player $i$.*

For any player $i$, define $\underline{v_i} := \inf\{v : \phi_{li}(v) > \max_{l\neq i} c_l\}$. Notice that $[\underline{v_i}, d_i]$ is the set of valuations where player $i$ has a positive probability of winning. Therefore, for any $v \in (\underline{v_i}, d_i]$, expected payoff $(v-\beta_i(v))\prod_{j\neq i} F_j(\phi_{ji}(v)) > 0$.

**Theorem 2.** *$\beta_i$ is continuous over the domain $(\underline{v_i}, d_i]$.*

*Proof.*

**Lemma 3.** *For any $\varepsilon > 0$, $v - \beta_i(v)$ is bounded away from $0$ on $[\underline{v_i}+\varepsilon, d]$.*

For any $\varepsilon > 0$, Lemma 3 implies that $\lim_{k\geq 1} \frac{1}{\frac{d\beta_i^k(v)}{dv}}$ is bounded above on the domain $[\underline{v_i}+\varepsilon, d_i]$,and hence $\lim_{k\geq 1} \frac{d\alpha_i^k(b)}{db}$ is uniformly bounded on the domain $[\beta_i(\underline{v_i}+\varepsilon), \beta_i(d_i)]$. Thus, $\alpha_i^k$ is Lipschitz continuous on $[\beta_i(\underline{v_i}+\varepsilon), \beta_i(d_i)]\ \forall\ k$ with some uniform Lipschitz constant. This implies that $\alpha_i$ is also Lipschitz continuous on the domain $[\beta_i(\underline{v_i}+\varepsilon), \beta_i(d_i)]$ with the same Lipschitz constant. By Dini's Theorem, it follows that $\{\alpha_i^k\}_{k\geq 1} \rightrightarrows \alpha_i$ on $[\beta_i(\underline{v_i}+\varepsilon), \beta_i(d_i)]$. Furthermore, $\lim_{k\geq 1} \frac{d\beta_i^k(v)}{dv}$ exists and is a.e. finite valued, which

[3]WLOG, we may take $\gamma$ to be such that $\gamma$ doesn't win with a tie and has a positive probability of the tie. This is so because then there is some deviation $\gamma'$ for player $i$ which is slightly higher than $\gamma$ and has no tie, thus yielding better expected payoff to $i$.

[4]The associated claims appear in the proof of Lemma 1 in the Appendix.

we prove in the Appendix under Claim 8. It follows that $\alpha_i$ is strictly monotonic on $[\beta_i(\underline{v_i}+\varepsilon), \beta_i(d_i)]$. Then Theorem 1 of Barvínek et al., 1991 implies that $\{\beta_i^k\}_{k\geq 1} \rightrightarrows \alpha_i^{-1}$. However, since $\{\beta_i^k\}_{k\geq 1} \to \beta_i$, the uniqueness property of limits in a Hausdorff space implies that $\alpha_i^{-1} = \beta_i$ on the domain $[\underline{v_i}+\varepsilon, d_i]$. Further, since $\alpha_i^{-1}$ is strictly monotone and continuous (inverse of a strictly monotone and a continuous function), it follows that $\beta_i$ is continuous and strictly monotone on the domain $[\underline{v_i}+\varepsilon, d_i]$. Letting $\varepsilon \to 0$, $\beta_i$ is continuous and strictly monotone on the domain $(\underline{v_i}, d_i]$. □

Define $\overline{v_i} := \min\{d_i, \sup\{v : \phi_{li}(v) < \min_{j\neq i} d_j\}\}$. Arguments from proof of Theorem 2 imply that if $v \in (\underline{v_i}, \overline{v_i})$, then $\phi_{li}(v) = \alpha_l(\beta_i(v))$ and is therefore a continuous function. This is so because $v \in (\underline{v_i}, \overline{v_i})$ is a sufficient condition for $\phi_{li}(v) = \alpha_l(\beta_i(v))$[5]. Define the set $\Lambda_i := \{l \in N : \phi_{li}(d_i) > d_l\}$. For each $l \in \Lambda_i$, define $v_i^l := \inf\{v : \phi_{li}(v) = d_l\}$. Define $\overline{v_i^j}$ to be the $j^{th}$ lowest value among all such $v_i^l$. Notice that $\overline{v_i} = \overline{v_i^1}$. Define $M_j := \{k : v_i^k < \overline{v_i^j}\}$ and define $m_j := \#(N \setminus M_j)$. Notice that if $v \in (\overline{v_i^j}, \overline{v_i^{j+1}})$, $M_j$ represents the set of players who are beaten by player $i$ with probability $1$, while all other players have a non-zero chance of beating player $i$.

**Theorem 3.** *For each player $i$, the equilibrium bid function is differentiable in the domain $[\underline{v_i}, d_i]$ except possibly at points $\{\overline{v_i^j}\}_{j\in J_i}$.*

*Proof.*

**Lemma 4.** *The functions $\langle \beta_i, \{\phi_{li}\}_{l\neq i}\rangle$ are differentiable on the domain $(\underline{v_i}, \overline{v_i})$ and are solutions to the following IVP (initial value problem)[6]:*

$$\gamma_i(\overline{v_i}) = \beta_i(\overline{v_i})$$

$$\psi_{li}(\overline{v_i}) = \phi_{li}(\overline{v_i})$$

$$\frac{d\gamma_i(v)}{dv} = (n-1)\frac{f_i(v)}{F_i(v)}\frac{1}{\frac{-(n-2)}{v-\gamma_i(v)} + \sum\limits_{j\neq i}\frac{1}{\psi_{ji}(v)-\gamma_i(v)}}$$

$$\frac{d\psi_{li}(v)}{dv} = \frac{f_i(v)}{F_i(v)}\frac{F_l(\psi_{li}(v))}{f_l(\psi_{li}(v))}\frac{\frac{-(n-2)}{\psi_{li}-\gamma_i(v)} + \sum\limits_{j\neq l}\frac{1}{\psi_{ji}(v)-\gamma_i(v)}}{\frac{-(n-2)}{v-\gamma_i(v)} + \sum\limits_{j\neq i}\frac{1}{\psi_{ji}(v)-\gamma_i(v)}};\ l\neq i$$

[5]If $v > \overline{v_i}$, then for the $l$ such that $\beta_i(v) > d_l$, $\alpha_l$ need not be $\beta_l^{-1}$ since $\beta_l$ need not be continuous or strictly monotone in $(d_l, d]$.

[6]Readers may notice that this is actually a BVP with values given at upper end. However, for mathematical analyses, we need not worry about this difference, since the Picard's existence and uniqueness theorem holds when running the ODE in the reverse direction. Numerically solving this problem is a different question entirely. Please see the next section for more discussion on numerical solving methods.

**Lemma 5.** *Given that $m_j > 2$, in the domain $(\overline{v_i^j}, \overline{v_i^{j+1}})$, the functions $\{\beta_i, \{\phi_{ki}\}_{k \notin M_j \cup \{i\}}\}$ are differentiable and solutions to the IVP:*

$$\gamma_i(\overline{v_i^j}) = \beta_i(\overline{v_i^j})$$

$$\psi_{li}(\overline{v_i^j}) = \phi_{li}(\overline{v_i^j})$$

$$\frac{d\gamma_i(v)}{dv} = (m_j - 1)\frac{f_i(v)}{F_i(v)} \frac{1}{\frac{-(m_j-2)}{v-\gamma_i(v)} + \sum\limits_{k \notin M_j \cup \{i\}} \frac{1}{\psi_{ki}(v)-\gamma_i(v)}}$$

$$\frac{d\psi_{li}(v)}{dv} = \frac{f_i(v)}{F_i(v)} \frac{F_l(\psi_{li}(v))}{f_l(\psi_{li}(v))} \frac{\frac{-(m_j-2)}{\psi_{li}-\gamma_i(v)} + \sum\limits_{k \notin M_j \cup \{l\}} \frac{1}{\psi_{ki}(v)-\gamma_i(v)}}{\frac{-(m_j-2)}{v-\gamma_i(v)} + \sum\limits_{k \notin M_j \cup \{i\}} \frac{1}{\psi_{ki}(v)-\gamma_i(v)}}$$

□

**Lemma 6.** *For each player $i$, there is a piece-wise differentiable, continuous bid function defined on the domain $[c_i, d_i]$. This is an equilibrium strategy profile for the game. Such an equilibrium strategy agrees with the limit of the equilibria in the set of valuations where player $i$ has a positive probability of winning the auction.*

Theorems 1, 2 and 3 together imply that for any IPVFPA with non disjoint but asymmetrical supports, there is an IPVFPA with common support whose BNE is the $\varepsilon$-equilibrium for the original IPVFPA. The common support may be defined as the union of all supports. We formalize this discussion as follows:

**Theorem 4.** *For any $\varepsilon > 0$, there exists $K_\varepsilon \in \mathbb{N}$ such that for any $k > K_\varepsilon$, the equilibrium $\{\beta_i^k\}_{i=1}^n$ of $\Gamma^k$ is an $\varepsilon$-equilibrium of $\Gamma$.*

*Proof.* Let $\varepsilon > 0$. Define $K_\varepsilon := \lceil \frac{2(d-\eta)}{\varepsilon} \rceil + 1$[7], where $\lceil . \rceil$ denotes the ceiling function. Then $\forall\ k > K_\varepsilon$, notice that $\varepsilon > \frac{2(d-\eta)}{(k-1)}$ and that $\forall\ v \in [\underline{v_i}, d_i]$, $\frac{2(d-\eta)}{k-1} \geq \frac{(v-\beta_i(v))}{k-1} + \frac{(v-\beta_i^k(v))}{k-1}$. Further notice that $\frac{1}{k-1} > \frac{1}{k} + \frac{1}{k^2} + \cdots + \frac{1}{k^{n-1}}$ and that $G_l(\phi_{li}(v)), G_l(\phi_{li}^k(v)) \leq 1$, and finally, notice that:

$$|G_l(\phi_{li}^k(v)) - G_l^k(\phi_{li}^k(v))| \leq \frac{1}{k}$$
$$|G_l(\phi_{li}(v)) - G_l^k(\phi_{li}(v))| \leq \frac{1}{k}$$

Thus, after appropriate algebraic manipulations, we can write:

[7] $\eta = \lim\limits_{k \to \infty} \eta^k$.

$$\frac{(v-\beta_i(v))}{k-1} \geq (v-\beta_i(v))\left[\prod_{l\neq i} G_l(\phi_{li}(v)) - \prod_{l\neq i} G_l^k(\phi_{li}(v))\right]$$
$$\frac{(v-\beta_i^k(v))}{k-1} \geq (v-\beta_i^k(v))\left[\prod_{l\neq i} G_l^k(\phi_{li}^k(v)) - \prod_{l\neq i} G_l(\phi_{li}^k(v))\right]$$

Summing up these two equations, we get:

$$\left[(v-\beta_i(v))\prod_{l\neq i} G_l(\phi_{li}(v)) - (v-\beta_i^k(v))\prod_{l\neq i} G_l(\phi_{li}^k(v))\right]+$$
$$\left[(v-\beta_i^k(v))\prod_{l\neq i} G_l^k(\phi_{li}^k(v)) - (v-\beta_i(v))\prod_{l\neq i} G_l^k(\phi_{li}(v))\right] < \varepsilon$$

For any player $i$, The term in the first square brackets is the loss in payoffs for the player $i$ when the strategy profile $\{\beta_j^k\}_{j=1}^n$ is played in the game $\Gamma$, and the term in the second square brackets represents the loss in payoffs for the player $i$ when the strategy profile $\{\beta_j\}_{j=1}^n$ is played in the game $\Gamma^k$. Both of these terms are non-negative. The proof is complete. □

Theorem 4 also allows us to state the following corollary.

**Corollary 2.** *Any limit of $\varepsilon$-equilibria of $\Gamma$ is the BNE of $\Gamma$.*

Theorem 4 and Corollary 2 form the basis of our numerical simulations, which we present in the next section.

# 4 Numerical Simulations

In light of Theorem 4, the computation of BNE of a general IPVFPA is effectively the same problem as the computation of BNE in IPVFPA with common support. This is effectively solving the two-point BVP a-la Lebrun, 1999. For sake of completeness, we mention the ODE system along with the initial and the boundary conditions here:

$$\frac{d\alpha_i(b)}{db} = \frac{1}{n-1}\frac{F_i(\alpha_i(b))}{f_i(\alpha_i(b))}\left(\frac{-(n-2)}{\alpha_i(b)-b} + \sum_{l\neq i}\frac{1}{\alpha_l(b)-b}\right)$$

$$\alpha_i(c) = c; \alpha_i(\eta) = d;\ \eta \in [\eta_1, \eta_2];\ \forall\, i$$

$$\eta_1 = d - \max_j \left\{ \int_c^d \prod_{l\neq j} F_l\Big(\Big\{F_j^{-1}\Big(F_l(v)\Big(\min_{v\leq w\leq d}\frac{F_j(w)}{F_l(w)}\Big)\Big)\Big\}^{-1}\Big)dv \right\}$$

$$\eta_2 = d - \min_i \left\{ \int_c^d F_i(v)^{n-1} \prod_{l \neq i} \min_{v \leq x \leq d} \frac{F_l(x)}{F_i(x)} dv \right\}$$

Observe that in symmetric cases, $\eta_1 = \eta_2$. For asymmetric cases, Lebrun, 2006 specifies that if there is some $x > 0$ such that $F_i$ is strictly log-concave over the interval $(\max_j c_j, \max_j c_j + x) \cap (c_i, d_i)$ for each player $i$, then $\eta_1 = \eta_2$. This turns out to be the condition of strictly decreasing reverse hazard rate. Fortunately, our examples satisfy the conditions imposed by Lebrun, 2006, thereby allowing us to undertake the computation without worrying about multiple $\eta$.

## 4.1 Discussion of Solvers for the ODE system

We have a first order two-point BVP with separable boundary conditions. It seems tempting to drop the boundary condition $\alpha_i(c) = c$ and just tackle this as an IVP with condition $\alpha_i(\eta) = d$, and run the ODE system in the reverse direction by executing a change of variables. This temptation turns out to be a grave sin, as we now discuss.

In reality, this problem is a two-point BVP. Therefore, the numerical value computed at any point $x \in (c, \eta)$ depends upon boundary conditions at both $c$ and $\eta$. IVP solvers do not take this fact into account and use only one boundary condition. An additional challenge to executing a change of variables and tackling this problem as an IVP arises from the fact that our ODE system might face a problem of *stiffness*. This means that $\frac{d\alpha_i(b)}{db}$ may change rapidly in some zones compared to other zones. The stiffness problem makes the traditional IVP solvers like RK45, RK23, etc. quite unstable because of endogenous step selection. The first problem, combined with the stiffness makes the problem unsolvable by using an IVP solver. Unfortunately, the very same issues which prevent the problem from getting solved as an IVP plague the shooting methods as well.

Thus, we are constrained to use a specialized BVP solver which relies upon an implicit Runge Kutta (IRK) method, which can tackle both of these problems in one shot. Such classes of Runge Kutta methods divide the interval $[c, \eta]$ into a finite number of mesh points $c \leq x_0 < x_1 < \cdots < x_n \leq \eta$, and generate a set of nonlinear equations using **all** of these mesh points simultaneously[8]. These methods then use the nonlinear equations generated using all of these mesh points and the Jacobian (evaluated at all the mesh points) to generate a solution (at all the mesh points). Subsequently, the solutions at the mesh points are interpolated by some smooth polynomial functions. Thus, the resultant solution is a continuous piece-wise polynomial. For a more detailed discussion of these issues, we refer the interested readers to Soetaert et al., 2012 and Ascher et al., 1995.

We use the MIRK $4$ algorithm proposed by Cash and Wright, 1991 and Cash, 1996. MIRK $4$ (Mono Implicit Runge Kutta of Order $4$) is an IRK method, which uses the Lobatto scheme ($c = x_0$, $x_n = \eta$). This algorithm is implementable in R, MATLAB,

[8]In contrast, the explicit Runge Kutta (ERK) methods solve a set linear equation at a particular mesh point $x_i$ using only the mesh points $x_0, x_1 \cdots, x_i$.

and Julia. This algorithm is implementable under the library **bvpSolve** in R (Mazzia et al., 2014); under the package **bvp4c** in MATLAB (Cash et al., 2013); and under the package **DifferentialEquations.jl** in Julia (Rackauckas and Nie, 2017). We use Julia to implement this solver for two major reasons. First, Julia's compiler is much faster than R's. Second, Julia is open-source. Julia uses Newton's methods to solve the equations generated at the mesh points and to find the interpolating polynomials.

## 4.2 Examples

### 4.2.1 Example 1

Let $N = 3$, $c_1 = c_2 = c_3 = 0$, $d_1 = d_2 = 10$, $d_3 = 8$. $F_1 = F_2 = U[0, 10]$, $F_3 = U[0, 8]$. We approximate $g_3$ as follows:

$$g_3^k(x) = \begin{cases} \frac{k-1}{8k} & x \in [0, 8) \\ P(x) & x \in [8, z] \\ \frac{1}{10k} & x \in [z, 10] \end{cases}$$

Here, $P(x)$ is a third degree polynomial of the form $P(t) = A \int (t-8)(t-z)dt + B$, where

1. $P(8) = \frac{k-1}{8k}$
2. $P(z) = \frac{1}{10k}$
3. $\int\limits_8^z P(x)dx = \frac{z}{10k}$

First two conditions follow from continuity of $g_3(x)$, while the last condition ensures that $g_3(x)$ is a valid density function. By construction, we are assured about two things:

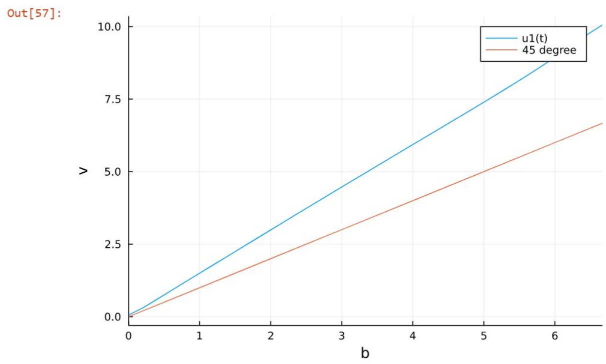
- $g_3^k(x)$ is continuously differentiable;
- $P(x) > 0$. This is so because by construction, $P(x)$ is a cubic polynomial with local maxima $8$ and local minima $z$ and is strictly monotone decreasing. Writing down the derivative of $P(x)$ makes this evident.

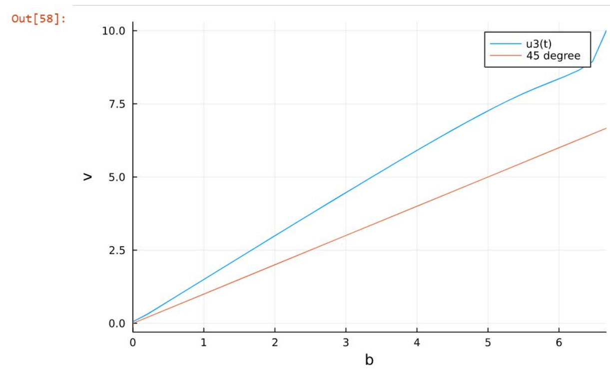
We set $k = 10$ directly instead of $\varepsilon$ and solve for $P(x)$ and $z$ using Newton's method in Julia. We compute $P(x) \approx 0.053361x^3 - 1.14726x^2 + 12.3666816x - 35.5595$, and $z \approx 9.561$. Thus, $g_3^k$ becomes:

$$g_3^{10}(x) = \begin{cases} \frac{9}{80} & x \in [0, 8) \\ 0.053361x^3 - 1.14726x^2 + 12.3666816x - 35.5595 & x \in [8, 9.561] \\ \frac{1}{100} & x \in [9.561, 10] \end{cases}$$

The inverse bid functions for players $1$ and $3$ in this example are shown below[9]:

[9]Player $2$ is identical to player $1$, and has the same inverse bid function. Therefore, we omit the inverse bid function for player $2$.



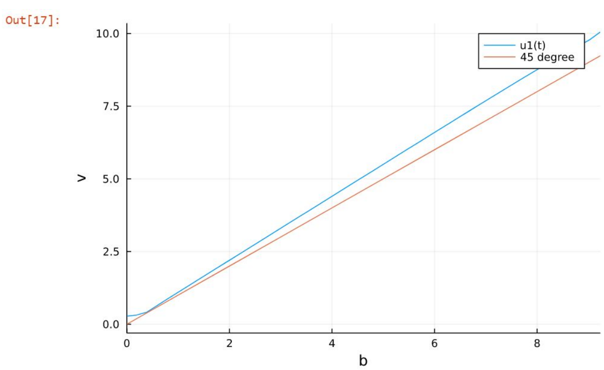



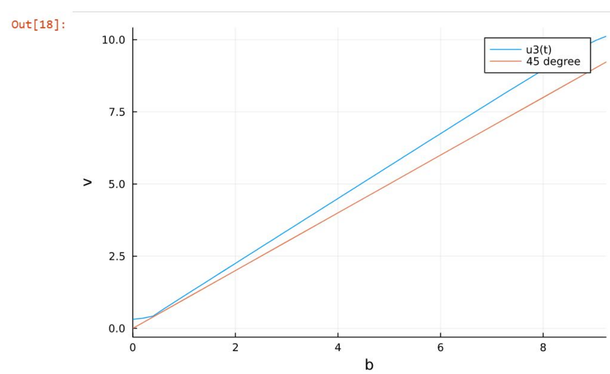

### 4.2.2 Example 2

Let $N = 3$, $c_1 = c_2 = c_3 = 0$, $d_1 = d_2 = 10$, and $d_3 = 8$. $F_1(x) = F_2(x) = \frac{x^2}{100}$, $F_3 = U[0, 8]$. Here again, since the distribution for player $3$ is the same as in previous example, we use the same approximation, viz.

$$g_3^{10}(x) = \begin{cases} \frac{9}{80} & x \in [0, 8) \\ 0.053361x^3 - 1.14726x^2 + 12.3666816x - 35.5595 & x \in [8, 9.561] \\ \frac{1}{100} & x \in [9.561, 10] \end{cases}$$

The inverse bid functions for players $1$ and $3$ in this example are[10]:




### 4.2.3 Example 3

Let $N = 3$, $c_1 = c_2 = 0$, $d_1 = d_2 = 10$, $c_3 = 2$ and $d_3 = 8$. $F_1 = F_2 = U[0, 10]$, $F_3 = U[2, 8]$. We again follow the same broad steps: first, we approximate the density of player $3$ by using the following distribution:

[10]Player $2$'s inverse bid function omitted for exact same reason as example $1$.

$$g_3^k(x) = \begin{cases} \frac{1}{10k} & x \in [0, z_1) \\ P_1(x) & x \in [z_1, 2] \\ \frac{k-1}{6k} & x \in (2, 8) \\ P_2(x) & v \in [8, z_2] \\ \frac{1}{10k} & x \in (z_2, 10] \end{cases}$$

Here, $P_1(t)$ and $P_2(t)$ are third degree polynomials of the forms $P_1(t) = A_1 \int (t-z)(t-2)dt + B_1$ and $P_2(t) = A_2 \int (t-8)(t-z)dt + B_2$, where

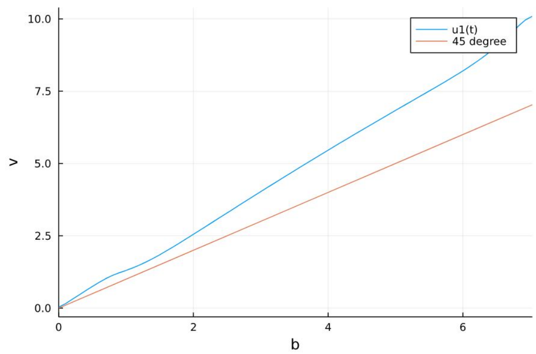
1. $P_1(z_1) = \frac{k-1}{10k}$
2. $P_1(2) = \frac{k-1}{6k}$
3. $P_2(8) = \frac{k-1}{6k}$
4. $P_2(z) = \frac{1}{10k}$
5. $\int\limits_0^{z_1} P_1(x)dx + \int\limits_8^{z_2} P_2(x)dx = \frac{z_2 - z_1}{10k}$

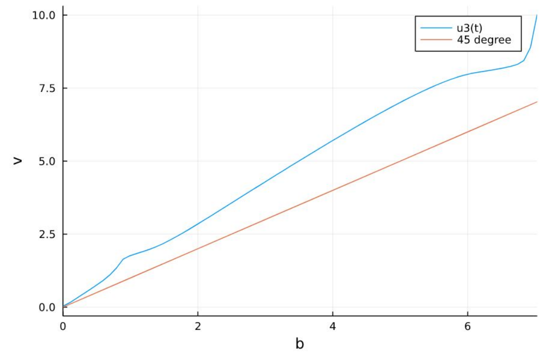
First four conditions again follow from continuity of $g_3(x)$, while the last condition ensures that $g_3(x)$ is a valid density function. However, notice that determining $P_1(x)$, $P_2(x)$, $z_1$ and $z_2$ involve finding out $6$ variables, viz. $A_1$, $A_2$, $B_1$, $B_2$, $z_1$ and $z_2$. But we only have $5$ non linear equations. Thus, we impose $A_1 + A_2 = 0$ to get the $6^{th}$ equation, so that we can solve our system of nonlinear equations. Again, by construction, we do not need to worry about differentiablility of $g_3(x)$, or about $P_1(x) \leq 0$ or $P_2(x) \leq 0$, because of the same logic as in the first example. We set $k = 10$ and solve for $P_1(x)$, $P_2(x)$, $z_1$ and $z_2$ using Newton's method in Julia. We compute $P_1(x) \approx -3.557x^3 + 19.0691x^2 - 33.53746x + 19.45963$; $P_2(x) \approx 3.557x^3 - 87.654635x^2 + 719.53016x - 1967.399617$; $z_1 \approx 1.57143$ and $z_2 \approx 8.42857$. Thus, $g_3^k$ becomes:

$$g_3^{10}(x) = \begin{cases} \frac{1}{100} & x \in [0, 1.57143) \\ -3.557x^3 + 19.0691x^2 - 33.53746x + 19.45963 & x \in [1.57143, 2] \\ \frac{9}{60} & x \in (2, 8) \\ 3.557x^3 - 87.654635x^2 + 719.53016x - 1967.399617 & v \in [8, 8.42857] \\ \frac{1}{100} & x \in (8.42857, 10] \end{cases}$$

The inverse bid functions for players $1$ and $3$ in this example are[11]:

[11] Player $2$'s inverse bid function omitted for exact same reason as example $1$.




# 5 Conclusion

We identify some major issues in the equilibrium characterization of Lebrun, 2006 for IPVFPA the asymmetric supports and provide a computationally feasible alternative characterization. In section 2, we present a perturbation technique that naturally extends Lebrun, 1999's result to the asymmetric support setting. In section 3 we use our perturbation method to prove the existence of a BNE in Theorem 1. We proceed to characterize the BNE in Theorems 2 (continuity result) and 3 (piece-wise differentiability result). Theorem 4 forms the important link between the $\varepsilon$-BNE of an IPVFPA with asymmetric supports and the BNE of the perturbed games with common support. This theorem forms the bedrock of our computation of the $\varepsilon$-BNE of an IPVFPA with asymmetric supports. Our proof of Theorem 2 rests on a novel theorem due to Barvínek et al., 1991, and this method of proof adds an important tool to the arsenal of Auction theorists. In section 4, we first discuss the important yet understated issue of numerically computing the $\varepsilon$-BNE for the IPVFPA with asymmetric supports. We state our reasons for using Julia and present the readers with some alternative options to undertake such computations. We then present some examples to illustrate the implementation of our characterization of the $\varepsilon$-BNE.

Some other theoretical questions remain. How crucial is the differentiability assumption on $f_i$? Lebrun, 1999's original result fails in the absence of this assumption. Would the monotonicity of inverse bid functions hold in the absence of this assumption? Is it possible to relax this assumption to Lipschitz continuity of $f_i$, or is this assumption indispensable? We conjecture that this assumption is dispensable and that the perturbation approach could support our conjecture. And then there is the question of $\eta$. Recall that in a setting with common support, $\eta$ is the bid that each player bids when the player's valuation hits the upper extremity. Lebrun, 1999 shows a way to compute $\eta$, and allows a possibility of multiple $\eta$'s. In Section 4, we mention these multiple $\eta'$s as contained in the interval $[\eta_1, \eta_2]$. Lebrun, 2006 provides some weak conditions which guarantee the uniqueness of $\eta$. For relatively simple cases, for instance, symmetric cases, this value of $\eta$ comes out to be unique, or $\eta_1 = \eta_2$. However, the question remains as to what happens when $\eta_1 \neq \eta_2$. It is pertinent to mention that there is absolutely no *a priori* reason as to why players should coordinate on a particular $\eta$. Distributions with strictly decreasing reverse hazard rates, while simple to analyze and common enough, are by no means universal. In such auctions, how do players select $\eta$? Moreover, what implications, if any, does it have for computation? We hope to see at least some of these questions answered in the future.

# Appendix

## A. Proof of Lemma 1

*Proof.* Suppose not. Consider a scenario wherein player $i$ wins the auction, but ties with $L-1$ players. By an abuse of notation, we denote by $L$ the set of players who tie with $i$. For any $\varepsilon > 0$, note that $X_{ij}(\beta_i, \beta_j, v, \varepsilon) \supseteq W_{ij}(\beta_i, \beta_j, v) \cup X_{ij}(\beta_i, \beta_j, v, 0)$ for every $\varepsilon > 0$. If $\varepsilon$ is chosen small enough, and if player $i$ bids $\beta_i(v) + \varepsilon$ instead of $\beta_i(v)$, then:

$$(v - \beta_i(v) - \varepsilon) \prod_{j \neq i} F_j(X_{ij}(\beta_i, \beta_j, v, \varepsilon))$$
$$-(v-\beta_i(v)) \left[ \prod_{j \neq i} F_j(X_{ij}(\beta_i, \beta_j, v, 0)) + \frac{1}{L} \prod_{l \in L} F_l(W_{il}(\beta_i, \beta_l, v)) \prod_{j \in N \setminus (\{i\} \cup L)} F_j(X_{ij}(\beta_i, \beta_j, v, 0)) \right] \geq 0 \tag{1}$$

If $v > \beta_i(v)$, it is always possible to find an $\varepsilon$ such that the above inequality is strict.

**Claim 1.** *If $v = \beta_i(v)$ and a player $i$ wins with positive probability, then there is a profitable deviation for the player $i$.*

*Proof.* Suppose player $i$ wins the auction, but has to bid his valuation to do so. Then, consider the situation that $\prod_{j \neq i} F_j(X_{ij}(\beta_i, \beta_j, v, 0)) > 0$. Since $F_j << \mu$, where $\mu$ is the Lebesgue measure, therefore, $\exists\ \varepsilon$ such that $\prod_{j \neq i} F_j(X_{ij}(\beta_i, \beta_j, v, -\varepsilon)) > 0$. This implies that there is a profitable deviation.

The only other case is that $L \neq \emptyset$, while $\prod_{j \neq i} F_j(X_{ij}(\beta_i, \beta_j, v, 0)) = 0$. In this situation, bidding slightly below the valuation yields the same payoff, viz. $0$. Therefore, bidding $\beta'(v) < \beta(v)$ is a profitable deviation for player $i$. $\square$

**Claim 2.** *If a player $i$ has a valuation $v$, then the limit of payoffs in the perturbed games exists and is equal to $(v - \beta_i(v)) \prod_{j \neq i} F_j(Z_{ij}(\beta_i, \beta_j, v))$*

*Proof.* Suppose $v_l \in \limsup_k X_{il}(\beta_i^k, \beta_l^k, v, 0)$. Then, there is some subsequence $\{\beta_l^{k_m}(v_l)\}_{m \geq 1}$ such that $\beta_l^{k_m}(v_l) < \beta_i^{k_m}(v)\ \forall\ m \geq 1$. Since $\{\beta_l^k(v_l)\}_{k \geq 1} \to \beta_l(v_l)$, this implies that $v_l \in Z_{il}(\beta_i, \beta_l, v)$. In turn, this implies that:

$$\limsup_k F_l(X_{il}(\beta_i^k, \beta_l^k, v, 0)) \leq F_l(\limsup_k X_{il}(\beta_i^k, \beta_l^k, v, 0)) \leq F_l(Z_{il}(\beta_i, \beta_l, v)) \tag{2}$$

$$\liminf_k F_l(X_{il}(\beta_i^k, \beta_l^k, v, 0)) \geq F_l(\liminf_k X_{il}(\beta_i^k, \beta_l^k, v, 0)) \geq F_l(X_{il}(\beta_i, \beta_l, v, 0)) \tag{3}$$

Notice that since $G_l^k(X_{il}(\beta_i^k, \beta_l^k, v, 0)) = F_l(X_{il}(\beta_i^k, \beta_l^k, v, 0)) + O(\frac{1}{k})$, the above two sets of inequalities imply the following:

$$F_l(X_{il}(\beta_i, \beta_l, v, 0)) \leq \liminf_k G_l^k(X_{il}(\beta_i^k, \beta_l^k, v, 0)) \leq \limsup_k G_l^k(X_{il}(\beta_i^k, \beta_l^k, v, 0)) \leq F_l(Z_{il}(\beta_i, \beta_l, v)) \tag{4}$$

and consequently:

$$\begin{aligned}(v - \beta_i(v)) \prod_{j \neq i} F_j(X_{ij}(\beta_i, \beta_j, v, 0)) &\leq \liminf_k (v - \beta_i^k(v)) \prod_{j \neq i} G_j^k(X_{il}(\beta_i^k, \beta_l^k, v, 0)) \\ \leq \limsup_k (v - \beta_i^k(v)) \prod_{j \neq i} G_j^k(X_{il}(\beta_i^k, \beta_l^k, v, 0)) &\leq (v - \beta_i(v)) \prod_{j \neq i} F_j(Z_{ij}(\beta_i, \beta_j, v))\end{aligned} \tag{5}$$

Thus, $(v - \beta_i(v)) \prod\limits_{j \neq i} F_j(Z_{ij}(\beta_i, \beta_j, v))$ is an upper bound for the limit of payoffs. Note that $\forall\, \varepsilon > 0$, since $\beta_k$ is a BNE for the perturbed game,

$$\begin{aligned}(v - \beta_i^k(v) - \varepsilon) \prod_{j \neq i} G_j^k(X_{ij}(\beta_i^k, \beta_j^k, v, \varepsilon)) &\leq (v - \beta_i^k(v)) \prod_{j \neq i} G_j^k(X_{ij}(\beta_i^k, \beta_j^k, v, 0)) \\ \implies (v - \beta_i(v) - \varepsilon) \prod_{j \neq i} F_j((X_{ij}(\beta_i, \beta_j, v, \varepsilon))) &\leq (v - \beta_i(v)) \prod_{j \neq i} F_j(Z_{ij}(\beta_i, \beta_j, v))\end{aligned}$$

The first term in the last inequality follows from the fact that if player $i$ had bid $\beta_i^k + \varepsilon$ along the sequence of perturbed games, while other players retained their bidding strategy $\beta_l^k$, then the $\liminf$ of player $i$'s payoffs under the bidding functions $\beta_i + \varepsilon$ should be at least as much as $(v - \beta_i(v) - \varepsilon) \prod\limits_{j \neq i} F_j((X_{ij}(\beta_i, \beta_j, v, \varepsilon)))$. This is so because $\beta_i^k \to \beta_i$. Further, notice that :

$$(v - \beta_i(v)) \prod_{j \neq i} F_j((X_{ij}(\beta_i, \beta_j, v, \varepsilon))) \geq (v - \beta_i(v)) \prod_{j \neq i} F_j(Z_{ij}(\beta_i, \beta_j, v))$$

Finally, we let $\varepsilon \to 0$. Sandwich theorem of limits completes the proof. $\square$

**Claim 3.** *Under the positive probability of a player $i$ winning, $\beta_i(v) < v$ and $\beta_l(\phi_{li}(v)) = \beta_i(v) < \phi_{li}(v)$.*

*Proof.* For the first part of the proof, suppose not. Then from Claim 2, the limit of payoffs is $0$. From Claim 1, there is a profitable deviation in bidding $\beta_i - \varepsilon$ in the limit game which yields a strictly positive payoff. Using equation $5$, this implies that for any such valuation $v$, $\exists\, K_v$ such that $\beta_i^k$ is not a best response to $\{\beta_j^k\}_{j \neq i}$ for any $k > K_v$. This is a contradiction. Arguing analogously for player $l$, we arrive at the second part of the claim. $\square$

Suppose that there is a winning bid $b = \beta_i(v)$ for some valuation $v$ of player $i$ which is tied with positive probability. Let $L$ be the set of those players who tie with $i$. For each player $l \in L$, a tie in the winning bid in the limit game implies that $F_l(W_{il}(\beta_i, \beta_l, v)) > 0$, which in turn implies that $\beta_l(v_l)$ is constant at bid $b = \beta_i(v)$. Thus, to show that a tie happens with $0$ probability, all we need to show is that $F_l(Z_{il}(\beta_i, \beta_l, v) \setminus X_{il}(\beta_i, \beta_l, v, 0)) = 0$. We show this in the next three claims.

**Claim 4.** $F_l(Z_{il}(\beta_i, \beta_l, v) \setminus \limsup_k X_{il}(\beta_i^k, \beta_l^k, v, 0)) = 0$

*Proof.* Follows from Claim 2. □

**Claim 5.** $F_l(\limsup_k(X_{il}(\beta_i^k, \beta_l^k, v, 0))) = F_l(\liminf_k(X_{il}(\beta_i^k, \beta_l^k, v, 0)))$.

*Proof.* Suppose not. Then a necessary condition for the supposition to hold would be that $v_3^l > v_2^l$. Note that $X_{il}(\beta_i^k, \beta_l^k, v, 0) = [c, \phi_{li}^k(v))$. Suppose that $x \in \limsup_k(X_{il}(\beta_i^k, \beta_l^k, v, 0)) \setminus \liminf_k(X_{il}(\beta_i^k, \beta_l^k, v, 0))$. Then, $x < \phi_{li}^k(v)$ i.o. In the limit, this implies that $x \leq \phi_{li}(v)$. Further, since $x \notin \liminf_k(X_{il}(\beta_i^k, \beta_l^k, v, 0))$, there is some subsequence such that $\beta_l^{k_m}(x) \geq \beta_i^{k_m}(v)$. At the limit, this implies that $x \geq \phi_{li}(v)$. Thus, $\limsup_k(X_{il}(\beta_i^k, \beta_l^k, v, 0)) \setminus \liminf_k(X_{il}(\beta_i^k, \beta_l^k, v, 0)) = \{\phi_{li}(v)\}$, and hence, $v_3^l = v_2^l = \phi_{li}(v)$.

□

**Claim 6.** $F_l(\liminf_k X_{il}(\beta_i^k, \beta_l^k, v, 0) \setminus X_{il}(\beta_i, \beta_l, v, 0)) = 0$

*Proof.* Suppose not. Then $v_1^l < v_2^l$. This in turn implies that $\beta_l(v_l) = \beta_i(v) \; \forall \; v_l \in (v_1^l, v_2^l)$. By Dini's theorem, $\beta_l^k \rightrightarrows \beta_l$ over $(v_1^l, v_2^l)$. Choose $\varepsilon > 0$ to be small enough.

Case 1: *$v$ is a continuity point of $\beta_i$:*

Let $x < v$ be such that $\beta_i(x) \in (\beta_i(v) - \varepsilon, \beta_i(v) - \frac{\varepsilon}{2})$. WLOG, there is some $K_\varepsilon$ such that $\forall \; k > K_\varepsilon$, $\sup_{v_l \in (v_1^l, v_2^l)} |\beta_i(v) - \beta_l^k(v_l)| < \frac{\varepsilon}{4}$ and $|\beta_i(v) - \beta_i^k(v)| < \frac{\varepsilon}{4}$. For player $i$ with valuation $x$, bidding $\beta_i^k(x) + \varepsilon$ beats the set of players $l$ with valuations $v_l \in Z_{il}(\beta_i, \beta_l, v)$. This implies that for $k$ large enough:

$$(x - \beta_i^k(x) - \varepsilon) \prod_{j \neq i} G_j^k(X_{ij}(\beta_i^k, \beta_j^k, x, \varepsilon)) > (x - \beta_i^k(x)) \prod_{j \neq i} G_j^k(X_{ij}(\beta_i^k, \beta_j^k, x, \varepsilon))$$

Case 2: *$v$ is a discontinuity point of $\beta_i$:*

Define $x_k = \sup\{y : \beta_i^k(y) < \beta_i^k(v) - \frac{\varepsilon}{2}\}$. Note that $x_k < v \; \forall \; k \geq 1$ and that $\{x_k\}_{k \geq 1}$ is a convergent sequence. Analogous arguments as for Case 1 hold for $x_k$ with $k$ large enough.

Thus, in both cases, we get a contradiction to $\beta_i^k$ being an equilibrium strategy for player $i$ in $\Gamma^k$. □

□

## B. Proof of Lemma 2

*Proof.* If $\beta_i$ is discontinuous, let $V = \{v_1, v_2 \cdots\}$ be the enumeration of points of discontinuity of $\beta_i$[12]. Then, for $\varepsilon$ small enough, define $U_\varepsilon := \bigcup\limits_{n\geq 1} B_{\frac{\varepsilon}{2^n}}(v_n) \setminus \{v_n\}$. $U_\varepsilon$ is an open set and $\beta_i$ is a continuous function on $[c,d] \setminus U_\varepsilon$. By Tietze extension theorem, let $\gamma_i$ be the continuous extension of $\beta_i$ on $[c,d]$. WLOG, $\gamma_i \neq \beta_i$ on $U_\varepsilon$ and $\gamma_i$ is monotonically increasing. Let $\{v_m\}_{m\geq 1}$ be a sequence of continuity points of $\beta_i$ in $(v_n - \frac{\varepsilon}{2^n}, v_n)$ converging to $v_n$, where $v_n \in V$. Then:

$$(v_m - \gamma_i(v_m)) \prod_{l\neq 1} F_l\{v_l \geq c : \beta_l(v_l) < \gamma_i(v_m)\} \leq (v_m - \beta_i(v_m)) \prod_{l\neq 1} F_l\{v_l \geq c : \beta_l(v_l) < \beta_i(v_m)\}$$

Suppose $\liminf\limits_m \beta_i(v_m) := \beta_{ix}(v) < \beta_i(v)$. Then:

$$(v - \beta(v)) \prod_{l\neq 1} F_l\{v_l \geq c : \beta_l(v_l) < \beta(v)\} \leq (v - \beta_{ix}(v)) \prod_{l\neq 1} F_l\{v_l \geq c : \beta_l(v_l) < \beta_{ix}(v)\}$$

Above equation is valid since $\Pi_i(v) := (v - \beta_i(v)) \prod\limits_{l\neq 1} F_l\{v_l \geq c : \beta_l(v_l) < \beta_i(v)\}$ is a continuous function of $v$ if $\beta_i$ is continuous, where continuity follows from the fact that all $\{\beta_l\}_{l\neq i}$ are monotonically increasing and can only have jump discontinuities.

If the above expression holds with strict inequality, we have a contradiction to $\beta_i$ being an equilibrium strategy. Hence, the only possible case is that $\beta_i$ yields the same payoff as a lower semi-continuous function which disagrees with $\beta_i$ only on its points of discontinuities. By analogous arguments, we can also define an upper semi-continuous function which disagrees with $\beta_i$ only on its points of discontinuities. □

## C. Proofs in Support of Theorem 2

**Claim 7.** $\lim\limits_{k\geq 1} \frac{d\beta_i^k(v)}{dv}$ *exists and is a.e. finite valued.*

*Proof.* The first part of the claim is immediate from the expression of $\frac{d\beta_i^k(v)}{dv}$:

$$\frac{d\beta_i^k(v)}{dv} = (n-1)\frac{g_i^k(v)}{G_i^k(v)} \frac{1}{\frac{-(n-2)}{v-\beta_i^k(v)} + \sum\limits_{j\neq i} \frac{1}{\phi_{ji}^k(v) - \beta_i^k(v)}}$$

For the second part of the claim, suppose not and that there is some set $X \subseteq [c,d]$ such that $\mu(X) > 0$ and $\lim\limits_{k\geq 1} \frac{d\beta_i^k(v)}{dv} = \infty$ on $X$. Then, by monotonicity of integration and Fatou's Lemma:

$$d - c > \lim_{k\geq 1}(\beta_i^k(d) - \beta_i^k(c)) \geq \lim_{k\geq 1} \int\limits_X \frac{d\beta_i^k(v)}{dv} d\mu \geq \int\limits_X \lim_{k\geq 1} \frac{d\beta_i^k(v)}{dv} d\mu = \infty$$

This is a contradiction. □

[12] V can be finite

**Proof of Lemma 3**

*Proof.* Suppose not. Then there is a sequence $\{v_m\}_{m\geq 1}$ such that $\{v_m - \beta_i(v_m)\}_{m\geq 1} \to 0$. Let $v' = \lim\limits_{m\to\infty} v_m$. Observe that $\{v_m\}_{m\geq 1}$ has a monotonically decreasing subsequence[13]. Passing on to this subsequence if necessary, notice that $\lim\limits_{v_m\to v'} (v_m - \beta_i(v_m)) \prod\limits_{j\neq i} F_j(\phi_{ji}(v_m)) = 0$, i.e. the limit of payoffs is $0$. Thus, if $\beta_i$ is replaced by $\hat{\beta}_i$ which differs from $\beta_i$ only at $v'$ and is upper semi-continuous at $v'$, Lemma 2 implies that $\hat{\beta}_i$ is also a pure strategy equilibrium of player $i$. But $\hat{\beta}_i$ yields a payoff of $0$. This is a contradiction to $\beta_i$ being an equilibrium strategy. □

## D. Proofs in Support of Theorem 3

**Proof of Lemma 4**

*Proof.* In the domain $(\underline{v_i}, \overline{v_i})$, bidding $\beta_i(v)$ is player $i$'s best response to other players playing $\beta_k$. Further, $\forall\ k \neq i$, $\beta_k$ is strictly monotonic and continuous, and hence, invertible in this domain. Further, $\beta_i(v)$ is the local maximizer of $(v-b)\prod\limits_{j\neq i} F_j(\beta_j^{-1}(b))$ and $i$'s maximized expected payoff in this scenario is $(v - \beta_i(v)) \prod\limits_{j\neq i} F_j(\phi_{ji}(v))$.

Notice that the above argument is equivalent to arguing that given any bid $b \in (\beta_i(\underline{v_i}), \beta_i(\overline{v_i}))$, and given the inverse bid functions $\alpha_j$ of players $j \neq i$, optimal inverse bid function of player $i$ is $\alpha_i$. This is so because in this domain, $\alpha_j = \beta_j^{-1}\ \forall\ j \neq i$, and $\phi_{ji}(v) = \alpha_j(\beta_i(v))$.

Since $\{\alpha_i\}_{i\in N}$ is Lipschitz, therefore it is absolutely continuous and of bounded variation. Thus, almost everywhere in the given domain:

$$\frac{d\ln\left((\alpha_i(b) - b)\prod\limits_{j\neq i} F_j(\alpha_j(b))\right)}{db} = 0$$

Arguing analogously for other players $j \neq i$, almost everywhere[14]:

$$\frac{d\alpha_j(b)}{db} = \frac{1}{n-1}\frac{F_j(\alpha_j(b))}{f_j(\alpha_j(b))}\left(\frac{-(n-2)}{\alpha_j(b) - b} + \sum_{l\neq j}\frac{1}{\alpha_l(b) - b}\right)$$

Notice that Lipschitz and hence absolutely continuous nature of $\{\alpha_i\}_{i\in N}$ implies $\alpha_i(b') - \alpha_i(b) = \int_b^{b'}\left(\frac{d\alpha_i}{db}\right) db\ \forall\ b, b' \in (\beta_i(\underline{v_i}), \beta_i(\overline{v_i}))$ , and hence $\{\alpha_i\}_{i\notin M_j}$ indeed solves the system of DE obtained. Since $\alpha_j(b) - b$ is bounded away from $0$, and since $f_j$ is differentiable in the domain $(\underline{v_i}, \overline{v_i})$, the RHS of the above system is Lipschitz continuous. We impose the boundary conditions obtained by values of $\{\alpha_i\}_{i\in N}$ at $\{\beta_i(\underline{v_i}), \beta_i(\overline{v_i})\}$. Therefore, the solution to the system must be continuously differentiable and unique by Picard's existence theorem.

[13] To see this, notice that there is some $M$ such that for all $m > M$, $\beta_i(v_m) > \beta_i(v')$.

[14] The qualifier a.e. follows from the fact that $\alpha_i, \beta_i, \{\phi_{ki}\}_{k\notin M_j\cup\{i\}}$ are almost everywhere differentiable as a result of their strict monotonicity.

**Claim 8.** $\frac{d\alpha_i}{db} > 0 \ \forall \ b \in (\beta_i(\underline{v_i} + \varepsilon), \beta_i(\overline{v_i}))$.

*Proof.* Suppose not and let $V := \{b : \frac{d\alpha_i}{db} = 0\}$. Invertibility and strict monotonicity of $\alpha_i$ imply that $V$ cannot contain an interval, while continuity of $\frac{d\alpha_i}{db}$ implies that $V$ cannot be dense in $(\beta_i(\underline{v_i}+\varepsilon), \beta_i(\overline{v_i}))$. Therefore, if there is some $\mu > 0$ such that $\frac{d\alpha_i}{db} = 0$ at $\beta_i(\overline{v_i} - \mu)$, it must always be possible to find $\hat{\mu} < \mu$ such that $\frac{d\alpha_i}{db} > 0$ at $\beta_i(\overline{v_i} - \hat{\mu})$. Lemma A2-2 of Lebrun, 1999 provides a contradiction and $V = \emptyset$. □

A simple change of variables, substituting $\beta_i = \alpha_i^{-1}$ and $\phi_{li} = \alpha_l(\alpha_i^{-1})$ on the domain $(\underline{v_i} + \varepsilon, \overline{v_i})$ with given boundary conditions and letting $\varepsilon \to 0$ completes the proof. □

### Proof of Lemma 5

*Proof.* The proof is similar to the proof of Lemma 4, with following modifications. Firstly, $F_m(\phi_{mi}(v)) = 1 \iff m \in M_j$, thus, players $m \in M_j$ drop out of the expression for payoff functions for players not in $M_j$. Secondly, Claim 8 is appropriately modified to hold for the domain $(\beta_i(\overline{v_i^j}), \beta_i(\overline{v_i^{j+1}}))$, and for players not in $M_j$. The rest of the proof is analogous and has been omitted. □

## E. Proofs from Final Discussion

### Proof of Lemma 6

*Proof.* Theorems 2 and 3 show that the lemma is true for $v \geq \underline{v_i}$. Notice that if $v < \underline{v_i}$, then the expected payoff of the player $i$ is $0$. Thus, in the interval $[c_i, \underline{v_i}]$, replacing the bid function $\beta_i$ with an affine function $\tilde{\beta}_i(v) := \beta_i(c_i) + \frac{v - c_i}{\underline{v_i}}(\beta_i(\underline{v_i}) - \beta_i(c_i))$ is also an equilibrium strategy for each player[15]. Notice that this affine function is dominated by the $45$ degree line. □

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[15]Notice that if $[c_i, \underline{v_i}] = \emptyset$ or $[c_i, \underline{v_i}] = \{c_i\}$, such a replacement is infructuous.

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