# Network Games Under Incomplete Information on Graph Realizations 

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#### Abstract

In most social and economic networks, participants are unaware of the full architecture of the network they are embedded in. Despite this, there has been little work on trying to understand rational behavior on networks under incomplete information. We study an incomplete information variant of Ballester et al. (2006) in which agents are endowed with beliefs over an ex-ante distribution over all unweighted and undirected networks on a given vertex set. Agents' information is restricted to the identity of their neighbors and is used to maximize interim linear-quadratic payoffs. We establish the existence and uniqueness of Bayesian-Nash equilibria in pure strategies and characterise their behavior. We find that when agents are endowed with beliefs about network topology itself, local connectivity provides information regarding indirect connectivity and agents will make use of it towards equilibrium play. In patricular, equilibrium actions are computable by the expected number of walks individuals have in the network.


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[^0]
## 1 Introduction

The majority of the work on network games with local complementarities assumes that agents have complete knowledge of the network architecture they are embedded in. This assumption is crucial for such games, as equilibrium behavior of agents is driven by metrics defined on the architecture of the entire network (Ballester et al. (2006)). In most real world social and economic networks, however, agents rarely have access to this information. In fact, as demostrated by Breza, Chandrasekhar, and Salehi (2018), agents are mostly aware of the identity of their immediate neighbors. Despite this, as well as the fact that network interactions are important drivers of behavior, there has been little work on trying to understand how this particular type of information drives equilibrium outcomes.

Within the confines of a game of incomplete information, the first question that arises is the following: as to what object is the information regarding agents neighbors informative of? In other words, if we presume that agent behavior within a network system is the result of equilibrium behavior in a game of incomplete information, over what object do agents have ex-ante beliefs? Given that agents can identify the number of agents they are connected to, the prevailing approach has been to assume that these ex-ante beliefs are representative of degree distributions. That is, existing models endow agents with beliefs about the number of connections that each individual may have in the network. An important network characteristic that this approach abstracts away from, however, is agent identity.

In this paper, we are motivated by the subset of complete information network games in which common knowledge of the entire network is assumed, and where the identity of individuals play a role in equilibrium behavior. We, therefore, take a different approach and assume the aforementioned ex-ante beliefs are representative of a distribution over networks themselves. In particular, we study an incomplete information variant of the popular quadratic game of Ballester et al. (2006) where ex-ante beliefs are prescribed by a probability mass function over the set of simple graphs on $n$ vertices, and in which agent's information is restricted to the identity of their neighbors only.

Our game proceeds as follows. Nature moves first and chooses an unweighted and undirected network on $n$ vertices from an ex-ante distribution over all such graphs. Agent $i$ 's type (in the Harsanyi (1967) sense) corresponds to the $i^{\text {th }}$ row of the adjacency representation of nature's chosen network. Agents are thus classified by the identity of their neighbors and are hence able to identify the agents from whom they will directly extract network complememtarities from. However, they are unaware of the types of these adjacent agents. Given their realized
type, and using Bayes rule, agents update their beliefs regarding these types and, therefore, their beliefs about the true topology of the network. They then proceed to simultaneously exert actions to maximize interim linear quadratic payoffs.

We establish existence and uniqueness of pure strategy Bayesian Nash equilibria (BNE) for arbitrary ex-ante distributions. These properties hold for a bound on the modularity parameter of cross activity that is identical to the complete information variant of the model.

Turning to the properties of the BNE, we show that agents can over or under exert actions when compared to the complete information Nash equilibrium induced over nature's chosen network. The extent to which they do so depends on the number of agents they are connected to (i.e. their degree), as well as the ex-ante distribution of networks. We show that in equilibrium, agents will use the information regarding their direct connections to make inferences about the complementarity strength of their actions with those of other agents. The streghth of this complementarity is computed by their ex-post expectation regarding the number of walks they have in the network. In this sense, the BNE results from a calculation similar to the one performed by agents in the complete information case, where the Nash equilibrium is proportional to the actual number of walks that agents have in the network (Balletser et al. (2006)) i.e. KB centrality.

It is important to note, however, that this expected sum of walks does not correspond to the agents ex-ante expected Katz-Bonacich (KB) centrality. This point raises an interesting question with regard to how network uncertainty should be gauged from an applied perspective. Motivated by results on complete information games and centrality driven outcomes, simulation based, and empirical approaches have been employed to deal with situations in which the network is not observed. The empirical approach has proposed estimators of network effects in environments in which researchers cannot observe the network (De Paula, Rasul, and Souza (2018) and Lewbel, Qu, and Tang (2021)). In the simulation based approaches, researchers will often employ random network models in an attempt to generate the distribution of the centrality metric that drives behavior in the network system of interest (Crucitti et al. (2006), Latora and Marchiori (2007)). There have also been analytical attempts towards this goal. Dasaratha (2020), for instance, characterizes the distributions of KB and eigenvalue centrality on large networks.

Our result, however, suggests that whenever network participants lack information about the true architecture of the network, their behavior is not consistent with complete information behavior, nor with expectations over complete information equilibrium outcomes. The caveat with such applied approaches is, therefore, that they do not internalize the fact
that the subjects themselves may not know the network. When this is the case, centrality distributions, or estimators of complete information network effects, may not be informative as to the true behavior of the network system of interest.

Next, we restrict attention to the role of degrees and impose the ex-ante distribution to be uniform. In this case, these expected walks, and consequently BNE actions, are directly proportional to agents realized degrees. In other words, the higher an agent's number of connections, the higher is their exerted action is in equilibrium. This result is closely related to the finding of Galeotti et al. (2010). They show that in an environment in which agents are endowed with information about their degree, but not the identity of the agents they are connected to, equilibrium actions are also driven by degrees. They do this by endowing agents with ex-ante beliefs on the degree distribution of the true networks. Degree distributions, however, are anonymous implying that agents do not employ local information towards equilibrium play. In contrast, we show that by endowing agents with beliefs about network topology itself, agents will internalize local information in order to form beliefs about the number of walks they have in the network.

Other than the large literature on linear-quadratic network games of complete information, there have been two other attempts that introduce incomplete information into the model. De Marti and Zenou (2015) study a linear quadratic game of incomplete information in which agents lack information regarding model parameters other than the network itself. These include the link complementarity strength, and the return to own action exertion. Unlike their work, we are interested in incomplete information on the network.

Closer to our model is Breza, Chandrasekhar, and Salehi (2018) who also employ a linear quadratic game in which agents lack complete information regarding the network itself. One of their crucial assumptions, however, is that the information set of any agent (i.e. the identity of their neighbors) doesn't provide any information about their indirect connections. In other words, their expectations regarding the existence of links between their neighbors and other agents is independent of the information they are endowed with. As a result, the equilibrium gets mapped to their ex-ante beliefs about the network. In contrast, we find that as long as agent are endowed beliefs about network topology itself, local connectivity provides information regarding indirect connectivity and agents will make use of it towards equilibrium play. This local information being different for each player in turn, implies that the equilibrium is no longer mapped to their ex-ante beliefs about the network.

The rest of the paper is structured as follows. Section 2 contains tool from network theory that will be throughout the paper and sets up the game. In section 3, we derive the Bayesian

Nash equilibrium and study its properties. Section 4 discusses welfare. Section 5 concludes. All proofs are relegated to the the appendix.

## 2 Model

### 2.1 Preliminaries

Let $N=\{1,2, \ldots, n\}$ denote the set of players. Letting $i \sim j$ denote a link between players $i$ and $j$, a network (graph) $\mathbf{g}$ is the collection of all pairwise links that exist between the players. The links are undirected such that $i \sim j \in \mathbf{g}$ implies $j \sim i \in \mathbf{g}$. The network can be represented via its adjacency matrix which, with some abuse of notation, is also denoted as $\mathbf{g}=\left[g_{i j}\right]$, where $g_{i j}=1$ if a link exists between players $i$ and $j$, and $g_{i j}=0$ otherwise. There are no self-loops and thus $g_{i i}=0$ for all $i \in N$. The fact that links are undirected implies $\mathbf{g}=\mathbf{g}^{T}$. We denote by $\mathcal{G}_{n}$ the set of all unweighted and undirected networks on $n$ vertices. We consider simple graphs, so the cardinality of $\mathcal{G}_{n}$ is $2^{\frac{n(n-1)}{2}}$.

Given the adjacency representation of a network $\mathbf{g} \in \mathcal{G}_{n}$ we let $\mathbf{g}_{i}$ denote its $i^{\text {th }}$ row. That is, $\mathbf{g}_{i}=\left(g_{i 1}, g_{i 2}, \ldots, g_{i n}\right) \in\{0,1\}^{n}$, where it is understood that $g_{i i}=0$. In the following section, it will be convenient to represent networks $\mathbf{g}$ by the rows of their adjacency matrix:

$$
\begin{equation*}
\mathbf{g}=\left(\mathbf{g}_{1}, \mathbf{g}_{2}, . ., \mathbf{g}_{n}\right) \tag{1}
\end{equation*}
$$

The neighborhood of player $i$ is the set of players to whom $i$ is linked and is denoted by: $\mathbf{N}\left(\mathbf{g}_{i}\right)=\left\{j: g_{i j}=1\right\}$. The size of this set is $i$ 's degree which counts the agent's direct connections: $d\left(\mathbf{g}_{i}\right) \equiv\left|\mathbf{N}\left(\mathbf{g}_{i}\right)\right|$. A network is called regular if all players have the same degree (which is also the degree of the network); otherwise, the network is irregular.

A walk of length $p$ from a node $i$ to a node $j_{p}$ is a sequence of links in the network $i \sim j_{1}$, $j_{1} \sim j_{2}, \ldots, j_{p-1} \sim j_{p}$. It is denoted by $i j_{1} j_{2} \ldots, j_{p}$. Given two nodes $i$ and $j_{p}$ there may exist more than one such walk. Using the adjacency representation, the number of walks of length $p$ from node $i$ to node $j_{p}$ can be computed by the $i j_{p}$ element of the matrix $\mathbf{g}^{p}$.

Finally, let $\mathbf{g}^{0}=\mathbf{I}$, then for a sufficiently small $\lambda>0$, the following influence matrix $\mathbf{M}(\mathbf{g}, \lambda)=\left[m_{i j}(\mathbf{g})\right]$ is well-defined and non-negative:

$$
\mathbf{M}(\mathbf{g}, \lambda) \equiv[\mathbf{I}-\lambda \mathbf{g}]^{-1}=\sum_{s=0}^{\infty} \lambda^{s} \mathbf{g}^{s}
$$

Each element $m_{i j}(\mathbf{g})$ measure the total number of walks of all lengths from agent $i$ to agent $j$. Given $\mathbf{M}(\mathbf{g}, \lambda)$, the Katz-Bonacich (KB) centrality of player $i, b_{i}(\mathbf{g})$, is the $i^{\text {th }}$-component of the vector $\mathbf{b}(\mathbf{g})=\mathbf{M}(\mathbf{g}, \lambda) \mathbf{1}$. It measures the total number of walks of all lengths originating from player $i$ to all other players in $\mathbf{g}$.

### 2.2 The Game

We study a variant of the simultaneous move local complementarities game of Ballester et al. (2006), in which agents have incomplete information about the full architecture of the network. In particular, agents are only aware of the identity of their immediate neighbors.

We follow Harsanyi's (1967) approach to games of incomplete information by introducing Nature as a non-strategic player who chooses a network out of set of all possible graphs on the number of vertices equal to the number of agents. The network is chosen from an exante distribution which is common knowledge to all agents. Following Nature's draw, players realize their direct connections (they can see the agents with who they are linked with) but fail to see the network's architecture beyond that. In other words they do not observe the links of the neighbors. Using the information on their direct connections, agents proceed to update their belief on which graph Nature chose according to Bayes' rule. Given these updates beliefs, agents will simultaneously exert actions to maximize their interim payoffs.

We proceed to describe the game formally.

## Agents and Types

$N$ is the set of players (nodes), with $|N|=n$. For each $i \in N$, we let $G_{i}$ denote the player's type set. We want agent types to be representative of their corresponding row in the adjacency representation of the network over which the game will be played. To this end, each player's types set is taken to be as follows:

$$
G_{i}=\left\{\left(g_{i 1}, g_{i 2}, \ldots ., g_{i n}\right)_{i} \in\{0,1\}^{n}: g_{i i}=0\right\}
$$

where we will think of $g_{i j}=1$ if player $i$ is connected to $j$ and 0 otherwise. ${ }^{1}$ The cardinality of each agent's type set is:

$$
\left|G_{i}\right| \equiv \gamma=2^{n-1}
$$

[^1]and we denote its elements by $\mathbf{g}_{i}^{t_{i}} \in G_{i} .{ }^{2}$ Given each player's type set, we can write down the type space of the game:
$$
G=\chi_{i \in N} G_{i}
$$

Observe that if we invoke network representation (1), an element $\mathbf{g} \in G$ may correspond to an adjacency matrix of an undirected and unweighted network. However, not all elements of $G$ are valid representations. As an example, consider the case with 3 players, $N=\{1,2,3\}$. The type set of each player is given by:

$$
\begin{aligned}
& G_{1}=\left\{(0,0,0)_{1},(0,1,0)_{1},(0,0,1)_{1},(0,1,1)_{1}\right\} \\
& G_{2}=\left\{(0,0,0)_{2},(1,0,0)_{2},(0,0,1)_{2},(1,0,1)_{2}\right\} \\
& G_{3}=\left\{(0,0,0)_{3},(1,0,0)_{3},(0,1,0)_{3},(1,1,0)_{3}\right\}
\end{aligned}
$$

with corresponding type space $G=G_{1} \times G_{2} \times G_{3}$. One element of $G$ is $\left((0,1,0)_{1},(0,0,1)_{2}\right.$, $\left.(0,1,0)_{3}\right)$, the vectors of which do not correspond to rows of the adjacency matrix of any undirected and un-weighted network. This is because agent 1 is connected to agent 2 while agent 2 is not connected to agent 1 , therefore, the corresponding adjacency matrix is not symmetric. In our model, we want to restrict attention to elements of $G$ that have valid network representations so that Nature's choice is reflective of a network. In what follows, we do so through the information structure.

## Ex-Ante Beliefs

We denote by, $p \in \Delta(G)$ the probability distribution over the type space, with $\Delta(G)$ denoting set of all probability distributions on $G$. In our game, Nature moves first and chooses an element of the type space $\mathbf{g} \in G$. As noted above, we want to restrict Nature's choice to those elements in $G$ that have valid network representations. Towards this, we define the following set of admissible distributions and impose the assumption that Nature draws a network according to a distribution in this set.

Definition 1. We say that the probability distribution $p \in \Delta(G)$ is admissible if it satisfies:

$$
p(\mathbf{g})=0 \forall \mathbf{g} \in G \text { s.t } \mathbf{g} \neq \mathbf{g}^{T}
$$

and denote set of all admissible distributions by $\Delta_{A}(G)$

Assumption 1: $p \in \Delta_{A}(G)$ and this is common knowledge.

[^2]Observe that the imposition of assumption 1 implies that $p(\mathbf{g})>0$ if and only if $\mathbf{g} \in \mathcal{G}_{n}$. Consequently, Nature will choose an unweighted and undirected network with certainty, and all agents are aware of this fact. In the following section we will make use of the uniform admissible distribution which is defined as follows:

Definition 2. The probability distribution $p \in \Delta_{A}(G)$ is uniform if it satisfies:

$$
p(\mathbf{g})= \begin{cases}\frac{1}{2^{\frac{n(n-1)}{2}}} & \text { if } \mathbf{g} \in \mathcal{G}_{n} \\ 0 & \text { if } \mathbf{g} \neq \mathbf{g}^{T}\end{cases}
$$

In the 3-player case, for instance, we have that $\left|\mathcal{G}_{3}\right|=8$ and thus Nature will choose any network with probability $p(\mathbf{g})=\frac{1}{8}$.

## Belief Updating

Given a assumption 1, agents know that Nature draws a network and proceed to update their beliefs regarding its true topology according to Bayes' Rule. These ex-post updated beliefs can be written as:

$$
\begin{equation*}
p\left(\mathbf{g}_{j} \mid \mathbf{g}_{i}\right)=\frac{p\left(\mathbf{g}_{i}, \mathbf{g}_{j}\right)}{p\left(\mathbf{g}_{i}\right)}=\frac{\sum_{q} p\left(G^{q}\right) \cdot \mathbb{I}\left\{\mathbf{g}_{i}, \mathbf{g}_{j} \in G^{q}\right\}}{\sum_{k} p\left(G^{k}\right) \cdot \mathbb{I}\left\{\mathbf{g}_{i} \in G^{k}\right\}} \forall i, j \in N \tag{2}
\end{equation*}
$$

where, $G^{q}, G^{k} \in G$. Intuitively, equation (3) states that agent $i$ who is of type $\mathbf{g}_{i} \in G_{i}$ will assign a probability to agent $j$ being of type $\mathbf{g}_{j} \in G_{j}$ according to (i) the number of states in the state space that contain both of these types, and (ii) their ex-ante probabilities. Given assumption 1, since agent types correspond to rows of an adjacency matrix, the probability the agent $i$ (whose row is $\mathbf{g}_{i}$ ) will assign to an agent having a row $\mathbf{g}_{j}$ will depend on the number of networks that themselves contain these rows and the probability that nature chooses them.

As an example, consider the 3-player case and suppose that after Nature's draw, agent 2 is of type $(1,0,0)_{2}$. In other words, agent 2 learns that it is connected to agent 1 but is not connected to agent 3 . Since players can only observe the neighbors, agent 2 does not know if agents 1 and 3 are connected between themselves and will thus have to form beliefs about the existence of a link between them. This is demonstrated in the left-most network in figure 1. However, the state space only contains 2 elements in which agent 2's type is admissible with a valid network representation and these are $\left((0,1,0)_{1},(1,0,0)_{2},(0,0,0)_{3}\right)$ and $\left((0,1,1)_{1},(1,0,0)_{2},(1,0,0)_{3}\right)$. In other words, there are only two graphs on 3 vertices
that contain the link $1 \sim 2$ and do not contain the link $2 \sim 3$. If we we took the ex-ante distribution to uniform, then agent 2 would assign a probability of $\frac{1}{2}$ that nature chose either of these.


Figure 1: Graphs of the types $(1,0,0)_{2},(1,1,0)_{3},(0,0,1)_{1}$ and $(0,0,0)_{2}$
Finally, note that assumption 1 implies implies that beliefs are consistent, in the sense that agents will give zero probability to others being of types that do not match the adjacency pattern induced by their own type. This is expressed formally in the proceeding remark. Remark 1. For all $g_{i} \in G_{i}, p\left(\mathbf{g}_{j} \mid \mathbf{g}_{i}\right)=0 \forall \mathbf{g}_{j} \in G_{j}$ for which $g_{i j} \neq g_{j i}, \forall i, j \in N$

## State Game and Equilibrium

Given the above, conditional on a state $\mathbf{g} \in G$ being realized, agents play the state game:

$$
s_{\mathbf{g}}=\left(N, A,\left(u_{i}\left(\mathbf{a}_{i}, \mathbf{a}_{-i}\right)\right)_{i \in N}\right)
$$

where the action set is the same for each agent and is equal to the positive real numbers $A \equiv \mathbb{R}_{+}$. Interim utilities assume a linear-quadratic form:

$$
\begin{equation*}
u_{i}\left(a_{i}\left(\mathbf{g}_{i}^{t_{i}}\right) ; \mathbf{a}_{i}\left(\mathbf{g}_{i}^{-t_{i}}\right), \mathbf{a}_{-i}\right)=a_{i}\left(\mathbf{g}_{i}^{t_{i}}\right)-\frac{1}{2} a_{i}\left(\mathbf{g}_{i}^{t_{i}}\right)^{2}+\lambda a_{i}\left(\mathbf{g}_{i}^{t_{i}}\right) \sum_{j=1}^{n} g_{i j}^{t_{i}} \sum_{\mathbf{g}_{j} \in G_{j}} p\left(\mathbf{g}_{j} \mid \mathbf{g}_{i}^{t_{i}}\right) a_{j}\left(\mathbf{g}_{j}\right) \tag{3}
\end{equation*}
$$

where $\mathbf{a}_{i}\left(\mathbf{g}_{i}^{-t_{i}}\right)=\left(a_{i}\left(\mathbf{g}_{i}^{1}\right), \ldots, a_{i}\left(\mathbf{g}_{i}^{t_{i}-1}\right), a_{i}\left(\mathbf{g}_{i}^{t_{i}+1}\right), \ldots, a_{i}\left(\mathbf{g}_{i}^{\gamma}\right)\right), \mathbf{a}_{-i}=\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_{n}\right)$ and $\mathbf{a}_{j}=\left(a_{j}\left(\mathbf{g}_{j}^{1}\right), \ldots, a_{j}\left(\mathbf{g}_{j}^{\gamma}\right)\right)$. As in Ballester et al. (2006) the first two term in the utility specification capture the cost and direct benefit to agent $i$ from exerting its own action. The third term captures local complementarities with the agents that the player is connected to, with $\lambda$ measuring the strength of this complementarity. Unlike Ballester et al. (2006) agents need to form beliefs about the actions of their adjacent agents. Agents simultaneously exert actions to maximize (3). For each agent $i$, a pure strategy $\sigma_{i}$ maps each possible type to an action. That is,

$$
\sigma_{i}=\left(a_{i}\left(\mathbf{g}_{i}^{1}\right), \ldots, a_{i}\left(\mathbf{g}_{i}^{\gamma}\right)\right)
$$

This is a simultaneous move game of incomplete information so we use the Bayesian-Nash equilibrium notion.

Definition 3. The pure strategy profile $\sigma^{*}=\left(\sigma_{i}^{*}, \sigma_{-i}^{*}\right)$ is a Bayesian-Nash equilibrium (BNE) if:

$$
a_{i}^{*}\left(\mathbf{g}_{i}^{t_{i}}\right)=\arg \max _{a_{i}\left(\mathbf{t}_{i}^{i}\right)} u_{i}\left(a_{i}\left(\mathbf{g}_{i}^{t_{i}}\right), \mathbf{a}_{i}^{*}\left(\mathbf{g}_{i}^{-t_{i}}\right), \sigma_{-i}^{*}\right) \forall i \in N, \forall \mathbf{g}_{i}^{t_{i}} \in G_{i}
$$

The game can be summarized according to the tuple:

$$
\begin{equation*}
\Gamma=\left\langle N,\left(G_{i}\right)_{i \in N}, p,\left(s_{\mathbf{g}}\right)_{\mathbf{g} \in G}\right\rangle \tag{4}
\end{equation*}
$$

## 3 Bayesian Nash Equilibrium

We proceed to characterize the BNE of $\boldsymbol{\Gamma}$ starting with best responses.

### 3.1 Best Responses

Given the payoff structure, the best response of the $i^{t h}$ player whose is of type $\mathbf{g}_{i}^{t_{i}}$ is given by:

$$
a_{i}\left(\mathbf{g}_{i}^{t_{i}}\right)=1+\lambda \sum_{j=1}^{n} g_{i j}^{t_{i}} \sum_{\mathbf{g}_{j} \in G_{j}} p\left(\mathbf{g}_{j} \mid \mathbf{g}_{i}^{t_{i}}\right) a_{j}\left(\mathbf{g}_{j}\right)
$$

The system of best responses can be written in vector notation as follows:

$$
\begin{equation*}
\mathbf{a}=\mathbf{1}_{n \gamma}+\lambda \mathbb{B} \mathbf{a} \tag{5}
\end{equation*}
$$

where $\mathbf{1}_{n \gamma}$ is the $n \gamma$-dimesnional column vector of 1 's, $\mathbf{a}=\left[\mathbf{a}_{i}\right]_{i=1}^{n}, \mathbf{a}_{i}=\left[a_{i}\left(\mathbf{g}_{i}^{t_{i}}\right)\right]_{t_{i}=1}^{\gamma}, \gamma=2^{n-1}$ is the total number of types of each player, and $\mathbb{B}$ is a block matrix that assumes the following form

$$
\mathbb{B}=\left(\begin{array}{cccc}
\mathbf{0} & G_{1 \sim 2} & \ldots & G_{1 \sim n} \\
G_{2 \sim 1} & \mathbf{0} & \ldots & G_{2 \sim n} \\
\ldots & \ldots & \ldots & \ldots \\
G_{n \sim 1} & G_{n \sim 2} & \ldots & \mathbf{0}
\end{array}\right)_{n \gamma \times n \gamma}
$$

with

$$
\left[G_{i \sim j}\right]_{t_{i} t_{j}}=g_{i j}^{t_{i}} p\left(\mathbf{g}_{j}^{t_{j}} \mid \mathbf{g}_{i}^{t_{i}}\right) \quad \forall t_{j}, t_{i}=1, . ., \gamma \quad \text { and } \quad \forall \mathbf{g}_{j}^{t_{j}} \in G_{j}, \mathbf{g}_{i}^{t_{i}} \in G_{i}
$$

It can be verified that if the ex-ante distribution satisfies $p(\mathbf{g})=1$ for some $\mathbf{g} \in \mathcal{G}_{n}$ and $p\left(\mathbf{g}^{\prime}\right)=0$ for all $\mathbf{g}^{\prime} \neq \mathbf{g}$, then $a_{i}\left(\mathbf{g}_{i}\right)=0$ for all $i \in N$ for which $\mathbf{g}_{i} \notin \mathbf{g}$. In this case, the system of best responses would reduce to the complete information Nash equilibrium (Ballester et al. (2006)):

$$
\begin{equation*}
\mathbf{a}^{c}=\mathbf{1}_{n}+\lambda \mathbf{g a}^{c} \tag{6}
\end{equation*}
$$

where $\mathbf{a}^{c}=\left(a_{1}\left(\mathbf{g}_{1}\right), . ., a_{n}\left(\mathbf{g}_{n}\right)\right)$ with $\mathbf{g}=\left(\mathbf{g}_{1}, . ., \mathbf{g}_{n}\right)$. In other words, in the complete information case, the matrix $\mathbb{B}$ would reduce to the actual network over which the game is played, and agents would best respond to the actions of their adjacent agents. In the incomplete information case, however, agents do not know the types of their neighbors and best respond to the updated beliefs regarding their neighbors actions. This is captured by the elements within the blocks of $\mathbb{B}$. For instance, consider agent $i$ and the block $\left[G_{i \sim j}\right]_{t_{i} t_{j}}$. Its elements are of the form $g_{i j}^{t_{i}} p\left(\mathbf{g}_{j}^{t_{j}} \mid \mathbf{g}_{i}^{t_{i}}\right)$, which state that if agent $i$ whose type $\mathbf{g}_{i}^{t_{i}}$ is such that it is connected to agent $j$, it will assign the probability to this agent being of type $\mathbf{g}_{j}^{t_{j}}$ equal to $p\left(\mathbf{g}_{j}^{t_{j}} \mid \mathbf{g}_{i}^{t_{i}}\right)$. Observe that this updating takes place as long as agent $i$ is connected to agent $j$, implying that agents form beliefs about other as long as a link exists between them. In this way, the matrix $\mathbb{B}$ may be interpreted in a similar way to the complete information case, but instead of adjacency over agents, it provides the adjacency pattern over network admissible types. In turn, this gives rise to a network between types themselves.

For example, suppose $n=3$ and let the underlying distribution be uniform on $\mathcal{G}_{3}$. This implies that updated beliefs are given by $p\left(\mathbf{g}_{j} \mid \mathbf{g}_{i}\right)=\frac{1}{2}, \forall \mathbf{g}_{j} \in G_{j}$. Setting $\lambda=0.1$, numerical
computation gives following characterization:

$$
\mathbf{a}=\left[\begin{array}{l}
a_{1}\left((0,0,0)_{1}\right) \\
a_{1}\left((0,1,0)_{1}\right) \\
a_{1}\left((0,0,1)_{1}\right) \\
a_{1}\left((0,1,1)_{1}\right) \\
a_{2}\left((0,0,0)_{2}\right) \\
a_{2}\left((1,0,0)_{2}\right) \\
a_{2}\left((0,0,1)_{2}\right) \\
a_{2}\left((1,0,1)_{2}\right) \\
a_{3}\left((0,0,0)_{3}\right) \\
a_{3}\left((1,0,0)_{3}\right) \\
a_{3}\left((0,1,0)_{3}\right) \\
a_{3}\left((1,1,0)_{3}\right)
\end{array}\right] \quad \mathbb{B}=\frac{1}{2}\left(\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}\right) \quad \mathbf{a}^{*}=\left[\begin{array}{c}
1 . \\
1.11765 \\
1.11765 \\
1.23529 \\
1 . \\
1.11765 \\
1.11765 \\
1.23529 \\
1 . \\
1.11765 \\
1.11765 \\
1.23529
\end{array}\right]
$$

If we think of $\mathbb{B}$ as an adjacency matrix whose entries are representative of links between network admissible types, it may then be visualized by the figure 2 .

Consider player 2 and suppose it has realized the type ( $1,0,0)_{2}$ (as visualized in the left-most network of figure 1). The player knows that it connected to player 1 , as $g_{21}=1$, and that it is not connected to 3 , as $g_{23}=0$. Therefore, agent 2 will form beliefs over agent 1 's types. Since the only types of agent 1 that are network admissible with the type $(1,0,0)_{2}$ are $(0,1,1)_{1}$ and $(0,1,0)_{1}$, then there exists a link between the types $(1,0,0)_{2}$ and $(0,1,1)_{1}$ as well as $(1,0,0)_{2}$ and $(0,1,0)_{1}$. A similar argument holds for all other agents and all of their possible types. This in turn produces figure 2.

Before we proceed, we note that closest to our best response characterization is the interaction structure considered in Golub and Morris (2020). Although the signal realizations of each agent in their model can be thought of as arising from a more general information structure (which could potentially allow for network signals themselves), the network realization itself is nonetheless common knowledge. In their general theory of networks and information, agent behavior is driven by an endowed interaction structure similar to our matrix $\mathbb{B}$. In our model, however, this is generated endogenously as a result of optimizing behavior.


Figure 2: Network between types of each players

### 3.2 Existence-Uniqueness

According to definition 3, the BNE is characterized by the fixed point of the system of equations in (5). We have the following classification:

Proposition 1. There exists an unique pure strategy $B N E$ for all $\lambda \in\left[0, \frac{1}{n-1}\right)$

Observe that the bound on the local complementarity parameter $\lambda$ which guarantees the
existence and uniqueness of an equilibrium is identical to the complete information bound. ${ }^{3}$ Formally, this holds because the elements in each row of $\mathbb{B}$ sum to at most $n-1$. This can be seen from the fact that the non-zero rows of its blocks $\left[G_{i \sim j}\right]_{k l}$ sum to 1 , as they correspond to conditional probability distributions between network admissible types.

Intuitively, $n-1$ represents the maximal number of agents that each individual can extract direct complementarities from. Since the complementarity strength arising from a single link is $\lambda$, the maximal direct complementarity that may be extracted by a single agent is $\lambda(n-1)$. Moreover, agents are embedded in a network, so they can also extract complementarities from their indirect connections. In the complete information case, the maximal complementarity that can be extracted by a single agent due to their $p^{t h}$ order indirect connections is $\lambda^{p}(n-$ $1)^{p}{ }^{4}$ Therefore, summing over all $p \in \mathbb{N}_{+}$gives the maximal complementarity that any agent can extract from any network, which in turn produces a bound on the strength $\lambda$ for actions to be bounded.

In the incomplete information case, a similar argument holds, but the bound on the maximal complementarity that may be extracted from the network is attained by decomposing it across states rather than links. For example, consider an agent $i$ who is of type $\mathbf{g}_{i}$, and who is connected to agent $j$. Given updated beliefs, agent $i$ assigns a probability $p\left(\mathbf{g}_{j} \mid \mathbf{g}_{i}\right)$ to agent $j$ being of type $\mathbf{g}_{j}$. This in turn induces a complementarity strength of $\lambda p\left(\mathbf{g}_{j} \mid \mathbf{g}_{i}\right)$ between the action of agent $i$ and that of an agent $j$ who is of the particular type $\mathbf{g}_{j}$. Since $\sum_{g_{j} \in G_{j}} p\left(\mathbf{g}_{j} \mid \mathbf{g}_{i}\right)=1$, then the maximal complementarity that can be extracted from a single neighbor is $\lambda$. A similar argument holds for indirect connections.

In other words, the complementarity an agent $i$ extracts from another $j$, is spread out across all of $j^{\prime} s$ types that are admissible with the realized type of agent $i$, In this sense, the model generates network externalities on the agent-state specific level rather than the agent specific level. This has important consequences for the properties of the BNE. We turn to these properties next.

[^3]
### 3.3 Walk Characterization

We now turn to the properties of the BNE. In what follows, we denote by $a_{i}^{*}\left(\mathbf{g}_{i}^{t_{i}}\right)$ the equilibrium action of agent $i$ who has realized the type $\mathbf{g}_{i}^{t_{i}} \in G_{i}$, and by $\mathbf{a}^{*}(\mathbf{g})$ the vector of equilibrium actions given that nature have drawn $\mathbf{g}=\left(\mathbf{g}_{1}^{t_{1}}, \ldots, \mathbf{g}_{n}^{t_{n}}\right) \in \mathcal{G}_{n}$. The following proposition characterizes the BNE for any ex-ante distribution and any realized network.

Proposition 2. For any $p \in \mathbb{N}_{+}$let $j_{1}, j_{2}, . ., j_{p}$ denote an arbitrary collection of $p$ indices. For any admissible probability distribution, and for any realized network $\mathbf{g} \in G$, the equilibrium actions of agents are given by:

$$
a_{i}^{*}\left(\mathbf{g}_{i}^{t_{i}}\right)=\sum_{p=0}^{\infty} \lambda^{p} \beta_{i, t_{i}}^{(p)} \quad \forall i \in N, \forall \mathbf{g}_{i}^{t_{i}} \in G_{i}
$$

where $\mathbf{g}_{i}^{t_{i}}=\left(g_{i j}^{t_{i}}\right)_{j \in N}$ is the realized type of agent $i$, and where:

$$
\beta_{i, t_{i}}^{(p)}=\sum_{j_{1}, j_{2}, \ldots, j_{p}=1}^{n} \sum_{t_{j_{1}, t_{j_{2}}, ., t_{j_{p-1}}=1}^{\gamma}}^{g_{i j_{1}}^{t_{i}} g_{j_{1} j_{2}}^{t_{j_{1}}} \ldots g_{j_{p-1}}^{t_{j_{p-1}}} p\left(\mathbf{g}_{j_{p-1}}^{t_{j_{p-1}}} \mid \mathbf{g}_{j_{p-2}}^{t_{j_{p-2}}}\right) p\left(\mathbf{g}_{j_{p-2}}^{t_{j_{p-2}}} \mid \mathbf{g}_{j_{p-3}}^{t_{j_{p-3}}}\right) \ldots p\left(\mathbf{g}_{j_{1}}^{t_{j_{1}}} \mid \mathbf{g}_{i}^{t_{i}}\right), ~ t_{i}}
$$

Proposition 2 is best understood when compared to the complete information Nash equilibrium over the same network:

$$
a_{i}^{c}\left(\mathbf{g}_{i}^{t_{i}}\right)=\sum_{p=0}^{\infty} \lambda^{p}\left[\sum_{j_{1}, j_{2}, \ldots, j_{p}=1}^{n} g_{i j_{1}}^{t_{i}} g_{j_{1} j_{2}}^{t_{j_{1}}} \ldots g_{j_{p-1}}^{t_{j_{p-1}}}\right] \equiv \sum_{p=1}^{\infty} \lambda^{p} d_{i}^{(p)}
$$

For each $p \in \mathbb{N}_{+}, d_{i}^{(p)}$ measures the total number of walks of length $p$ originating from player $i$ to all others (including $i$ itself). In the complete information scenario, each agent has knowledge of the full architecture of the network and can thus compute these walks for all lengths $p$. Intuitively, each of these walks $i j_{1} j_{2}, . ., j_{p}$, captures the complementarity of agent $i^{\prime} s$ action with that of agent $j_{p}$ due the existence of a particular sequence of intermediate links $i \sim j_{1}, j_{1} \sim j_{2}, . ., j_{p-1} \sim j_{p}$ connecting them. Thus, each agent will take into account all of these complementarities and exert an action equal to their total strength. In turn this produces a Nash equilibrium equal to the KB centrality vector.

In the incomplete information case, knowledge of these walks is limited to those that are of first order, as agents can only identify their neighbors. Even though information is limited,
agents are, nonetheless, aware of the fact that they participate in a network, and hence, internalize the fact that walks of arbitrary orders may exist between them and all other agents. Since these walks capture complementarity strengths, that in turn dictate the magnitude of actions, agents will need to form expectations as to what their actual strength is. In the statement of proposition 2, each term $\beta_{i, t_{i}}^{(p)}$ captures this expected measure for all walks of the particular order $p$.

To describe this expected measure in more detail, consider the case $p=3$. With a slight rearranging of terms we can write

$$
\begin{aligned}
\beta_{i, t_{i}}^{(3)} & =\sum_{j, k, l=1}^{n} \sum_{t_{j}, t_{k}=1}^{\gamma} g_{i j}^{t_{i}} g_{j k}^{t_{j}} g_{k l}^{t_{k}} p\left(\mathbf{g}_{k}^{t_{k}} \mid \mathbf{g}_{j}^{t_{j}}\right) p\left(\mathbf{g}_{j}^{t_{j}} \mid \mathbf{g}_{i}^{t_{i}}\right) \\
& =\sum_{j, k=1}^{n} \sum_{l=1}^{n} g_{i j}^{t_{i}} \sum_{t_{j}=1}^{\gamma} g_{j k}^{t_{j}} p\left(\mathbf{g}_{j}^{t_{j}} \mid \mathbf{g}_{i}^{t_{i}}\right) \sum_{t_{k}=1}^{\gamma} g_{k l}^{t_{k}} p\left(\mathbf{g}_{k}^{t_{k}} \mid \mathbf{g}_{j}^{t_{j}}\right)
\end{aligned}
$$

As per the timing of events in the game, agent $i$ gets to know its type $\mathbf{g}_{i}^{t_{i}}$ and hence has full knowledge of the links $g_{i j}^{t_{i}}$. The player is thus aware of the agents through which it can form a walk of length three. To fix ideas, suppose that player $i$ wants to form of a belief about the complementarity strength of its action with that of agent $l$ due to the particular walk $i j k l$. As for the reasons mentioned earlier, agent $i$ has complete information about $g_{i j}^{t_{i}}$, i.e. the link between it and agent $j$. However, it does not have complete information about $j^{\prime} s$ type, nor about $j^{\prime} s$ neighbors' neighbors' type which in turn may or may not include a link with agent $k$ through which the walk of interest $i j k l$ reaches agent $l$.

The expectation regarding the strength of this complementarity is formed in three steps. First, the agent will condition on the fact that it has a link with agent $j$. This occurs with probability $p\left(\mathbf{g}_{i}^{t_{i}} \mid \mathbf{g}_{i}^{t_{i}}\right)=1$ (since we are assuming that $g_{i j}^{t_{i}}=1$ ) and thus, we may think of $g_{i j}^{t_{i}}$ as the expected number of ways that agent $i$ can reach agent $j$. Second, the agent internalizes its own type through $p\left(\mathbf{g}_{j}^{t_{j}} \mid \mathbf{g}_{i}^{t_{i}}\right)$, to compute expectations over the links between its neighbor $j$ and its neighbors' neighbor $k$. Using this, the agent counts the expected number of ways it can reach $k$ through $j$, given the link $g_{i j}^{t_{i}}$ exists i.e. given it is of type $\mathbf{g}_{i}^{t_{i}}$. This is given by $\sum_{t_{j}=1}^{\gamma} g_{j k}^{t_{j}} p\left(\mathbf{g}_{j}^{t_{j}} \mid \mathbf{g}_{i}^{t_{i}}\right)$. Third, the agent internalizes the information about the possible types of its neighbor $j$ through $p\left(\mathbf{g}_{k}^{t_{k}} \mid \mathbf{g}_{j}^{t_{j}}\right)$, to compute the expectations over the links between its neighbors' neighbor $k$ and agent $l$, (who is its neighbors' neighbors' neighbor). Using this, the agent counts the expected number of possible ways it can reach $l$ through $k$, conditional on the existence of the link $g_{j k}^{t_{j}}$ i.e. given that $j$ is of type $\mathbf{g}_{j}^{t_{j}}$. This is given by $\sum_{t_{k}=1}^{\gamma} g_{k l}^{t_{k}} p\left(\mathbf{g}_{k}^{t_{k}} \mid \mathbf{g}_{j}^{t_{j}}\right)$.


Figure 3: Expected walk of length 3

Given the above, the expected total number of ways player $i$ can reach player $l$ via a walk of length three is given by the product of (i) the actual link between it and $j$, (ii) the number of ways it can reach $k$ from $j$ given the previous link $g_{i j}^{t_{i}}$ exists and (iii) the number of ways it can reach $l$ from $k$ given $g_{j k}^{t_{j}}$ exists. Repeating this process for all possible walks of length three which start from agent $i$, and summing over all possible values of $j, k$ and $l$, gives the expected complementarity strength of agent $i^{\prime} s$ action with all other agents due to walks of length three $\beta_{i, t_{i}}^{(3)}$.

There are a couple of remarks that we make with regard to the nature of the preceding expected complementarity calculation. First, we note that, $\beta_{i, t_{i}}^{(p)} \neq \mathbb{E}\left(d_{i}^{(p)}\right)$. That is, the expected complementarity arising from walks of length $p$ does not equal to the ex-ante expectation of these walks. This to be expected as the equilibrium is an interim notion, allowing for belief-updating. An important consequence of this, is that the BNE equilibrium of this game does not equal the ex-ante expectation of KB centrality. Motivated by complete information equilibrium notions, applied work has tried to estimate network effects in environments in which researchers cannot observe the network. These approaches implicitly presume that although the researcher does not have information about the network, the agents themselves have this information. The proposed estimators are reflective of this, as they correspond to ex-ante expectations of complete information outcomes. As demonstrated by Breza et al. (2018), however, the assumption that a researcher is unaware of the network while the subjects are aware of it, may in some cases be inconsistent. If so, and as Proposition 2
suggests, agent behavior under such information settings would not correspond to ex-ante expectations over complete information outcomes.

Next, we also note that, $\beta_{i, t_{i}}^{(p)} \neq \mathbb{E}\left(d_{i}^{(p)} \mid \mathbf{g}_{i}^{t_{i}}\right)$. To shed more light into this, consider the case of $p=3$.

$$
\begin{aligned}
\mathbb{E}\left(d_{i}^{(3)} \mid \mathbf{g}_{i}^{t_{i}}\right) & =\mathbb{E}\left(\sum_{j, k=1}^{n} \sum_{l=1}^{n} g_{i j} g_{j k} g_{k l} \mid \mathbf{g}_{i}^{t_{i}}\right) \\
& =\sum_{j, k=1}^{n} \sum_{l=1}^{n} \mathbb{E}\left(g_{i j} g_{j k} g_{k l} \mid \mathbf{g}_{i}^{t_{i}}\right) \\
& =\sum_{j, k=1}^{n} \sum_{l=1}^{n} \sum_{t_{j}=1}^{\gamma} \sum_{t_{k}=1}^{\gamma} g_{i j}^{t_{i}} g_{j k}^{t_{j}} g_{k l}^{t_{k}} p\left(g_{j k}^{t_{j}} g_{k l}^{t_{k}} \mid \mathbf{g}_{i}^{t_{i}}\right)
\end{aligned}
$$

This hypothetical ex-post expectation calculation, fails to capture the dynamics by which the agent $i$ internalizes the possible types of its neighbors, its neighbors' neighbors and so on. In other words, only conditioning on their own type $\mathbf{g}_{i}^{t_{i}}$, makes it a more restricted measure. On the other hand, $\beta_{i, t_{i}}^{(p)}$ gives us the process by which agent $i$ internalizes the possible types of its neighbors, its neighbors' neighbors and so on. The interim pay-off structure being linear quadratic, implies that the payoff to agent $i$ is dependent on their neighbors' actions. As a result, the equilibrium action strategy given by $\sum_{p=0}^{\infty} \lambda^{p} \beta_{i, t_{i}}^{(p)}$, enables us to understand the dynamics of how the agents internalizes the information about the possible types of others, and how they are employed towards equilibrium play.

### 3.4 The Uniform Case

Before we proceed to welfare, we discuss the special case in which the ex-ante distribution is uniform over the set of all networks. We have the following characterization.

Proposition 3. When the underlying probability distribution is uniform, the equilibrium actions of the agents for any realized graph are given by:

$$
a^{*}(d)=1+\frac{\lambda d}{1-\frac{n \lambda}{2}}, \quad \forall d \in\{0,1,2, \ldots,(n-1)\}
$$

where $d$ is the degree of an agent in the realized network.

Proposition 3 is similar to the finding of Galleotti et al. (2010) (Proposition 2). They study a reduced form game of incomplete information in which agents are endowed with beliefs about
degree distributions, establishing the existence of a BNE in which the actions of agents are monotonically increasing in their degrees. The result is as consequence of two assumptions. The first is anonymity, which states that while agents are aware of the number of agents they are connected to, they are unaware of the identity of these adjacent agents. The second is independence, which states that the degree of any agent who is connected to another is independent of the latter's degree.

While our result provides the same quantitative insight as their finding, it hold due to a different belief structure. First, we do not assume anonymity. As seen from Proposition 2 , our BNE is the result of an expected walk calculation. These walks are computed for all possible sequences of nodes, which in turn require agent identity to be accounted for. Clearly, these expected walks would be different for different ex-ante distributions that place higher probability mass on specific sets of networks which contain specific sets of walks. While anonymity is not imposed in our model, Proposition 3 provides a condition for it to appear as if it arises in equilibrium. This appearance is due to the fact that equilibrium actions are completely characterized by agent degrees, which in turn may be thought of as abstracting away from the computation of possible walks. However, this is a consequence of the uniform assumption and the corresponding expected walks it induces. As the following lemma suggests, the uniform case has the special property that the ex-post expectation of any agent regarding any other's (who is either connected or non connected to the former) degree is the same for all agents.

Lemma 1. For any player $i$ denote by $\mathbb{E}_{i}\left(d_{j_{1}}\right), \mathbb{E}_{i}\left(d_{j_{2}}\right)$,. the agent's ex-post expectations of any of its neighbor's degree, any of its possible neighbor's neighbor's degree and so on. When the underlying probability distribution is uniform, we have:

$$
\mathbb{E}_{i}\left[d_{j_{1}}\right]=\mathbb{E}_{i}\left[d_{j_{2}}\right]=\ldots \ldots=\frac{n}{2}
$$

Consequently,

$$
\beta_{i, t_{i}}^{(p)}=\left(\frac{n}{2}\right)^{p}
$$

Uniformity in ex-ante beliefs provides the least amount of information with respect to the identification of which walks are present in the network, inducing the trivial belief that each other agent has a degree equal to $\frac{n}{2}$. Consequently, each agent expects the complementarity strength of its action with any other agent due to a walk of length $p$ to be equal to $\left(\frac{n}{2}\right)^{p}$.

Lemma 1 also speaks to the second assumption in Galleotti et al. (2010) regarding degree independence. In their set up, independence of degrees implies that the belief that a player
who is of degree $d$, and that of another who is of degree $d+1$, regarding the degrees of each of their neighbors are the same. In our case, the same property holds, but it arises endogenously as a result of ex-ante uniformity.

## 4 Welfare

Let $W^{*}(\mathbf{g}), W^{I}(\mathbf{g})$ and $W^{C}(\mathbf{g})$ respectively denote ex-post, interim, and complete information welfare, given that nature has chosen some $\mathbf{g} \in \mathcal{G}_{n}$. These are given by:

$$
\begin{aligned}
W^{*}(\mathbf{g}) & =a_{i}^{*}\left(\mathbf{g}_{i}\right)-\frac{1}{2} a_{i}^{*}\left(\mathbf{g}_{i}\right)^{2}+\lambda a_{i}^{*}\left(\mathbf{g}_{i}\right) \sum_{j} g_{i j} a_{j}^{*}\left(\mathbf{g}_{j}\right) \\
W^{I}(\mathbf{g}) & =\frac{1}{2} \sum_{i=1}^{n} a_{i}^{*}\left(\mathbf{g}_{i}\right)^{2} \\
W^{C}(\mathbf{g}) & =\frac{1}{2} \sum_{i=1}^{n} a_{i}^{c}\left(\mathbf{g}_{i}\right)^{2}
\end{aligned}
$$

where $a_{i}^{*}\left(\mathbf{g}_{i}\right)$ is the BNE action of agent $i$ who is of type $\mathbf{g}_{i}$ and $a_{i}^{c}\left(\mathbf{g}_{i}\right)$ is complete information Nash equilibrium action of agent $i$ whose row is also $\mathbf{g}_{i}$. Note that the difference between ex-post and interim payoffs lies in the manner by which network externalities are accounted for. Interim welfare considers these externalities in expectation, while ex-post computes the actual level of externalities produced in equilibrium given that nature has chosen a particular network.

Proposition 4. If the underlying probability distribution is uniform, then for a d-regular graph with $d \leq \frac{n}{2}$ we have

$$
W^{C}(\mathbf{g}) \leq W^{*}(\mathbf{g})
$$

Since the underlying distribution is assumed to uniform, Lemma 1 suggests that each agent expects the complementarity of its action due to walks of length $p$ to be $\left(\frac{n}{2}\right)^{p}$, regardless of the realization of their type. In other words, even though the degree induced by their realized type is less $\frac{n}{2}$, agents still expect that they will have $\left(\frac{n}{2}\right)^{p}$ walks of length $p$ which is less than what they actually have in the realized network. Consequently, this induces over-exertion of actions relative to the complete information Nash equilibrium leading to larger welfare.

## 5 Conclusion

We study a linear quadratic network game of incomplete information in which agents information is restricted to the identity of their neighbors only. We characterize Bayesian-Nash equilibria, demonstrating that agents will make use of local information in order to form beliefs about the number of walks they have in the network and consequently the complementarity strength of their action with all other agents.

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## Appendix

## Proof of Proposition 1

Define a mapping $P: \mathbb{R}^{\gamma} \longrightarrow \mathbb{R}^{\gamma}$, such that

$$
P(\mathbf{a})=\mathbf{1}+\lambda(n) \cdot \mathbb{B}(n) \mathbf{a}
$$

where,

$$
[\mathbb{B}(n)]_{i j}=\left\{\begin{array}{lll}
\mathbf{0}_{\gamma \times \gamma} & \text { if } & i=j \\
G_{i \sim j} & \text { if } & i \neq j
\end{array}\right.
$$

and

$$
\left[G_{i \sim j}\right]_{k l}=g_{i j}^{k} p\left(\mathbf{g}_{j}^{l} \mid \mathbf{g}_{i}^{k}\right) \quad \forall l, k=1(1) \gamma \quad \text { and } \quad \forall \mathbf{g}_{j}^{l} \in G_{j}, \mathbf{g}_{i}^{k} \in G_{i}
$$

Let $\left(\mathbb{R}^{\gamma},\|\cdot\|_{\infty}\right)$ be a metric space with $\|\cdot\|_{\infty}$ being the sup-norm defined on $\mathbb{R}^{\gamma}$, i.e.

$$
\|\mathbf{a}\|_{\infty}=\max \left\{\left|a_{j}\right|: j \in N\right\}
$$

Hence, we can write:

$$
\begin{aligned}
\|P(\mathbf{x})-P(\mathbf{y})\|_{\infty} & =|\lambda(n)|\|\mathbb{B}(n) \cdot(x-y)\|_{\infty} \\
& \leq \lambda(n)(n-1)\|x-y\|_{\infty} \\
& =r\|x-y\|_{\infty}
\end{aligned}
$$

where the first inequality results from the fact that $\|\mathbb{B}(n) \cdot \mathbf{a}\|_{\infty} \leq(n-1)\|\mathbf{a}\|_{\infty}$ for $\mathbb{B}(n) \in$ $\{0,1\}^{n \gamma}$. Thus, we get

$$
\|P(\mathbf{x})-P(\mathbf{y})\|_{\infty} \leq r\|x-y\|_{\infty}
$$

so that $P$ is a contraction mapping on $\mathbb{R}^{\gamma}$ as long as $r \in[0,1)$. This holds if:

$$
0 \leq \lambda(n) \cdot(n-1) \leq 1 \Rightarrow 0 \leq \lambda(n)<\frac{1}{n-1}
$$

Hence, for $0 \leq \lambda(n)<\frac{1}{n-1}, P$ is a contraction mapping on $\mathbb{R}^{\gamma}$ and $\left(\mathbb{R}^{\gamma},\|\cdot\|_{\infty}\right)$ is a complete metric space. Therefore, by the Banach fixed point theorem, there exists an unique $\mathbf{a}^{*} \in \mathbb{R}^{\gamma}$, such that

$$
P\left(\mathbf{a}^{*}\right)=\mathbf{a}^{*} \Rightarrow \mathbf{a}^{*}=\mathbf{1}+\lambda(n) \cdot \mathbb{B}(n) \mathbf{a}^{*}
$$

Consequently, there exists an unique pure strategy BNE for the game $\boldsymbol{\Gamma}$ whenever $\lambda(n) \in$ $\left[0, \frac{1}{n-1}\right)$.

## Proof of Proposotion 2

The equilibrium actions can be written from (5) as,

$$
\begin{aligned}
\mathbf{a}^{*} & =(\mathbf{I}-\lambda(n) \mathbb{B}(n))^{-1} \cdot \mathbf{1} \\
& =\mathbf{1}+\lambda \mathbb{B} \cdot \mathbf{1}+\lambda^{2} \mathbb{B}^{2} \cdot \mathbf{1}+\ldots \ldots
\end{aligned}
$$

For an agent $i \in N$ of type $\mathbf{g}_{i}^{t_{i}}$,

$$
\mathbf{a}^{*}=1+\lambda[\mathbb{B} \cdot \mathbf{1}]_{i, t_{i}}+\lambda^{2}\left[\mathbb{B}^{2} \cdot \mathbf{1}\right]_{i, t_{i}}+\ldots \ldots
$$

Then,

$$
\begin{aligned}
\beta_{i, t_{i}}^{(1)} & =[\mathbb{B} \cdot \mathbf{1}]_{i, t_{i}} \\
& =\sum_{j=1}^{n} \sum_{t_{j} \in G_{j}}\left[G_{i \sim j}\right]_{t_{i}, t_{j}} \\
& =\sum_{j=1}^{n} \sum_{t_{j} \in G_{j}} g_{i j}^{t_{i}} p\left(\mathbf{g}_{j}^{t_{j}} \mid \mathbf{g}_{i}^{t_{i}}\right) \\
& =\sum_{j=1}^{n} g_{i j}^{t_{i}} \sum_{t_{j} \in G_{j}} p\left(\mathbf{g}_{j}^{t_{j}} \mid \mathbf{g}_{i}^{t_{i}}\right) \\
& =\sum_{j=1}^{n} g_{i j}^{t_{i}}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\beta_{i, t_{i}}^{(2)} & =\left[\mathbb{B}^{2} \cdot \mathbf{1}\right]_{i, t_{i}} \\
& =[\mathbb{B} \cdot(\mathbb{B} \cdot \mathbf{1})]_{i, t_{i}} \\
& =\sum_{j=1}^{n} \sum_{t_{j} \in G_{j}}\left[G_{i \sim j}\right]_{t_{i}, t_{j}} \beta_{j, t_{j}}^{(1)} \\
& =\sum_{j=1}^{n} \sum_{t_{j} \in G_{j}} g_{i j}^{t_{i}} p\left(\mathbf{g}_{j}^{t_{j}} \mid \mathbf{g}_{i}^{t_{i}}\right) \sum_{k=1}^{n} g_{j k}^{t_{j}} \\
& =\sum_{j=1}^{n} \sum_{t_{j} \in G_{j}} \sum_{k=1}^{n} g_{i j}^{t_{i}} g_{j k}^{t_{j}} p\left(\mathbf{g}_{j}^{t_{j}} \mid \mathbf{g}_{i}^{t_{i}}\right)
\end{aligned}
$$

This calculation generalizes for $\beta_{i, t_{i}}^{(m)}$.

## Proof of Lemma 1

Fix an agent $i \in N$, then, for any $j$ a uniform ex-ante distribution implies

$$
p\left(d_{j_{1}}=d\right)=\frac{\binom{n-2}{d-1} 2^{\frac{(n-1)(n-2)}{2}-(n-2)}}{2^{\frac{(n-1)(n-2)}{2}}}=\frac{\binom{n-2}{d-1}}{2^{n-2}}
$$

Similarly, we can write

$$
p\left(d_{j_{2}}=d\right)=\frac{\binom{n-2}{d-1} 2^{\frac{(n-1)(n-2)}{2}}-(n-2)}{2^{\frac{(n-1)(n-2)}{2}}}=\frac{\binom{n-2}{d-1}}{2^{n-2}}
$$

and so on. Therefore,

$$
\begin{aligned}
\mathbb{E}_{i}\left[d_{j_{2}}\right] & =\sum_{d=1}^{n-1} p\left(d_{j_{1}}=d\right) d \\
& =\sum_{d=1}^{n-1} \frac{\binom{n-2}{d-1}}{2^{n-2}} d \\
& =\frac{n}{2}
\end{aligned}
$$

Similarly, we can show the rest.

## Proof of Proposition 3

To prove proposition 3, if we characterise the actions of each agents with respect to degrees, then first order condition can be re-written as:

$$
\begin{gather*}
a(d)=1+\lambda d \sum_{d=1}^{n-1} \frac{\binom{n-2}{d-1}}{2^{n-2}} a(d)  \tag{7}\\
\therefore\binom{n-2}{d-1} a(d)=\binom{n-2}{d-1}+\left[\frac{\lambda}{2^{n-2}} \sum_{d=1}^{n-1}\binom{n-2}{d-1} a(d)\right]\binom{n-2}{d-1} d \\
\Rightarrow \sum_{d=1}^{n-1}\binom{n-2}{d-1} a(d)=\sum_{d=1}^{n-1}\binom{n-2}{d-1}+\left[\frac{\lambda}{2^{n-2}} \sum_{d=1}^{n-1}\binom{n-2}{d-1} a(d)\right] \sum_{d=1}^{n-1}\binom{n-2}{d-1} d \\
\Rightarrow \sum_{d=1}^{n-1}\binom{n-2}{d-1} a(d)=2^{n-2}+\left[\frac{\lambda}{2^{n-2}} \sum_{d=1}^{n-1}\binom{n-2}{d-1} a(d)\right] \sum_{d=1}^{n-1}\binom{n-2}{d-1} d
\end{gather*}
$$

We have

$$
\begin{aligned}
\sum_{d=1}^{n-1}\binom{n-2}{d-1} d & =\sum_{d=1}^{n-1}(d-1)\binom{n-2}{d-1}+\sum_{d=1}^{n-1}\binom{n-2}{d-1} \\
& =\sum_{d=1}^{n-1} \frac{(n-2)!}{(d-2)!(n-2-\overline{d-1})!}+2^{n-2} \\
& =(n-2) \sum_{d=2}^{n-1}\binom{n-3}{d-2}+2^{n-2} \\
& =(n-2) 2^{n-3}+2^{n-2}
\end{aligned}
$$

so we can write,

$$
\begin{aligned}
\sum_{d=1}^{n-1}\binom{n-2}{d-1} a(d) & =2^{n-2}+\left[\frac{\lambda}{2^{n-2}} \sum_{d=1}^{n-1}\binom{n-2}{d-1} a(d)\right]\left((n-2) 2^{n-3}+2^{n-2}\right) \\
& =\frac{2^{n-2}}{1-\frac{n \lambda}{2}}
\end{aligned}
$$

Thus from equation (8),

$$
\begin{equation*}
a(d)=1+\frac{\lambda d}{1-\frac{n \lambda}{2}}, \quad \forall d \in\{0,1,2, \ldots,(n-1)\} \tag{8}
\end{equation*}
$$

## Proof of Proposition 4

For the complete information case, the equilibrium actions of each agent in a realized dregular graph $\mathbf{g} \in G$ is given by:

$$
a_{i}^{c}(\mathbf{g})=1+\frac{\lambda d}{1-\lambda d} \quad \forall i \in N
$$

Since $d \leq \frac{n}{2}$, we get that

$$
\begin{aligned}
1-\lambda d & \geq 1-\lambda \frac{n}{2} \\
\Longrightarrow 1+\frac{\lambda d}{1-\lambda d} & \leq 1+\frac{\lambda d}{1-\frac{n \lambda}{2}} \\
\Longrightarrow a_{i}^{c}(\mathbf{g}) & \leq a_{i}^{*}(\mathbf{g}), \quad \forall i \in N
\end{aligned}
$$

Hence the ex-post utilities for each players,

$$
\begin{aligned}
u_{i}^{c}(\mathbf{g})-u_{i}^{*}(\mathbf{g}) & =\frac{1}{2} a_{i}^{c}(\mathbf{g})^{2}-a_{i}^{*}(\mathbf{g})+\frac{1}{2} a_{i}^{*}(\mathbf{g})^{2}-\lambda a_{i}^{*}(\mathbf{g}) \sum_{j=1}^{n} g_{i j} a_{j}^{*}(\mathbf{g}) \\
& =\frac{1}{2} a_{i}^{c}(\mathbf{g})^{2}-a_{i}^{*}(\mathbf{g})+\frac{1}{2} a_{i}^{*}(\mathbf{g})^{2}-\lambda d a_{i}^{*}(\mathbf{g})^{2} \\
& \leq \frac{1}{2} a_{i}^{c}(\mathbf{g})^{2}-a_{i}^{c}(\mathbf{g})-\left(\lambda d-\frac{1}{2}\right) a_{i}^{c}(\mathbf{g})^{2} \\
& =(1-\lambda d) a_{i}^{c}(\mathbf{g})^{2}-a_{i}^{c}(\mathbf{g}) \\
& =\left((1-\lambda d) a_{i}^{c}(\mathbf{g})-1\right) a_{i}^{c}(\mathbf{g}) \\
& =0
\end{aligned}
$$

Where the second equality is due to the fact that for a d-regular graph realization, $a_{i}^{*}(\mathbf{g})=$ $a_{j}^{*}(\mathbf{g}), \forall i, j$ and the third inequality is due to the fact that $a_{i}^{c}(\mathbf{g}) \leq a_{i}^{*}(\mathbf{g}), \forall i \in N$. Thus we have

$$
u_{i}^{c}(\mathbf{g})-u_{i}^{*}(\mathbf{g}) \leq 0, \quad \forall i \in N
$$

Summing over all agents gives the result.


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[^1]:    ${ }^{1}$ Note that the subscript in $\left(g_{i 1}, g_{i 2}, \ldots, g_{i n}\right)_{i}$ is imposed to differentiate between agents whose types consist of the same sequence of 0 's and 1 's. For instance, if $n=3$ it differentiates between the type $(0,1,0)_{1}$ for agent 1 and $(0,1,0)_{3}$ for agent 3 .

[^2]:    ${ }^{2}$ Whenever the context is clear and we need not enumerate the elements of each type set we suppress the superscript.

[^3]:    ${ }^{3}$ Note that a more general bound for the complete information case is $\lambda \rho(\mathbf{g})<1$ where $\rho(\mathbf{g})$ is the spectral radius of the adjacency matrix $\mathbf{g}$. The bound $\lambda<\frac{1}{n-1}$ is the tightest possible, since that maximal spectral radius for any graph $\mathbf{g} \in \mathcal{G}_{n}$ is $n-1$ and this corresponds to the complete network. We use the bound $\lambda<\frac{1}{n-1}$ so that comparisons between complete and incomplete information is possible throughout the paper without having to consider heterogeneous $\lambda$ for the complete and incomplete information cases.
    ${ }^{4}$ Since any agent is connected to at most $n-1$ others so that $p$ links aways from any node are at most $(n-1)^{p}$ other nodes from which a complementarity strength of $\lambda^{p}$ is extracted.

