On Blocking Mechanisms in economies with club goods

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Abstract

The paper talks about various forms of blocking by coalitions and how they characterise equilibrium allocations in an economy consisting of both private and club goods, where club goods are treated as articles of choice just like private goods. Clubs in this framework are described by the characteristic of their members and the local project the club endorses. We show that the set of core allocations can be achieved even if one restrictis to coalitions of any sgiven size between zero and measure of the grand coalition. We further introduce mixed economy in our scenario and establish equivalence between core allocations of such an economy with the core allocations of it's associated continuum economy. Equivalence between the set of equilibrium aloccations between these economies for a restricted class of allocations helps us achieve core-equivalence in a mixed economy for such restricted class of allocations. At the end we provide another characterisation of club equilibrium in continuum economies by considering veto power of the grand coalition in infinitely many economies obtained by perturbation of initial endowments in the economy.

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1 Introduction

Clubs are discretionary groups set up by individuals to support social consumptions. In recent years the term "clubs" has been broadened to incorporate organisations and projects such as toll roads, private parks, satelliteus, television network connection, golf clubs etc. Club goods stand roughly in the middle of the spectrum that ranges form pure public to pure private goods. Club goods in sharp distinctionto public goods are exclusive to only the members of the club. Individual members often willingly or unwillingly imposes congestion cost on other members thereby adding to their disutility. In such settings the club literature has mainly emphasized that individuals calculate the marginal utility or disutility from joining a club and such optimising behavior on behalf of individuals in terms determine the optimal sizes of the clubs in equilibrium. Seminal works in club literature ranging from that by Tiebout[29], Wiseman[31], Buchanan[9] and others aimed at finding "optimal club sizes" . However, such findings based on individual choices was possible because of the finite number of indiviiduals. This coupled with the core "indivisible" nature of club projects rendered models devoid of "perfect competition".

Ellickson et al[12] first introduced a framework that tackled issues at both the fronts. They adapt the continuum framework introduced by Aumann[2] by introducing clubs and club memberships in parallel to a continuum of agents. Club sizes are limited in our framework, so any particular club of a certain type (gyms, swimming pools, libraries etc.) can only have finite number of members, although number of clubs for a certain type may be large enough. Thus clubs are significant with respect to individuals but infinitesimal with respect to the market. Each club is identified through the non-Samuelson public project it provides and the characteristics of the members of the club. Individuals acts as members to the clubs and are bestowed with some external characteristics upon them. These characteristics are external since they are not only observable to other agents but also affects other agents utilities. As clubs have limited size compared to the market the externalities arising from such member charcacteristics are internalised within the clubs. However, trading of private goods is not restricted to within clubs as the number of private goods are more than one and individuals are members to more than one club. The model becomes further robust from the parallel treatment of private goods and club memberships. Ever membership embodies in itself a description of all relevant aspects such as the profile of charcteristics of other members, members in question, purpose of the club and resources necessary to form the club. The price for these memberships are contingent upon characteristics of the member, characteristics of other members of the club and the club projects. Thus prices for club memberships reflect the externalities within clubs.

Despite such paralell treatments there exists some fundamental differences between club memberships and private goods. Prices of public goods are always positive but that of memberships can be positive, negative or even zero. The other obvious difference arises from the indivisble nature of club memberships as opposed to private goods which are purely divisble in nature .The main difference between the two lies however in the feasibility condition. Feasibility of private goods implies equality of demand and supply. For club memberships, feasibility entails that given a particular proportion of members for a particular club type, number of clubs of the given type should be such that the proportion remains intact in the aggregate. They not only establish that competitive equilibrium exists in their setup but also can be decentralised by means of core allocations under some reasonable assumptions.

Since it's inception general equilibrium studies have focused mainly on two avenues of studying equilibrium in the economy. One is through the market economy and such allocations are referred to as Walrasian Equilibria. The other avenue is that of the set of allocations arrived through cooperative behaviour amongst the agents. We term such set of allocations as the Core allocations. Edgeworth[11] in his seminal work conjectured that as the number of agents tends to infinity one can expect the set of core allocations to merge with the set of Walrasian equilibrium allocations and was later validated by Aumann in his paper Aumann[2] . Aumann claimed that with an infinite number of agents the number of possible coalitions increases and also market power of individual agents becomes negligible which guarantees the equivalence. Shitovitz[28] in 1973 presented an opposition to Aumann's framework and claimed that no market is entirely competitive. To that extent he introduced a market with large traders and called such economies as "mixed economies". Shitovitz in his paper showed that unless and until there exists atleast two large traders with similar initial endowments and preferences the set of competitive equilibrium fails to converge to the set of core allocations. We prove an extension of the main result in Greenberg and Shitovitz[18]. Thus we show that the core of a mixed club economy and that of an associated continuum economy are equivalent. We adapt an assumption made by Basile et al[3] in order to prove the same. Further establishment of equivalence between the set of equilibrium allocations across the two economies helps us extend the core equivalence result in

Ellickson et al[12] to the case of mixed economies.

In general forming coalitions require communication between individuals forming a coalition, which however at times maybe quite costly. So charcterising core allocations with respected to coalitions of certain sizes have been studied quite extensively in the literature. Vind[30] added a very insightful remark to Aumann[2] by claiming that it is sufficient for one to concentrate only on coalitions of a fixed size lying between zero and size of the grand coalition to guarantee the equivalence between the set of equilibrium allocations and core allocations. The finite dimensional version of this theorem is guaranteed by the application of Lyapnov's convexity theorem. However, immediate extension of this result does not follow for infinite dimensional spaces. Work by Beloso et al[21] extends the Vind's theorem for economies where agents have myopic utility functions and the commodity space is the space of bounded sequences l_∞. Bhowmik and Cao[6] provides an extension where the commodity space is an ordered Banach space with non-empty positive cone. Later works by Bhowmik and Cao[6] and Bhowmik and Graziano[8] further extends the literature where commodity space is a Banch lattice with empty interior and economies with large agents and generalised coalitions respectively. We extend the Vind's theorem in the context of club economies where despite the non-convexity of club memberships the use of finite dimensional Lyapnov's convexity theorem remains validated.

One special characteristation of Walrasian equilibrium allocations for an atomless economy was posited by Hervés-Beloso and Moreno- García[20]. Instead of exercising the veto power of infinitely many coalitions in a single economy they exercised the veto power of the grand coalition in infinitely many peturbed economies. Such economies were constructed by peturbing the initial endowments of a coalition of agents. The choice of the size of such coalitions may be arbitrarily large, arbitrarily small or may be of a fixed given size. Hervés-Beloso and Moreno- García showed that the set of Walrasian equilibrium allocations are equivalent to those that are non dominated in any of the peturbed economies. They referred to them as "Robustly Efficient" set of allocations. Later, Bhowmik and Cao[7] in their paper developed the notion of robustly efficient allocations for a mixed economy with an infinite dimension commodity space. Graziano and Romaniello[17] in their paper showed that for an economy with infinitely many public goods, characterising linear cost share equilibrium in terms of non-dominated allocations in infinitely many peturbed economies does away with the dependency on the cost distribution scheme, cost share function unlike core. This followed from the fact that the grand coalition always contributed share of one to the formation of public projects. Hervés-Beloso and Moreno- García in their seminal work showed that the second welfare theorem follows directly from their main result and that the second welfare theorem fails to hold in clubs framework has been established in Ellickson et al[12] . Thus, charcterising club equilibrium in terms of robustly efficient allocation for club economies is not possible. Bhowmik and Kaur[4] attempts to find an approximation for robustly efficient allocations by making a assumption that the net trade in club memberships belong to the class of cosistent club memberships. They show that set of club equilibrium is a subset of such an approximate class of robustly efficient allocations under some stringent conditions. We basically attempt to find an approximation for which we can show that the set of club equilibrium are a subset of them without any such stringent conditions. The major difference in our approximate notion with that of Bhowmik and Kaur[4] is that we do away with the assumption on net trades of club membership. Furthermore, compared to Hervés-Beloso and Moreno-García for an allocation to be dominated, it needs to be dominated in a sequence of economies and not just one.

2 Economic Model

We assume that the set of agents for our economy is a positive, complete and finite measure space. We denote it by (A, Σ, λ) with A as the set of agents, $\mathscr{F}$ as the corresponding σ-*algebra* and λ the associated lebesgue measure.. We decompose A into two parts. A_0 is the atomless part or the set of small traders in the economy. A_1 denotes the other part which is the set of large traders. For any $T \subset A_1$, and $B \subset T$ either $\lambda(B) = 0$ or $\lambda(T \setminus B) = 0$. The atomic part is represented as union of a countable collection of atoms. Such a collection is represented as $\{T_1, T_2, T_3, \cdots\}$. The economy is said to be atomless if $A = A_0$. Now let N denote the set of private commodities . We assume that the commodities are perfectly divisible[1].The space of private goods is described the n-dimensional real valed space, i.e. $\mathbb{R}^N$. The consumption of private commodities for each agent is encompassed by the non-negative orthrant $\mathbb{R}^N_+$. Furthermore, let $\mathbb{R}^N_{++}$ denote the strictly positive elements of $\mathbb{R}^N$. For any two commodity bundles $x, y \in \mathbb{R}^N_+$, $x \geq y$ implies $x_i \geq y_i$ for all i, $x > y$ implies that $x_i \geq y_i$, however $x \neq y$ and $x >> y$ implies that $x_i > y_i$ for each i. We denote $||x|| = \sum_{n=1}^{N} x_n$.

2.1 Clubs

Each potential member of a club, as in Ellickson et al[12] is bestowed with some external characteristics. These characteristics are external in nature to the extent that they are

[1] Without loss of generality we assume that N also denotes the cardinality for the set commodities.

observable to other members and also creates externality within clubs. Examples of such characteristics can be sex, appearance, religion etc. To capture such externalities we define a broad set of finite characteristics to which an agent may accrue from. Let Ω denote the set of such characteristics. An element $\omega \in \Omega$ denotes the characteristic of an individual agent relevant for other members. Each club can be characterised by the composition of its members where the composition is defined as what proportion of the members belong to a given characteristic. For that we define, a map $\pi : \Omega \to \mathbb{Z}_+$, $\mathbb{Z}_+$ being the set of non-negative integers. We identify the composition of a club with such a map , and term it as profile of a club. Thus for any $\omega \in \Omega$ the number $\pi(\omega)$ denotes the number of individuals having characteristic ω. The total number of such members is denoted by $|\pi| = \sum_{\omega\in\Omega} \pi(\omega)$.

Now each club endorses a public project (local to the club). Such projects are part of finite set of club activities available to the profile of agents. Projects are part of an abstract set as in Mas-Colell[23]. The set is abstarct in the sense that their does not exist a common pre-defined ordering over these set of activities and ranking are entirely subjective to individual members. We denote the set of such activities by Γ. Activities are not traded and ranking amongst them maybe influenced by private goods consumption. Now we define a club completely as a composition of its members and activities. So a club is defined as a pair $\{\mathcal{C} = (\pi, \gamma)\}$. Now club projects are to be financed by members of the clubs only. In absence of notion of money in our model, projects are to be financed collaboratively by members. Thus requirement of inputs to such activities are denoted by $inp\,(\pi, \gamma)$, a vector in $\mathbb{R}^N_+$.

Club types in this framework is defined as a comboniation of the profile of the club and the local project it endorses. Memberships in general grants right of admission to individuals for clubs. So for an agent of external characteristic $\omega \in \Omega$, a club membership is basically a triplet in the form of his own characteristic, the profile of the club and the associated project of the club. We denote such a triplet by $m = (\omega, \pi, \gamma)$. Now if and only if $\pi(\omega) \geq 1$, then only an agent of characteristic $\omega \in \Omega$ will subscribe to the club. The set of all club memberships is denoted by $\mathscr{M}$. An agent may subscribe to one or more clubs and also can purchase multiple memberships of one particular. We define a map $\mathscr{L} : \mathscr{M} \to \{0, 1, 2, \dots\}$, $\mathscr{L}\,(\omega, \pi, \gamma)$ of a membership $m = (\omega, \pi, \gamma)$ denotes the number of that membership being bought. We term the above defined map as *list*. Now we denote the set of all such possible *list* by the following notation :-

$$Lists = \{\mathscr{L} : \mathscr{L} \text{ is a } list\ \}.$$

Let us now define all possible maps from the set $\mathscr{M}$ to the real line and denote it by

$\mathscr{R}^{\mathscr{M}}$. Now, it is obvious that memberships purchased are not only integer amounts but also non-negative. Thus the set *Lists* are those maps from $\mathscr{M} \to \mathbb{R}$ for which the range is restricted to the set of non-negative integers $\mathbb{Z}_+$

2.2 Club Economy

We now can complete the definition of our economy. The club economy is composed of agents, their private goods consumption, their external characteristics and clubs. Each agent posesses an initial endowmnent of priavte goods denoted by e_a. Now it is quite trivial that club goods has an embodied notion of excludability to them. Therefore certain agents may be barred from certain clubs. Thus the possible set of lists for an individual $a \in A$, denoted by $Lists_a$, a strict subset of $Lists$. The entire consumption set of an agent is hereafter denoted as $X_a = X_+^N \times Lists_a$. Let $u_a : X_a \to \mathbb{R}$ denote the utility of agent $a \in A$. As is obvious, agents derive utilities from both private goods and club memberships.

We now state some assumptions and notions pertaining to our club economy $\mathcal{E}$ which is mapping $a \mapsto (\omega_a, X_a, e_a, u_a)$:

A.1 The utility function is continuous over private goods consumption.

A.2 The external characteristic mapping $a \mapsto \omega_a$ is measurable [2].

A.3 The endowment mapping $a \mapsto e_a$ is an integrable function. Also, individual endowments e_a belong to the space X_{++}^N.

A.4 The consumption correspondence of an agent denoted by $a : A \rightrightarrows X_a$ is a measurable one.

A.5 The utility mapping $(a, x_a, l_a) \mapsto u_a(x_a, l_a)$ is jointly measurable.

A.6 Given any consumption pair (x_a, γ_a) , where γ_a is the club membership vector for agent a, utility for the agent is strictly monotone in private goods . In other words, $u_a\left(x_a + y_a, \gamma_a\right) > u_a\left(x_a, \gamma_a\right)$ whenever $y_a \in Pr_{\left(\mathbb{R}_+^N \setminus 0\right)} X_a$.

[2] One must note that the mapping is not a correspondence as agents can posses only one external characteristic.

2.3 States , Club Consistency and Transfers

Now we state the definition of club consistency. In everyday life , club memberships are indivisible and hence the need for a consistency requirement. Clubs in our framework are such that there sizes are limited and they have no market power.Therefore, juxtaposed to continuum of agents in our model, the above requirement translates to club sizes being finite. Since clubs are composed of members, individual memberships to clubs must be bounded and finite.[3] All these basically makes clubs infinitesimal relative to the society. Also, external characteristics as stated earlier inflicts externalities, but such externalities are confined within the clubs, thereby enabling the model to remain competitive.

Any states in the above defined economy $\mathcal{E}$, as noted earlier is basically a pair. For any agent $a \in A$ the first entry denotes the amount of private good consumption and the second the membership vector for the individual. Any such pair (x_a, γ_a) is said to be feasible if $(x_a, \gamma_a) \in X_a$. In standard general equilibrium model social feasibility just requires market clearance for private goods. However, in clubs framework in addition consistency is required in terms of matching of agents. To model consistency we basically decompose a club by the proportion of members the club holds for each $\omega \in \Omega$. Then when we aggregate over similar club types and check if the proportion remains intact or not.

Definition 2.1. Given an aggregate membership vector $\overline{\gamma} \in \mathscr{R}^{\mathscr{M}}$, if for each club type (κ, υ), there exists a number $\psi(\kappa, \upsilon) \in \mathbb{Z}_+ \setminus \{0\}$ such that

$$\overline{\gamma}(\omega, \pi, \gamma) = \psi(\pi, \gamma)\, \pi(\omega)$$

For all $\omega \in \Omega$ .Then we call such a vector $\overline{\gamma}$ **consistent**.

Therefore, for any coalition B, a **subscription** or a **choice function** denoted by $\gamma : B \to \mathit{Lists}$ is *consistent* if the corresponding aggregate subscription vector $\overline{\gamma}_B = \int_B \gamma_a d\mu(a)$ in $\mathscr{R}^{\mathscr{M}}$ is *consistent*. Such vectors form a subspace of $\mathscr{R}^{\mathscr{M}}$ and are strictly restricted to the positive orthant of $\mathscr{R}^{\mathscr{M}}$. The subspace is written as :

$$\mathscr{C}ons = \left\{ \overline{\gamma} \in \mathscr{R}^{\mathscr{M}} : \overline{\gamma} \text{ is consistent} \right\}$$

Now, we will define conditions under which a state is feasible to the society as a whole. Over and above the already defined conditions of consistency and individual feasibility, private goods need to achieve clearance. This is guaranteed by material

[3] All these restrictions on club sizes and individual memberships to be bounded along with finite number of public goods makes the choices finite dimensional as pointed by [12]

balance. One part of material balance is basically inputs to club activities or projects for each agent $a \in A$. We define the allocation rule as in [12] and it takes the form given by $\frac{1}{|\pi|} inp\left(\pi, \gamma\right)$. We assume that any allocation rule defined as above is **measurable with respect to the agents space**.

Definition 2.2. A state (x, γ) is feasible for a set $B \in A$ such that $\lambda\left(B\right) > 0$ if it abides by the following conditions:

- **Individual Feasibility**: $(x_a, \gamma_a) \in X_a$ for each $a \in B$,
- **Material Balance**:

$$\int_B x_a d\lambda(a) + \int_B \sum_{(\omega,\pi,\gamma)} \frac{1}{|\pi|} inp\left(\pi, \gamma\right) l_a\left(\omega, \pi, \gamma\right) d\lambda(a) = \int_B e_a d\lambda(a),$$

- **Consistency**: $\overline{\gamma}_B$ is consistent.

If the coalition $B = A$ then we simply call it feasible.

Definition 2.3. We say that $q \in \mathbb{R}^{\mathscr{M}}$ is a pure transfer if $q \in \mathcal{T}rans$, where

$$\mathcal{T}rans = \left\{q \in \mathbb{R}^{\mathscr{M}} : q.\gamma = 0 \;\; for \;\; each \;\; \gamma \in \mathscr{C}ons\right\}.$$

Thus for each club type (π, γ) and $q \in \mathcal{T}rans$, $\sum_{\omega\in\Omega} \pi(\omega) q\left(\omega, \pi, \gamma\right) = 0$.

2.4 Equilibrium and Optimality

In this chapter we shall layout some definitions. As in every general equilibrium model we shall begin with Pareto optimality and then outline the co-operative behavior of the individuals. While doing so, we shall resort to both strong and weak notions of such concepts. After refining the objection mechanism to a two step objection, counter-objection one we will highlight one of the main concepts in our framework by defining the bargaining set.

Definition 2.4. A club equilibrium consists of a feasible state (x, γ) and prices $(p, q) \in \mathbb{R}^N_+ \times \mathbb{R}^{\mathscr{M}}$, $p \neq 0$, such that :

- **Budget Feasibility for Individuals** : For almost all $a \in A$,

$$(p, q).(x_a, \gamma_a) = p.x_a + q.\gamma_a \leq p.e_a$$

- **Optimisation**: For almost all $a \in A$,

$$(x'_a, \gamma'_a) \in X_a \quad and$$

$$u_a(x'_a, \gamma'_a) > u_a(x_a, \gamma_a) \Rightarrow p.x'_a + q.\gamma'_a > p.e_a.$$

- **Budget Balance for Club types** : For each $(\pi, \gamma) \in \mathcal{C}lubs$,

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p.inp(\pi, \gamma).$$

$\rightarrow$ A club quasi-equilibrium satisfies the first, third condition as a club equilibrium and instead of the second condition it satisfies :

- Quasi-Optimisation: For almost all $a \in A$,

$$(x'_a, \gamma'_a) \in X_a \quad and$$

$$u_a(x'_a, \gamma'_a) > u_a(x_a, \gamma_a) \Rightarrow p.x'_a + q.\gamma'_a \geq p.e_a.$$

Definition 2.5. A pure-transfer club equilibrium consists of a feasible state (x, γ) and prices $(p, q) \in \mathbb{R}_+^N \times \mathbb{R}^{\mathscr{M}}$, $p \neq 0$, such that :

- **Budget Feasibility for Individuals** : For almost all $a \in A$,

$$p.x_a + \sum_{(\omega,\pi,\gamma)} p.\frac{1}{|\pi|} inp(\pi, \gamma) l_a(\omega, \pi, \gamma) + q.\gamma_a \leq p.e_a$$

- **Optimisation**: For almost all $a \in A$,

$$(x'_a, \gamma'_a) \in X_a \quad and$$

$$u_a(x'_a, \gamma'_a) > u_a(x_a, \gamma_a) \Rightarrow p.x'_a + \sum_{(\omega,\pi,\gamma)} p.\frac{1}{|\pi|} inp(\pi, \gamma) l'_a(\omega, \pi, \gamma) + q.\gamma'_a > p.e_a.$$

- **Pure Transfers** : $q \in \mathcal{T}rans$.

Definition 2.6. A feasible state (x, γ) of the economy $\mathcal{E}$ is said to be :

(i) **weakly Pareto optimal** if there does not another feasible state (x', γ') such that $u_a(x'_a, \gamma'_a) > u_a(x_a, \gamma_a)$ for almost all $a \in A$.

(ii) **strongly Pareto optimal** if there does not exist another feasible state (x', γ') such that $u_a(x', \gamma') \geq u_a(x, \gamma)$ for almost all $a \in A$ and $u_a(x'_a, \gamma'_a) > u_a(x_a, \gamma_a)$ for some $A' \subset A$ such that $\lambda(A') > 0$.

Definition 2.7. : A feasible state (x, γ) of the economy $\mathcal{E}$ is said to be :

(i) **weakly objected** if there exists some coalition $B \subset A$, $\lambda(B') > 0$ and a feasible state (x', γ') associated with B such that $u_a(x'_a, \gamma'_a) > u_a(x_a, \gamma_a)$ for almost all $a \in B$.

(ii) **strongly objected** if there exists some coalition $B \subset A$, $\lambda(B) > 0$ and a feasible state (x', γ') associated with B such that $u_a(x'_a, \gamma'_a) \geq u_a(x_a, \gamma_a)$ for almost all $a \in B$ and $u_a(x'_a, \gamma'_a) > u_a(x_a, \gamma_a)$ for all agents in some positive measure subset B' of B.

A feasible state (x, γ) is said to be in the **weak core** of the economy $\mathcal{E}$ if it is not weakly objected. A feasible state (x, γ) is similarly said to be in the **strong core** of the economy $\mathcal{E}$ if it is not strongly objected.

3 Vind's Theorem

Vind remarked that while considering allocations outside the core one can consider blocking by coalitions of any size between zero and the grand coalition. This includes blocking by coalitions of arbitrarily large sizes, thus emphasizing that core outcomes can be charcterised also as an outcome of a majority voting rule. The applicability of the Lyapnov's convexity theorem remains validated by $\mathcal{C}ons$ being a linear subspace. Before we move on to the main theorem we present a lemma which will be useful in proving the main theorem for this section.

Lemma 3.1. *Let (f, l) and (g, l') be two states of $\mathscr{E}$ such that $u_a(g_a, l'_a) > u_a(f_a, l_a)$ λ-a.e. on S for some coalition S. Then given any $0 < \alpha < 1$ there is a state (h, l'') such that*

(i) $u_a(h_a, l''_a) > u_a(f_a, l_a)$ *μ-a.e. on S;*

(ii) $\int_S h_a d\lambda = \int_S (\alpha g_a + (1-\alpha) f_a) d\lambda$*; and*

(iii) $\int_S l''_a d\lambda = \int_S (\alpha l'_a + (1-\alpha) l_a) d\lambda$.

Proof. Consider a vector measure $\lambda : \Sigma_S \to \mathbb{R}^{N+1} \times \mathbb{R}^{\mathscr{M}}$ such that

$$\lambda(R) := \left\{ \left(\lambda(R), \int_R (g_a - f_a) d\lambda, \int_R (l'_a - l_a) d\lambda \right) : R \in \Sigma_S \right\}.$$

Pick any $\alpha \in (0,1)$. In view of Lyapunov's Convexity theorem, there exits a sub-coalition R of S such that

(a) $\lambda(R) = \alpha \lambda(S)$;

(b) $\int_R (g_a - f_a) d\lambda = \alpha \int_S (g_a - f_a) d\lambda$; and

(c) $\int_R (l'_a - l_a)\, d\lambda = \alpha \int_S (l'_a - l_a) d\lambda$.

It follows from the continuity and measurability of utilities that there exists a function $\widetilde{g} : S \to \mathbb{R}^N_+$ and some $z \in \mathbb{R}^N_+ \setminus \{0\}$ such that $u_a(\widetilde{g}_a, l'_a) > u_a(f_a, l_a)$ λ-a.e. on R and

$$\int_R \widetilde{g}_a d\lambda = \int_R g_a d\lambda - z.$$

Finally, define $h : S \to \mathbb{R}^N_+$ and $l'' : S \to \mathbb{R}^{\mathscr{M}}$ by letting

$$h_a := \begin{cases} \widetilde{g}_a & \text{if } a \in R\,; \\ f_a + \frac{z}{\lambda(S \setminus R)} & \text{if } a \in S \setminus R\,, \end{cases}$$

and

$$l''_a := \begin{cases} l'_a & \text{if } a \in R\,; \\ l_a & \text{if } a \in S \setminus R\,. \end{cases}$$

From Assumption **A.7**, it follows directly that $u_a(h_a, l''_a) > u_a(f_a, l_a)$ λ-a.e. on S. Further, in the presence of (a) and (b), we can readily verify that

$$\int_S h_a d\lambda = \int_S (\alpha g_a + (1-\alpha) f_a)\, d\lambda$$

and

$$\int_S l''_a d\lambda = \int_S (\alpha l'_a + (1-\alpha) l_a)\, d\lambda.$$

This completes the proof. □

Now we present the main theorem for this section. The only private good version of this theorem was proposed by [30].

Theorem 3.2. *Let (f,l) be a feasible state not belonging to the weak core of the club economy $\mathscr{E}$. Then for any $0 < \varepsilon < \lambda(A)$, there exist a coalition R such that $\lambda(R) = \varepsilon$ and (f,l) is blocked by R.*

Proof. Suppose that (f, l) is a feasible state not belonging to the weak core of $\mathscr{E}$. Thus, there exist a coalition S and a state (g, l') such that

(i) $u_a(g_a, l'_a) > u_a(f_a, l_a)$ λ-a.e. on S;

(ii) $\int_S g_a d\lambda + \int_S \tau(l'_a) d\lambda = \int_S e_a d\lambda$; and

(iii) $\int_S l'_a d\lambda \in \mathcal{C}ons$.

Take any $\varepsilon \in (0, \lambda(S))$. Choose some $\alpha \in (0, 1)$ such that $\varepsilon = \alpha\lambda(S)$. By the Lyapunov convexity theorem, there exist sub-coalition B of S such that $\lambda(B) = \alpha\lambda(S)$ and

$$\int_B [g_a + \tau(l'_a) - e_a] \, d\lambda = \alpha \int_S [g_a + \tau(l'_a) - e_a] \, d\lambda = 0.$$

Consequently, (f, l) is blocked by B whose measure is ε. Next, let $\lambda(S) < \varepsilon < \lambda(A)$. Let $\delta \in (0, 1)$ be an element such that

$$\delta = 1 - \frac{\epsilon - \lambda(S)}{\lambda(A \setminus S)}.$$

By the continuity and measurability of utility functions, we can choose a function $\widetilde{g} : S \to \mathbb{R}^N_+$ such that $u_a(\widetilde{g}_a, l'_a) > u_a(f_a, l_a)$ λ-a.e. on S and

$$\int_S \widetilde{g}_a d\lambda = \int_S g_a d\lambda - z.$$

By Lemma 3.1, there exists some state (h, l'') such that

(iv) $u_a(h_a, l''_a) > u_a(f_a, l_a)$;

(v) $\int_S h_a d\lambda = \int_S (\delta\widetilde{g}_a + (1 - \delta) f_a) \, d\lambda$; and

(vi) $\int_S l''_a d\lambda = \int_S (\delta l'_a + (1 - \delta) l_a) \, d\lambda$.

Another use of Lyapunov's convexity theorem ensures the existence of a sub-coalition C of $A \setminus S$ such that

(vii) $\lambda(C) = (1 - \delta)\lambda(A \setminus S)$;

(viii) $\int_C [f_a + \tau(l_a) - e_a] d\lambda = (1 - \delta) \int_{A\setminus S} [f_a + \tau(l_a) - e_a] d\lambda$; and

(ix) $\int_C l_a d\lambda = (1 - \delta) \int_{A\setminus S} l_a d\lambda$.

Lastly, let us define $R := S \cup C$ and a state $(y, \psi) : A \to \mathbb{R}_+^N \times \mathscr{R}^{\mathscr{M}}$ such that

$$(y_a, \psi_a) = \begin{cases} (h_a, l''_a), & \text{if } a \in S, \\ \left(f_a + \frac{z\delta}{\lambda(C)}, l_a\right), & \text{otherwise.} \end{cases}$$

It follows that $\lambda(R) = \varepsilon$. By Assumption **A.7**, $u_a(h_a, l''_a) > u_a(f_a, l_a)$ λ-a.e. on R. In the presence of (vi) and (ix), it can readily verified that

$$\int_R \psi_a d\lambda(a) = \delta \int_S l'_a d\lambda + (1-\delta) \int_A l_a d\lambda.$$

Since $\int_S l'_a d\lambda \in \mathcal{C}ons$ and $\int_A l_a d\lambda \in \mathcal{C}ons$ we have $\int_R \psi_a d\lambda(a) \in \mathcal{C}ons$. Furthermore, using (ii), (v) and (viii), we deduce that

$$\int_R y_a d\lambda + \int_R \tau(\psi_a) d\lambda - \int_R e_a d\lambda = 0.$$

Therefore, we have a coalition R that blocks the state (f, l) through the state (y, ψ). $\square$

4 Interpretation via continuum economy

Before we prove our next theorem we need to introduce the associated continuum economy $\mathscr{E}^*$ to our mixed economy $\mathscr{E}$. We associate with the set of large agents A_1, an atomless positive measure space $\left(A_1^*, \Sigma_{A_1}^*, \lambda_{A_1}^*\right)$ such that $A_0 \cap A_1^* = \emptyset$ and $\lambda(A_1) = \lambda^*(A_1^*)$. For every large agent T_n there exists a one-to-one correspondence with a measurable subset T_n^* such that $\lambda(T_n) = \lambda^*(T_n^*)$. Thus $A_1^* = \bigcup \{T_n : n \geq 1\}$. We basically identify the interval $[\lambda(A_0), \lambda(A_1)]$ with A_1^* which is union of countably many disjoint intervals T_n^* where $T_1^* = (\lambda(A_0), \lambda(A_1))$ and for any $n \in \mathbb{N}$, $T_n^* = \left[\lambda(A_0) + \lambda\left(\bigcup_{i=1}^{n-1} T_i\right), \lambda(A_0) + \lambda\left(\bigcup_{i=1}^{n} T_i\right)\right]$. We then define $A^* = A_0 \cup A_1^*$ and the associated σ-algebra as the direct sum of the two σ- algebras i.e.

$$\Sigma^* = \left\{C \cup D : C \cap D = \emptyset, C \in \Sigma_{A_0}, D \in \Sigma_{A_1}^*\right\}$$

and the associated measure $\lambda^* : \Sigma^* \to \mathbb{R}_+$ such that for any $C \in \Sigma^*$:

$$\lambda(C^*) = \lambda_{A_0}(C \cap A_0) + \lambda_{A_1}^*(C \cap A_1^*)$$

Thus the measure space $(A^*, \Sigma^*, \lambda^*)$ is obtained. Also, each individual small agent $a \in T$ has the same characteristics that of the large agent T. Thus for every $a \in T_n^*$ we define

$$\begin{array}{lll} e_a^* = e_a; & u_a^* = u_a; & \text{if } a \in A_0 \\ e_a^* = e_n = e(T_n); & u_a^* = u_n = u(T_n); & \text{if } a \in T_n^*, n \geq 1 \end{array}$$

Now given any allocation $(f, l) \in \mathscr{E}$ we define an allocation $(f^*, l^*) = \Xi((f, l))$ for the associated continuum economy $\mathscr{E}^*$ as :

$$(f_a^*, l_a^*) = \begin{cases} (f_a, l_a), & \text{if } a \in A_0 \text{ ;} \\ (f(T_n), l(T_n)), & \text{if } a \in T_n^*, n \geq 1. \end{cases}$$

Similarly, given any allocation (f^*, l^*) in $\mathscr{E}^*$ we define the corresponding allocation $(f, l) = \varphi(f^*, l^*)$ for $\mathscr{E}$ as:

$$(f_a, l_a) = \begin{cases} (f_a^*, l_a^*), & \text{if } a \in A_0 \text{ ;} \\ \left(\frac{1}{\lambda^*(T_n^*)} \int_{T_n^*} f_a^* d\lambda^*, \frac{1}{\lambda^*(T_n^*)} \int_{T_n^*} l_a^* d\lambda^* \right), & \text{if } a = T_n, n \geq 1. \end{cases}$$

Lemma 4.1. *Let $\mathscr{E}$ be a mixed club economy. Suppose that R is a coalition containing all large agents and that for $t, s \in R$ we have* (i) $u_t = u_s$; (ii) $e(t) = e(s)$. *Assume further that $\lambda(R \setminus A_1) > 0$. Let (f, l) be a state of $\mathscr{E}$ belonging to $\mathscr{C}(\mathscr{E})$ such that*

$$(\widetilde{f}_R, \widetilde{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda, \ \frac{1}{\lambda(R)} \int_R l_a d\lambda \right)$$

exists. Then $u_a(f_a, l_a) = u_a(\widetilde{f}_R, \widetilde{l}_R)$ μ-a.e on R.

Proof. We first assume that u_R and $e(R)$ the common values of u_a and $e(a)$, respectively. Define

$$B := \left\{ a \in R : u_R(\widetilde{f}_R, \widetilde{l}_R) > u_R(f_a, l_a) \right\}$$

and

$$C := \left\{ a \in R : u_R(\widetilde{f}_R, \widetilde{l}_R) < u_R(f_a, l_a) \right\}.$$

Recognized that B and C are Σ-measurable sets. We shall complete the proof by showing that none of these sets has positive measure. To this end, we first assume that $\lambda(B) > 0$. By the continuity of $\widetilde{u}_R$, there exist a sub-coalition D of B and some $z \in \mathbb{R}_+^N \setminus \{0\}$ such that

$$u_R(\widetilde{f}_R - z, \widetilde{l}_R) > u_R(f_a, l_a)$$

for all $a \in D$. Define $r_0 \in (0, 1]$ by letting $r_0 := \frac{\lambda(D)}{\lambda(R)}$. By Lyapnov's convexity theorem, there exists a sub-coalition E of $A \setminus R$ such that

(i) $\int_E (f_a - e_a) d\lambda = r_0 \int_{A \setminus R} (f_a - e_a) d\lambda$; and

(ii) $\int_E l_a d\lambda = r_0 \int_{A \setminus R} l_a d\lambda$.

Let $S := D \cup E$. Define $g : A \to \mathbb{R}_+^N$ by

$$g_a := \begin{cases} \widetilde{f}_R - z, \ a \in D; \\ \\ f_a + \frac{z\lambda(D)}{\lambda(E)}, \ \text{otherwise.} \end{cases}$$

It is claimed that (f, l) is blocked by S via (g, l). To see this, first note that

$$\int_S l_a d\lambda = r_0 \int_A l_a d\lambda \in \mathcal{C}ons.$$

As a consequence of this, we have

$$\int_S \tau(l_a) d\lambda = r_0 \int_A \tau(l_a) d\lambda.$$

As $\int_D (\widetilde{f}_R - e(R)) d\lambda = r_0 \int_R (\widetilde{f}_R - e(R)) d\lambda$, it is just routine to verify that

$$\int_S (g_a + \tau(l_a) - e_a) d\lambda = r_0 \int_A (f_a + \tau(l_a) - e_a) d\lambda = 0.$$

Therefore, $(f, l) \notin \mathscr{C}(\mathscr{E})$, which leads to a contradiction. Thus, we conclude that $\lambda(B) = 0$, which means that $u_R(\widetilde{f}_R, \widetilde{l}_R) < u_R(f_a, l_a)$ λ-a.e. on R. We now assume that $\lambda(C) > 0$. By Jensen's inequality, one obtains

$$u_R\left(\frac{1}{\lambda(C)} \int_C (f, l) d\lambda\right) > u_R(\widetilde{f}_R, \widetilde{l}_R)$$

and

$$u_R\left(\frac{1}{\lambda(R \setminus C)} \int_{R \setminus C} (f, l) d\lambda\right) > u_R(\widetilde{f}_R, \widetilde{l}_R).$$

Let $\alpha = \frac{\lambda(C)}{\lambda(R)}$. By Lemma 5.28 in Aliprantis and Border (2005), one has

$$\begin{aligned} u_R(\widetilde{f}_R, \widetilde{l}_R) &= u_R\left(\frac{\alpha}{\lambda(C)} \int_C (f, l) d\lambda + \frac{1-\alpha}{\lambda(R \setminus C)} \int_{R \setminus C} (f, l) d\lambda\right) \\ &> u_R(\widetilde{f}_R, \widetilde{l}_R), \end{aligned}$$

which is a contradiction. Therefore, we have $\lambda(C) = 0$. Hence, we conclude that $u_a(f_a, l_a) = u_a(\widetilde{f}_R, \widetilde{l}_R)$ μ-a.e on R □

We now present the main theorem for this section. Denote by $\mathscr{C}^*(\mathscr{E}^*)$ the core of the continuum economy. The following theorem is an extension of **Proposition 6** of Basil et al[3] to the case of impure public goods.

Following from the competitiveness assumption embodied in the model the number of possible club types are finite .Also, the number of club memberships one particular individual can buy of all types combined is bounded above by a finite number M. So the number of possible combinations of membership demand that one individual can choose from is finite. Thus, the mapping from the set of agents to the set of possible demand vectors for club memberships is a many to one mapping. Thus we can partition the set the agents based on the above reasoning into finitely many disjoint sets $\{K_1, K_2, \cdots, K_l\}$ such that $\lambda(K_j) > 0$ for all $j = 1, 2, \cdots, l$.

Theorem 4.2. *Let $\mathscr{E}$ be a mixed club economy. Suppose that R is a coalition containing all large agents and that for $t, s \in R$ we have* (i) $u_t = u_s$; (ii) $e(t) = e(s)$. *Assume further that $\lambda(R \setminus A_1) > 0$. Let (f, l) be a state of $\mathscr{E}$ belonging to $\mathscr{C}(\mathscr{E})$ such that*

$$(\widetilde{f}_R, \widetilde{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda,\ \frac{1}{\lambda(R)} \int_R l_a d\lambda \right)$$

exists. Then $(f^*, l^*) \in \mathscr{C}(\mathscr{E}^*)$, where $(f^*, l^*) := \Xi((f, l))$.

Proof. Let $(f, l) \in \mathscr{C}(\mathscr{E})$. We show that $(f^*, l^*) \in \mathscr{C}(\mathscr{E}^*)$. Suppose this is not true, i.e, $(f^*, l^*) \notin \mathscr{C}(\mathscr{E}^*)$. Thus, there exists a coalition $S \in \Sigma^*$ and a state (g, γ) such that

$$\int_{S \cap A_0^*} (g_a - e_a + \tau(\gamma_a))\, d\lambda^* + \int_{S \cap A_1^*} (g_a - e_a + \tau(\gamma_a))\, d\lambda^* = 0 \tag{4.1}$$

If $\lambda^*(S \cap A_1^*) = 0$ then we immediately arrive at a contradiction. So we assume that $\lambda^*(S \cap A_1^*) > 0$, and note that γ only takes finitely many values.[4] Let the range of γ be $\{\gamma^1, \cdots, \gamma^l\}$. For each $1 \leq j \leq l$, define $K_j := \{a \in A^* : \gamma_a = \gamma^j\}$. Notice that K_j is Σ^*-measurable for all $1 \leq j \leq l$. Define

$$\mathbb{J} := \{j : 1 \leq j \leq l \text{ and } \lambda^*(S \cap A_1^* \cap K_j) > 0\}.$$

For each $j \in \mathbb{J}$, denote $F_j := S \cap A_1^* \cap K_j$. Consequently, we have

$$S \cap A_1^* = \bigcup \{F_j : j \in \mathbb{J}\}.$$

[4]It follows from the fact that there an uniform upper bound M on the number of club memberships that each agent can have.

Thus, from equation (4.1) we have

$$\int_{S\cap A_0^*} (g_a - e_a + \tau(\gamma_a))\, d\lambda^* + \sum_{j\in\mathbb{J}} \lambda^*(F_j)\left(g_j - e_a + \tau\left(\gamma^j\right)\right) d\lambda^* = 0, \tag{4.2}$$

where $g_j := \frac{1}{\lambda^*(F_j)} \int_{F_j} g_a d\lambda^*$. We denote by $G \subseteq A_0$, the coalition $R \setminus A_1$.

Case 1. $\lambda^*(S \cap A_1^*) \leq \lambda(G)$. In this case, we can choose a sub-coalition G_1 of G such that $\lambda(G_1) = \lambda^*(F_1)$. Since $\lambda^*(R \cap A_1^* \setminus F_1) \leq \lambda(G \setminus G_1)$, we can analogously choose another sub-coalition $G_2 \subseteq (G \setminus G_1)$ such that $\lambda(G_2) = \lambda^*(F_2)$. Continuing this way, for each $j \geq 2$, we can choose a sub-coalition $G_j \subseteq G \setminus (G_1 \cup G_2, \cdots \cup G_{j-1})$ such that $\lambda(G_j) = \lambda^*(F_j)$. Thus, Equation (4.2) boils down to

$$\int_{S\cap A_0} (g_a - e_a + \tau(\gamma_a))\, d\lambda + \sum_{j\in\mathbb{J}} \lambda(G_j)\left(g_j - e_a + \tau\left(\gamma^j\right)\right) d\lambda = 0 \tag{4.3}$$

Case 2. $\lambda^*(S \cap A_1^*) > \lambda(G)$. In this case, we first choose $\alpha \in (0, 1]$ such that $\lambda(G) = \alpha\lambda^*(S \cap A_1^*)$. As in **Case 1**, there is a partition $\{\widehat{G}_j : j \in \mathbb{J}\}$ of G such that $\lambda(\widehat{G}_j) = \alpha\lambda^*(F_j)$ for all $j \in \mathbb{I}$. Applying the Lyapunov convexity theorem, we can find a sub-coalition E of $S \cap A_0$ such that

$$\int_E (g_a - e_a + \tau(\gamma_a))\, d\lambda = \alpha \int_{S\cap A_0} (g_a - e_a + \tau(\gamma_a))\, d\lambda$$

and

$$\int_E \gamma_a d\lambda = \alpha \int_{S\cap A_0} \gamma_a d\lambda.$$

Thus, it follows from (4.2) that

$$\int_E (g_a - e_a + \tau(\gamma_a))\, d\lambda + \sum_{j\in\mathbb{J}} \lambda(\widehat{G}_j)\left(g_j - e_a + \tau\left(\gamma^j\right)\right) d\lambda = 0. \tag{4.4}$$

Therefore, by Equation (4.3) and Equation (4.4), there exist a coalition $B_0 \subseteq S \cap A_0$ and a sequence $\{\widetilde{G}_j : j \in \mathbb{J}\} \subseteq \Sigma_{A_0}$ of pairwise disjoint coalitions such that

$$\int_{B_0} (g_a - e_a + \tau(\gamma_a))\, d\lambda + \sum_{j\in\mathbb{J}} \lambda(\widetilde{G}_j)\left(g_j - e_a + \tau\left(\gamma^j\right)\right) d\lambda = 0. \tag{4.5}$$

Define $G_0 = \bigcup\{\widetilde{G}_j : j \in \mathbb{J}\}$ and $(\widetilde{g}, \widetilde{\gamma}_a) : G_0 \to R_+^N \times \mathbb{R}_+^{\mathcal{M}}$ by letting $(\widetilde{g}_a, \widetilde{\gamma}_a) = (g_j, \gamma^j)$, if $a \in \widetilde{G}_j$. By Lemma 4.1, we have $u_a(\widetilde{g}_a, \widetilde{\gamma}_a) > u_a(f_a, l_a)$ λ-a.e. on G_0. If $\lambda(B_0 \cap G_0) = 0$, then is evident from Equation (4.5) that $B_0 \cup G_0$ blocks (f, l) via $(h, \hat{l})$, which is defined as

$$(h_a, \hat{l}_a) := \begin{cases} (\widetilde{g}_a, \widetilde{\gamma}_a), & \text{if } a \in G_0; \\ \\ (g_a, \gamma_a), & \text{otherwise.} \end{cases}$$

This is a contradiction. Thus, we assume that $\lambda(B_0 \cap G_0) \neq 0$. We define a measurable set $C := (B_0 \setminus G_0) \cup (G_0 \setminus B_0)$. By the Lyapunov convexity theorem, there is some $C_0 \in \Sigma_C$ such that

$$\int_{C_0} (h_a - e_a + \tau(\hat{l}_a)) d\lambda = \frac{1}{2} \int_C (h_a - e_a + \tau(\hat{l}_a)) d\lambda$$

and

$$\int_{C_0} \hat{l}_a d\lambda = \frac{1}{2} \int_C \hat{l}_a d\lambda.$$

By Lemma 3.1, there a state (φ, l'') such that

$$\int_{B_0 \cap G_0} \varphi_a d\lambda = \frac{1}{2} \int_{B_0 \cap G_0} (g_a + \widetilde{g}_a) d\lambda$$

and

$$\int_{B_0 \cap G_0} l'' d\lambda = \frac{1}{2} \int_{B_0 \cap G_0} (\gamma_a + \widetilde{\gamma}_a) d\lambda.$$

It follows that

$$\int_{B_0 \cap G_0} \tau(l'') d\lambda = \frac{1}{2} \int_{B_0 \cap G_0} (\tau(\gamma_a) + \tau(\widetilde{\gamma}_a)) d\lambda.$$

Therefore, we conclude that

$$\int_{B_0 \cap G_0} (\varphi_a - e_a + \tau(l''_a)) \, d\lambda = \frac{1}{2} \int_{B_0 \cap G_0} (g_a + \widetilde{g}_a - 2e_a + \tau(\gamma_a) + \tau(\widetilde{\gamma}_a)) \, d\lambda.$$

In view of above, we can re-write Equation (4.5) as follows

$$\int_{C_0} (h_a - e_a + \tau(\hat{l}_a)) d\lambda + \int_{B_0 \cap G_0} (\varphi_a - e_a + \tau(l''_a)) \, d\lambda = 0 \tag{4.6}$$

It can be readily verified that

$$\int_{C_0} \hat{l}_a d\lambda + \int_{B_0 \cap G_0} l''_a d\lambda = \frac{1}{2} \int_{B_0} \gamma_a d\lambda + \frac{1}{2} \sum_{j \in \mathbb{J}} \lambda(\widetilde{G}_i) \gamma^j,$$

which is equal to

$$\begin{cases} \frac{1}{2}\int_S \gamma_a d\lambda^*, & \text{if } \lambda^*(S\cap A_1^*) \le \lambda(G); \\ \\ \frac{\alpha}{2}\int_S \gamma_a d\lambda^*, & \text{otherwise.} \end{cases}$$

This belongs to $\mathcal{C}ons$. Therefore, (f,l) is blocked by $C_0 \cup (B_0 \cap G_0)$ via the state (y,ξ), where (y,ξ) is defined by

$$(y_a,\xi_a) := \begin{cases} \left(h_a, \hat{l}_a\right), & \text{if } a \in C_0; \\ \\ (\varphi_a, l''_a), & \text{otherwise.} \end{cases}$$

This completes the proof. □

Theorem 4.3. *Let $\mathscr{E}$ be a mixed club economy and (f^*,l^*) be a state of the associated continuum economy $\mathscr{E}^*$ such that $\varphi(f^*,l^*)$ exists. Then given that $(f^*,l^*) \in \mathscr{C}(\mathscr{E}^*)$ we have $(f,l) := \varphi(f^*,l^*)$ belongs to $\mathscr{C}(\mathscr{E})$.*

Proof. Let $(f^*,l^*) \in \mathscr{C}(\mathscr{E}^*)$. On the contrary, let us assume that $(f,l) := \varphi(f^*,l^*) \notin \mathscr{C}(\mathscr{E})$. Then there exist a coalition S and a state (y,μ) such that:

(i) $\int_S y_a d\lambda + \int_S \tau(\mu_a) d\lambda = \int_S e_a d\lambda$;

(ii) $u_a(y_a,\mu_a) > u_a(f_a,l_a)$ for each $a \in S$; and

(iii) $\int_S \mu_a d\lambda \in \mathcal{C}ons$.

We define $(y^*,\mu^*) := \Xi(y,\mu)$ and $S^* := (S \cap A_0) \cup \bigcup\{T_i^* : T_i \in S\}$. It is claimed that (f^*,l^*) is blocked by S^* via (g^*,μ^*). Indeed, for all $a \in T_i^*$ with $T_i \in S$, we have

$$u_a(y_a^*,\mu_a^*) = u_{T_i}(y_{T_i},\mu_{T_i}) > u_{T_i}(f_{T_i},l_{T_i}) \ge \frac{1}{\lambda^*(T_i^*)}\int_{T_i^*} u_{T_i}(f_a^*,l_a^*) d\lambda^*.$$

Pick an arbitrary $T_i \in S$. By Lemma 4.1 to an atomless economy $\mathscr{E}$ with $A_1 = \emptyset$ with $R \cap A_0 = T_i^*$, we have $u_a(f_a^*,l_a^*)$ is μ-a.e. constant on A_i^*. This implies that $u_a(y_a^*,\mu_a^*) > u_a(f_a^*,l_a^*)$ λ-a.e. on T_i^*. Therefore, (f^*,l^*) is blocked by S^* via (y^*,μ^*), which leads to a contradiction. Hence, our supposition was wrong and (f,l) belongs to $\mathscr{C}(\mathscr{E})$. □

Therefore, we establish that under our assumption that there exists a positive measurable subset of negligible agents representing the atomic sector, the core of the mixed economy and that of the associated continuum economy are equivalent. Now we shall try and replicate the core equivalence theorem for the mixed economy $\mathscr{E}$. But, before that we shall show the set of equilibrium allocations are equialent. The following theorem summarizes that.

Theorem 4.4. *$\mathscr{W}(\mathscr{E}^*)$ is equivalent to $\mathscr{W}(\mathscr{E})$ i.e.*

(i) $(f,l) \in \mathscr{W}(\mathscr{E}) \implies (f^*,l^*) = \Xi(f,l) \in \mathscr{W}(\mathscr{E}^*)$;

(ii) $(f^*,l^*) \in \mathscr{W}(\mathscr{E}^*) \implies (f,l) = \varphi(f^*,l^*) \in \mathscr{W}(\mathscr{E})$.

Proof. Let $(f,l) \in \mathscr{W}(\mathscr{E})$ corresponding to equilibrium prices (p,q). Thus we can infer that $u_a(f_a,l_a) \geq u_a(z_a,\mu_a)$ for all $(z_a,\mu_a) \in B_a(p,q,e_a)$. No we define the corresponding continuum economy allocation $(f^*,l^*) = \Xi(f,l)$. For every $a \in T_n^*$, and $n \geq 1$ the endowments and preferences are same as that of T_n and for the small agents the endowments and preferences are same as that in $\mathscr{E}$. Thus one can automatically claim that $(f^*,l^*) \in \mathscr{W}(\mathscr{E}^*)$.

Let $(f^*,l^*) \in \mathscr{W}(\mathscr{E}^*)$ for the price system (p,q), which implies that (f_a^*,l_a^*) is the maximal element in individual a's budget set for λ-a.e. on A^*. Now since individual agents in each of $T_n^*, n \geq 1$ are endowed with the same preference their demand must be indifferent to each other implying $u_a(f_a^*,l_a^*) = u_{a'}(f_{a'}^*,l_{a'}^*)$ for all $a,a' \in T_n^*; n \geq 1$. Now from the quasi-concavity of the utility function we can say that for any $a = T_n$ we have that $u_n(f,l) \geq u_a(f_a^*,l_a^*); a \in T_n^*$, where $(f,l) = \varphi(f^*,l^*)$. Again since preference of T_n are similar to that of any non-negligible agent in T_n^* we can say that $(f,l) \in D_{T_n}(p,q,e_n)$ for all $n \geq 1$. Thus $(f,l) \in \mathscr{W}(\mathscr{E})$. □

Combining theorem 4.2 and theorem 4.4 one can establsih the equivalence between the set of core allocations and equilibrium allocations in the mixed economy $\mathscr{E}$.

Theorem 4.5. *Let $\mathscr{E}$ be a mixed club economy. Suppose that R is a coalition containing all large agents and that for $t,s \in R$ we have* (i) $u_t = u_s$; (ii) $e(t) = e(s)$. *Assume further that $\lambda(R \setminus A_1) > 0$. Let (f,l) be a state of $\mathscr{E}$ belonging to $\mathscr{C}(\mathscr{E})$ such that*

$$(\widetilde{f}_R, \widetilde{l}_R) := \left(\frac{1}{\lambda(R)} \int_R f_a d\lambda,\ \frac{1}{\lambda(R)} \int_R l_a d\lambda \right)$$

exists. Then the set of Walrasian equilibrium allocations coincides with that of the core of the economy.

Proof. One side of the proof is quite immediate as $\mathscr{W}(\mathscr{E}) \subseteq \mathscr{C}(\mathscr{E})$ from the first fundamental theorem. Now for the other side let $(f,l) \in \mathscr{C}(\mathscr{E})$. Then from theorem 4.2

we can say $(f^*, l^*) = \Xi((f, l))$ belongs to $\mathscr{C}(\mathscr{E}^*)$. Now from theorem 5.1 of Ellickson et al[12], we can infer that (f^*, l^*) belongs to $\mathscr{W}(\mathscr{E}^*)$. Thus from theorem 4.4 we can infer that $(f, l) = \varphi(f^*, l^*)$ belongs to $\mathscr{W}(\mathscr{E})$. This completes the proof.

□

5 Approximate Robust Efficiency

In this section we drop the assumption on consistency of net trade of club memberships adapted by Bhowmik and Kaur and introduce a new notion of robust efficiency.To begin with, we assume that while considering blocking we assume that the final club consumption and the initial one over a certain subset of agents is consistent. We present the definitions and the results below.

Definition 5.1. A state (f, l) is said to be **dominated by a state** (g, γ) in an economy $\mathscr{E}(S, B, f, l, \alpha)$ if

(i) $u_a(g_a, \gamma_a) > u_a(f_a, l_a)\, \mu$ a.e. on A;

(ii) $\int_A g_a d\lambda + \int_A \tau(\gamma_a)\, d\lambda = \int_A e(S, f, \alpha)\, d\lambda + \int_B \tau(l_a)\, d\lambda$; and

(iii) $\int_A \gamma_a d\lambda,\ \int_B l_a d\lambda \in \mathcal{C}ons$.

A state (f, l) is termed as **robustly efficient** if it is not dominated by any other state.

Definition 5.2. A state (f, l) is said to be **sequentially ε-dominated** if there exist a sequences $\{\mathscr{E}(S_n, B_n, f, l, \alpha_n) : n \geq 1\}$ of economies and a sequence $\{(g^n, \gamma^n) : n \geq 1\}$ of states such that (f, l) is dominated by (g^n, γ^n) in $\mathscr{E}(S_n, B_n, f, l, \alpha_n)$ and the following conditions are satisfied:

(i) there is a coalition R such that $u_a(h_a^n, \gamma_a^n) > u_a(f_a, l_a)$ for all $h_a^n \in g_a^n + \mathbb{B}(0, \varepsilon)$ with $a \in R$ and $n \geq 1$; and

(ii) $\mathbb{I}_{B_n} = \mathbb{I}_{S_n}$ and $\lambda(B_n^i) \geq \alpha_n . \lambda(S_n^i)$ for all $n \geq 1$ and $i \in \mathbb{I}_{S_n}$; and

(iii) $\{(\alpha_n, \lambda(B_n)) : n \geq 1\}$ converges to $(0, 0)$.

A state (f, l) is called **ε-robustly efficient** if it is not sequentially ε-dominated. Furthermore, an allocation (f, l) is said to be **approximate robustly efficient** if it is ε-robustly efficient for all $\varepsilon > 0$.

If we denote by $\mathrm{RE}_\varepsilon(\mathscr{E})$ the set of ε-robustly efficient states and $\widetilde{\mathrm{RE}}(\mathscr{E})$ the set of robustly efficient states then $\{\mathrm{RE}_\varepsilon(\mathscr{E}) : \varepsilon > 0\}$ is accending sequence and satisfying

$$\widetilde{\mathrm{RE}}(\mathscr{E}) = \bigcap \{\mathrm{RE}_\varepsilon(\mathscr{E}) : \varepsilon > 0\}.$$

Theorem 5.3. *Let (f,l) be a feasible allocation. Then (f,l) is a club equilibrium allocation if and only if it is approximate robustly efficient allocation, where utility from any allocation pair of the form $(0_a, \gamma_a) \in X_a$ is assumed to be zero, for all $a \in A$.*

Proof. Assume that (f,l) is a club equilibrium allocation. Let (p,q) be a corresponding equilibrium price. Without loss of generality, we assume that $\|p\| = 1$. Suppose by the way of contradiction that (f,l) is not an ε-robustly effecient allocation for some $\varepsilon > 0$. This implies that there exist there exist a sequences $\{\mathscr{E}(S_n, B_n, f, l, \alpha_n) : n \geq 1\}$ of economies and a sequence $\{(g^n, \gamma^n) : n \geq 1\}$ of allocations such that (f,l) is dominated by (g^n, γ^n) in $\mathscr{E}(S_n, B_n, f, l, \alpha_n)$, which means

(i) $u_a(g_a^n, \gamma_a^n) > u_a(f_a, l_a)\, \mu$ a.e. on A;

(ii) $\int_A g_a^n d\lambda + \int_A \tau(\gamma_a^n)\, d\lambda = \int_A e(S_n, f, \alpha_n)\, d\lambda + \int_{B_n} \tau(l_a)\, d\lambda$; and

(iii) $\int_A \gamma_a^n d\lambda,\ \int_B l_a d\lambda \in \mathcal{C}ons.$

In addition, the following conditions are satisfied:

(iv) there is a coalition R such that $u_a(h_a^n, \gamma_a^n) > u_a(f_a, l_a)$ for all $h_a^n \in g_a^n + \mathbb{B}(0, \varepsilon)$ with $a \in R$ and $n \geq 1$; and

(v) $\mathbb{I}_{B_n} = \mathbb{I}_{S_n}$ and $\lambda(B_n^i) \geq \alpha_n \lambda(S_n^i)$ for all $n \geq 1$ and $i \in \mathbb{I}_{S_n}$; and

(vi) $\{(\alpha_n, \lambda(B_n)) : n \geq 1\}$ converges to $(0,0)$.

For each $i \in \mathbb{I}_S$, there is a sub-coalition C_n of B_n such that $\lambda(C_n^i) = \alpha_n \lambda(S_n^i)$ for all $n \geq 1$. Thus, we have

$$\int_{B_n} l_a d\lambda - \alpha_n \int_{S_n} l_a d\lambda = \int_{B_n \setminus C_n} l_a d\lambda.$$

Since $\{\lambda(B_n) : n \geq 1\}$ converges to 0, we have $\{q \cdot \int_{B_n \setminus C_n} l_a d\lambda : n \geq 1\}$ converges to 0. Let $n_0 \geq 1$ be an integer such that

$$q \cdot \int_{B_{n_0} \setminus C_{n_0}} l_a d\lambda < \frac{\varepsilon \lambda(R)}{2N}.$$

Letting

$$\delta := \frac{2q}{\lambda(R)} \int_{B_{n_0} \setminus C_{n_0}} l_a d\lambda,$$

we note that $\delta < \frac{\varepsilon}{2N}$. It follows that $z_0 := (\delta, \cdots, \delta) \in \mathbb{B}(0, \varepsilon)$. Thus we consider $\widetilde{h} : A \to \mathbb{R}_+^N$ such that

$$\widetilde{h}_a = \begin{cases} g_a^{n_0} - z_0, \text{ if } a \in R; \\ \\ g_a^{n_0}; \text{ otherwise }. \end{cases}$$

As a consequence, we have

$$\int_A \widetilde{h}_a d\lambda = \int_A g_a^{n_0} d\lambda - \lambda(R) z_0.$$

It follows from (i) and (iv) that

$$p \cdot \widetilde{h}_a + q \cdot \gamma_a^{n_0} > p \cdot e_a \geq p \cdot f_a + q \cdot l_a$$

λ-a.e. on A. Consequently,

$$\int_{S_{n_0}} \left(p \cdot \widetilde{h}_a + q \cdot \gamma_a^{n_0} \right) d\lambda > \int_{S_{n_0}} p \cdot e(S_{n_0}, f, \alpha_{n_0}) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot l_a d\lambda,$$

which further implies that

$$\int_A \left(p \cdot \widetilde{h}_a + q \cdot \gamma_a^{n_0} \right) d\lambda > \int_A p \cdot e(S_{n_0}, f, \alpha_{n_0}) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot l_a d\lambda.$$

This immediately yields that

$$\int_A (p \cdot g_a^{n_0} + q \cdot \gamma_a^{n_0}) d\lambda - \lambda(R)\delta > \int_A p \cdot e(S_{n_0}, f, \alpha_{n_0}) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot l_a d\lambda,$$

which is equivalent to

$$\int_A (p \cdot g_a^{n_0} + q \cdot \gamma_a^{n_0}) d\lambda > \int_A p \cdot e(S_{n_0}, f, \alpha_{n_0}) d\lambda + \alpha_{n_0} \int_{S_{n_0}} q \cdot l_a d\lambda + \lambda(R)\delta.$$

Thus we have that

$$\int_A (p \cdot g_a^{n_0} + q \cdot \gamma_a^{n_0}) d\lambda > \int_A p \cdot e(S_{n_0}, f, \alpha_{n_0}) d\lambda + \int_{B_{n_0}} q \cdot l_a d\lambda. \tag{5.1}$$

In view of (iii), we have

$$\int_A p \cdot [\tau(\gamma_a^{n_0}) - \tau(l_a)] d\lambda = \int_A q \cdot [\gamma_a^{n_0} - l_a] d\lambda.$$

Thus, it follows from (ii) that

$$\int_A (p \cdot g_a^{n_0} + q \cdot \gamma_a^{n_0})\, d\lambda = \int_A p \cdot e\,(S_{n_0}, f, \alpha_{n_0})\, d\lambda + \int_{B_{n_0}} q \cdot l_a d\lambda.$$

This contradicts (5.1). □

Remark 5.4.

The converse of the above theorem however fails to hold. As already pointed out earlier dominating an allocation requires it to be dominated to in a sequence of economies compared to only one in Hervés-Beloso and Moreno-García. Thus the notion of blocking is much weaker in our case compared to the original definition of robust efficiency. Thus, the set of allocations that can be blocked reduces yielding that our class of approximate robustly efficient allocations is a super set of the set of robustly efficient set of allocations and hence a super-set of the club equilibrium allocations.

Remark 5.5.

As the sequence $\{\alpha_n\}_{n \geq 1}$ tends towards zero, the set of agents over which the initial club consumption is assumed to be consistent gets smaller and smaller. Thus, asymptotically again our consistency condition for blocking kind of tends towards where only the final consumption of club membership needs to be consistent as the one defined in Ellickson et al[12].

6 Conclusion

We provide some further concluding remarks to our analysis done in this paper and also posit a few possible extensions to our work in this section.

Remark 6.1. The paper by Ellickson et al[12] was one of the seminal works in club literature that focused on building a competitive model of club economy and not remain restricted on determining the optimal club sizes. They extend Aumann's[2] result of the classical core equivalence theorem to their setting. Vind[30] later characterised the core of a continuum economy as in Aumann's framework by restricting the size of coalitions to any arbitrary size greater than zero and less than the grand coalition. In section 3 we provide a similar charcterisation of the core in line with Vind. This further strengths the decentralisation of equilibrium further in two ways. First, forming large coalitions can be costly as it requires establishing communication between large number of agents. Thus even if one concentrates in such cases to coalitons of small sizes one can still guarantee the equivalence result in Ellickson et al[12]. Secondly, by concentrating on a class of colaitions $\left\{D \subset A : \lambda(D) > \frac{\lambda(A)}{2}\right\}$, it can be inferred that core allocations can also be cahracterised as outcomes from a majority voting rule.

Remark 6.2. Bhowmik and Kaur[4] in their work provided a first ever charcterisation of club equilibrium in terms of robustly efficient allocations. Hervés-Beloso and Moreno-García[20] in their seminal work in 2008 first introduced such allocations. Since, the set of core allocations are equivalent to the set of competitive allocations, robustly efficient allocations provided another charcterisation of the core. As pointed by Ellickson et al in their work, such charcterisation fails to hold. Thus Bhowmik and Kaur showed that only under stringent conditions can equilibrium allocations be arobustly efficient allocation. We departed from any such stringent assumption and established in the previous section that for a weaker version of ronustly efficient allocations, namely "ϵ-robustly efficient allocations" each club equilibrium allocation can be supported as an ϵ-robustly efficient allocation. However, as emphasized in Remark 5.4 our notion of blocking is much weaker comapred to that of Hervés-Beloso and Moreno-García the reverse inclusion stands not true. Thus, one important extension to our work can be finding a notion of robust efficiency in between that of approximate robust efficiency and ϵ-robust efficiency such that the equivalence result of Hervés-Beloso and Moreno-García can be established.

Remark 6.3. Shitovitz[28] in his paper conjectured that the core of an economy with large traders coincides with the set of equilibrium allocation only when there exist atleast two agents of similar characteristics. Now even with such set of large traders our equivalence result holds only for a special class of allocations. Given an associated continuum economy to our mixed economy, we restrict ourselves to allocations in the continuum economy for which the average consumption bundle is defined. We show that for these restricted allocations the set of equiibrium allocations in the continuum economy is equivalent to the set of equilibrium allocations of the mixed economy. We also establish the main result in Greenberg and Shitovitz[18] only under assumptions adapted from Basile et al[3]. The finite possible club types in such contexts helps us partition agents in such a way that memberships are constant over each partition which further enables defining average consumption. Thus given a core allocation of the mixed economy, by our Theorem 4.2 we can claim that it belongs to the core of the continuum economy. By the core equivalence theorem of Ellickson et al[12] one can infer that such an allocation is an equilibrium allocation of the continuum economy. Further Theorem 4.4 enables us to conclude that the corresponding mixed economy allocation belongs to the set of equilibrium allocations of the economy.

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