Auctions with resale at a later date

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Abstract

Consider the following two period game with two risk neutral bidders for the sale of one indivisible object. In period 1, a sealed bid auction occurs. In period 2, the winner (and potential reseller) of the auction may resell the object to the loser (buyer) *via* a single take-it-or-leave-it offer. There is a fixed time delay between the two periods which impacts the bidders' valuations. In particular, the winner may obtain some value from depleting the object, either by consuming it or exploiting it, over the interim period and the loser may lose some value by virtue of the object being depleted by the winner. Our main result is that fixed time delays lead to asymmetric bid distributions. For a special family of probability distributions, we show that the first-price auction is revenue superior to the second-price auction.

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1 Introduction

Consider the sale of an indivisible object. Two risk neutral bidders, one strong and one weak,[1] with private values play the following two-period game. In period 1, the original seller of the object conducts a sealed-bid auction. In period 2, the winner (and potential reseller) of the object may make a single take-it-or-leave-it offer to the loser (buyer). There is a fixed time delay between the two periods which impacts the bidders' valuations. In particular, the winner may obtain some value from depleting the object, either by consuming it or exploiting it, over the interim period and the loser may lose some value by virtue of the object being depleted by the winner.

The characteristics of the object determine whether the object depletes. The situation wherein the characteristics of the object do not change with

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[1]A bidder is *strong* (resp., *weak*) if he is more likely to get a higher (resp., lower) value than the other bidder.

time, for example, arts, artifacts, antiques, jewelries, etc., is called the *one-sided case*. In this case, the loser does not lose any value in the interim period. The complementary case wherein the characteristics of the object change with time, for example, natural resources, spectrum licenses, used cars, carbon emissions, livestock, etc., is called the *two-sided case*. In this case, the loser loses some value in the interim period. In both the cases, the winner obtains some value in the interim period.

Resale markets are forbidden usually for various objects such as emission rights, spectrum licenses, oil drilling, etc. Nonetheless, resale still happens indirectly *via* takeovers, mergers, etc. However, these indirect procedures do not happen immediately after the object is won in an auction, it happens with a time delay. Therefore, the winner may deplete the object by consuming it for a short period of time before reselling it. For other objects such as arts, artifacts, used cars, etc., it is natural to consider that the object is resold with a time delay as is seen in the real world.

The inefficiency of the first-price auction leads to the origins of *auctions with resale* literature. That is, as a lower valuation bidder may win the object, he can gain by reselling the object to a higher valuation bidder. This trade may increase efficiency. In the present paper, even if the object is allocated efficiently, bidders can still have potential mutual gains, provided that the object does not deplete and the bidders' valuations are close to each other. This is because the winner's valuation may be more than the loser's valuation, but it may be the case that the winner's *reservation valuation* is less that the loser' valuation.

Proposition 2 of this paper characterizes perfect Bayesian equilibria of the first-price auction, presuming that equilibria exist. The necessary and sufficient conditions are represented by a boundary value problem. The first-order conditions of period 1 suggest that the "marginal utility" of a bidder is inversely proportional to his opponent's reverse hazard rate of bid. Importantly, the marginal utility of a bidder depends both on the resale price and the opponent's inverse bid function.

The main result of this paper is that, with two risk neutral bidders, the property of bid symmetrization[2] does not hold whenever resale happens after a fixed time delay (Theorem 1). Specifically, the weak bidder wins the auction with a higher probability than the strong bidder, as long as the object does not deplete over time. Moreover, the strong bidder wins the auction with a higher probability than the weak bidder, as long as the object depletes over time.

The intuitive idea of the above result is as follows. In the resale market, the weak bidder is always a reseller and the strong bidder is always a buyer. The marginal utility of the weak bidder comprises of the resale price and the value obtained by consuming the object in the interim, net of the bid

[2] Bid symmetrization means that all the bidders win the auction with equal probability.

payments. The marginal utility of the strong bidder comprises of the resale price and the value lost in the interim, net of the bid payments.

If the object does not deplete over time, the marginal utility of the weak bidder exceeds that of the strong bidder. This is because the strong bidder does not lose any value in the interim. As a result, the weak bidder's bid distribution dominates the strong bidder's bid distribution. If the object depletes over time, the marginal utility of the strong bidder exceeds that of the weak bidder. This is because the value obtained by the weak bidder in the interim is more than the value lost by the strong bidder, which follows from the fact that the weak bidder bids more aggressively than the strong bidder (Corollary 2). As a result, the strong bidder's bid distribution dominates the weak bidder's bid distribution.

We also compare the bid behavior of the present paper with two standard models in the literature: (a) resale without a time delay[3] (Theorem 2), and (b) no resale[4] (Theorem 3). In all the three aforementioned models, the bid functions are characterized by two differential equations, which does not allow us to directly compare the bid functions of any two models. Nevertheless, we adopt an alternate approach by constructing "bid-equivalence functions" which captures the relative bid behavior of a bidder. We show that, in the one-sided case, the bid functions are more asymmetric when resale happens after a fixed time delay than when resale happens without a time delay, and in the two-sided case, the bid functions are less asymmetric when resale happens after a fixed time delay than when resale happens without a time delay. A key implication of Theorem 2 is that the bid functions are more closer to each other when the object depletes as compared to when the object does not deplete (Corollary 3).

Consider the second-price auction. Whenever resale happens without a time delay, bidders bid their own values.[5] Furthermore, the object is allocated efficiently. As a result, bidders never resell the object. On the contrary, whenever resale happens with a fixed time delay and the object does not deplete, bid-your-own-value is *not* an equilibrium strategy. Furthermore, bidders bid their own values as long as the object depletes (Theorem 4).

In particular, whenever the object does not deplete, the bidder who has realized the lowest valuation has an incentive to bid more than his own valuation, provided that the highest valuation bidder bids his valuation. This is because the lowest valuation bidder can resell the object to the highest valuation bidder in such a way that the highest valuation bidder incurs a zero surplus, and the lowest valuation bidder can extract a positive surplus by consuming the object in the interim.

For a special family of probability distributions and the two-sided case,

[3]See Hafalir and Krishna [4].

[4]See Lebrun [6]; Maskin and Riley [9].

[5]See Hafalir and Krishna [4].

we derive closed-form bid functions and study some important characteristics of equilibrium. We also show that the seller derives more expected revenues from the first-price auction than from the second-price auction (Theorem 5).

1.1 The literature

Auctions with resale has been substantially studied in Garratt and Tröger [3]; Hafalir and Krishna [4, 5]; Lebrun [7, 8]; Cheng and Tan [2]; Cheng [1]; and Virág [11, 12] among others. The literature has implicitly assumed that resale happens immediately after the auction ends. However, it may be possible that there is a time delay between the auction and resale. This paper captures the bidding behavior when resale happens with a fixed time delay.

Hafalir and Krishna [4] characterizes and proves the existence of a unique equilibrium in the first-price auction by considering two risk neutral bidders. They show that bid symmetrization holds, i.e., both the bidders win the auction with equal probability. They also provide a general revenue ranking between the first- and second-price auctions. Specifically, they show that the first-price auction is revenue stronger to the second-price auction. Virág [12] introduces reserve prices with two risk neutral bidders and shows that bid symmetrization does not hold. They also show that the second-price auction may produce more revenues for the seller than the first-price auction. Virág [11] considers more than two risk neutral bidders and provide characterization and existence of equilibria. They also show that bid symmetrization does not hold. Specifically, the bidder with a stronger value distribution produces a stronger bid distribution than the other bidders.

The outline of the paper is as follows. In section 2, we formalize the model. In section 3, we characterize the equilibria. In section 4, we compare the bid distributions. In section 5, we study other comparative results. In section 6, we derive closed form solutions and compare expected revenues between the first- and second-price auction for a special family of probability distributions. In section 7, we conclude.

2 Economic environment

Let there be one indivisible object for sale *via* the first-price auction with sealed bids. Denote the set of bidders by $N = \{s, w\}$ where s is the "strong" bidder and w is the "weak" bidder. Denote the value space by $T_i = [0, a_i]$ for every $i \in N$ where $a_s > a_w > 0$. The values are private information and the prior distributions of the random variables $\mathcal{T}_s, \mathcal{T}_w$ are given by $F_i : T_i \to \Re_+$ for every $i \in N$. The prior distributions are statistically independent and twice continuously differentiable, and their density functions, f_s and f_w, are strictly positive everywhere on the value space. Bidders and the seller are assumed to be risk neutral.

The game is played in two periods. Period 1 is the *bid period* where the seller conducts the first-price auction. Period 2 is the *resale period* where bid period's winner may resell the object to the loser *via* a single take-it-or-leave-it offer.[6] The game ends after period 2 and there is no further resale of the object.[7] There is a fixed time delay between period 1 and period 2, which is exogenously given. This time delay impacts bidders' valuations of the object, which may deplete the object.

In particular, the reseller consumes the object in the interim, and obtains a fraction α_R of his realized value. The loser loses a fraction α_B of his realized value, as the object is being consumed by the reseller. Note that $\alpha_R \in (0, 1)$ and $\alpha_B \in [0, 1)$. Formally, the payoffs of the bidders are as follows.

(A) If a bidder with value t wins the object by bidding b in period 1 and is able to resell in period 2 for a price p, then his utility is $\alpha_R t + p - b$. In this case, $\alpha_R t$ is the value generated from the object during the interim period.

(B) If a bidder with value t wins the object by bidding b in period 1 and is not able to resell in period 2, then his utility is $t - b$.

(C) If a bidder with value t loses the object by bidding b in period 1 and is able to purchase in period 2 for a price p, then his utility is $(1-\alpha_B)t-p$. In this case, $\alpha_B t$ is the value lost during the interim period.

(D) If a bidder with value t loses the object by bidding b in period 1 and is not able to purchase in period 2, then his utility is 0.

The characteristics of the object determines whether the object depletes. The case wherein the characteristics of the object does not change over time is called the *one-sided case*. In this case, $\alpha_R > 0$ and $\alpha_B = 0$, i.e., the loser does not lose any value in the interim. The complementary case wherein the characteristics of the object change with time is called the *two-sided case*. In this case, $\alpha_R = \alpha_B \equiv \alpha > 0$, i.e., the loser loses value in the interim.

Assumptions

(A1) F_s dominates F_w in terms of reverse hazard rate;

(A2) The hazard rate of bidder s, $f_s/(1 - F_s)$, is non-decreasing in value;

(A3) For every $\rho > 0$ and for every $t \in T_w$, $F_w(\rho t)/f_w(\rho t) = \rho F_w(t)/f_w(t)$;

(A4) Either $\alpha_R > \alpha_B = 0$ or $\alpha_R = \alpha_B$;

(A5) $(1 - \alpha_R)a_s > (1 - \alpha_B)a_w$.

The first assumption says that bidder s is more likely to get a greater

[6] In the literature, this is known as the *monopoly mechanism*.

[7] Losing bids are not revealed after period 1. So, in period 2, the winner does not know the realized value of loser.

value than bidder w.[8] For this reason, bidder s is called the *strong bidder* and bidder w is called the *weak bidder*. The second assumption confirms uniqueness of the resale price in equilibria. The third assumption restricts the family of distribution functions. The fourth assumption describes the two cases, i.e., the *one-sided case* and *two-sided case*. The fifth assumption is a parametric condition which is required to characterize the equilibria. Note that (A5) implies $(1-\alpha_B)a_s > (1-\alpha_R)a_w$ as $a_s > a_w$ and $\alpha_R \geq \alpha_B$.

3 Equilibria

Direction of resale

Let the bid and inverse bid functions be denoted by β_i and ϕ_i respectively. We restrict ourselves to the family of strictly increasing and continuous bid functions. It is verified in Lemma B.2 that $\phi_s(0) = \phi_w(0) = 0$, $\phi_s(\bar{b}) = a_s$ and $\phi_w(\bar{b}) = a_w$ for some $\bar{b} > 0$. Therefore, $\beta_i : [0, a_i] \to [0, \bar{b}]$ and $\phi_i : [0, \bar{b}] \to [0, a_i]$ for every $i \in N$. In order to establish the direction of resale and perhaps to simplify the analysis, we claim the following properties.

(A) If bidder w wins the auction with bid b and $(1-\alpha_B)\phi_s(b) > (1-\alpha_R)\phi_w(b)$, then he makes a resale offer.

(B) If bidder s wins the auction with bid b and $(1-\alpha_R)\phi_s(b) > (1-\alpha_B)\phi_w(b)$, then he does not make a resale offer.

To see this, first consider property (A) and bidder w with value t_w. Suppose he bids b and wins the object. Then it must be true that $t_w = \phi_w(b)$. Therefore, $(1-\alpha_B)\phi_s(b) > (1-\alpha_R)\phi_w(b)$ implies $(1-\alpha_R)t_w < (1-\alpha_B)\phi_s \circ \phi_w^{-1}(t_w)$. Since he wins, it must be true that $b > \phi_s^{-1}(t_s)$. This implies $(1-\alpha_B)t_s < (1-\alpha_B)\phi_s \circ \phi_w^{-1}(t_w)$. Therefore, with a strictly positive probability, $(1-\alpha_R)t_w < (1-\alpha_B)t_s$ and thus there are returns from resale if bidder w makes a resale offer.

Now, consider property (B) and bidder s with value t_s. Suppose he bids b and wins the object. Then it must be true that $t_s = \phi_s(b)$. Therefore, $(1-\alpha_R)\phi_s(b) > (1-\alpha_B)\phi_w(b)$ implies $(1-\alpha_R)t_s > (1-\alpha_B)\phi_w \circ \phi_s^{-1}(t_s)$. Since he wins, it must be true that $b > \phi_w^{-1}(t_w)$. This implies $(1-\alpha_B)t_w < (1-\alpha_B)\phi_w \circ \phi_s^{-1}(t_s)$. Therefore, with probability 1, $(1-\alpha_R)t_s > (1-\alpha_B)t_w$ and thus there are no returns from resale if bidder s makes a resale offer.

As a starting point, we assume that $(1-\alpha_R)\phi_s(b) > (1-\alpha_B)\phi_w(b)$ for every b. It is important to note that $(1-\alpha_R)\phi_s(b) > (1-\alpha_B)\phi_w(b)$ implies $(1-\alpha_B)\phi_s(b) > (1-\alpha_R)\phi_w(b)$. Later we shall prove the assumption as a Proposition.

[8] F_s is said to dominate F_w in terms of reverse hazard rate if

$$\frac{F_s(t)}{f_s(t)} < \frac{F_w(t)}{f_w(t)}$$

for every $t \in (0, a_w)$.

As a result, if resale happens in period 2, then the direction of resale has to be from bidder w to bidder s.

The game is solved by backward induction.

Resale period

From the above discussion, it is clear that bidder s does not have any optimization problem in the resale period. For the ease of notation, let $z = 1/(1-\alpha_B)$. Consider bidder w with value t_w. Since bidder w is finding the optimum resale price in the resale period, it must be the case that he has won the auction. Suppose bidder w wins by bidding b. Then it must be true that $b > \phi_s^{-1}(\mathcal{T}_s)$, or equivalently, $\mathcal{T}_s < \phi_s(b)$. Bidder w's offer is rejected if $\mathcal{T}_s < zp$ conditional on the event that $\mathcal{T}_s < \phi_s(b)$. In this case, the object is retained with him which gives him a utility of $t_w - b$. Bidder w's offer, p, is accepted if $\mathcal{T}_s > zp$ conditional on the event that $\mathcal{T}_s < \phi_s(b)$. In this case, he gets a utility of $p + \alpha_R t_w - b$ where $\alpha_R t_w$ is the utility obtained from consuming the object during the interim period. Therefore the optimization problem of bidder w is

$$\max_p \Pr(\mathcal{T}_s < zp | \mathcal{T}_s < \phi_s(b))(t_w - b) + \Pr(\mathcal{T}_s > zp | \mathcal{T}_s < \phi_s(b))(p + \alpha_R t_w - b).$$

The optimization problem can be rewritten as

$$\max_p \frac{F_s(zp)}{F_s \circ \phi_s(b)}(t_w - b) + \frac{F_s \circ \phi_s(b) - F_s(zp)}{F_s \circ \phi_s(b)}(p + \alpha_R t_w - b).$$

This is equivalent to solving the following problem

$$\max_p F_s(zp)(t_w - b) + [F_s \circ \phi_s(b) - F_s(zp)](p + \alpha_R t_w - b).$$

The first-order necessary condition gives

$$(1 - \alpha_R)t_w = p - \frac{F_s \circ \phi_s(b) - F_s(zp)}{z f_s(zp)}. \tag{1}$$

From (A2), it can be shown that the right-hand side of (1) is strictly increasing in p[9] and therefore the equation can be read as some monotone function which equals a constant term. Thus a unique value of p exists which satisfies (1). Furthermore, (A2) guarantees that the unique solution of (1) is the maximum.[10]

[9] The proof of this claim is given in Appendix B (Lemma B.2).

[10] The proof of this claim is given in Appendix B (Lemma B.3).

Bid period

Let the optimum value of resale price be denoted as $p(t_w, b)$. From (1), it follows that $p(t_w, b)$ is strictly increasing in t_w and b. Furthermore, $(1-\alpha_B)\phi_s(b) > p(t_w, b) > (1-\alpha_R)t_w$. Let bidder w with value t_w bid b and bidder s bid according to his bid function. Bidder w wins if $b > \phi_s^{-1}(\mathcal{T}_s)$, or equivalently, $\mathcal{T}_s < \phi_s(b)$. If he wins, then he makes a resale offer. His offer is rejected by bidder s if $p > (1-\alpha_B)\mathcal{T}_s$, or equivalently, $\mathcal{T}_s < zp$. In this case, the object remains with bidder w incurring him a utility of $t_w - b$. On the other hand, his offer is accepted if $p < (1-\alpha_B)\mathcal{T}_s$, or equivalently $\mathcal{T}_s > zp$. In this case, bidder w incurs a utility of $p + \alpha_R t_w - b$, where $\alpha_R t_w$ is the value he gets from consuming the object during the interim period. Therefore the expected utility function of bidder w is

$$\begin{aligned} U_w(t_w, p(t_w, b), b) =& F_s(zp(t_w, b))(t_w - b) + \\ & [F_s \circ \phi_s(b) - F_s(zp(t_w, b))](p(t_w, b) + \alpha_R t_w - b). \end{aligned}$$

Maximizing w.r.t. b and using the fact that, in equilibrium, $t_w = \phi_w(b)$, we have the following first-order differential equation[11]

$$\frac{F_s \circ \phi_s(b)}{\mathrm{D}F_s \circ \phi_s(b)} = p(\phi_w(b), b) - b + \alpha_R \phi_w(b). \tag{2}$$

Let bidder s with value t_s bid b and bidder w bid according to his inverse bid function. Bidder s wins if $b > \phi_w^{-1}(\mathcal{T}_w)$, or equivalently, $\mathcal{T}_w < \phi_w(b)$. If he wins, then he does not make a resale offer and incurs a utility of $t_s - b$. Bidder s loses if $b < \phi_w^{-1}(\mathcal{T}_w)$, or equivalently, $\mathcal{T}_w > \phi_w(b)$. If he loses, he has an option of purchasing the object in the resale market. He accepts the offer if $(1-\alpha_B)t_s > p$ and gets a utility of $(1-\alpha_B)t_s - p$ where $\alpha_B t_s$ is the value lost between the two periods. He rejects the offer if $(1-\alpha_B)t_s < p$. Therefore the expected utility function of bidder s is

$$\begin{aligned} U_s(t_s, p(\phi_w(b), b), b) = & F_w \circ \phi_w(b)(t_s - b) + \\ & \int_{\phi_w(b)}^{a_w} \max\{(1-\alpha_B)t_s - p(\omega, \phi_w^{-1}(\omega)), 0\} f_w(\omega) \mathrm{d}\omega. \end{aligned}$$

Maximizing w.r.t. b and using the fact that, in equilibrium, $t_s = \phi_s(b)$, we have the following first-order differential equation[12]

$$\frac{F_w \circ \phi_w(b)}{\mathrm{D}F_w \circ \phi_w(b)} = p(\phi_w(b), b) - b + \alpha_B \phi_s(b). \tag{3}$$

[11] Envelope theorem has been used to obtain (2).

[12] Leibniz integral rule has been used to obtain (3).

Proposition 1. *Let the triple* (ϕ_s, ϕ_w, p) *be a perfect Bayesian equilibrium and let Assumptions (A1)-(A5) hold. Then,*

$$(1-\alpha_R)\phi_s(b) > (1-\alpha_B)\phi_w(b)$$

for every $b \in (0, \bar{b}]$.

The above result ensures that bidder s is always a buyer in period 2. Two immediate corollaries are as follows.

Corollary 1. *Let the primitives of Proposition 1 be satisfied. Then*

$$(1-\alpha_B)\phi_s(b) > (1-\alpha_R)\phi_w(b)$$

for every $b \in (0, \bar{b}]$.

The above result ensures that bidder w is always a reseller in period 2.

Corollary 2. *Let the primitives of Proposition 1 be satisfied. Then, bidder* w *bids more aggressively than bidder* s.

The above result states that, in both the one- and two-sided cases, bidder w bids more aggressively than bidder s.

The following result characterizes the equilibria.

Proposition 2. *Let Assumptions (A1)-(A5) be satisfied. A triple* (ϕ_s, ϕ_w, p) *is a perfect Bayesian equilibrium if and only if it solves the following Dirichlet problem*

$$\begin{aligned}
\frac{F_s \circ \phi_s(b)}{\mathrm{D}F_s \circ \phi_s(b)} &= p(\phi_w(b), b) - b + \alpha_R \phi_w(b) \\
\frac{F_w \circ \phi_w(b)}{\mathrm{D}F_w \circ \phi_w(b)} &= p(\phi_w(b), b) - b + \alpha_B \phi_s(b) \\
(1-\alpha_R)\phi_w(b) &= p(\phi_w(b), b) - \frac{F_s \circ \phi_s(b) - F_s(zp(\phi_w(b), b))}{z f_s(zp(\phi_w(b), b))} \\
\phi_s(0) = \phi_w(0) &= 0, \phi_s(\bar{b}) = a_s \ \textit{and} \ \phi_w(\bar{b}) = a_w \ \textit{for some} \ \bar{b} > 0.
\end{aligned} \tag{4}$$

The above result provides the necessary and sufficient conditions of the equilibria. It says that if a profile of measurable functions is known to be an equilibrium profile, then it must solve the Dirichlet problem given by (4). In addition, if any profile of measurable functions solves the Dirichlet problem given by (4), then it must be the case that it is an equilibrium profile. To establish sufficiency, we show that unilateral deviations are not desirable for both the bidders.

Few Remarks are as follows.

Remark 1. *In the two-sided case, Assumption (A3) is not necessary to characterize the equilibria and deriving other properties.*

Remark 2. *When resale happens after a fixed time delay and the object does not deplete, the object may be resold even if it is allocated efficiently. In particular, if the valuations are close enough and the highest valuation bidder wins, then he has an incentive to resell the object to the lowest valuation bidder.*

4 Bid distributions

It is well-known that bidder s produces a stronger bid distribution than bidder w when there are no resale markets (see Maskin and Riley [9]; Lebrun [6]), and both the bidders produce same bid distribution when resale happens without a delay (see Hafalir and Krishna [4]). In the following result, we show that fixed time delays lead to asymmetric bid distributions.

Theorem 1. *Let the triple (ϕ_s, ϕ_w, p) be a perfect Bayesian equilibrium and let Assumptions (A1)-(A5) hold.*
(A) If $\alpha_B = 0$ and $\alpha_R > 0$, then $F_w \circ \phi_w(b) < F_s \circ \phi_s(b)$ for every $b \in (0, \bar{b})$.
(B) If $\alpha_B = \alpha_R \equiv \alpha$, then $F_s \circ \phi_s(b) < F_w \circ \phi_w(b)$ for every $b \in (0, \bar{b})$.

Part (A) says that, in the one-sided case, the bid distribution function of bidder w first-order stochastically dominates that of bidder s. In other words, bidder w has higher chances of winning the object than bidder s. Part (B) says that, in the two-sided case, the bid distribution function of bidder s first-order stochastically dominates that of bidder w. In other words, bidder s has higher chances of winning the object than bidder w.

The intuition of the above result is as follows. Consider bidder w with value t_w. Suppose he wins by bidding b and loses if he reduces the bid by a small margin $\epsilon > 0$. His utility, $u_w : \Re^2_+ \to \Re_+$, when he bids $b - \epsilon$ is $u_w(\phi_w(b-\epsilon), b-\epsilon) = 0$. This is because if he loses then he cannot buy in the resale market. If he bids b, then he is able to resell the object in period 2[13] and incurs a utility of $u_w(\phi_w(b), b) = p(\phi_w(b), b) - b + \alpha_R \phi_w(b)$. The change in utility is $\Delta_w = u_w(\phi_w(b), b) - u_w(\phi_w(b-\epsilon), b-\epsilon) = p(\phi_w(b), b) - b + \alpha_R \phi_w(b)$. In particular, Δ_w comprises of two terms: the first term, $p(\phi_w(b), b) - b$, is the earning from payments, and the second term, $\alpha_R \phi_w(b)$, is the valuation obtained from the object.

Now, consider bidder s with value t_s. Suppose he wins by bidding b and loses if he reduces the bid by a small margin $\epsilon > 0$. His utility, $u_s : \Re^2_+ \to \Re_+$, when he bids b is $u_s(\phi_s(b), b) = \phi_s(b) - b$. This is because if he wins then he does not resell the object in period 2. If he bids $b - \epsilon$, then he is able

[13] The reason that bidder w is able to resell is as follows. As bidder w loses with $b - \epsilon$ and wins with b, it must be the case that bidder s bids between $b - \epsilon$ and b. Then, the realized value of bidder s lies between $\phi_s(b - \epsilon)$ and $\phi_s(b)$. As $(1 - \alpha_B)\phi_s(c) > p(\phi_w(b), b)$ for every $c \in (b - \epsilon, b)$, bidder s accepts the resale offer.

to buy the object in period 2[14] and incurs a utility of $u_s(\phi_s(b-\epsilon), b-\epsilon) = (1-\alpha_B)\phi_s(b-\epsilon) - p(\phi_w(b), b)$. The change in utility is $\Delta_s = u_s(\phi_s(b), b) - u_s(\phi_s(b-\epsilon), b-\epsilon)$ which approximately equals $p(\phi_w(b), b) - b + \alpha_B\phi_s(b)$. In particular, Δ_s comprises of two terms: the first term, $p(\phi_w(b), b) - b$, is the earning on payments, and the second term, $\alpha_B\phi_s(b)$, is the valuation obtained from the object.

The first-order differential equation for bidder w may be rewritten as

$$\lim_{\epsilon \downarrow 0} \left[\frac{\int_{b-\epsilon}^{b} \mathrm{D}\phi_s(y) f_s \circ \phi_s(y) \mathrm{d}y}{\epsilon F_s \circ \phi_s(b)} \right]^{-1} = \Delta_w.$$

The above equation may be interpreted as the inverse of reverse hazard rate of bid for bidder s equals change in utility for bidder w. For a given $\epsilon > 0$, the first-order condition for bidder w may be rewritten as

$$\left[\frac{\epsilon \mathrm{D}\phi_s(b) f_s \circ \phi_s(b)}{F_s \circ \phi_s(b)} \right]^{-1} = \frac{\Delta_w}{\epsilon}. \tag{5}$$

Since $\int_{b-\epsilon}^{b} \mathrm{D}\phi_s(y) f_s \circ \phi_s(y) \mathrm{d}y \approx \epsilon \mathrm{D}\phi_s(b) f_s \circ \phi_s(b)$, the left hand side is interpreted as the inverse of the probability that bidder s bids at least $b - \epsilon$ given that his bid does not exceed b. The right hand side is the marginal utility of bidder w.

Similarly, the first-order differential equation for bidder s may be rewritten as

$$\lim_{\epsilon \downarrow 0} \left[\frac{\int_{b-\epsilon}^{b} \mathrm{D}\phi_w(y) f_w \circ \phi_w(y) \mathrm{d}y}{\epsilon F_w \circ \phi_w(b)} \right]^{-1} = \Delta_s.$$

The above equation may be interpreted as the inverse of reverse hazard rate of bid for bidder w equals change in utility for bidder s. For a given $\epsilon > 0$, the first-order condition for bidder s may be rewritten as

$$\left[\frac{\epsilon \mathrm{D}\phi_w(b) f_w \circ \phi_w(b)}{F_w \circ \phi_w(b)} \right]^{-1} = \frac{\Delta_s}{\epsilon}. \tag{6}$$

Since $\int_{b-\epsilon}^{b} \mathrm{D}\phi_w(y) f_w \circ \phi_w(y) \mathrm{d}y \approx \epsilon \mathrm{D}\phi_w(b) f_w \circ \phi_w(b)$, the left hand side is interpreted as the inverse of the probability that bidder w bids at least $b - \epsilon$ given that his bid does not exceed b. The right hand side is the marginal utility of bidder s.

In the one-sided case, from (5) and (6), the marginal utility of bidder w exceeds that of bidder s. Therefore, the probability that bidder s bids very close to the winning bid is less than the probability that bidder w bids

[14]The reason that bidder s is able to buy is as follows. As bidder s loses with $b - \epsilon$ and wins with b, it must be the case that bidder w bids between $b - \epsilon$ and b. Then, the realized value of bidder w lies between $\phi_w(b-\epsilon)$ and $\phi_w(b)$, and his resale offer lies between $p(\phi_w(b-\epsilon), b-\epsilon)$ and $p(\phi_w(b), b)$. As $p(\phi_w(c), c) > (1-\alpha_R)\phi_s(b-\epsilon)$ for every $c \in (b-\epsilon, b)$, bidder s accepts the resale offer.

very close to the winning bid. In other words, the marginal probability of winning is more for bidder w than bidder s. As a result, bidder w produces a stronger bid distribution than bidder s.

In the two-sided case, from (5) and (6), the marginal utility of bidder s exceeds that of bidder w. Therefore, the marginal probability of winning is more for bidder s than bidder w. As a result, bidder s produces a stronger bid distribution than bidder w.

5 Comparative results

In this section, we study a number of comparative results. Our aim is to determine the impact of fixed time delays on the bid behavior. We do so by comparing the bid behavior in our model with two standard models in the literature, i.e., (1) resale without a delay, and (2) no resale. Given the asymmetric structure of differential equations in our model, (1), and (2), it becomes difficult to compare the bid functions between any two models. Nonetheless, it is possible to compare the relative bid functions of our model with (1) and (2) *via* a "bid-equivalence function". Furthermore, we compare the bid behavior of our model with the standard symmetric auctions model (see, Riley and Samuelson [10]).

Let $\Psi_s : T_w \to T_s$ be defined as $\Psi_s(t) := \phi_s \circ \phi_w^{-1}(t)$. The function Ψ_s may be called a *bid-equivalence function*. Given a value t of bidder w, $\Psi_s(t)$ may be interpreted as the value required by bidder s such that the bids of both the bidders are equal.

In the Hafalir and Krishna [4] (H-K) model where resale happens without a delay, let the bid-equivalence function of bidder s be denoted by Θ_s. The following result compares the bid-equivalence functions of our model and the H-K model.

Theorem 2. *Let Ψ_s be the bid-equivalence function of bidder s when resale happens with a fixed time delay. Let Θ_s be the bid-equivalence function of bidder s when resale takes place without a delay.*
(A) If $\alpha_B = 0$ and $\alpha_R > 0$, then $\Psi_s(t) > \Theta_s(t)$ for every $t \in (0, a_w)$.
(B) If $\alpha_B = \alpha_R \equiv \alpha$, then $\Psi_s(t) < \Theta_s(t)$ for every $t \in (0, a_w)$.

Part (A) tells that, in the one-sided case, the bid functions are more asymmetric when resale happens with a fixed time delay than when it happens without a delay. In other words, fixed time delays increase bidder w's aggression against bidder s. Part (B) tells that, in the two-sided case, the bid functions are less asymmetric when resale happens with a fixed time delay than when it happens without a delay. In other words, fixed time delays decrease bidder w's aggression against bidder s.

A key implication of the above result is given in the following Corollary.

Corollary 3. *The bid functions are more asymmetric in the one-sided case than they are in the two-sided case.*

Let Λ_s be the bid-equivalence function of bidder s when there are no resale markets. This model has been studied in Maskin and Riley [9] and Lebrun [6] (M-R-L). In the following result, we compare the bid-equivalence functions of our model with the M-R-L model.

Theorem 3. *Let Ψ_s be the bid-equivalence function of bidder s when resale happens after a fixed time delay. Let Λ_s be the bid-equivalence function of bidder s when there are no resale markets. If $\alpha_B = 0$ and $\alpha_R > 0$, then*

$$\Psi_s(t) > \Lambda_s(t)$$

for every $t \in (0, a_w)$.

The above result states that, in the one-sided case, the bid functions are more asymmetric when resale happens with a fixed time delay than when there are no resale markets. In other words, bidder w's aggression against bidder s is more when there are resale markets with a fixed time delay.

In the next two results, we compare bid functions of our model with the standard symmetric auctions without resale model.

Proposition 3. *Let (ϕ_s, ϕ_w, p) be a perfect Bayesian equilibrium when resale happens after a fixed time delay. Let (μ_s, μ_s) be a symmetric Bayesian equilibrium when there are no resale markets. Let $F_s(0) > 0$. Let $\alpha_B = \alpha_R \equiv \alpha$. Then $\phi_s(b) > \mu_s(b)$ for every $b \in (0, \bar{b}]$.*

The above result conveys that bidder s bids more aggressively if he has an opponent of his own kind and there are no resale markets than if his opponent is bidder w and there are resale markets after a fixed time delay.

Proposition 4. *Let (ϕ_s, ϕ_w, p) be a perfect Bayesian equilibrium when resale happens after a fixed time delay. Let (μ_w, μ_w) be a symmetric Bayesian equilibrium when there are no resale markets. Let $F_w(0) > 0$. Let $\alpha_B = \alpha_R \equiv \alpha$. Then $\phi_w(b) < \mu_w(b)$ for every $b \in (0, \mu_w^{-1}(a_w)]$.*

The above result conveys that bidder w bids less aggressively if he has an opponent of his own kind and there are no resale markets than if his opponent is bidder s and there are resale markets after a fixed time delay.

6 Revenues

In this section, we analyze the second-price auction for the one- and two-sided cases. For the first-price auction and the two-sided case, we derive closed-form solutions by considering a special family of probability distributions, and study important properties of equilibrium. Furthermore, we compare seller's *ex-ante* expected revenues between the first- and second-price auctions.

It is well-known that, in second-price auctions without resale, bid-your-own-value is a weakly dominant strategy for every bidder.[15] Unfortunately, the result does not extend when resale opportunities are introduced. Nonetheless, bid-your-your-value still remains an equilibrium strategy when resale happens without a delay. In the following result, we show that if resale happens after a fixed time delay, then bid-your-own-value is *not* an equilibrium strategy in the one-sided case whereas bid-your-own-value is an equilibrium strategy in the two-sided case.

Theorem 4. *Let the auction format be the second-price auction.*

(A) If $\alpha_B = 0$ and $\alpha_R > 0$, then there does not exists any equilibrium such that bidders bid their own value.

(B) If $\alpha_B = \alpha_R \equiv \alpha$, then bid-your-own-value is an equilibrium strategy for every bidder.

When resale happens without a delay, the second-price auction allocates the object allocated efficiently. As a result, the game never reaches the resale period. This result extends when resale happens with a fixed time delay and the object depletes.

When the object does not deplete, bid-your-own-value is not an equilibrium. In particular, the lowest valuation bidder always has an incentive to bid more than his valuation. To understand this idea, suppose the highest valuation bidder bids his valuation. If the lowest valuation bidder also bids his valuation, then the highest valuation bidder wins. If the valuations are close to each other, then the highest valuation bidder has an incentive to resell the object; otherwise not. If he resells, then he extracts all the surplus from the lowest valuation bidder, as the losing bid gives him information about the exact value of the lowest valuation bidder. In either case, the lowest valuation bidder gets a utility of zero.

The lowest valuation bidder can extract a positive surplus by increasing his bid above the highest valuation bidder. This is because the lowest valuation bidder gets a positive surplus by consuming the object in the interim period, and covers his payments by reselling the object to the highest valuation bidder.

The highest valuation bidder may have an incentive to bid less than his valuation. This case is discussed in the proof of Theorem 4.

For the rest of this section, assume that $\alpha_R = \alpha_B \equiv \alpha$. Let $\mathcal{F}$ be the family of probability distributions of the form

$$F_i(t) = \left(\frac{t}{a_i}\right)^{\frac{1}{\tau_i}}$$

[15] The result is true for both symmetric and asymmetric cases.

where $\tau_i > 0$ for every $i \in N$. The collection $\mathcal{F}$ is called *the family of power distributions* where $1/\tau_i$ reflects the power of bidder i. We assume $\tau_s = 1$, i.e., F_s follows uniform distribution. Therefore, τ_w reflects the relative power of bidder s, i.e., greater the value of τ_w, stronger is bidder s.

Proposition 5. *Let $F_s, F_w \in \mathcal{F}$ such that $\tau_w > \tau_s = 1$. Let $\alpha > (\tau_w - 1)/(\tau_w + 3)$ and $a_s(3\alpha + 1 + \tau_w\alpha - \tau_w) = a_w(3\alpha - 1 + \tau_w\alpha + \tau_w)$. Then the closed-form solution of (ϕ_s, ϕ_w, p) is given by*

$$\begin{aligned} &\phi_s(b) = \frac{(\tau_w + 3)\alpha + \tau_w - 1}{2\alpha} b, \quad \phi_w(b) = \frac{(\tau_w + 3)\alpha + 1 - \tau_w}{2\alpha} b, \\ &p(\phi_w(b), b) = \frac{(\tau_w + 3)(1 - \alpha)}{2} b. \end{aligned} \tag{7}$$

Few observations are as follows. First, fixing the values of α and a_w, an increase in τ_w increases a_s and ϕ_s and decreases ϕ_w. That is, as the degree of asymmetry in probability distribution increases, bidder w bids more aggressively and bidder s bids less aggressively. Put differently, a higher degree of asymmetry in probability distributions asymmetrizes the bid functions. Second, fixing the values of τ_w and a_w, an increase in α decreases a_s and ϕ_s and increases ϕ_w. That is, a higher rate at which bidder's value changes due to a delay in resale and a decrease in the degree of asymmetry in probability distributions reduces the degree of asymmetry of the bid functions.

The bid distributions are as follows.

$$F_s \circ \phi_s(b) = \frac{(\tau_w + 3)\alpha + \tau_w - 1}{2\alpha a_s} b, \quad F_w \circ \phi_w(b) = \left[\frac{(\tau_w + 3)\alpha + 1 - \tau_w}{2\alpha a_w} b\right]^{\frac{1}{\tau_w}}.$$

It may be verified that $F_s \circ \phi_s(b) < F_w \circ \phi_w(b)$ for every $b \in (0, \bar{b})$.

The following Lemma gives us explicit expressions for expected revenues in the first- and second-price auctions.

Lemma 1. *Let the primitives of Proposition 5 be satisfied and let $a_w = 1$. Then the seller's* ex-ante *expected revenues in the first- and second-price auctions are given by*

$$\begin{aligned} R^I(\alpha, \tau_w) &= \frac{2\alpha(1 + \tau_w)}{(2\tau_w + 1)(3\alpha + 1 + \tau_w\alpha - \tau_w)} \text{ and} \\ R^{II}(\alpha, \tau_w) &= \frac{1}{\tau_w + 1} - \frac{3\alpha + 1 + \tau_w\alpha - \tau_w}{2(2\tau_w + 1)(3\alpha - 1 + \tau_w\alpha + \tau_w)} \end{aligned} \tag{8}$$

respectively.

In the following result, we compare expected revenues between the two auction formats.

Theorem 5. *Let $F_s, F_w \in \mathcal{F}$ such that $\tau_w > \tau_s = 1$. Let $\alpha > (\tau_w - 1)/(\tau_w + 3)$, $a_w = 1$, and $a_s(3\alpha + 1 + \tau_w\alpha - \tau_w) = 3\alpha - 1 + \tau_w\alpha + \tau_w$. Then the first-price auction is revenue superior to the second-price auction.*

To prove the above result, we first argue that as $\alpha \uparrow 1$, the difference between the revenues in the first- and second-price auctions is positive. We then argue that this difference increases as the value of α decreases.

We conclude this section by a numerical example.

Example 1. *Let $\alpha = 3/5$, $a_w = 1$, $a_s = 2$ and $\tau_w = 2$. Then $\phi_s(b) = \frac{10}{3}b$, $\phi_w(b) = \frac{5}{3}b$ and $p(\phi_w(b), b) = b$. The seller's expected revenues in the first- and second-price auction are $9/25$ and $17/60$ respectively.*

7 Conclusion

In this paper, we have considered that resale in auctions happens after a fixed time delay which impacts the bidders' valuations. To the best of our knowledge, this is the first attempt to consider time delays in auctions with resale. We have shown that inclusions of fixed time delays in auctions with resale lead to the failure of bid symmetrization property. In particular, if the object does not deplete, then the bidder with a weaker value distribution wins more often, and if the object depletes, then the bidder with a stronger value distribution wins more often. We have also studied the impact of fixed time delays on the bid behavior. For a special family of probability distributions, we have derived closed-form solution of bid functions and shown that the first-price auction generates more expected revenues than the second-price auction, as long as the object depletes.

A Appendix A: Proofs

Proof of Proposition 1. As $(1-\alpha_R)\phi_s(\bar{b}) = (1-\alpha_R)a_s > (1-\alpha_B)a_w = (1-\alpha_B)\phi_w(\bar{b})$, it follows that there exists $\epsilon > 0$ such that $(1-\alpha_R)\phi_s(b) > (1-\alpha_B)\phi_w(b)$ for every $b \in (\bar{b}-\epsilon, \bar{b}]$. We show that the two inverse bid functions never cross each other. We show by contradiction. Suppose, if possible, there exists $b^* > 0$ such that $(1-\alpha_R)\phi_s(b^*) = (1-\alpha_B)\phi_w(b^*)$[16] and $(1-\alpha_R)\phi_s(b) > (1-\alpha_B)\phi_w(b)$ for every $b \in (b^*, \bar{b}]$. Let $k = (1-\alpha_R)/(1-\alpha_B)$.

[16]Under this scenario, the direction of trade is discussed in Lemma B.4.

Then $k \le 1$. So, from (2) and (3), we have

$$\begin{aligned}(1-\alpha_B)\phi_w'(b^*) &= \frac{F_w \circ \phi_w(b^*)}{f_w \circ \phi_w(b^*)} \frac{1-\alpha_B}{p(\phi_w(b^*), b^*) - b^* + \alpha_B \phi_s(b^*)} \\ &\ge \frac{F_w \circ \phi_w(b^*)}{f_w \circ \phi_w(b^*)} \frac{1-\alpha_B}{p(\phi_w(b^*), b^*) - b^* + \alpha_R \phi_w(b^*)} \\ &= \frac{F_w \circ [k\phi_s(b^*)]}{f_w \circ [k\phi_s(b^*)]} \frac{1-\alpha_B}{p(\phi_w(b^*), b^*) - b^* + \alpha_R \phi_w(b^*)} \\ &= \frac{F_w \circ \phi_s(b^*)}{f_w \circ \phi_s(b^*)} \frac{k(1-\alpha_B)}{p(\phi_w(b^*), b^*) - b^* + \alpha_R \phi_w(b^*)} \\ &> \frac{F_s \circ \phi_s(b^*)}{f_s \circ \phi_s(b^*)} \frac{1-\alpha_R}{p(\phi_w(b^*), b^*) - b^* + \alpha_R \phi_w(b^*)} \\ &= (1-\alpha_R)\phi_s'(b^*).\end{aligned}$$

The second step follows from the fact that $\alpha_B \phi_s(b^*) \le \alpha_R \phi_w(b^*)$, the fourth step follows from assumption (A3), and the fifth step follows from assumption (A1). Therefore there exists $\delta > 0$ such that $(1-\alpha_B)\phi_w(b^*+\delta) > (1-\alpha_R)\phi_s(b^*+\delta)$, which is a contradiction. Hence $(1-\alpha_B)\phi_s(b) > (1-\alpha_R)\phi_w(b)$ for every $b \in (0, \bar{b}]$. ■

Proof of Proposition 2. The necessity of equilibria has already been established in the main body of the paper. We establish the sufficiency of equilibria. Suppose (ϕ_s, ϕ_w, p) solves the Dirichlet problem. Let bidder w with value t_w overbids to c such that $\phi_w(c) > t_w$. Then, it must be the case that $p(\phi_w(c), c) > p(t_w, c)$. Differentiating (3), we have

$$\begin{aligned}\mathrm{D}_c U_w(t_w, p(t_w, c), c) &= \mathrm{D}_c F_s \circ \phi_s(c)[p(t_w, c) - c + \alpha_R t_w] - F_s \circ \phi_s(c) \\ &< \mathrm{D}_c F_s \circ \phi_s(c)[p(\phi_w(c), c) - c + \alpha_R \phi_w(c)] - F_s \circ \phi_s(c) \\ &= 0 = \mathrm{D}_c U_w(\phi_w(c), p(\phi_w(c), c), c).\end{aligned}$$

Therefore, overbidding is not optimum for bidder w. Let bidder w with value t_w underbids to c such that $\phi_w(c) < t_w$. Then, it must be the case that $p(\phi_w(c), c) < p(t_w, c)$. Differentiating (3), we have

$$\begin{aligned}\mathrm{D}_c U_w(t_w, p(t_w, c), c) &= \mathrm{D}_c F_s \circ \phi_s(c)[p(t_w, c) - c + \alpha_R t_w] - F_s \circ \phi_s(c) \\ &> \mathrm{D}_c F_s \circ \phi_s(c)[p(\phi_w(c), c) - c + \alpha_R \phi_w(c)] - F_s \circ \phi_s(c) \\ &= 0 = \mathrm{D}_c U_w(\phi_w(c), p(\phi_w(c), c), c).\end{aligned}$$

Therefore underbidding is also not optimum for bidder w.

Let bidder s with value t_s overbids to c such that $\phi_s(c) > t_s$. Differentiating (3), we have

$$\begin{aligned}\mathrm{D}_c U_s(t_s, p(\phi_w(c), c), c) &= \mathrm{D}_c F_w \circ \phi_w(c)(t_s - c) - F_w \circ \phi_w(c) \\ &\quad - \max\{(1-\alpha_B)t_s - p(\phi_w(c), c), 0\}\mathrm{D}_c F_w \circ \phi_w(c)\end{aligned}$$

Using the fact that $\max\{(1-\alpha_B)t_s - p(\phi_w(c), c), 0\} \geq (1-\alpha_B)t_s - p(\phi_w(c), c)$, we have

$$\begin{aligned}
\mathrm{D}_c U_s(t_s, p(\phi_w(c), c), c) &\leq \mathrm{D}_c F_w \circ \phi_w(c)[p(\phi_w(c), c) - c + \alpha_B t_s] - F_w \circ \phi_w(c) \\
&< \mathrm{D}_c F_w \circ \phi_w(c)[p(\phi_w(c), c) - c + \alpha_B \phi_s(c)] - F_w \circ \phi_w(c) \\
&= 0 = \mathrm{D}_c U_s(\phi_s(c), p(\phi_w(c), c), c).
\end{aligned}$$

Therefore, overbidding is not optimum for bidder s. Let bidder s with value t_s underbids to c such that $\phi_s(c) < t_s$. Differentiating (3), we have

$$\begin{aligned}
\mathrm{D}_c U_s(t_s, p(\phi_w(c), c), c) = \mathrm{D}_c F_w \circ \phi_w(c)(t_s - c) &- F_w \circ \phi_w(c) \\
&- \max\{(1-\alpha_B)t_s - p(\phi_w(c), c), 0\} \mathrm{D}_c F_w \circ \phi_w(c)
\end{aligned}$$

Using the fact that $(1-\alpha_B)\phi_s(c) > p(\phi_w(c), c)$, we have

$$\begin{aligned}
\mathrm{D}_c U_s(t_s, p(\phi_w(c), c), c) &= \mathrm{D}_c F_w \circ \phi_w(c)[p(\phi_w(c), c) - c + \alpha_B t_s] - F_w \circ \phi_w(c) \\
&> \mathrm{D}_c F_w \circ \phi_w(c)[p(\phi_w(c), c) - c + \alpha_B \phi_s(c)] - F_w \circ \phi_w(c) \\
&= 0 = \mathrm{D}_c U_s(\phi_s(c), p(\phi_w(c), c), c).
\end{aligned}$$

Therefore underbidding is also not optimum for bidder s. Thus (ϕ_s, ϕ_w, p) is an equilibrium profile. ■

Proof of Theorem 1. We first show (A). Suppose $\alpha_B = 0$ and $\alpha_R > 0$. Then, from (3), we have

$$\frac{F_s \circ \phi_s(b)}{\mathrm{D}F_s \circ \phi_s(b)} = p(\phi_w(b), b) - b + \alpha_R \phi_w(b) > p(\phi_w(b), b) - b = \frac{F_w \circ \phi_w(b)}{\mathrm{D}F_w \circ \phi_w(b)}.$$

This implies

$$\mathrm{D}\left[\frac{F_w \circ \phi_w(b)}{F_s \circ \phi_s(b)}\right] > 0.$$

As $F_s \circ \phi_s(\bar{b}) = F_w \circ \phi_w(\bar{b}) = 1$ and the above inequality holds, the desired result follows. We now show (B). Suppose $\alpha_B = \alpha_R \equiv \alpha$. Then, from (3), we have

$$\begin{aligned}
\frac{F_s \circ \phi_s(b)}{\mathrm{D}F_s \circ \phi_s(b)} &= p(\phi_w(b), b) - b + \alpha_R \phi_w(b) \\
&< p(\phi_w(b), b) - b + \alpha_B \phi_s(b) \\
&= \frac{F_w \circ \phi_w(b)}{\mathrm{D}F_w \circ \phi_w(b)}.
\end{aligned}$$

This implies

$$\mathrm{D}\left[\frac{F_s \circ \phi_s(b)}{F_w \circ \phi_w(b)}\right] > 0.$$

As $F_s \circ \phi_s(\bar{b}) = F_w \circ \phi_w(\bar{b}) = 1$ and the above inequality holds, the desired result follows. ■

Proof of Theorem 2. For a given t, we have $\Psi_s(t) = \phi_s \circ \phi_w^{-1}(t)$. Differentiating w.r.t. t and using (4), we have

$$\mathrm{D}\Psi_s(t) = \frac{F_s \circ \Psi_s(t)}{f_s \circ \Psi_s(t)} \frac{f_w(t)}{F_w(t)} \frac{p(t, \phi_w^{-1}(t)) - \phi_w^{-1}(t) + \alpha_B \Psi_s(t)}{p(t, \phi_w^{-1}(t)) - \phi_w^{-1}(t) + \alpha_R t}. \tag{9}$$

From H-K model, we have

$$\mathrm{D}\Theta_s(t) = \frac{F_s \circ \Theta_s(t)}{f_s \circ \Theta_s(t)} \frac{f_w(t)}{F_w(t)}.$$

Note that $\Psi_s(a_w) = \Theta_s(a_w) = a_s$. We first show (A). Suppose $\alpha_B = 0$ and $\alpha_R > 0$. Then, from (9), we have

$$\mathrm{D}\Psi_s(t) < \frac{F_s \circ \Psi_s(t)}{f_s \circ \Psi_s(t)} \frac{f_w(t)}{F_w(t)}.$$

As $\mathrm{D}\Psi_s(a_w) < f_w(a_w)/f_s(a_s) = \mathrm{D}\Theta_s(a_w)$, it follows there exists $\epsilon > 0$ such that $\Psi_s(t) > \Theta_s(t)$ for every $t \in (a_w - \epsilon, a_w)$. We show that the two bid-equivalence functions never cross each other. We show by contradiction. Suppose there exists $t^* > 0$ such that $\Psi_s(t^*) = \Theta_s(t^*)$ and $\Psi_s(t) > \Theta_s(t)$ for every $t \in (t^*, a_w)$. Then, we have

$$\begin{aligned} \mathrm{D}\Psi_s(t^*) &< \frac{F_s \circ \Psi_s(t^*)}{f_s \circ \Psi_s(t^*)} \frac{f_w(t^*)}{F_w(t^*)} \\ &= \frac{F_s \circ \Theta_s(t^*)}{f_s \circ \Theta_s(t^*)} \frac{f_w(t^*)}{F_w(t^*)} \\ &= \mathrm{D}\Theta_s(t^*). \end{aligned}$$

Therefore there exists $\delta > 0$ such that $\Psi_s(t^* + \delta) < \Theta_s(t^* + \delta)$, which is a contradiction. Hence $\Psi_s(t) > \Theta_s(t)$ for every $t \in (0, a_w)$.

We now show (B). Suppose $\alpha_B = \alpha_R \equiv \alpha$. Then, from (9) and the fact that $\Psi_s(t) > t$ (this follows from Corollary 2), we have

$$\mathrm{D}\Psi_s(t) > \frac{F_s \circ \Psi_s(t)}{f_s \circ \Psi_s(t)} \frac{f_w(t)}{F_w(t)}.$$

As $\mathrm{D}\Psi_s(a_w) > f_w(a_w)/f_s(a_s) = \mathrm{D}\Theta_s(a_w)$, it follows there exists $\epsilon > 0$ such that $\Psi_s(t) < \Theta_s(t)$ for every $t \in (a_w - \epsilon, a_w)$. We show that the two bid-equivalence functions never cross each other. We show by contradiction. Suppose there exists $t^* > 0$ such that $\Psi_s(t^*) = \Theta_s(t^*)$ and $\Psi_s(t) < \Theta_s(t)$ for every $t \in (t^*, a_w)$. Then, we have

$$\begin{aligned} \mathrm{D}\Psi_s(t^*) &> \frac{F_s \circ \Psi_s(t^*)}{f_s \circ \Psi_s(t^*)} \frac{f_w(t^*)}{F_w(t^*)} \\ &= \frac{F_s \circ \Theta_s(t^*)}{f_s \circ \Theta_s(t^*)} \frac{f_w(t^*)}{F_w(t^*)} \\ &= \mathrm{D}\Theta_s(t^*). \end{aligned}$$

Therefore there exists $\delta > 0$ such that $\Psi_s(t^* + \delta) > \Theta_s(t^* + \delta)$, which is a contradiction. Hence $\Psi_s(t) < \Theta_s(t)$ for every $t \in (0, a_w)$. ■

Proof of Theorem 3. When there are no resale markets, let the inverse bid functions be denoted by γ_s and γ_w. The characterization of equilibria is

$$\begin{aligned} \mathrm{D}\gamma_w(b) &= \frac{F_w \circ \gamma_w(b)}{f_w \circ \gamma_w(b)} \frac{1}{\gamma_s(b) - b} \\ \mathrm{D}\gamma_s(b) &= \frac{F_s \circ \gamma_s(b)}{f_s \circ \gamma_s(b)} \frac{1}{\gamma_w(b) - b} \end{aligned}$$

The above characterization can be found in Lebrun [6] and Maskin and Riley [9].

Use the above characterization to obtain

$$\mathrm{D}\Lambda_s(t) = \frac{F_s \circ \Lambda_s(t)}{f_s \circ \Lambda_s(t)} \frac{f_w(t)}{F_w(t)} \frac{\Lambda_s(t) - \gamma_w^{-1}(t)}{t - \gamma_w^{-1}(t)}.$$

Note that $\Psi_s(a_w) = \Theta_s(a_w) = a_s$ and $\Lambda_s(t) > t$ (Maskin and Riley [9], Proposition 3.5). As $\mathrm{D}\Psi_s(a_w) < f_w(a_w)/f_s(a_s) < \mathrm{D}\Lambda_s(a_w)$, it follows there exists $\epsilon > 0$ such that $\Psi_s(t) > \Lambda_s(t)$ for every $t \in (a_w - \epsilon, a_w)$. We show that the two bid-equivalence functions never cross each other. We show by contradiction. Suppose there exists $t^* > 0$ such that $\Psi_s(t^*) = \Lambda_s(t^*)$ and $\Psi_s(t) > \Theta_s(t)$ for every $t \in (t^*, a_w)$. Then, we have

$$\begin{aligned} \mathrm{D}\Psi(t^*) &< \frac{F_s \circ \Psi_s(t^*)}{f_s \circ \Psi_s(t^*)} \frac{f_w(t^*)}{F_w(t^*)} \\ &= \frac{F_s \circ \Lambda_s(t^*)}{f_s \circ \Lambda_s(t^*)} \frac{f_w(t^*)}{F_w(t^*)} \\ &< \frac{F_s \circ \Lambda_s(t^*)}{f_s \circ \Lambda_s(t^*)} \frac{f_w(t^*)}{F_w(t^*)} \frac{\Lambda_s(t^*) - \gamma_w^{-1}(t^*)}{t - \gamma_w^{-1}(t^*)} \\ &= \mathrm{D}\Lambda_s(t^*). \end{aligned}$$

Therefore there exists $\delta > 0$ such that $\Psi_s(t^* + \delta) < \Lambda_s(t^* + \delta)$, which is a contradiction. Hence $\Psi_s(t) > \Lambda_s(t)$ for every $t \in (0, a_w)$. ■

Proof of Proposition 3. We prove in two steps. In step 1, we show that $\phi_s > \mu_s$ in the neighborhood of 0. In step 2, we show that the two functions do not intersect.

When bidders are symmetric with the probability distribution pair (F_s, F_s), the equilibria is characterized as

$$\begin{aligned} \mathrm{D}\mu_s(b) &= \frac{F_s \circ \mu_s(b)}{f_s \circ \mu_s(b)} \frac{1}{\mu_s(b) - b} \\ \mu_s(0) &= 0, \mu_s(\bar{b}_s) = a_s \text{ for some } \bar{b}_s > 0. \end{aligned} \tag{10}$$

By contradiction, we show that $\phi_s > \mu_s$ in the neighborhood of 0. Suppose there exists $\epsilon > 0$ such that $\phi_s(b) \leq \mu_s(b)$ for every $b \in (0, \epsilon)$. Then, from (4) and (10), we have

$$\begin{aligned}
\frac{F_s \circ \phi_s(b)}{\mathrm{D}F_s \circ \phi_s(b)} &= p(\phi_w(b), b) - b + \alpha\phi_w(b) \\
&< (1-\alpha)\phi_s(b) - b + \alpha\phi_s(b) \\
&= \phi_s(b) - b \\
&\leq \mu_s(b) - b \\
&= \frac{F_s \circ \mu_s(b)}{\mathrm{D}F_s \circ \mu_s(b)}.
\end{aligned}$$

This implies

$$\mathrm{D}\left[\frac{F_s \circ \phi_s(b)}{F_s \circ \mu_s(b)}\right] > 0.$$

Since $\phi_s(0) = \mu_s(0) = 0$, $F_s(0) > 0$ and the above fact holds, it follows that $\phi_s(b) > \mu_s(b)$ for every $b \in (0, \epsilon)$. Therefore, $\phi_s > \mu_s$ in the neighborhood of 0.

By contradiction, we now show that the two functions do not intersect. Suppose there exists $b^* > 0$ such that $\phi_s(b^*) = \mu_s(b^*)$ and $\phi_s(b) > \mu_s(b)$ for every $b \in (0, b^*)$. Then, from (4) and (10), we have

$$\begin{aligned}
\phi_s'(b^*) &= \frac{F_s \circ \phi_s(b^*)}{f_s \circ \phi_s(b^*)} \frac{1}{p(\phi_w(b^*), b^*) - b^* + \alpha\phi_w(b^*)} \\
&> \frac{F_s \circ \phi_s(b^*)}{f_s \circ \phi_s(b^*)} \frac{1}{p(\phi_w(b^*), b^*) - b^* + \alpha\phi_s(b^*)} \\
&> \frac{F_s \circ \phi_s(b^*)}{f_s \circ \phi_s(b^*)} \frac{1}{(1-\alpha)\phi_s(b^*) - b^* + \alpha\phi_s(b^*)} \\
&= \frac{F_s \circ \phi_s(b^*)}{f_s \circ \phi_s(b^*)} \frac{1}{\phi_s(b^*) - b^*} \\
&= \frac{F_s \circ \mu_s(b^*)}{f_s \circ \mu_s(b^*)} \frac{1}{\mu_s(b^*) - b^*} \\
&= \mu_s'(b^*).
\end{aligned}$$

Thus there exists $\delta > 0$ such that $\phi_s(b^* - \delta) < \mu_s(b^* - \delta)$, a contradiction. Hence $\phi_s(b) > \mu_s(b)$ for every $b \in (0, \bar{b}]$.

■

Proof of Proposition 4. We prove in two steps. In step 1, we show that $\phi_w < \mu_w$ in the neighborhood of 0. In step 2, we show that the two functions do not intersect.

When bidders are symmetric with the probability distribution pair (F_w, F_w), the equilibria is characterized as

$$\begin{aligned}\mathrm{D}\mu_w(b) &= \frac{F_w \circ \mu_w(b)}{f_w \circ \mu_w(b)} \frac{1}{\mu_w(b) - b} \\ \mu_w(0) &= 0, \mu_w(\bar{b}_w) = a_s \text{ for some } \bar{b}_w > 0.\end{aligned} \tag{11}$$

By contradiction, we show that $\phi_w < \mu_w$ in the neighborhood of 0. Suppose there exists $\epsilon > 0$ such that $\phi_w(b) \geq \mu_w(b)$ for every $b \in (0, \epsilon)$. Then, from (4) and (11), we have

$$\begin{aligned}\frac{F_w \circ \phi_w(b)}{\mathrm{D}F_w \circ \phi_w(b)} &= p(\phi_w(b), b) - b + \alpha\phi_s(b) \\ &> (1-\alpha)\phi_w(b) - b + \alpha\phi_w(b) \\ &= \phi_w(b) - b \\ &\geq \mu_w(b) - b \\ &= \frac{F_w \circ \mu_w(b)}{\mathrm{D}F_w \circ \mu_w(b)}.\end{aligned}$$

This implies

$$\mathrm{D}\left[\frac{F_w \circ \mu_w(b)}{F_w \circ \phi_w(b)}\right] > 0.$$

Since $\phi_w(0) = \mu_w(0) = 0$, $F_w(0) > 0$ and the above fact holds, it follows that $\phi_w(b) < \mu_w(b)$ for every $b \in (0, \epsilon)$. Therefore, $\phi_w > \mu_w$ in the neighborhood of 0.

By contradiction, we now show that the two functions do not intersect. Suppose there exists $b^* > 0$ such that $\phi_w(b^*) = \mu_w(b^*)$ and $\phi_w(b) < \mu_w(b)$ for every $b \in (0, b^*)$. Then, from (4) and (11), we have

$$\begin{aligned}\phi_w'(b^*) &= \frac{F_w \circ \phi_w(b^*)}{f_w \circ \phi_w(b^*)} \frac{1}{p(\phi_w(b^*), b^*) - b^* + \alpha\phi_s(b^*)} \\ &< \frac{F_w \circ \phi_w(b^*)}{f_w \circ \phi_w(b^*)} \frac{1}{\phi_w(b^*) - b^*} \\ &= \frac{F_w \circ \mu_w(b^*)}{f_w \circ \mu_w(b^*)} \frac{1}{\mu_w(b^*) - b^*} \\ &= \mu_w'(b^*).\end{aligned}$$

Thus there exists $\delta > 0$ such that $\phi_w(b^* - \delta) > \mu_w(b^* - \delta)$, a contradiction. Hence $\phi_w(b) < \mu_w(b)$ for every $b \in (0, \mu_w^{-1}(a_w)]$. ■

Proof of 4. We first show (A). We show by contradiction. Without loss of generality, fix bidder w with value t_w who follows truth-telling strategy. Let $(1 - \alpha_R)t_w < (1 - \alpha_R)t_s < t_w < t_s$. If bidder s follows truth-telling strategy, then his optimal choice is to resell which gives him a utility of

$\alpha_R t_s + t_w - t_w = \alpha_R t_s$. We show that underbidding is desirable for bidder s. Suppose bidder s with value t_s underbids to b_s such that $(1-\alpha_R)t_w < b_s < (1-\alpha_R)t_s < t_w < t_s$. Then bidder s loses in period 1. If bidder w resells in period 2, then his utility is $\alpha_R t_w + b_s - b_s = \alpha_R t_w$. If bidder w does not resell in period 2, then his utility is $t_w - b_s$. Therefore it is optimal for bidder w to resell which incurs bidder s a utility of $t_s - b_s > \alpha_R t_s$. Thus deviation is desirable.

We now show (B). Without loss of generality, fix bidder w with value t_w who follows truth-telling strategy. Suppose bidder s with value t_s underbids to b_s such that $b_s < t_s$. There can be two potential cases—Case 1: $t_w < t_s$ and Case 2: $t_w > t_s$. In case 1, if bidder s follows truth-telling strategy, then his utility is $t_s - t_w > 0$. In case 2, if bidder s follows truth-telling strategy, then his utility is 0. We show that underbidding is not desirable for bidder s.

Consider Case 1. There can be two potential sub-cases—Case 1.1: $t_w < b_s < t_s$ and Case 1.2: $b_s < t_w < t_s$. Consider Case 1.1. Then bidder s wins in period 1. If he resells in period 2, then his utility is $\alpha t_s + (1-\alpha)t_w - t_w = \alpha(t_s - t_w)$. If he does not resell in period 2, then his utility is $t_s - t_w$. Therefore it is optimal for bidder s not to resell which incurs him a utility of $t_s - t_w$. Hence deviation from t_s is not desirable in this case.

Consider case 1.2. Then bidder s loses in period 1. If bidder w resells in period 2, then his utility is $\alpha t_w + (1-\alpha)b_s - b_s = \alpha(t_w - b_s)$. If he does not resell in period 2, then his utility is $t_w - b_s$. Therefore it is optimal for bidder w not to resell which incurs bidder s a utility of 0. Hence deviation from t_s is not desirable in this case.

Consider case 2. Then $b_s < t_s < t_w$ implies that bidder s loses in period 1. If bidder w resells in period 2, then his utility is $\alpha t_w + (1-\alpha)b_s - b_s = \alpha(t_w - b_s)$. If he does not resell in period 2, then his utility is $t_w - b_s$. Therefore it is optimal for bidder w not to resell which incurs bidder s a utility of 0. Hence deviation from t_s is not desirable in this case.

Now, suppose bidder s with value t_s overbids to b_s such that $b_s > t_s$. There can be two potential cases—Case 1: $t_w < t_s$ and Case 2: $t_w > t_s$. In case 1, if bidder s follows truth-telling strategy, then his utility is $t_s - t_w > 0$. In case 2, if bidder s follows truth-telling strategy, then his utility is 0. We show that overbidding is not desirable for bidder s. Consider case 1. Then $t_w < t_s < b_s$ implies that bidder s wins in period 1. If he resells in period 2, then his utility is $\alpha t_s + (1-\alpha)t_w - t_w = \alpha(t_s - t_w)$. If he does not resell in period 2, then his utility is $t_s - t_w$. Therefore it is optimal for bidder s not to resell which incurs him a utility of $t_s - t_w$. Hence deviation from t_s is not desirable in this case.

Consider case 2. There can be two potential sub-cases—Case 2.1: $t_w > b_s > t_s$ and Case 2.2: $b_s > t_w > t_s$. Consider case 2.1. Then bidder s loses in period 1. If bidder w resells in period 2, then his utility is $\alpha t_w + (1-\alpha)b_s - b_s = \alpha(t_w - b_s)$. If he does not resell in period 2, then his utility is

$t_w - b_s$. Therefore it is optimal for bidder w not to resell which incurs bidder s a utility of 0. Hence deviation from t_s is not desirable in this case.

Consider case 2.2. Then bidder s wins in period 1. If he resells in period 2, then his utility is $\alpha t_s + (1-\alpha)t_w - t_w = \alpha(t_s - t_w)$. If he does not resell in period 2, then his utility is $t_s - t_w$. Therefore it is optimal for bidder s is to resell which incurs him a utility of $\alpha(t_s - t_w) < 0$. Hence deviation from t_s is not desirable in this case. ∎

Proof of Proposition 5. We can rewrite (4) as

$$\begin{aligned}
\phi_s'(b) &= \frac{\phi_s(b)}{p(\phi_w(b), b) - b + \alpha\phi_s(b)} \\
\phi_w'(b) &= \frac{\tau_w\phi_w(b)}{p(\phi_w(b), b) - b + \alpha\phi_w(b)} \\
(1-\alpha)\phi_w(b) &= p(\phi_w(b), b) - \frac{\phi_s(b) - zp(\phi_w(b), b)}{z}.
\end{aligned}$$

Let $\phi_s(b) = \bar{\phi}_s b$, $\phi_w(b) = \bar{\phi}_w b$ and $p(\phi_w(b), b) = \bar{p}b$. Then, from the above system of equations, we have

$$\bar{p} = 2 - \alpha\bar{\phi}_w = 1 + \tau_w - \alpha\bar{\phi}_s, \quad \bar{\phi}_s + \bar{\phi}_w = 2z\bar{p}.$$

Solving the above equations, we get the desired result. ∎

Proof of Lemma 1. The seller's expected revenues in the first-price auction are

$$R^I = \int_0^{\bar{b}} bF_w \circ \phi_w(b) F_s \circ \phi_s(\mathrm{d}b) + \int_0^{\bar{b}} bF_s \circ \phi_s(b) F_w \circ \phi_w(\mathrm{d}b)$$

The first term is the expected revenue generated from bidder w and the second term is the expected revenue generated from bidder s. Using the expressions of inverse bid functions from Proposition 5, we have

$$\begin{aligned}
R^I &= \int_0^{\bar{b}} \frac{3\alpha + \alpha\tau_w + \tau_w - 1}{2\alpha a_s} \frac{1}{\tau_w} \left(\frac{3\alpha + \alpha\tau_w + 1 - \tau_w}{2\alpha}\right)^{\frac{1}{\tau_w}} b^{1+\frac{1}{\tau_w}} \mathrm{d}b \\
&\quad + \int_0^{\bar{b}} \left(\frac{3\alpha + \alpha\tau_w + 1 - \tau_w}{2\alpha}\right)^{\frac{1}{\tau_w}} \frac{3\alpha + \alpha\tau_w + \tau_w - 1}{2\alpha a_s} b^{1+\frac{1}{\tau_w}} \mathrm{d}b \\
&= \frac{3\alpha + \alpha\tau_w + \tau_w - 1}{2\alpha a_s} \left(\frac{3\alpha + \alpha\tau_w + 1 - \tau_w}{2\alpha}\right)^{\frac{1}{\tau_w}} \left(1 + \frac{1}{\tau_w}\right) \int_0^{\bar{b}} b^{1+\frac{1}{\tau_w}} \mathrm{d}b \\
&= \frac{1 + \tau_w}{2\tau_w + 1} \frac{3\alpha + \alpha\tau_w + \tau_w - 1}{2\alpha a_s} \left(\frac{3\alpha + \alpha\tau_w + 1 - \tau_w}{2\alpha}\right)^{\frac{1}{\tau_w}} \bar{b}^{\frac{2\tau_w + 1}{\tau_w}}
\end{aligned}$$

Using the value of $\bar{b} = (3\alpha + \alpha\tau_w + 1 - \tau_w)/(2\alpha)$ and the fact that

$$\frac{3\alpha + \alpha\tau_w + 1 - \tau_w}{2\alpha} = \frac{3\alpha + \alpha\tau_w + \tau_w - 1}{2\alpha a_s},$$

we have

$$
\begin{aligned}
R^I &= \frac{1+\tau_w}{2\tau_w+1}\frac{3\alpha+\alpha\tau_w+1-\tau_w}{2\alpha}\left(\frac{3\alpha+\alpha\tau_w+1-\tau_w}{2\alpha}\right)^{\frac{1}{\tau_w}} \\
&\quad \left(\frac{3\alpha+\alpha\tau_w+1-\tau_w}{2\alpha}\right)^{\frac{2\tau_w+1}{\tau_w}} \\
&= \frac{1+\tau_w}{2\tau_w+1}\frac{2\alpha}{3\alpha+\alpha\tau_w+1-\tau_w}.
\end{aligned}
$$

The seller's expected revenues in the second-price auction are

$$
\begin{aligned}
R^{II} &= \int_0^1 [1-F_s(t)][1-F_w(t)]\mathrm{d}t \\
&= \int_0^1 \left(1-\frac{t}{a_s}\right)(1-t^{\frac{1}{\tau_w}})\mathrm{d}t \\
&= \int_0^1 \left(1-t^{\frac{1}{\tau_w}}-\frac{t}{a_s}+\frac{t^{\frac{1+\tau_w}{\tau_w}}}{a_s}\right)\mathrm{d}t \\
&= \frac{1}{1+\tau_w}-\frac{1}{2a_s(2\tau_w+1)} \\
&= \frac{1}{1+\tau_w}-\frac{3\alpha+1+\alpha\tau_w-\tau_w}{2(2\tau_w+1)(3\alpha-1+\alpha\tau_w+\tau_w)}.
\end{aligned}
$$

■

Proof of Theorem 5. Let $\Delta(\alpha,\tau_w) := R^I(\alpha,\tau_w) - R^{II}(\alpha,\tau_w)$. We show in two steps. In step 1, we show that $\Delta(1,\tau_w) > 0$. In step 2, we show that $\mathrm{D}_\alpha\Delta(\alpha,\tau_w) < 0$ for every $\alpha < 1$.

From (8), we have

$$
R^I(1,\tau_w) = \frac{1+\tau_w}{2(2\tau_w+1)}, \quad R^{II}(1,\tau_w) = \frac{2\tau_w}{(2\tau_w+1)(\tau_w+1)}.
$$

This implies

$$
\begin{aligned}
\Delta(1,\tau_w) &= \frac{1+\tau_w}{2(2\tau_w+1)} - \frac{2\tau_w}{(2\tau_w+1)(\tau_w+1)} \\
&= \frac{(\tau_w-1)^2}{2(2\tau_w+1)(\tau_w+1)} > 0.
\end{aligned}
$$

Fix any $\alpha < 1$. From (8), we have

$$
\begin{aligned}
\Delta(\alpha,\tau_w) = &\frac{2\alpha(1+\tau_w)}{(2\tau_w+1)(3\alpha+1+\tau_w\alpha-\tau_w)} + \frac{3\alpha+1+\tau_w\alpha-\tau_w}{2(2\tau_w+1)(3\alpha-1+\tau_w\alpha+\tau_w)} \\
&- \frac{1}{\tau_w+1}.
\end{aligned}
$$

The first-order derivative w.r.t. α gives

$$\begin{aligned}
\mathrm{D}_\alpha \Delta(\alpha, \tau_w) &= \frac{\tau_w - 1}{2\tau_w + 1}\left[\frac{3+\tau_w}{(3\alpha - 1 + \tau_w\alpha + \tau_w)^2} - \frac{2(1+\tau_w)}{(3\alpha + 1 + \tau_w\alpha - \tau_w)^2}\right] \\
&< \frac{\tau_w - 1}{2\tau_w + 1}\left[\frac{3+\tau_w}{(3\alpha + 1 + \tau_w\alpha - \tau_w)^2} - \frac{2(1+\tau_w)}{(3\alpha + 1 + \tau_w\alpha - \tau_w)^2}\right] \\
&= \frac{-(\tau_w - 1)^2}{(3\alpha + 1 + \tau_w\alpha - \tau_w)^2} \\
&< 0.
\end{aligned}$$

Therefore, $\Delta(\alpha, \tau_w) > 0$ for every $\alpha < 1$. ■

B Appendix B: Lemmas

Lemma B.1. *$\phi_s(0) = \phi_w(0) = 0$ and there exists $\bar{b} > 0$ such that $\phi_s(\bar{b}) = a_s$ and $\phi_w(\bar{b}) = a_w$.*

Proof. We show $\phi_s^{-1}(0) = \phi_w^{-1}(0) = 0$. We show in two steps. In step 1, we show $\phi_s^{-1}(0) = \phi_w^{-1}(0)$. In step 2, we show $\phi_s^{-1}(0) = \phi_w^{-1}(0) = 0$.

We begin by showing $\phi_s^{-1}(0) = \phi_w^{-1}(0)$. For contradiction, without loss of generality, assume $\phi_w^{-1}(0) > \phi_s^{-1}(0)$. Consider bidder w with value 0. If he wins, then he will try to resell at price p such that $p > \phi_w^{-1}(0) - \alpha_R.0 > \phi_s^{-1}(0)$. Consider bidder s with value zp where $z = 1/(1 - \alpha_B)$. If he accepts the offer, then his utility is $(1 - \alpha_B)zp - p = 0$. Therefore, he is indifferent between accepting and not accepting the offer which gives him a utility of 0. However, if he deviates from his bid to $\phi_w^{-1}(0) + \epsilon$ such that $0 < \epsilon < p - \phi_w^{-1}(0)$, then it is profitable for him as he now gets a utility of $zp - \phi_w^{-1}(0) - \epsilon > 0$. Thus, $\phi_s^{-1}(0) = \phi_w^{-1}(0)$.

We now show $\phi_s^{-1}(0) = \phi_w^{-1}(0) = 0$. For contradiction, assume that $\phi_s^{-1}(0) = \phi_w^{-1}(0) > 0$. Without loss of generality, consider bidder w with value 0. If he wins, then he will make a resale offer of p such that $p > \phi_w^{-1}(0) - \alpha_R.0 > 0$; otherwise the object remains with him which gives a utility of $0 - \phi_w^{-1}(0) < 0$. Also, if he wins, then it must be the case that $\phi_w^{-1}(0) \geq \phi_s^{-1}(t_s)$, or, equivalently $t_s \leq \phi_s \circ \phi_w^{-1}(0)$. Since $\phi_s^{-1}(0) = \phi_w^{-1}(0)$, it follows that $t_s = 0$. This implies that the offer is not accepted by bidder s as his utility would have been $(1-\alpha_B).0-p < 0$. On the other hand, if bidder w loses the auction, then it must be the case that $t_s \geq \phi_s \circ \phi_w^{-1}(0) = 0$. If bidder s makes a resale offer, then the resale price p is such that $p > (1-\alpha_R)t_s \geq 0$. Therefore, the offer will not be accepted by bidder w which gives him a utility of 0. Thus, the expected utility of bidder w is negative. Hence, $\phi_s^{-1}(0) = \phi_w^{-1}(0) = 0$.

We show $\phi_s^{-1}(a_s) = \phi_w^{-1}(a_w) \equiv \bar{b}$. For contradiction, without loss of generality, assume that $\phi_w^{-1}(a_w) > \phi_s^{-1}(a_s)$. Consider bidder w with value a_w. Then, he wins with certainty. If he resells, then the resale price p is such that $p > (1-\alpha_R)a_w$. If his offer is accepted, then he gets a utility of $\alpha_R a_w + p - \phi_w^{-1}(a_w)$. For a small enough $\epsilon > 0$, a small downward deviation from $\phi_w^{-1}(a_w)$ to $\phi_w^{-1}(a_w) - \epsilon$ still guarantees his win, but increases his utility. If his offer is rejected or he does not make a resale offer, then the object remains with him and gives a utility of $a_w - \phi_w^{-1}(a_w)$. Again, a small downward deviation is desirable. Therefore, $\phi_s^{-1}(a_s) = \phi_w^{-1}(a_w) \equiv \bar{b}$. ■

Lemma B.2. *Suppose Assumption (A2) is satisfied. Then*

$$\frac{zf_s(zp)}{F_s \circ \phi_s(b) - F_s(zp)}$$

is non-decreasing in p.

Proof. Let $p, q \in \Re_{++}$ such that $p > q$. From assumption A2, it follows that

$$\frac{f_s(p)}{1-F_s(p)} \geq \frac{f_s(q)}{1-F_s(q)},$$

or, equivalently

$$f_s(zp)[1-F_s(zq)] \geq f_s(zq)[1-F_s(zp)].$$

We show

$$\frac{f_s(zp)}{F_s \circ \phi_s(b) - F_s(zp)} \geq \frac{f_s(zq)}{F_s \circ \phi_s(b) - F_s(zq)},$$

or, equivalently

$$f_s(zp)[F_s \circ \phi_s(b) - F_s(zq)] \geq f_s(zq)[F_s \circ \phi_s(b) - F_s(zp)].$$

We consider two cases. Case 1: $f_s(p) > f_s(q)$ and Case 2: $f_s(p) \leq f_s(q)$. Consider case 1. This implies $f_s(zp) > f_s(zq)$. Since $F_s \circ \phi_s(b) - F_s(zq) > F_s \circ \phi_s(zp) - F_s(zp)$, the result follows.

Now consider case 2. This implies $f_s(zp) \leq f_s(zq)$. Let

$$A := f_s(zp)[F_s \circ \phi_s(b) - F_s(zq)] - f_s(zq)[F_s \circ \phi_s(b) - F_s(zp)].$$

Then $(\mathrm{d}/\mathrm{d}F_s \circ \phi_s(b))A = f_s(zp) - f_s(zq) \leq 0$. Since $F_s \circ \phi_s(b) < 1$, it follows that

$$A \geq f_s(zp)[1-F_s(zq)] - f_s(zq)[1-F_s(zp)] \geq 0.$$

■

Lemma B.3. *Suppose Assumption (A2) is satisfied. Then the unique value p that solves (1) is the maximum.*

Proof. We have

$$\mathrm{D}_p U_w(t_w, p, b) = -z f_s(zp)\Big[p - (1-\alpha_R)t_w - \frac{F_s \circ \phi_s(b) - F_s(zp)}{z f_s(zp)}\Big].$$

In equilibrium $\mathrm{D}_p U_w(t_w, p, b) = 0$. The second-order derivative is

$$\begin{aligned}
\mathrm{D}^2_{pp} U_w(t_w, p, b) &= -z^2 f_s'(zp)\Big[p - (1-\alpha_R)t_w - \frac{F_s \circ \phi_s(b) - F_s(zp)}{z f_s(zp)}\Big] - \\
&\qquad z f_s(zp)\mathrm{D}_p\Big[p - (1-\alpha_R)t_w - \frac{F_s \circ \phi_s(b) - F_s(zp)}{z f_s(zp)}\Big] \\
&= -z f_s(zp) + z f_s(zp)\mathrm{D}_p\Big[\frac{F_s \circ \phi_s(b) - F_s(zp)}{z f_s(zp)}\Big] \\
&< 0.
\end{aligned}$$

■

Lemma B.4. *Suppose, for some $b^* > 0$, $(1-\alpha_R)\phi_s(b^*) = (1-\alpha_B)\phi_w(b^*)$. Then (2) and (3) are satisfied.*

Proof. First, let us consider the one-sided case. The equality is equivalent to $(1-\alpha_R)\phi_s(b^*) = \phi_w(b^*)$. Suppose bidder s with value t_s wins the auction by bidding b^*. Then $t_s = \phi_s(b^*)$ and $b^* > \phi_w^{-1}(t_w)$. This implies $t_w < \phi_w \circ \phi_s^{-1}(t_s) = (1-\alpha_R)t_s$. Hence there are no gains from trade if bidder s makes a resale offer. Now, suppose bidder w with value t_w wins the auction by bidding b^*. Note that $(1-\alpha_R)\phi_s(b^*) = \phi_w(b^*)$ implies $\phi_s(b^*) > (1-\alpha_R)\phi_w(b^*)$. We have $t_w = \phi_w(b^*)$ and $b^* > \phi_s^{-1}(t_s)$. This implies $t_s < \phi_s \circ \phi_w^{-1}(t_w)$ and $(1-\alpha_R)t_w < \phi_s \circ \phi_w^{-1}(t_w)$. Therefore, with positive probability, $(1-\alpha_R)t_w < t_s$. Hence there are gains from trade if bidder s makes a resale offer. In this case, bidder w makes a resale offer of $p(\phi_w(b^*), b^*)$ such that $\phi_s(b^*) > p(\phi_w(b^*), b^*) > (1-\alpha_R)\phi_w(b^*)$.

Second, let us consider the two-sided-case. The equality is equivalent to $\phi_s(b^*) = \phi_w(b^*)$. Suppose bidder s with value t_s wins the auction by bidding b^*. Then $t_s = \phi_s(b^*)$ and $b^* > \phi_w^{-1}(t_w)$. This implies $t_w < \phi_w \circ \phi_s^{-1}(t_s) = t_s$. Hence there are no gains from trade if bidder s makes a resale offer. Suppose bidder w with value t_w wins the auction by bidding b^*. Then $t_w = \phi_w(b^*)$ and $b^* > \phi_s^{-1}(t_s)$. This implies $t_s < \phi_s \circ \phi_w^{-1}(t_w) = t_w$. Hence there are no gains from trade if bidder w makes a resale offer. In this case, without loss of generality, we assume that whoever wins shall make a resale offer of $p(\phi_w(b^*), b^*) = (1-\alpha)\phi_s(b^*) = (1-\alpha)\phi_w(b^*)$. ■

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