Stochastic Optimal Growth through State-Dependent Probabilities *

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Abstract

We extend the classical discrete time stochastic one-sector optimal growth model with logarithmic utility and Cobb-Douglas production á-la Brock and Mirman (1972) to allow probabilities to be state-dependent. In this setting the probability of occurrence of a given shock depends on the capital stock, thus as the economy accumulates more capital the probability of occurrence of different shocks changes over time. We explicitly determine the optimal policy and its relation with state-dependent probabilities in two alternative scenarios in which the probability function, assumed to take a logarithmic form, is either decreasing or increasing with capital. We show that, by affecting the optimal policy, state-dependent probabilities act as an engine of capital accumulation, which, through its effects on the probability of shocks realization, impacts the evolution of economic inequality. In particular, whenever the probability is decreasing (increasing) in the capital stock the probability of the most (least) favorable shock increases, and this incentivizes the planner to increase (decrease) his capital investment, which in turn will generate a widening (reduction) in economic inequalities over time. We then show that the optimal solution can be converted into an affine iterated function system with affine state-dependent probabilities which converges to an invariant self-similar measure supported on a compact (eventually fractal) attractor. We also characterize the properties of such an invariant self-similar measure in terms of singularity and absolutely continuity with respect to the Lebesgue measure, which ultimately depends on the magnitude of the capital share.

Keywords: Brock and Mirman Model; Iterated Function Systems; Optimal Growth; State-Dependent Probabilities
JEL Classification: C61, C63, O40

*In memory of Tapan Mitra, with whom Fabio Privileggi coauthored some early pioneering works on stochastic one-sector optimal growth models and the possible appearance of Cantor-like attractors supporting singular invariant probability distributions.

1 Introduction

Over the last decades, following the seminal work by Brock and Mirman (1972), a large and growing number of works has tried to characterize the implications of stochasticity on macroeconomic dynamics and economic growth (see Olson and Roy, 2005, for a survey). Several of these studies analyze the eventual fractal nature of the steady state in traditional macroeconomic frameworks, which are now well known to give rise to random dynamics possibly converging to invariant measures supported on fractal sets (Montrucchio and Privileggi, 1999). Indeed, in a classical discrete time one-sector Ramsey (1928) model with logarithmic utility and Cobb-Douglas production in which output is affected by binary random shocks, the optimal economic dynamics can be converted into affine iterated function systems converging to invariant probability measures, which may turn out to be either singular and supported on a Cantor-like set or absolutely continuous (Montrucchio and Privileggi, 1999; Mitra et al., 2003; Mitra and Privileggi, 2004; 2006; 2009; La Torre et al., 2015). Several extensions of the standard setup have been developed over the years in order to consider multisector frameworks, to allow for sustained endogenous growth, to permit shocks to affect factor shares, and to account for pollution externalities, showing that even in such contexts similar results apply apart from the fact that the support of the invariant probability measure may be some other fractal set, like the Sierpinski gasket or the Barnsley's fern (La Torre et al., 2011, 2015, 2018b, 2018c).

To the best of our knowledge, all the refinements and extensions of the classical stochastic optimal growth model rely upon the assumption that the probability with which shocks occur is constant. Even if this setting is useful to characterize macroeconomic dynamics in a simple and intuitive way, it limits the analysis of the implications of important phenomena like economic inequality. Economic inequalities are a pervasive phenomenon in modern economies and its mutual relation with macroeconomic outcomes has been known since Kuznets (1955). Several studies discuss the extent to which inequalities, both within and between countries, have largely widened over the last decades generating important social problems (see Banerjee and Duflo, 2003; Piketty, 2015; Scheidel, 2017). The recent economic history clearly shows that economic inequality, and in particular income inequality, is characterized by path-dependency: the richer (poorer) countries or individuals initially are, the richer (poorer) they will tend to become over time (Piketty and Saez, 2014; Alvaredo et al., 2017). Understanding thus the implications of path-dependency for inequalities and macroeconomic outcomes is crucial to develop a realistic theory of economic development. This paper wishes to make a first contribution in this direction by extending the classical optimal stochastic growth model to allow probabilities to be state-dependent, that is to depend on the level of the capital stock, giving rise to path-dependency and eventually economic inequalities. State-dependent probabilities are a straightforward (but nontrivial) generalization of constant probabilities which allow to explain the path-dependency phenomenon and to enrich the set of possible model's outcomes, shedding some lights on the mutual links between macroeconomic dynamics and economic inequalities (see, *e.g.*, Cozzi and Privileggi, 2009).

Specifically, we extend the classical discrete time stochastic one-sector optimal growth model with logarithmic utility and Cobb-Douglas production á-la Brock and Mirman (1972) to allow probabilities to be state-dependent. We assume that the probability of occurrence of different shocks depends on the capital stock, and thus as the economy accumulates capital the probability of realization of a given stock endogenously changes. We consider the state-dependent probability to be a monotonic function of capital, analyzing how results may change in situations in which the probability increases or decreases with capital. By assuming that the probability function takes a logarithmic form, we are able to explicitly characterize the optimal solution of such an extended optimal growth model, showing that by affecting the optimal policy state-dependent probabilities act as an engine of capital accumulation, which through its effects on the probability of shocks realization crucially drives the evolution of economic inequality. In particular, whenever the probability is decreasing (increasing) in the capital stock the probability of the most (least) favorable shock increases, and this incentivize the planner to increase (decrease) capital investment, which in turn will generate a widening (reduction) in economic inequalities over time. This result generalizes those traditionally discussed in the stochastic optimal growth literature (Brock and Mirman, 1972; Montrucchio and Privileggi, 1999; Mitra et al., 2003), as the optimal policy boils down to the standard policy under constant probability whenever the probability function does not depend on the capital stock. We also show that the optimal dynamics can be converted into a contractive affine iterated function system (IFS) with affine state-dependent probabilities (SDP) which, under rather general conditions, converges to an invariant self-similar measure supported on a (possibly fractal) compact attractor. This result generalizes those presented in the fractal steady state and stochastic optimal growth literature (Montrucchio and Privileggi, 1999; Mitra et al., 2003; La Torre et al., 2015), which has shown that under constant probabilities the optimal dynamics can be transformed in a traditional IFS (without state-dependent-probabilities). Moreover, we present a new result, more general than those discussed in extant literature (Mitra et al., 2003; Shmerkin, 2014), determining sufficient conditions for the invariant self-similar measure associated with our affine iterated function system with state-dependent probabilities (IFSSDP) to be either singular or absolutely continuous with respect to the Lebesgue measure, showing that this ultimately depends on the magnitude of the capital share.

Despite the fact that the probability of shocks realization may depend on the level of some state variable is a very intuitive and natural framework to consider, the role of state-dependent probabilities has not been explored in depth thus far. State-dependent probabilities and in particular IFSSDP have received much attention in the mathematics literature (Barnsley et al., 1988; Stenflo, 2002), but they have only seldom been discussed in economics (La Torre et al., 2019). To the best of our knowledge, the only paper analyzing the role of state-dependent probabilities in an economic setup is La Torre et al.'s (2019). They discuss the implications of state-dependent probabilities on the possible steady state outcome in a purely dynamic economic growth model with health capital (abstracting completely from optimizing behavior) in which the probability of shocks depends on the relative abundance of health capital with respect to physical capital. Unlike them we consider an optimal growth framework in which the social planner endogenously determines the level of investment in capital accumulation, which thus requires him to account for how the future capital level will impact the probability of shocks occurrence.

The paper proceeds as follows. Section 2 reviews some well-known concepts on the IFS theory and it focuses in particular on the theory of IFSSDP, deriving also a novel theoretical result which will allow us to characterize singularity vs. absolute continuity of the self-similar measure associated with the IFSSDP derived from our stochastic optimal growth model. Section 3 introduces our extended Brock and Mirman's (1972) model with state-dependent probabilities, distinguishing between situations in which the probabilities are either decreasing or increasing with the capital stock. Section 4 derives the optimal solution of such a model discussing how the optimal policy changes (with respect to the standard one under constant probabilities) because of the presence of state-dependent probabilities. It also shows that the optimal dynamics can be converted into an affine IFSSDP from which it is possible to investigate the singularity vs. absolute continuity properties of the self-similar measure associated with such an IFSSDP. Section 5 presents some specific examples by numerically approximating the invariant distribution of our IFSSDP to illustrate how it may change under different assumptions regarding the shape of the probability function. Section 6 as usual presents concluding remarks and highlights directions for future research. All the proofs of our main results are presented in appendix A.

2 Iterated Function Systems

We now review some basic concepts and the main results in the IFS theory. We first recall the case of IFS with constant probabilities (see also Kunze et al., 2012), and then we move to the case of IFS with state-dependent probabilities discussing with more depth the implications of such an extension. In particular, we discuss more thoroughly a sufficient condition for the existence of a unique fixed point for state-dependent probabilities and we present two novel examples that show the existence of multiple equilibria. Then we provide a new result regarding the properties (in terms of singularity vs. absolute continuity) of the self-similar measure associated with IFS with state-dependent probabilities in the case of affine maps (Theorem 4), which will allow us in the next sections to characterize the steady state of our stochastic optimal growth model under state-dependent probabilities.

2.1 Constant Probabilities

In the following, we denote by (X, d) a compact metric space. An N-map *Iterated Function System* (IFS) on X, $\mathbf{w} = \{w_1, \ldots, w_N\}$, is a set of N contraction mappings on X, *i.e.*, $w_i : X \to X$, $i = 1, \ldots, N$, with contraction factors $c_i \in [0, 1)$ (see Hutchinson, 1981; Barnsley, 1989; Kunze et al., 2012). Associated with an N-map IFS, there is the following set-valued mapping $\hat{\mathbf{w}}$ defined on the space $\mathcal{H}(X)$ of nonempty compact subsets of X:

$$\hat{\mathbf{w}}(S) := \bigcup_{i=1}^{N} w_i(S), \qquad S \in \mathcal{H}(X).$$

For any pair of elements $\mathcal{H}(X)$ the distance between them is measured by means of the classical Hausdorff distance h defined as:

$$h(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x, y), \sup_{x \in B} \inf_{y \in A} d(x, y) \right\}.$$

It can be proved that $(\mathcal{H}(X), h)$ is a complete metric space (Hutchinson, 1981; Barnsley, 1989). One reason for choosing the Hausdorff distance is that it allows the contractivity of the IFS maps w_i to be translated into contractivity of $\hat{\mathbf{w}}$ on $(\mathcal{H}(X), h)$, that is, for $A, B \in \mathcal{H}(X)$, the following holds (see Hutchinson, 1981):

$$h(\hat{\mathbf{w}}(A), \hat{\mathbf{w}}(B)) \leq cH(A, B) \qquad \text{where } c = \max_{1 \leq i \leq N} c_i < 1.$$

This result implies the existence and uniqueness of a fixed point A such that $\hat{\mathbf{w}}(A) = A$. Moreover, A is self-similar, that is, it is the union of distorted copies of itself. It is possible to prove (see Hutchinson, 1981) that there exists a unique set $A \in \mathcal{H}(X)$, the *attractor* of the IFS $\mathbf{w}$, such that:

$$A = \hat{\mathbf{w}}(A) = \bigcup_{i=1}^{N} w_i(A).$$

Moreover, for any $B \in \mathcal{H}(X)$, $h(A, \hat{\mathbf{w}}^t B) \to 0$ as $t \to \infty$.

An N-map *iterated function system with (constant) probabilities* $(\mathbf{w}, \mathbf{p})$ is an N-map IFS $\mathbf{w}$ with associated probabilities $\mathbf{p} = \{p_1, \ldots, p_N\}$, $\sum_{i=1}^N p_i = 1$. Denote by $\mathcal{M}(X)$ the set of probability measures on (Borel subsets of) X. Then an N-map IFSP induces a *Markov operator* $M : \mathcal{M} \to \mathcal{M}$ defined as follows. For any $\mu \in \mathcal{M}(X)$ and any measurable set $S \subset X$, let us denote by $\nu(S) = (M\mu)(S)$ the following set function:

$$\nu(S) = (M\mu)(S) = \sum_{i=1}^{N} p_i \mu\left(w_i^{-1}(S)\right).$$

We introduce the *Monge-Kantorovich* distance on $\mathcal{M}(X)$, defined as follows. For any pair of probability measures $\mu, \nu \in \mathcal{M}(X)$, we have

$$d_{MK}(\mu, \nu) = \sup_{f \in Lip_1(X)} \left[\int f d\mu - \int f d\nu\right],$$

where $Lip_1(X) = \{f : X \to \mathbb{R} : |f(x) - f(y)| \leq d(x, y)\}$. Again, we can prove that the metric space $(\mathcal{M}(X), d_{MK})$ is complete (Hutchinson, 1981; Barnsley, 1989).

As with the Hausdorff distance, a very useful feature of the Monge-Kantorovich metric is that it results in M being a contraction mapping over $\mathcal{M}(X)$. It is possible to prove (see Hutchinson, 1981) that for $\mu, \nu \in \mathcal{M}(X)$,

$$d_{MK}(M\mu, M\nu) \leq c d_{MK}(\mu, \nu).$$

This implies that there exists a unique measure $\bar{\mu} \in \mathcal{M}(X)$, the *invariant measure* of the IFSP $(\mathbf{w}, \mathbf{p})$, such that:

$$\bar{\mu}(S) = (M\bar{\mu})(S) = \sum_{i=1}^{N} p_i \bar{\mu}\left(w_i^{-1}(S)\right) \quad \text{where} \quad c = \max_{1 \leq i \leq N} c_i.$$

Moreover, for any $\nu \in \mathcal{M}(X)$, $d_{MK}(\bar{\mu}, M^t \nu) \to 0$ as $t \to \infty$.

There is a link between the invariant set of an IFS with the invariant probability measure of an IFSP. It can be proved (see Hutchinson, 1981) that the support of the invariant measure $\bar{\mu}$ of an N-map IFSP $(\mathbf{w}, \mathbf{p})$ is the attractor A of the IFS $\mathbf{w}' = \{w_i : p_i > 0\}$, *i.e.*,

$$\operatorname{supp} \bar{\mu} = A.$$

For instance, the following two-map IFS on $X = [0, 1]$,

$$w_1(x) = \frac{1}{3}x, \qquad w_2(x) = \frac{1}{3}x + \frac{2}{3},$$

has attractor the Cantor set $C \subset [0, 1]$. Let $p_1 \equiv p_2 \equiv 1/2$. It is well known that the invariant measure $\bar{\mu}$ of this IFSP is a (uniform) measure on the Cantor set whose cumulative distribution function is the famous *Devil's staircase* function.

We note that while we assume in this paper that all the maps w_i in the IFS are contractive, the contractivity of M only requires *average contractivity*, $\sum_i p_i c_i < 1$. Moreover, it is also possible to relax the compactness condition on X.

2.2 State-Dependent Probabilities

We now consider the case in which the probabilities, $p_i, 1 \leq i \leq N$, associated with an N-map IFS $\mathbf{w}$ are state-dependent, *i.e.*, $p_i : X \to [0,1]$ such that:

$$\sum_{i=1}^{N} p_i(x) = 1, \qquad \text{for all } x \in X. \tag{1}$$

The result is an N-map *IFS with state-dependent probabilities* (IFSSDP).

Example 1 (Affine probability functions) *In the special case $X = [0,1] \subset \mathbb{R}$ with affine probabilities $p_i(x) = \alpha_i x + \beta_i$, substitution into (1) along with the fact that the functions x and 1 are linearly independent over [0,1] yields the following conditions on the α_i and β_i,*

$$\sum_{i=1}^{N} \alpha_i = 0\,, \qquad \sum_{i=1}^{N} \beta_i = 1.$$

Only two other conditions must be imposed, namely, (i) $0 \leq p_i(0) \leq 1$ and $0 \leq p_i(1) \leq 1$ for $1 \leq i \leq N$, which lead to the following additional constraints,

$$0 \leq \beta_i \leq 1, \qquad 0 \leq \alpha_i + \beta_i \leq 1, \qquad 1 \leq i \leq N.$$

These constraints also imply that $-1 \leq \alpha_i \leq 1$. In the special case $\alpha_i = 0$, $1 \leq i \leq N$, the IFSSDP reduces to an IFSP with constant probabilities $p_i = \beta_i$, $1 \leq i \leq N$.

The Markov operator $M : \mathcal{M}(X) \to \mathcal{M}(X)$ associated with an N-map IFSSDP, $(\mathbf{w}, \mathbf{p})$, is defined as:

$$\nu(S) = M\mu(S) = \sum_i \int_{w_i^{-1}(S)} p_i(x)\, d\mu(x)\,, \tag{2}$$

where $\mu \in \mathcal{M}(X)$ and $S \subset X$ is a Borel set.

Theorem 1 (La Torre et al., 2018a) *Given M as defined in equation (2), then M maps $\mathcal{M}(X)$ to itself. In other words, if $\mu \in \mathcal{M}(X)$, then $\nu = M\mu \in \mathcal{M}(X)$.*

Under appropriate conditions, the above Markov operator can be contractive with respect to the Monge-Kantorovich metric.

Theorem 2 (La Torre et al., 2018a) *Let (X, d) be a compact metric space and $(\mathbf{w}, \mathbf{p})$ an N-map IFSSDP with IFS maps $w_i : X \to X$ with contraction factors $c_i \in [0,1)$. Furthermore, assume that the probabilities $p_i : X \to \mathbb{R}$ are Lipschitz functions, with Lipschitz constants $K_i \geq 0$. Let $M : \mathcal{M}(X) \to \mathcal{M}(X)$ be the Markov operator associated with this IFSSDP, as defined in (2). Then for any $\mu, \nu \in \mathcal{M}(X)$,*

$$d_{MK}(M\mu, M\nu) \leq (c + KDN)\, d_{MK}(\mu, \nu)\,,$$

where $c = \max_i c_i$, $K = \max_i K_i$ and $D = \operatorname{diam}(X) < \infty$.

We note, however, that it is not necessary for M to be contractive with respect to the Monge-Kantorovich metric in order to have a fixed point. In fact, by the Schauder fixed point theorem, as long as all the $p_i(x)$s are continuous there is at least one invariant measure for M. The following examples exhibit more than one invariant measure.

Example 2 *Take the IFS on $X = [0, 1]$ with:*

$$w_1(x) = \frac{1}{2}x, \qquad w_2(x) = \frac{1}{2}x + \frac{1}{2},$$

and consider the two state-dependent probability functions $p_1(x) = 1 - x^4$ and $p_2(x) = x^4$. In this case the two Dirac measures δ_0 and δ_1, concentrated at the points 0 and 1 respectively, are both fixed points and thus it is not possible for the Markov operator to be contractive with respect to any metric on $\mathcal{M}([0, 1])$. Moreover, for any $\xi \in [0, 1]$, the measure $\mu = \xi\delta_0 + (1 - \xi)\delta_1$ is also a fixed point of the Markov operator.

Example 3 *Let*

$$w_1(x) = \frac{1}{2}x, \qquad w_2(x) = \frac{1}{2}x + \frac{1}{2},$$

and for any $x \in [0, 1]$, define the function $\tau(x)$ to be a binary sequence in $\{0, 1\}^{\mathbb{N}}$ given by

$$\tau(x)_i = \lfloor 2^i x \rfloor \bmod 2.$$

(so that $\tau(x)_i$ is the i^{th} digit in the binary expansion of x, choosing the terminating one if necessary). Next define the function $\phi : [0, 1] \to [0, 1]$ by

$$\phi(x) = \limsup \frac{\#\{1 \le i \le n : \tau(x)_i = 0\}}{n} = 1 - \liminf \frac{1}{n}\sum_{i=1}^{n} \tau(x)_i$$

(so $\phi(x)$ is something like the asymptotic fraction of digits which are 0). Notice that $\phi(x) = \phi(w_i(x))$ for $i = 0, 1$ and also that $\phi(x) = \phi(2x)$ if $x \in [0, 1/2]$ and $\phi(x) = \phi(2x - 1)$ for $1/2 \le x \le 1$. Further notice that ϕ is Borel-measurable since it is the limit supremum of a sequence of measurable functions. It is, however, discontinuous everywhere in $[0, 1]$ with a graph which is dense in $[0, 1]^2$.

For each $p \in (0, 1)$ define the set $B_p \subset [0, 1]$ by

$$B_p = \{x \in [0, 1] : \phi(x) = p\}.$$

We see from the definition that $B_p \cap B_q = \varnothing$ if $p \ne q$. Finally, define the IFS with state-dependent probabilities W by

$$\{w_0(x), w_1(x), \phi(x), 1 - \phi(x)\}. \tag{3}$$

We now argue that there are uncountably many different invariant measures. To see this, for each $p \in (0, 1)$ let μ_p be the invariant measure for the IFS with probabilities W_p given by

$$\{w_0(x), w_1(x), p, 1 - p\}.$$

It is well-known that $\mu_p(B_p) = 1$; in fact, it has full measure on the smaller set given by

$$\left\{x \in [0, 1] : \lim \frac{\#\{1 \le i \le n : \tau(x)_i = 0\}}{n} = p\right\}.$$

This means that for any Borel set $S \subset [0, 1]$ we have $\mu_p(S) = \mu_p(S \cap B_p)$.

Let $S \subset [0,1]$ be given. First we assume that $S \subset [0,1/2]$, the case $S \subset [1/2,1]$ is similar and the general case follows from additivity and these two. Using M to denote the Markov operator associated with (3), we see that

$$\begin{aligned}
M\mu_p(S) &= \int_{2S} \phi(x)\ d\mu_p(x) = \int_S \phi(2x)\ d\mu_p(2x) = \int_S \phi(x)\, d\mu_p(2x) \\
&= \int_{S\cap B_p} \phi(x)\, d\mu_p(2x) = p \int_{S\cap B_p} d\mu_p(2x) = p\,\mu_p\left(w_0^{-1}(S\cap B_p)\right) \\
&= \frac{p}{p}\mu_p(S\cap B_p) = \mu_p(S).
\end{aligned}$$

In the second line, we have used the fact that $\phi(x) = p$ for all $x \in B_p$ and in the third we have used $\mu_p(A) = p \circ \mu_p\left(w_0^{-1}(A)\right)$ whenever $A \subset [0,1/2] = w_0([0,1])$.

For $S \subset [1/2,1]$ we have

$$\begin{aligned}
M\mu_p(S) &= \int_{2S-1} (1-\phi(x))\ d\mu_p(x) = \int_S (1-\phi(2x-1))\ d\mu_p(2x-1) \\
&= \int_{S\cap B_p} d\mu_p(2x-1) - \int_{S\cap B_p} \phi(x)\ d\mu_p(2x-1) \\
&= \frac{1}{1-p}\mu_p(S) - p\int_{w_1^{-1}(S\cap B_p)} d\mu_p(x) \\
&= \frac{1}{1-p}\mu_p(S) - \frac{p}{1-p}\mu_p(S) = \mu_p(S).
\end{aligned}$$

Thus for all $p \in (0,1)$ we have that μ_p is invariant under this state-dependent IFS with probabilities.

We now describe the so-called *Chaos Game* for an IFS with probabilities. Start with $x_0 \in X$, and define the sequence $x_t \in X$ by:

$$x_{t+1} = w_{\sigma_t}(x_t),$$

where $\sigma_t \in \{1,2,\ldots,N\}$ is chosen according to the probabilities $p_i(x_t)$ (that is, $P[\sigma_t = i] = p_i(x_t)$). We note that the sequence (x_t) is a Markov chain with values in X. The following theorem (from results in Elton, 1987; and Barnsley et al., 1988) gives conditions as to when an IFSSDP has a unique stationary distribution μ and the Chaos Game "converges" to μ in a distributional sense.

Theorem 3 (Elton, 1987; Barnsley et al., 1988) *Suppose that there is a $\delta > 0$ so that $p_i(x) > \delta$ for all $x \in X$ and $i = 1,2,\ldots,N$ and suppose further that the moduli of continuity of the p_is satisfy Dini's condition (see Elton, 1987; and Barnsley et al., 1988). Then there is a unique stationary distribution $\bar{\mu}$ for the Markov operator. Furthermore, for each continuous function $f : X \to \mathbb{R}$,*

$$\frac{1}{t+1}\sum_{i=0}^{t} f(x_i) \to \int_X f(x)\ d\bar{\mu}(x). \tag{4}$$

Theorem 3 can be used to show the following result.

Corollary 1 *Suppose that the IFSSDP $\{\mathbf{w}, p_i\}$ satisfies the hypothesis of Theorem 3. Then the support of the invariant measure $\bar{\mu}$ of the N-map IFSSDP $(\mathbf{w},\mathbf{p})$ is the attractor A of the IFS $\mathbf{w}$, i.e.,*

$$\operatorname{supp}\bar{\mu} = A.$$

Example 4 *Modifying Example 2 slightly by using*

$$p_1(x) = 1 - \beta - (1 - 2\beta)\, x^4, \qquad \text{and} \qquad p_2(x) = (1 - 2\beta)\, x^4 + \beta,$$

for $0 < \beta < 1$ we obtain an IFSSDP which satisfies the conditions of Theorem 3 and thus has a unique invariant distribution. Notice that the probability functions from Example 2 correspond to the case $\beta = 0$.

Notice that if p_i are not continuous then the IFSSDP might have more than one invariant measure (in fact, continuity is not enough; see Stenflo, 2002).

2.3 Singularity vs. Absolute Continuity

We now derive some novel results which allow us to characterize singularity vs. absolute continuity of the self-similar measure associated with IFS with state-dependent probabilities in the case of affine maps, generalizing what has been discussed in extant literature. Recall that an absolutely continuous invariant measure can be represented by a density, and thus admits a full representation depending only on a few parameters. Conversely, a singular invariant measure does not have a simple and effective representation, unless one states its value on every point of its domain.

Theorem 4 *Take the two-map IFS $\{\alpha x, \alpha x + 1 - \alpha\}$ on $X = [0, 1]$ where $\alpha \in [0, 1)$ along with the two probability functions $p_1(x) = p(x)$ and $p_2(x) = 1 - p(x)$ on $[0, 1]$. Assume that $\inf\{p(x) : 0 \le x \le 1\} > 0$ and that p is Hölder continuous. Let μ_α be the invariant measure of this state-dependent IFS.*

1. *If $0 \le \alpha < 1/2$ then μ_α is singular with respect to Lebesgue measure.*
2. *If $\alpha = 1/2$ then μ_α is either singular with respect to Lebesgue measure or is equal to Lebesgue measure on $[0, 1]$ and $p(x) = 1 - p(x) \equiv 1/2$.*
3. *For each $\alpha > 1/2$, let h_α be defined by*

   $$h_\alpha = -\int_0^1 \{p(x) \ln[p(x)] + [1 - p(x)] \ln[1 - p(x)]\}\, d\mu_\alpha.$$

   Then μ_α is absolutely continuous with respect to Lebesgue measure for Lebesgue almost every α such that $\alpha > e^{-h_\alpha}$. Moreover, μ_α is singular for almost every $\alpha < e^{-h_\alpha}$.

Theorem 4 characterizes the singularity vs. absolutely continuity properties of the invariant measure according to the value of the parameter α, which measures the slope of the maps of the IFSSDP. Note that its third statement determines the absolutely continuity property in terms of the invariant measure μ_α itself; as in most cases the expression of μ_α is unknown, verifying the condition at point 3 might be difficult. In order overcome this issue, Corollary 2 provides a sufficient condition based on the sup and the inf of the function $p(x)$ over the interval $[0, 1]$, which does not require a priori knowledge of the invariant measure μ_α.

Corollary 2 *Under the hypotheses of Theorem 4, let us define $p_{\inf} := \inf\{p(x) : 0 \le x \le 1\} > 0$ and $p_{\sup} := \sup\{p(x) : 0 \le x \le 1\} < 1$ and the quantity:*

$$\Theta := \max\{p_{\sup} \ln(p_{\sup}), p_{\inf} \ln(p_{\inf})\} + \max\{(1 - p_{\sup}) \ln(1 - p_{\sup}), (1 - p_{\inf}) \ln(1 - p_{\inf})\} < 0$$

Then for Lebesgue almost every $\alpha > e^{\Theta}$, μ_α is absolutely continuous with respect to Lebesgue measure. If $p(x)$ is constant, $p(x) \equiv p$ for any $x \in [0,1]$, then:

$$\Theta = p \ln(p) + (1-p) \ln(1-p) = \ln\left(p^p (1-p)^{1-p}\right)$$

and μ_α is absolutely continuous with respect to Lebesgue measure when $\alpha > e^{\Theta} = p^p (1-p)^{1-p}$.

Corollary 2 determines a sufficient condition which allows to directly determine whether the invariant measure is absolutely continuous in the case of affine IFSSDP. Note that the last result is consistent with what has been shown in the case of constant probabilities by Mitra et al. (2003) for intermediate values of the constant probability p (*i.e.*, $1/3 \leq p \leq 2/3$) and by Shmerkin (2014) for smaller and larger values (*i.e.*, $p < 1/3$ and $p > 2/3$). Theorem 4 and Corollary 2 will allow us in the next sections to analyze the characteristics of the steady state of our state-dependent-probability extended Brock and Mirman's (1972) model.

3 The Model

We extend the classical discrete time stochastic one-sector growth model á-la Brock and Mirman (1972), with logarithmic utility and Cobb-Douglas production function, to allow probabilities to be state-dependent. This can be described by a social planner's problem summarized by the following stochastic dynamic programming model:

$$\begin{aligned} V(k_0, z_0) = \max_{c_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \\ \text{s.t. } k_{t+1} = z_t k_t^\alpha - c_t \\ k_0 > 0 \text{ and } z_0 \in \{r, 1\} \text{ given,} \end{aligned} \tag{5}$$

where $\mathbb{E}_0$ is the expectation operator at time $t = 0$, k_t capital, c_t consumption, $0 < \alpha < 1$ the capital share, $0 < \beta < 1$ the discount factor, and $\{z_t\}_{t=0}^{\infty}$ a Bernoulli process taking values $0 < r < 1$ and 1 with probabilities $p(k_t)$ and $1 - p(k_t)$, respectively. Therefore, at each time t, z_t can take only two values with state-dependent probabilities, and in particular the fact that probabilities depend on the capital level implies that the realization of shocks is related to the past evolution of capital, implying a path-dependency in macroeconomic dynamics with important consequences on the evolution of economic inequalities. Intuitively, whenever $p' < 0$ capital accumulation will naturally imply a widening of economic inequalities (i.e., the probability of the best shock realization, $z_t = 1$, given by $1 - p(k_t)$ increases with the capital stock), while, if $p' > 0$, a reduction in inequalities (i.e., the probability of the best shock realization decreases with capital). Note that this setting boils down to the classical Brock and Mirman's (1972) model whenever $p(k_t) \equiv p$, that is, probabilities are constant. In order to understand how the macroeconomic outcomes might respond to different state-dependent probabilities and uncover the underlying mechanisms, in the remainder of the paper we will focus on a framework in which the relation between p and k_t is monotonic analyzing how the results may change when either $p' \leq 0$ or $p' \geq 0$.

The reduced problem associated with (5) can be stated as follows:

$$\begin{aligned} V(k_0, z_0) = \max_{k_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln(z_t k_t^\alpha - k_{t+1}) \\ \text{s.t. } 0 \leq k_{t+1} \leq z_t k_t^\alpha \\ k_0 > 0 \text{ and } z_0 \in \{r, 1\} \text{ are given.} \end{aligned} \tag{6}$$

Note that the probability $p(k_t)$ determines the occurrence of the random shock z_t at the same time t in which the actual amount of capital k_t is employed in production; in this scenario production occurs after the shock z_t is realized, and its occurrence is controlled by the state-dependent probability $p(k_t)$ depending on the actual availability of the stock of capital k_t in the same period t. However, as the amount of capital available at time t corresponds to the investment decision made at time $t-1$, such an assumption actually determines in essence a Markov-type stochastic dynamic for capital, in which the probability of the random variable z_t at time t depends on a choice made in the previous period $t-1$.

It is straightforward to verify that (6) is a *concave problem* as the z_t-sections of the graph $G = \{(k_t, k_{t+1}, z_t) : k_{t+1} \in \Gamma(k_t, z_t)\}$ of the optimal correspondence $\Gamma(k_t, z_t) = \{k_{t+1} : 0 \leq k_{t+1} \leq z_t k_t^{\alpha}\}$ are convex sets. Moreover, the dynamic constraint $\Gamma(k_t, z_t)$ eventually (monotonically) leads any feasible trajectory $\{k_t\}_{t=1}^{\infty}$ inside the interval $[0,1]$ as time elapses, because $z_t k_t^{\alpha} \leq k_t^{\alpha} < k_t$ for any value $k_t > 1$. That is, the trapping region for the dynamics that are admissible for the problem (6) is the interval $[0,1]$, so that, without loss of generality, by assuming that the initial capital value k_0 lies in such an interval, any trajectory will have values that remain confined in it.

The Bellman equation associated to (6) reads as:

$$V(k,z) = \max_{0 \leq y \leq zk^{\alpha}} \left[\ln(zk^{\alpha} - y) + \beta \mathbb{E}_y V(y, z')\right],$$

where $\mathbb{E}_y$ denotes the expectation operator that depends on the probabilities of both realizations of the random variable z' occurring in the next period, itself depending on the saving choice y, which corresponds to the capital available in the next period, that is, $\Pr(z' = r) = p(y)$, while $\Pr(z' = 1) = 1 - p(y)$ — recall that, for given y, the random variable z' is independent of past realizations. Then, the expectation $\mathbb{E}_y$ can be directly evaluated and the above equation can be rewritten in the following form:

$$V(k,z) = \max_{0 \leq y \leq zk^{\alpha}} \left\{\ln(zk^{\alpha} - y) + \beta p(y) V(y,r) + \beta\left[1 - p(y)\right] V(y,1)\right\}. \tag{7}$$

Our goal is to find a closed-form solution for the Bellman equation that generalizes the well known results discussed under the assumption that the probability of the shock is constant (see, *e.g.*, Montrucchio and Privileggi, 1999; Mitra et al., 2003). Following the cited literature, in order to do so we apply the "Guess and Verify" Method (Stokey and Lucas, 1989; Bethmann, 2007; La Torre et al., 2015); to this purpose, however, we must explicitly compute the derivative with respect to y in the RHS of (7), which, in turn requires also an explicit, non trivial functional form for the state-dependent probability $p(y)$. Having in mind that such a functional form must allow for the explicit calculation of the FOC in the RHS of (7), and reminding that the standard approach to the log-Cobb-Douglas Brock-Mirman model is to assume (guess) a logarithmic form for the value function, we proceed by assuming a logarithmic form of the type $p(y) = A + B \ln y$ for the state-dependent probability as well. Of course, any such logarithmic forms turn out to be unbounded over the interval $(0,1]$, while state-dependent probabilities must satisfy $0 < p(k) < 1$ for any feasible state value k. We overcome such an issue by opting for a piecewise functional form that is constant for k values close to 0 while taking the form $p(k) = A + B \ln k$ for larger k values, so to keep the probability bounded between 0 and 1.

Recall that, by assuming that the initial capital value k_0 lies in the interval $[0,1]$, any trajectory have values that remain confined in it. Under such an assumption we can introduce the following two piecewise-logarithmic forms for the state-dependent probability, one decreasing and one increasing in k, defined for $k \in [0,1]$:

$$p(k) = \begin{cases} 1-\delta & \text{if } 0 \leq k < e^{-\frac{1-\delta-\gamma}{\varepsilon}} \\ \gamma - \varepsilon \ln k & \text{if } e^{-\frac{1-\delta-\gamma}{\varepsilon}} \leq k \leq 1 \end{cases} \tag{8}$$

$$p(k) = \begin{cases} \delta & \text{if } 0 \leq k < e^{-\frac{1-\delta-\gamma}{\varepsilon}} \\ 1-\gamma + \varepsilon \ln k & \text{if } e^{-\frac{1-\delta-\gamma}{\varepsilon}} \leq k \leq 1, \end{cases} \tag{9}$$

with $\delta, \gamma > 0$ such that $\delta + \gamma < 1$ and $\varepsilon > 0$ sufficiently small.

Clearly, as $k \leq 1$, (8) defines a (Lipschitz) continuous state-dependent probability which satisfies $0 < p(k) < 1$ for all $0 \leq k \leq 1$, is constant over $\left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ and strictly decreasing in k over $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$, while (9) defines a continuous state-dependent probability which again satisfies $0 < p(k) < 1$ for all $0 \leq k \leq 1$, is constant over $\left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ and strictly increasing in k over $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$. Note that, for any fixed $\delta, \gamma > 0$ satisfying $\delta + \gamma < 1$, ε can be chosen small enough so to have the (more relevant) interval $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$ arbitrarily large; we shall return on this property later on. Figure 1 shows an example of the probability functions according to (8) and (9) for $\delta = \gamma = 0.01$ and $\varepsilon = 0.1756$; for such parameters' values the kink point turns out to be $e^{-\frac{1-\delta-\gamma}{\varepsilon}} = 0.0038$.

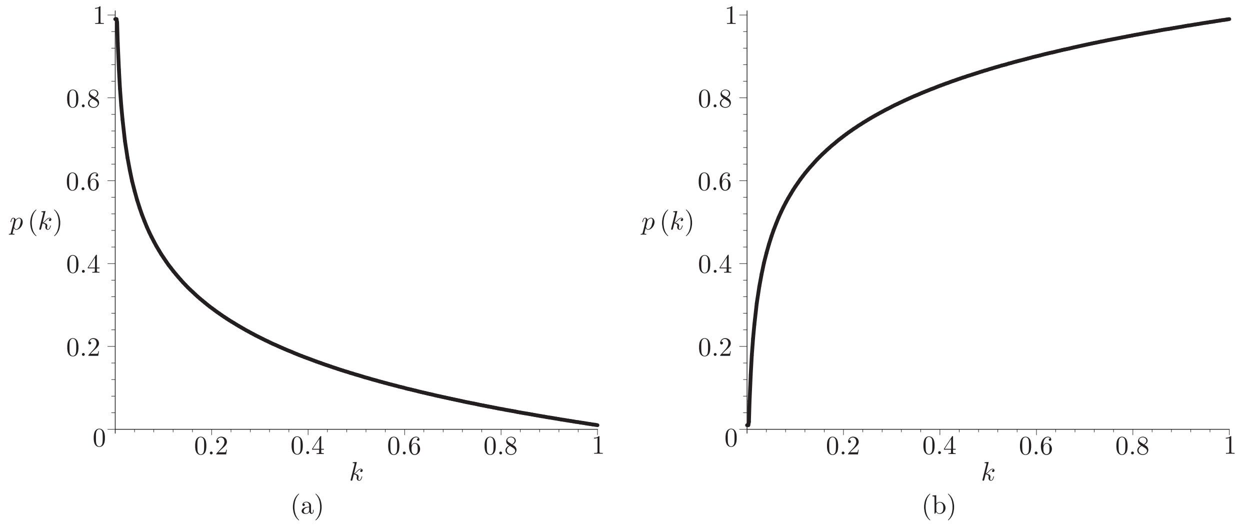

FIGURE 1: state-dependent probabilities for $\delta = \gamma = 0.01$ and $\varepsilon = 0.1756$; a) as defined in (8), b) as defined in (9).

In the next section we will characterize the optimal solution of problem (6) by determining a closed-form expression for the value function in (7) in situations in which $p(k_t)$ is defined as either in (8) or in (9).

4 Optimality

In order to search for a closed-form solution of our optimization problem, following previous literature we guess the following form for the value function in (7):

$$V(k,z) = A + B \ln k + C \ln z,$$

where A, B and C are constants to be determined. For such a logarithmic guess the Bellman equation in (7) becomes:

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{0\le y\le zk^{\alpha}} \left[\ln\left(zk^{\alpha}-y\right)+\beta\left(A+B\ln y\right)+\beta p\left(y\right)C\ln r\right]. \end{aligned} \tag{10}$$

Both state-dependent probabilities $p(y)$ with the forms defined either in (8) or in (9) are not differentiable at $y = e^{-\frac{1-\delta-\gamma}{\varepsilon}}$; hence, provided that $zk^{\alpha} \ge rk^{\alpha} > e^{-\frac{1-\delta-\gamma}{\varepsilon}}$, in both cases we must consider two different Bellman equations of the type in (10) depending on whether $y \in \left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ or $y \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, zk^{\alpha}\right]$. Specifically, when $p(y)$ is defined according to (8), the above equation becomes:

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{0\le y< e^{-\frac{1-\delta-\gamma}{\varepsilon}}} \left[\ln\left(zk^{\alpha}-y\right)+\beta A+\beta B\ln y+\beta\left(1-\delta\right)C\ln r\right], \end{aligned} \tag{11}$$

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{e^{-\frac{1-\delta-\gamma}{\varepsilon}}\le y\le zk^{\alpha}} \left[\ln\left(zk^{\alpha}-y\right)+\beta A+\beta B\ln y+\beta\left(\gamma-\varepsilon\ln y\right)C\ln r\right], \end{aligned} \tag{12}$$

while when $p(y)$ is defined according to (9), it takes the form:

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{0\le y< e^{-\frac{1-\delta-\gamma}{\varepsilon}}} \left[\ln\left(zk^{\alpha}-y\right)+\beta A+\beta B\ln y+\beta\delta C\ln r\right], \end{aligned} \tag{13}$$

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{e^{-\frac{1-\delta-\gamma}{\varepsilon}}\le y\le zk^{\alpha}} \left[\ln\left(zk^{\alpha}-y\right)+\beta A+\beta B\ln y+\beta\left(1-\gamma+\varepsilon\ln y\right)C\ln r\right]. \end{aligned} \tag{14}$$

Equations (11) and (13) represent problems that keep probabilities constant forever ($p = 1-\delta$ and $1-p=\delta$ in the former, $p=\delta$ and $1-p=1-\delta$ in the latter); however, if, after a finite number of iterations, k_t becomes larger than $e^{-\frac{1-\delta-\gamma}{\varepsilon}}$, the relevant Bellman equations become those defined in (12) and (14). Therefore, equations (11) and (13) turn out to be completely useless unless we can guarantee that $k_t < e^{-\frac{1-\delta-\gamma}{\varepsilon}}$ forever, that is, for every $t \ge 0$. Because $\delta+\gamma<1$, for ε sufficiently small the term $e^{-\frac{1-\delta-\gamma}{\varepsilon}}$ can be made arbitrarily small, which, in turn, implies that the possibility of k_t jumping above the level $e^{-\frac{1-\delta-\gamma}{\varepsilon}}$ after a finite number of iterations becomes likely. As a matter of fact, the Inada conditions exhibited by the lower Cobb-Douglas production function, rk_t^{α}, invites the social planner to choose investment levels k_{t+1} much larger than the actual stock of capital k_t available at time t when the latter is very close to the left-end point 0 of the feasible set $[0,1]$, thus easily leading to a value $k_{t+1} > e^{-\frac{1-\delta-\gamma}{\varepsilon}}$.

For k values in $\left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ problem (6) turns out to be a standard stochastic intertemporal model with constant probabilities, either $p=1-\delta$ and $1-p=\delta$ or $p=\delta$ and $1-p=1-\delta$. Hence, in this scenario we can invoke the well known result for this class of problems and easily find that, whenever $k \in \left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ the optimal policy yields the optimal investment given by (see, e.g., Mitra et al., 2003; Stokey and Lucas, 1989):

$$y^* = h(k,z) = \alpha\beta zk^{\alpha}. \tag{15}$$

Now, if ε is chosen sufficiently small with respect to parameters α, β and r, after a finite number τ of iterations of the policy (15) the optimal short-run trajectory will reach a value $k_{\tau} > e^{-\frac{1-\delta-\gamma}{\varepsilon}}$. The next assumption identifies such a threshold value for ε.

A. 1 *Parameters $\delta, \gamma, \varepsilon$ satisfy $\delta, \gamma, \varepsilon > 0$ and $\delta + \gamma < 1$. Moreover ε is small enough to satisfy:*

$$\varepsilon < -\frac{(1-\alpha)(1-\delta-\gamma)}{\ln(\alpha\beta r)}. \tag{16}$$

Note that the RHS in (16) is positive as $1-\delta-\gamma > 0$ and $\ln(\alpha\beta r) < 0$.

Lemma 1 *Under Assumption A.1—specifically, condition (16)—the regime represented by both Bellman equations in (11) and (13) cannot be sustained over time, as there exist a finite number of iterations $\tau \geq 0$ such that the optimal capital value in that iteration satisfies $k_\tau \geq e^{-\frac{1-\delta-\gamma}{\varepsilon}}$.*

In view of Lemma 1, in the following we shall assume that Assumption A.1 holds and that the initial capital stock satisfies $k_0 \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$, and focus exclusively on the (truly) state-dependent case represented by the second-type Bellman equations (12) and (14) over the (compact) interval $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$. The next Propositions 1 and 2 will establish that, under such assumptions, the optimal capital trajectory k_t^* remains confined in the interval $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$ for all $t \geq 0$ indeed, thus justifying the focus exclusively on the relevant Bellman equations (12) and (14).

We consider first the case characterized by the decreasing state-dependent probability defined in (8) for $y \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$: $p(y) = \gamma - \varepsilon \ln y$. In this case the (relevant) Bellman equation (12) reads as:

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{e^{-\frac{1-\delta-\gamma}{\varepsilon}} \leq y \leq zk^\alpha} \left[\ln(zk^\alpha - y) + \beta(A + \gamma C\ln r) + \beta(B - \varepsilon C\ln r)\ln y\right]. \end{aligned} \tag{17}$$

It is then possible to prove the following result.

Proposition 1 *Under Assumption A.1 and for $k_0 \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$, the solution of the Bellman equation (17) is the function:*

$$V(k,z) = A + B\ln k + C\ln z$$

where:

$$A = \frac{\ln[1-\beta(\alpha-\varepsilon\ln r)]}{1-\beta} + \frac{\beta(\alpha-\varepsilon\ln r)\ln[\beta(\alpha-\varepsilon\ln r)] + \beta\gamma\ln r}{(1-\beta)[1-\beta(\alpha-\varepsilon\ln r)]}, \tag{18}$$

$$B = \frac{\alpha}{1-\beta(\alpha-\varepsilon\ln r)}, \tag{19}$$

$$C = \frac{1}{1-\beta(\alpha-\varepsilon\ln r)}; \tag{20}$$

the optimal policy for capital is given by:

$$k_{t+1}^* = h(k_t^*, z_t) = \beta(\alpha - \varepsilon\ln r)\, z_t (k_t^*)^\alpha, \tag{21}$$

while the corresponding optimal policy for consumption is given by:

$$c_t^* = [1-\beta(\alpha-\varepsilon\ln r)]\, z_t (k_t^*)^\alpha. \tag{22}$$

It is possible to show (see Appendix A) that $\beta(\alpha - \varepsilon \ln r) < 1$, which ensures that: i) coefficients B in (19) and C in (20) are strictly positive, which, in turn, imply that the value function $V(k,z)$ solving equation (17) is strictly concave in k and that the RHS is strictly concave in y, so that the optimal policy in (21) is unique; and ii) the optimal consumption in (22) is strictly positive. Therefore, Proposition 1 determines the unique optimal policy associated with our extended Brock and Mirman's (1972) model with decreasing state-dependent probabilities. We can note that the optimal policy in (21) differs from the standard (under constant probability) optimal policy $k^*_{t+1} = h(k,z) = \alpha\beta z k^\alpha$ as in (15) because of the role of the state-dependent probability $p(k) = \gamma - \varepsilon \ln k$ as in (8). Specifically, the positive term added to the original multiplicative coefficient α appearing in (15) (i.e., $-\varepsilon \ln r$) takes into account that, as $p(k) = \gamma - \varepsilon \ln k$ is decreasing in k, investing more in future capital increases the probability $1 - p(k)$ of having future favorable shocks $z_t = 1$. Clearly, if $\varepsilon = 0$, that is, the probability $p(k)$ does no longer depend on capital, the optimal policy (21) perfectly coincides with the standard one in (15).

We now move to the case of an increasing state-dependent probability defined in (9) for $y \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$: $p(y) = 1 - \gamma + \varepsilon \ln y$. In this case the (relevant) Bellman equation (14) reads as:

$$\begin{aligned} V(k,z) &= A + B\ln k + C\ln z \\ &= \max_{e^{-\frac{1-\delta-\gamma}{\varepsilon}} \le y \le zk^\alpha} \left[\ln(zk^\alpha - y) + \beta\left[A + (1-\gamma)C\ln r\right] + \beta(B + \varepsilon C\ln r)\ln y\right]. \end{aligned} \tag{23}$$

Unlike the case with decreasing state-dependent probabilities, now we need an additional condition on parameter ε in the definition of probability in (9) – the following condition (24) – that guarantees interiority of the optimal policy (28) determined in the next Proposition 2 whenever its argument $k^*_t \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$. In fact, now the term $\varepsilon \ln r < 0$ indicates that, when the state-dependent probability is increasing, the optimal choice on investment turns out to be strictly lower than that prescribed by the standard optimal policy (15). This property requires that the upper bound for parameter ε in condition (16) is further restricted in order to assure that the optimal trajectory generated by (28) remains trapped in the (open) interval $\left(e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right)$ for all $t \geq 0$.

A. 2 *Under Assumption A.1, suppose that ε is sufficiently small to satisfy:*

$$e^{-\frac{(1-\alpha)(1-\delta-\gamma)}{\varepsilon}} - (\beta r \ln r)\,\varepsilon < \alpha\beta r. \tag{24}$$

Condition (24), although stated in implicit form with respect to ε, is meaningful, as the RHS is strictly positive and the LHS is strictly positive, strictly increasing in ε and tends to 0 as $\varepsilon \to 0^+$. In other words, for any choice for $0 < \alpha, \beta, \delta, \gamma, r < 1$ satisfying all our assumptions, there always exist some values $\varepsilon > 0$ satisfying (24). Its threshold upper bound value is the unique $\varepsilon > 0$ satisfying (24) with equality. Moreover, as $-(\beta r \ln r)\,\varepsilon > 0$, condition (24) is stricter than (*i.e.*, implies) condition (16); indeed, $e^{-\frac{(1-\alpha)(1-\delta-\gamma)}{\varepsilon}} - (\beta r \ln r)\,\varepsilon < \alpha\beta r \Longrightarrow e^{-\frac{(1-\alpha)(1-\delta-\gamma)}{\varepsilon}} < \alpha\beta r$, where the last inequality is equivalent to (16). Therefore, Lemma 1 always holds true under Assumption A.2.

Proposition 2 *Under Assumption A.2 and for $k_0 \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$, the solution of the Bellman equation (23) is the function:*

$$V(k,z) = A + B\ln k + C\ln z$$

where:

$$A = \frac{\ln\left[1-\beta\left(\alpha+\varepsilon\ln r\right)\right]}{1-\beta} + \frac{\beta\left(\alpha+\varepsilon\ln r\right)\ln\left[\beta\left(\alpha+\varepsilon\ln r\right)\right]+\beta\left(1-\gamma\right)\ln r}{\left(1-\beta\right)\left[1-\beta\left(\alpha+\varepsilon\ln r\right)\right]}, \tag{25}$$

$$B = \frac{\alpha}{1-\beta\left(\alpha+\varepsilon\ln r\right)}, \tag{26}$$

$$C = \frac{1}{1-\beta\left(\alpha+\varepsilon\ln r\right)}; \tag{27}$$

the optimal policy for capital is given by:

$$k_{t+1}^* = h\left(k_t^*, z_t\right) = \beta\left(\alpha+\varepsilon\ln r\right) z_t \left(k_t^*\right)^\alpha, \tag{28}$$

while the corresponding optimal policy for consumption is given by:

$$c_t^* = \left[1-\beta\left(\alpha+\varepsilon\ln r\right)\right] z_t \left(k_t^*\right)^\alpha. \tag{29}$$

Also in this case it is possible to show (Appendix A) that under condition (24) $0 < \beta\left(\alpha+\varepsilon\ln r\right) < 1$ holds, thus assuring that: i) the optimal investment in (28) is strictly positive; ii) the optimal consumption in (29) is strictly positive; and iii) coefficients B in (26) and C in (27) are both strictly positive, which, in turn, together with the property $\beta\left(\alpha+\varepsilon\ln r\right) > 0$, imply that the value function $V\left(k,z\right)$ solving equation (23) is strictly concave in k and that the RHS is strictly concave in y, so that the optimal policy in (28) is unique. Therefore, Proposition 2 determines the unique optimal policy associated with our model with increasing state-dependent probabilities. We can note that also in the case of increasing probability the optimal policy in (28) differs from the standard (under constant probability) optimal policy $k_{t+1}^* = h\left(k,z\right) = \alpha\beta z k^\alpha$ in (15) because of the effects of the state-dependent probability $p\left(k\right) = 1-\gamma+\varepsilon\ln k$ as in (9). The negative term added to the original multiplicative coefficient α appearing in (15) (i.e., $+\varepsilon\ln r$), emphasizes the fact that, as $p\left(k\right) = 1-\gamma+\varepsilon\ln k$ is increasing in k, the decision maker takes into account that too large an investment increases the probability $p\left(k\right)$ of bad shocks $z_t = r$ occurring in subsequent times that will cause a reduction in the future capital stock. Also in this case, whenever $\varepsilon = 0$ the probability $p\left(k\right)$ turns out not to depend on capital any longer, and thus the optimal policy (28) perfectly coincides with the standard one in (15).

We wish to stress that, independently of whether the probability increases or decreases with the capital stock, the optimal policy under state-dependent probability crucially depends on the shocks probability $p(k_t)$. Therefore, by affecting the optimal capital dynamics, state-dependent probabilities act as an engine of capital accumulation, which through its effects on the probability of shocks realization impacts the evolution of economic inequality. Such effects are completely absent under the standard constant probability assumption, explaining why the standard Brock and Mirman's (1972) setup cannot say anything about the source of path-dependent outcomes, limiting thus its ability to explain the relation between macroeconomic dynamics and economic inequalities. If macroeconomic outcomes and inequalities are related (and inequalities are characterized by path-dependency) as confirmed by several works (Piketty, 2015; Piketty and Saez, 2014; Alvaredo et al., 2017), we need to allow probabilities to be state-dependent in order to account for the nature of such a relation. Comparing the optimal policies (21) and (28) under decreasing and increasing state-dependent probabilities respectively, it is straightforward to notice that they differ only for the additive term $\varepsilon\ln r$, whose sign is positive in the former case and negative in the latter case, such that the optimal policy prescribes a larger (smaller) investment whenever the probability decreases (increases) with the capital stock.

The optimal policies (21) and (28) derived in Propositions 1 and 2 can be rewritten as follows, respectively:

$$k_{t+1} = \theta_1 z_t k_t^\alpha \qquad \text{and} \qquad k_{t+1} = \theta_2 z_t k_t^\alpha,$$

where $\theta_1 = \beta(\alpha - \varepsilon \ln r)$ and $\theta_2 = \beta(\alpha + \varepsilon \ln r)$. Consistent with extant literature (Montrucchio and Privileggi, 1999; Mitra et al., 2003; La Torre et al., 2019), the following log-linear transformations:

$$x_t = -\frac{1-\alpha}{\ln r}\ln k_t + 1 + \frac{\ln \theta_i}{\ln r}, \qquad \text{for } i = 1,2, \tag{30}$$

define affine topological conjugate dynamics in the new variable x_t over the interval $[0,1]$. As $0 < r < 1$ and, by Propositions 1 and 2, $0 < \theta_i < 1$ for $i = 1,2$ as well, (30) are increasing affine transformations of $\ln k_t$ for $i = 1,2$. Specifically, they are invertible and each of them establish a one-to-one correspondence between the nonlinear dynamics of k_t defined by maps (21) and (28) and the affine dynamics of the new variable x_t according to:

$$x_{t+1} = \alpha x_t + (1-\alpha)\left(1 - \frac{\ln z_t}{\ln r}\right),$$

which, in turn, can be rewritten in terms of the following IFSSDP:

$$x_{t+1} = \begin{cases} \alpha x_t & \text{with probability } \tilde{p}_i(x_t) \\ \alpha x_t + (1-\alpha) & \text{with probability } 1 - \tilde{p}_i(x_t) \end{cases} \tag{31}$$

where the *conjugate state-dependent probabilities* $\tilde{p}_i(x) : [0,1] \to [0,1]$ are the affine functions defined in the next proposition.

Proposition 3 *For the probabilities defined in (8) and (9), under Assumptions A.1 and A.2 the conjugate state-dependent probabilities* $\tilde{p}_i(x) : [0,1] \to [0,1]$ *associated to the IFSSDP (31) for* $i = 1,2$ *are:*

$$\tilde{p}_1(x) = \gamma - \frac{\varepsilon}{1-\alpha}\ln(\theta_1 r) + \frac{\varepsilon \ln r}{1-\alpha}x \tag{32}$$

$$\tilde{p}_2(x) = 1 - \gamma + \frac{\varepsilon}{1-\alpha}\ln(\theta_2 r) - \frac{\varepsilon \ln r}{1-\alpha}x. \tag{33}$$

Both satisfy $0 < \tilde{p}_i(x) < 1$ *for all* $x \in [0,1]$, $\tilde{p}_1(x)$ *is strictly decreasing while* $\tilde{p}_2(x)$ *is strictly increasing.*

Different from what happens under constant probabilities, Proposition 3 states that with state-dependent probabilities also the probability function needs to be converted in an affine function in order to derive a topologically equivalent transformation of the original dynamical system. Thanks to this transformation the IFSSDP (31) with associated state-dependent probabilities (32) and (33) can be analyzed through the tools described in Section 2, which ensure the existence of a unique stationary distribution μ for such an IFSSDP. Moreover, since our IFSSDP is characterized by affine maps, Theorem 4 and Corollary 2 directly apply allowing us to determine the singularity vs. absolute continuity properties of the self-similar measure associated with our IFSSDP. According to Theorem 4 and Corollary 2 the invariant distribution turns out be either singular if the capital share α is small (i.e., $\alpha \leq 1/2$) or absolutely continuous if it is large (i.e., $\alpha > 1/2$). Therefore, the capital share plays an important role in the determination of the steady state of our state-dependent-probability extended Brock and Mirman's (1972) model as its magnitude drives the singularity vs. absolute continuity properties of the invariant distribution, and, as we are going to see through some specific examples in the next section, different values of the capital share have important implications not only for the long run but also for the short run macroeconomic dynamics and for the evolution of inequalities.

5 Numerical Examples

We now consider a few examples of the optimal growth model (6) based on the two state-dependent probabilities defined in (8) and (9). Specifically, we numerically approximate the evolution of a given probability distribution over time according to the affine IFSSDP (31) where state-dependent probabilities $\tilde{p}_i(x)$ are the affine functions defined in (32) and (33), that is, they are obtained as log-linearization of the state-dependent probabilities associated to the optimal policies (21) and (28), which solve (6) in our two scenarios: 1) when $p(k)$ is decreasing and defined according to (8), and 2) when $p(k)$ is increasing and defined according to (9). To this purpose, we apply a Maple algorithm[1] that approximates successive iterations of the Markov operator (2) associated with the IFSSDP (31) based on Algorithm 1 in La Torre et al. (2019), in order to have a qualitative idea on what the invariant distribution μ_α may look like. The novelty with respect to the previous Algorithm is that this update is capable of handling the case in which the two maps in (31) are allowed to overlap, thus paving the way for the exploration of models characterized by high capital share values α, under which, as we have seen from Theorem 4 and Corollary 2, the invariant distribution μ_α may be absolutely continuous.

We fix the following parameters' values:

$$\beta = 0.96, \qquad r = 0.25, \qquad \text{and} \qquad \delta = \gamma = 0.01, \tag{34}$$

and then consider the following values for the capital share:

$$\alpha = 0.33, \qquad \alpha = 0.5, \qquad \text{and} \qquad \alpha = 0.8. \tag{35}$$

To each of the α values in (35), a value for parameter ε satisfying the most restrictive condition (24)—but very close to its upper bound—will be associated, both for the decreasing and the increasing state-dependent probabilities scenarios.

In order to illustrate the evolution over time of some arbitrary initial distribution μ_0 supported over the interval $[0,1]$ according to the Markov operator (2) associated with the IFSSDP (31) along with how it is affected by the fact that the probabilities $\tilde{p}_i(x)$ are state-dependent, we start with an example in which $\alpha = 0.5$. Such a feature envisages the whole interval $[0,1]$ as the support of the invariant distribution, while, at the same time, the images of the two maps $w_1(x) = \alpha x$ and $w_2(x) = \alpha x + (1-\alpha)$ in (31) (almost) do not overlap, except for the zero-Lebesgue measure point $w_1(1) = w_2(0)$. In this scenario, the full effect of the state-dependent probabilities on the probability distribution μ_t at time t can be neatly appreciated as, in order to build $\mu_{t+1} = M\mu_t(x)$, fractions of its mass are being distributed between each of the two sub-intervals appearing from each interval belonging to the pre-fractal in t after the t^{th} iteration of (2). When $\alpha < 0.5$ such sub-intervals shorten too fast, thus reducing the visual magnitude of the state-dependent probabilities effects, while if $\alpha > 0.5$ the overlapping images of the w_i maps in (31) introduce a distortion that somewhat hides the full effect of the probabilities $\tilde{p}_i(x)$ and $1 - \tilde{p}_i(x)$.

For $\alpha = 0.5$ the ε value satisfying condition (24) with equality is 0.1757, so that we set $\varepsilon = 0.1756$, that is, 0.0001 less that its upper bound. Hence, the optimal policies (21) and (28) turn out to be $k_{t+1} = \beta(\alpha - \varepsilon \ln r) z_t k_t^\alpha = (0.7137) z_t k_t^\alpha$ and $k_{t+1} = \beta(\alpha + \varepsilon \ln r) z_t k_t^\alpha = (0.2463) z_t k_t^\alpha$ respectively. The plots of the state-dependent probabilities $p(k)$ for these parameters' values, both for the decreasing and for the increasing probability, are those reported in Figure 1 for the relevant interval $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right] = [0.0038, 1]$; to such probabilities correspond

[1]The detailed code is available upon request.

the log-linearized affine probabilities (32) and (33) defined as $\tilde{p}_1(x) = 0.6154 - (0.4870)\,x$ and $\tilde{p}_2(x) = 0.0107 + (0.48780)\,x$ respectively. Figure 2 shows the first $n = 8$ iterations of our Maple algorithm for the state-dependent probabilities $\tilde{p}_1(x)$ and $1 - \tilde{p}_1(x)$, *i.e.*, when the (decreasing) probability of the shock $z = r$ is defined according to $\tilde{p}_1(x) = 0.6154 - (0.4870)\,x$, starting from the uniform initial distribution $\mu_0(x) \equiv 1$. Figure 3 shows the first $n = 8$ iterations of our Maple algorithm for the state-dependent probabilities $\tilde{p}_2(x)$ and $1 - \tilde{p}_2(x)$, *i.e.*, when the (increasing) probability of the shock $z = r$ is defined according to $\tilde{p}_2(x) = 0.0107 + (0.48780)\,x$, starting from the uniform initial distribution $\mu_0(x) \equiv 1$.

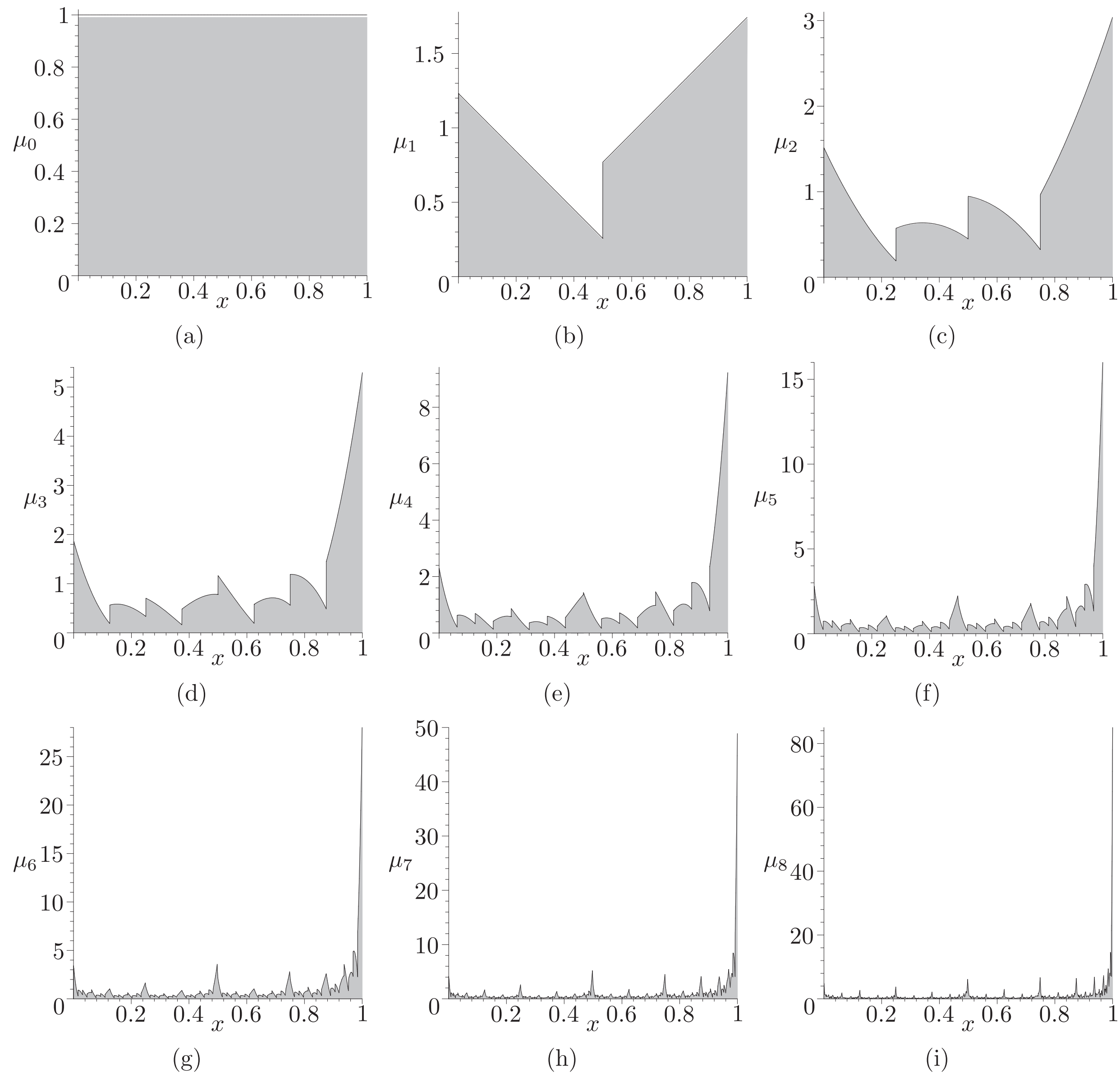

Figure 2: First 8 iterations of our Algorithm to approximate the Markov operator (2) associated to the IFSSDP (31) for $\beta = 0.96$, $r = 0.25$, $\delta = \gamma = 0.01$, $\alpha = 0.5$, $\varepsilon = 0.1756$ when the state-dependent probability is decreasing and defined by $\tilde{p}_1(x) = 0.614 - (0.4870)\,x$.

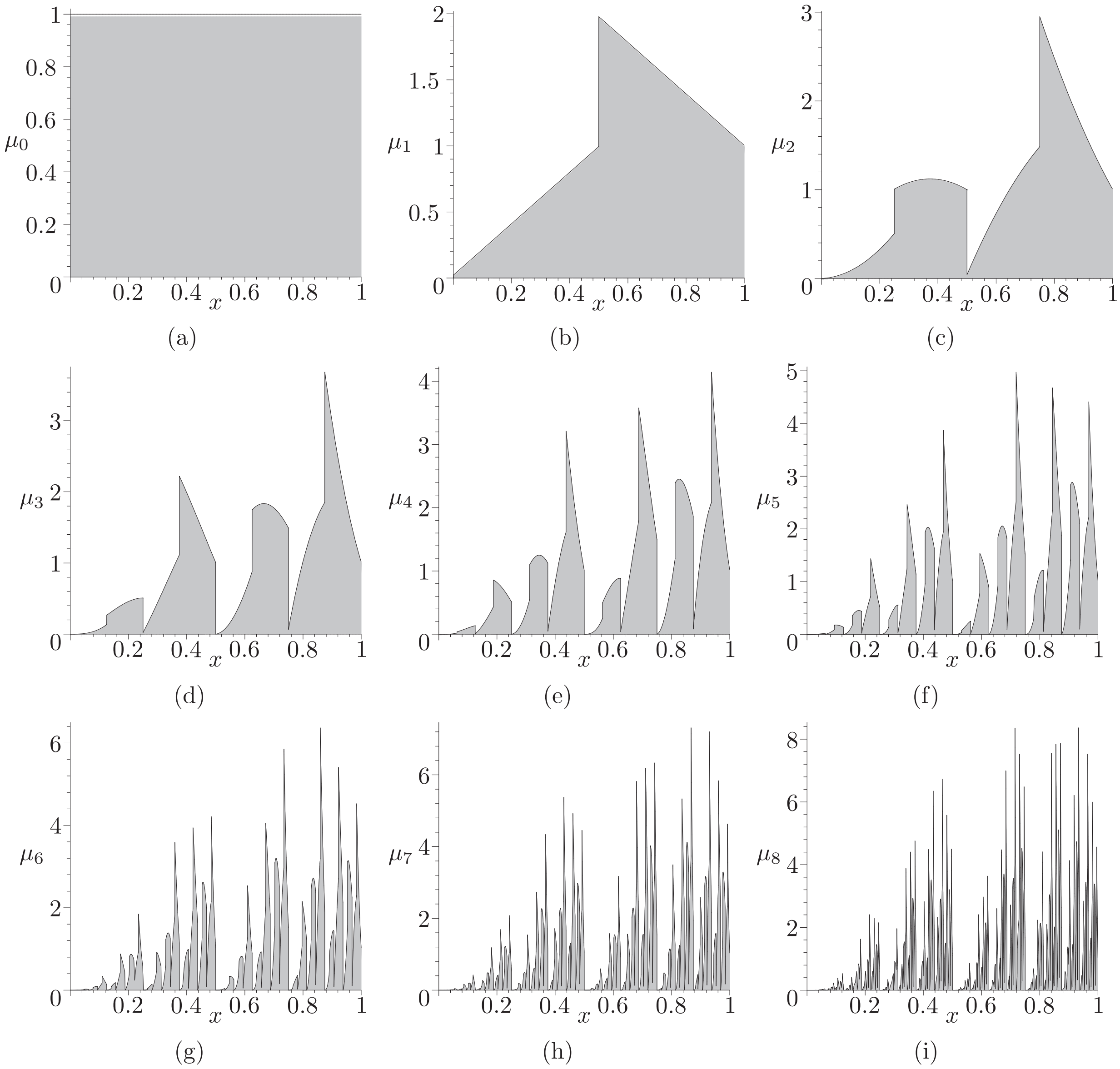

FIGURE 3: First 8 iterations of our Algorithm to approximate the Markov operator (2) associated to the IFSSDP (31) for $\beta = 0.96$, $r = 0.25$, $\delta = \gamma = 0.01$, $\alpha = 0.5$, $\varepsilon = 0.1756$ when the state-dependent probability is increasing and defined by $\tilde{p}_2(x) = 0.0107 + (0.4870)\,x$.

Assuming that Figures 2(i) and 3(i) provide a reasonably meaningful approximation of the invariant distribution μ_α in both models, from Figure 2 we learn that, as expected, μ_α tends to preserve a high degree of inequality by allocating most of the mass on x values close to the endpoints of $[0,1]$, especially on the best event $x = 1$ (notice the very high spike close to $x = 1$ in Figure 2(i), corresponding to the largest probability value for the best shock $z = 1$, $1 - \tilde{p}_1(1) = 1 - 0.1285 = 0.8715$). This is explained by the fact that a decreasing probability $\tilde{p}_1(x)$ introduces a conservative pattern for the x values, with a higher probability to remain close to $x = 0$ if the system is already there, and (much) higher probability to remain close to $x = 1$ if the system is already in that area. A larger spike close to $x = 1$ than close to $x = 0$ is determined by the property that $\tilde{p}_1(x) = 0.6154 - (0.4870)\,x$ implies uniformly larger values for $1 - \tilde{p}_1(x)$ than for $\tilde{p}_1(x)$ for all $x \in [0,1]$, which translates in having always a higher probability associated with the best shock $z = 1$. Conversely, Figure 3 shows that an increasing probability like $\tilde{p}_2(x) = 0.0107 + (0.48780)\,x$ tends to concentrate more mass in the middle of $[0,1]$, that is, future values of x are more likely to jump (almost) anywhere in the interval $[0,1]$ than in the previous case. Again the justification of this pattern originates from the increasing probability $\tilde{p}_2(x)$ that raises the chance of the occurrence of the best shock $z=1$ when x is small and viceversa. As also in this case $\tilde{p}_2(x) = 0.0107 + (0.48780)\,x$ implies uniformly larger values for $1-\tilde{p}_2(x)$ than for $\tilde{p}_2(x)$ for all $x \in [0,1]$, again the probability associated with the best shock $z=1$ is always higher, thus explaining the presence of a slightly larger mass in the right half of the interval $[0,1]$ in Figure 3(i) (notice the largest probability value for the best shock $z=1$ is reached in $x=0$, as $1-\tilde{p}_2(0) = 1-0.0107 = 0.9893$). Finally, the height and irregularity of the spikes in both figures 2(i) and 3(i) are consistent with statement 2 in Theorem 4 establishing that, when $\alpha = 0.5$, our state-dependent probabilities $\tilde{p}_1(x)$ and $\tilde{p}_2(x)$ imply that the invariant distribution μ_α must be singular with respect to Lebesgue measure.

For $\alpha = 0.33$ the ε value satisfying condition (24) with equality is 0.1720, so that we set $\varepsilon = 0.1719$, that is, 0.0001 less that its upper bound. Hence, the optimal policies (21) and (28) turn out to be $k_{t+1} = \beta(\alpha - \varepsilon \ln r)\, z_t k_t^\alpha = (0.5456)\, z_t k_t^\alpha$ and $k_{t+1} = \beta(\alpha + \varepsilon \ln r)\, z_t k_t^\alpha = (0.0880)\, z_t k_t^\alpha$ respectively and the relevant interval becomes $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right] = [0.0033, 1]$; to such probabilities correspond the log-linearized affine probabilities (32) and (33) defined as $\tilde{p}_1(x) = 0.5211 - (0.3557)\,x$ and $\tilde{p}_2(x) = 0.0110 + (0.3557)\,x$ respectively. Figure 4(a) shows the 5^{th} iteration of our Maple algorithm for the state-dependent probabilities $\tilde{p}_1(x)$ and $1-\tilde{p}_1(x)$, where the former is decreasing, starting from the uniform initial distribution $\mu_0(x) \equiv 1$. Figure 4(b) shows the 5^{th} iteration of our Maple algorithm for the state-dependent probabilities $\tilde{p}_2(x)$ and $1-\tilde{p}_2(x)$, where the former is increasing, starting from the uniform initial distribution $\mu_0(x) \equiv 1$. Both invariant distributions in Figure 4 have the Ternary Cantor set as attractor, so that they concentrate on a much thinner and sparser set than the invariant distributions in Figures 2 and 3; besides this feature, the general pattern exhibited by the approximations of Figures 4(a) and 4(b) seems consistent with that already discussed for the approximations in Figures, 2 and 3. Clearly, consistently with statement 1 in Theorem 4, both invariant distributions must be singular with respect to Lebesgue measure.

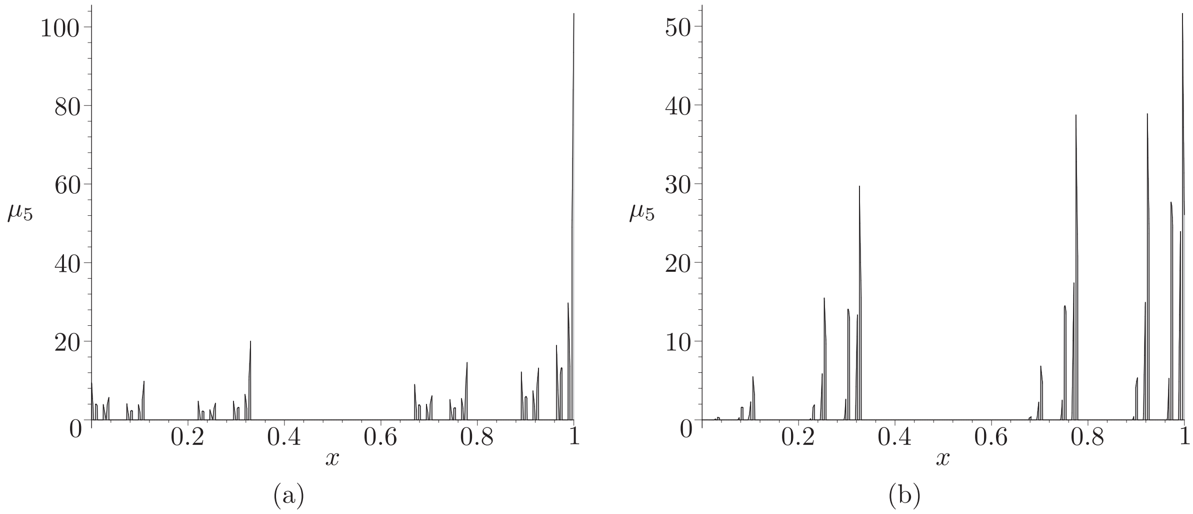

FIGURE 4: 5^{th} iteration of our Algorithm to approximate the Markov operator (2) associated to the IFSSDP (31) for $\beta = 0.96$, $r = 0.25$, $\delta = \gamma = 0.01$, $\alpha = 0.33$, $\varepsilon = 0.1719$; a) when the state-dependent probability is decreasing and defined by $\tilde{p}_1(x) = 0.5211 - (0.3557)\,x$; b) when the state-dependent probability is increasing and defined by $\tilde{p}_2(x) = 0.0110 + (0.3557)\,x$.

If $\alpha = 0.8$ the ε value satisfying condition (24) with equality is 0.1058, so that we set $\varepsilon = 0.1057$, that is, 0.0001 less that its upper bound. Hence, the optimal policies (21) and (28) turn out to be $k_{t+1} = \beta(\alpha - \varepsilon \ln r) z_t k_t^\alpha = (0.9087) z_t k_t^\alpha$ and $k_{t+1} = \beta(\alpha + \varepsilon \ln r) z_t k_t^\alpha = (0.6273) z_t k_t^\alpha$ respectively and the relevant interval becomes $\left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right] = [0.0001, 1]$; to such probabilities correspond the log-linearized affine probabilities (32) and (33) defined as $\tilde{p}_1(x) = 0.7932 - (0.7326)x$ and $\tilde{p}_2(x) = 0.0110 + (0.7326)x$ respectively. Figure 5(a) shows the 10^{th} iteration of our Maple algorithm for the state-dependent probabilities $\tilde{p}_1(x)$ and $1 - \tilde{p}_1(x)$, where the former is decreasing, starting from the uniform initial distribution $\mu_0(x) \equiv 1$. Figure 5(b) shows the 10^{th} iteration of our Maple algorithm for the state-dependent probabilities $\tilde{p}_2(x)$ and $1 - \tilde{p}_2(x)$, where the former is increasing, starting from the uniform initial distribution $\mu_0(x) \equiv 1$. Figure 5 considers an IFSSDP in which the images of the two maps $w_1(x) = \alpha x$ and $w_2(x) = \alpha x_t + (1-\alpha)$ exhibit a large overlapping region, with magnitude of 0.6. Such a property, consistently with Theorem 4, implies that the invariant distribution μ_α is more likely to be smooth, a feature clearly apparent from both Figures 5(a) and 5(b). More precisely, the invariant distribution approximated in Figure 5(a) satisfies the sufficient condition of Corollary 2, and thus it is *almost surely* absolutely continuous. In fact, the term Θ turns out to be $\Theta = -0.2287$, so that $e^\Theta = e^{-0.2287} = 0.7956 < 0.8 = \alpha$. In other words, the spikes present in the finite-time approximation of μ_α in Figure 5(a) are likely to be asymptotically smoothed out as the number of iterations approaches infinity. A similar sufficient condition does not hold for the invariant distribution approximated in Figure 5(b), as in this case $\Theta = -0.0289$, so that $e^\Theta = e^{-0.0289} = 0.9715 > 0.8 = \alpha$.

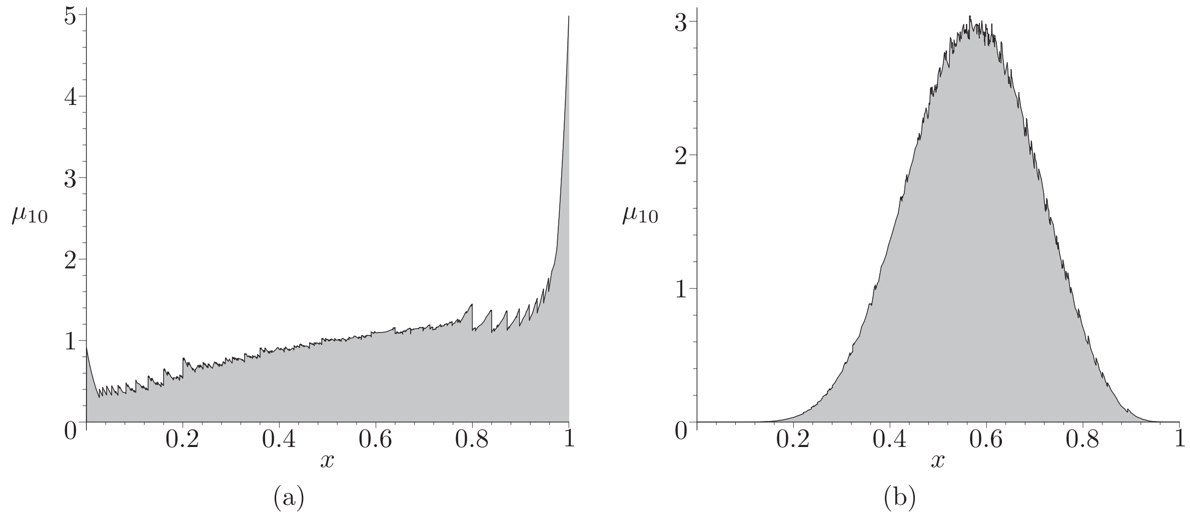

FIGURE 5: 10^{th} iteration of our Algorithm to approximate the Markov operator (2) associated to the IFSSDP (31) for $\beta = 0.96$, $r = 0.25$, $\delta = \gamma = 0.01$, $\alpha = 0.8$, $\varepsilon = 0.1057$; a) when the state-dependent probability is decreasing and defined by $\tilde{p}_1(x) = 0.7932 - (0.7326)x$; b) when the state-dependent probability is increasing and defined by $\tilde{p}_2(x) = 0.0110 + (0.7326)x$.

A careful comparison between the pair of Figures 2(i), 3(i) and the pair of Figures 5(a), 5(b), easily allows to appreciate that the general characteristics of the invariant distributions generated by decreasing vs. increasing state-dependent probabilities are the same, regardless on whether they turn out to be singular or absolutely continuous. Specifically, both plots in Figures 2(i) and 5(a) exhibit a certain degree of inequality with a higher spike close to $x = 1$, with the approximation in Figure 5(a) characterized by a slightly larger mass concentration around the middle of $[0,1]$ due to the overlapping images of the maps w_i. Similarly, both Figures 3(i) and 5(b) exhibit enough mass in the middle of $[0,1]$, which in both cases happens to be slightly shifted to the right half of the interval $[0,1]$; again, Figure 5(b) concentrates more mass in the middle of $[0,1]$ due to the overlapping images of the maps w_i.

As a curiosity, Figure 6 reports the approximations of the invariant distribution of the original nonlinear random dynamics defined by the optimal policies (21) and (28) for the first two models considered in Figures 2 and 3, *i.e.*, for the parameters' values as in (34), $\alpha = 0.5$ and $\varepsilon = 0.1756$, so that, for the decreasing probability $p(k) = \gamma - \varepsilon \ln k_t$ as in (8), the dynamics are generated by $k_{t+1} = \theta_1 z_t k_t^\alpha$, while, for the increasing probability $p(k) = 1 - \gamma + \varepsilon \ln k_t$ as in (9), the dynamics are generated by $k_{t+1} = \theta_2 z_t k_t^\alpha$ where $\theta_1 = \beta(\alpha - \varepsilon \ln r)$ and $\theta_2 = \beta(\alpha + \varepsilon \ln r)$. Specifically, the IFSSDP describing the dynamics of the former model turns out to be:

$$k_{t+1} = \begin{cases} (0.1784)\, k_t^{0.5} & \text{with probability } p(k_t) = 0.01 - (0.1756) \ln k_t \\ (0.7137)\, k_t^{0.5} & \text{with probability } 1 - p(k_t) = 0.99 + (0.1756) \ln k_t, \end{cases} \tag{36}$$

defined over the interval $\left[(\theta_1 r)^2, \theta_1^2\right] = [0.0318, 0.5094]$ whose endpoints are the fixed points of the two maps $w_1(k) = \theta_1 r k_t^\alpha$ and $w_2(k) = \theta_1 k_t^\alpha$, while the IFSSDP describing the dynamics of the latter model is:

$$k_{t+1} = \begin{cases} (0.0616)\, k_t^{0.5} & \text{with probability } p(k_t) = 0.01 - (0.1756) \ln k_t \\ (0.2463)\, k_t^{0.5} & \text{with probability } 1 - p(k_t) = 0.99 + (0.1756) \ln k_t, \end{cases} \tag{37}$$

defined over the interval $\left[(\theta_2 r)^2, \theta_2^2\right] = [0.0038, 0.0606]$ whose endpoints are the fixed points of the two maps $w_1(k) = \theta_2 r k_t^\alpha$ and $w_2(k) = \theta_2 k_t^\alpha$.

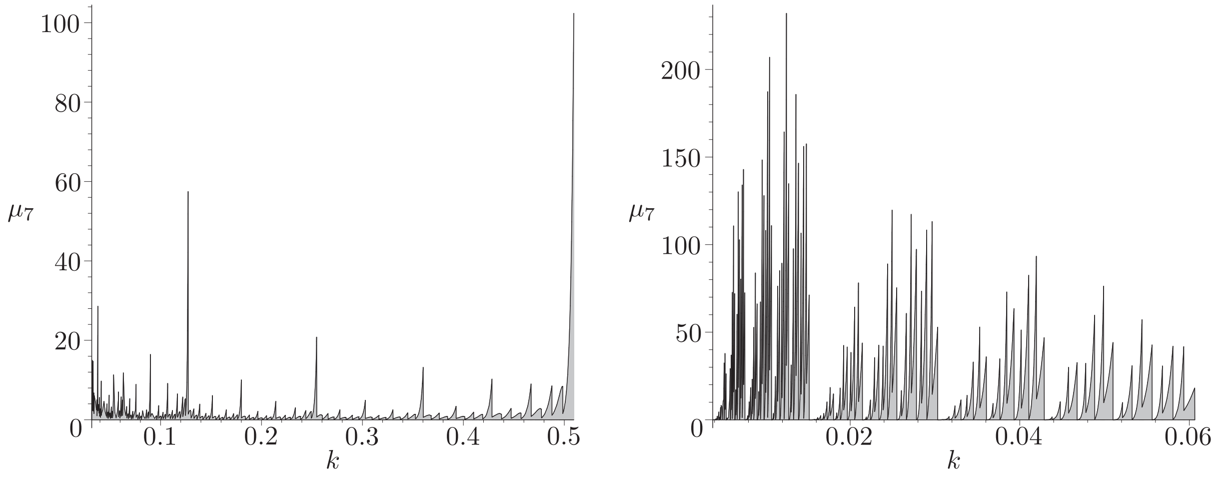

FIGURE 6: 7^{th} iteration of our Algorithm to approximate the Markov operator (2) associated to the nonlinear IFSSDP defined by a) (36) over $[0.0318, 0.5094]$, and b) (37) over $[0.0038, 0.0606]$.

As in both IFSSDPs defined in (36) and (37) the derivative of the higher map $w_2(k) = \theta_i k_t^\alpha$ evaluated at the left endpoint, $(\theta_i r)^2$, of their attractor $\left[(\theta_i r)^2, \theta_i^2\right]$ is larger than 1, both IFSSDPs are not contractive. However, convergence to a unique invariant distribution supported over the whole intervals $\left[(\theta_i r)^2, \theta_i^2\right]$ is guaranteed by the fact that they are topologically conjugate of the affine IFSSDP (31), which, being a contraction, converges to a unique invariant distribution that, when $\alpha = 0.5$, is supported on the full interval $[0,1]$. Moreover, as occurs to the IFSSDP (31), for $\alpha = 0.5$, the images of $w_1(k)$ and $w_2(k)$ (almost) do not overlap, except for the zero-Lebesgue measure points $w_1(\theta_i^2) = w_2\left[(\theta_i r)^2\right]$, for $i = 1, 2$. In other words, the invariant distributions of the IFSSDPs (36) and (37) are just transformations of the invariant distributions whose approximations are plotted in Figures 2(i) and 3(i). This is confirmed by Figures 6(a) and 6(b), where the 7^{th} iteration of our Maple algorithm to approximate the Markov operator (2) associated with the nonlinear IFSSDPs (36) and (37) starting from the uniform distribution $\mu_0 \equiv 1/\left[\theta_i^2 - (\theta_i r)^2\right]$ over the interval $\left[(\theta_i r)^2, \theta_i^2\right]$ are shown respectively. As a matter of fact, the latter approximations exhibit a similar pattern of those appearing in Figures 2(i) and 3(i), or, more appropriately, in Figures 2(h) and 3(h), only with a shift, together with higher spikes, of mass toward smaller values for the variable k; more precisely, in Figure 6(a) the general inequality features of Figures 2(h) and 2(i) are kept, but with higher spikes appearing closer to the left endpoint of the support, while Figure 6(b) maintains the general pattern of Figures 3(h) and 3(i), only with the overall mass shifted to the left half of the support.

6 Conclusion

We extend the classical discrete time stochastic one-sector growth model with logarithmic utility and Cobb-Douglas production function á-la Brock and Mirman (1972) to allow probabilities to be state-dependent. Under state-dependent probabilities the probability of occurrence of a given shock depends on the capital stock, thus as the economy accumulates more capital along its process of economic development the probability of occurrence of different shocks changes over time. As the social planner in making his investment decisions needs to account for how the future capital stock level will impact these probabilities, the optimal policy critically depends on the characteristics of the state-dependent probability function. Therefore, state-dependent probabilities act as an engine of capital accumulation, which through its effects on the probability of shocks realization impacts the evolution of economic inequality. We show that whenever the probability (assumed to take a logarithmic form) is decreasing (increasing) in the capital stock the probability of the most (least) favorable shock increases, and this incentivize the planner to increase (decrease) his capital investment, which in turn will generate a widening (reduction) in economic inequalities over time. We also show that the optimal solution can be converted into a contractive affine IFS with affine SDP which, under rather general conditions, converges to an invariant self-similar measure supported on a (possibly fractal) compact attractor. Moreover, we characterize the properties of the invariant self-similar measure associated with our IFSSDP in terms of singularity and absolutely continuity with respect to the Lebesgue measure, showing that this is ultimately related to the magnitude of the capital share.

To the best of our knowledge, ours is the first attempt to analyze the role of state-dependent probabilities in optimal stochastic economic growth settings. Therefore, several interesting issues associated with the role of state-dependent probabilities on macroeconomic dynamics still need to be uncovered. We have considered only the situation in which the probability function monotonically depends on the capital stock, thus it is natural to wonder how results may change in more general settings in which the probability may be non-monotonic in capital. We have also analyzed only the centralized outcome in which the social planner internalizes the dependence of the shocks probability on capital, thus it may be interesting to understand how results would change in a decentralized setting and how to eventually decentralize the social optimum. The analysis of these further issues is left for future research.

Acknowledgments

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A Proofs of the Main Results

A.1 Proof of Theorem 4

1) The first conclusion is clear since whenever $\alpha < 1/2$ the measure μ_α is supported on a Cantor set with zero Lebesgue measure. 3) By the results in Peres and Solomyak (1996) the IFS $\{\alpha x, \alpha x + 1 - \alpha\}$ satisfies the transversality condition for all $\alpha > 1/2$ and then the conclusion follows by Theorem 1.1 in Bárány (2015). 2) Since $\alpha = 1/2$ is fixed, we use μ rather than μ_α to avoid extraneous clutter on our notation. Recall that the "Markov operator" is given by

$$M\nu(S) = \sum_i \int_{w_i^{-1}(S)} p_i(x)\, d\nu(x) = \sum_i \int_S p_i\left(w_i^{-1}(x)\right) d\nu\left(w_i^{-1}(x)\right)$$

and that μ is the unique fixed point of M. Suppose that μ is absolutely continuous with density function $f(x)$. Then we obtain the equation

$$\begin{aligned}\int_S f(x)\,dx &= \sum_i \int_{w^{-1}(S)} p_i(y) f(y)\,dy = 2\sum_i \int_S f\left(w_i^{-1}(x)\right) p_i\left(w_i^{-1}(x)\right) dx \\ &= 2\int_S \left[p_1(2x) f(2x) \chi_{[0,1/2]}(x) + p_2(2x-1) f(2x-1) \chi_{[1/2,1]}(x)\right] dx\end{aligned}$$

where $\chi_A(x)$ is the characteristic function of the set A. For this to be true for all Borel sets S we must have that, for almost every x, the two equations

$$f(x) = 2f(2x) p_1(2x) \quad 0 \le x \le 1/2,$$

and

$$f(x) = 2f(2x-1) p_2(2x-1) \quad 1/2 \le x \le 1.$$

Doing a simple change of variable these become for $0 \le y \le 1$

$$f(y/2) = 2f(y) p_1(y) \quad \Longrightarrow \quad p_1(y) = \frac{f(y/2)}{2f(y)}$$

and

$$f(y/2 + 1/2) = 2f(y) p_2(y) \quad \Longrightarrow \quad p_2(y) = \frac{f(y/2+1/2)}{2f(y)}.$$

Then the condition that $p_1(y) + p_2(y) = 1$ implies that

$$2f(y) = f(y/2) + f(y/2 + 1/2) \tag{38}$$

for Lebesgue almost every $y \in [0,1]$. Thus $f(x)$ is the fixed point of the operator

$$T(g)(x) = \frac{g(x/2) + g(x/2 + 1/2)}{2}.$$

We show that the only fixed point of T which is a density function is the constant function $g(x) = 1$. It is easy to see that Tg is a density if g is a density. Suppose that $g \in C^1[0,1]$. Then $(Tg)'(x) = g'(x/2)/4 + g'(x/2+1/2)/4$ and so $\|(Tg)'\|_\infty \leq \frac{1}{2}\|g'\|_\infty$. By induction this means that

$$\|(T^n g)'\|_\infty \leq \frac{1}{2^n}\|g'\|_\infty. \tag{39}$$

Next, suppose that we have a density function $g \in C^1[0,1]$ with $|g'(x)| \leq m$ for all $x \in [0,1]$. Then for all $x \in [0,1]$ we have

$$g(0) - mx \leq g(x) \leq g(0) + mx$$

and thus, integrating over $[0,1]$, we have

$$g(0) - m/2 \leq 1 \leq g(0) + m/2 \Rightarrow |g(0) - 1| \leq m/2$$

and so

$$|g(x) - 1| \leq mx + m/2 \leq \frac{3}{2}m \Rightarrow \|g-1\|_\infty \leq \frac{3}{2}m. \tag{40}$$

Next for two functions $f, g \in L^1[0,1]$, integrating the inequality

$$|T(f)(x) - T(g)(x)| \leq \frac{1}{2}|f(x/2) - g(x/2)| + \frac{1}{2}|f(x/2+1/2) - g(x/2+1/2)|$$

over $[0,1]$ we get

$$\begin{aligned}
&\int_0^1 |T(f)(x) - T(g)(x)|\,dx \\
&\qquad \leq \frac{1}{2}\int_0^1 |f(x/2) - g(x/2)|\,dx + \frac{1}{2}\int_0^1 |f(x/2+1/2) - g(x/2+1/2)|\,dx \\
&\qquad\qquad = \int_0^{1/2} |f(u) - g(u)|\,du + \int_{1/2}^1 |f(u) - g(u)|\,du = \int_0^1 |f(u) - g(u)|\,du,
\end{aligned}$$

and thus $\|T(f) - T(g)\|_1 \leq \|f - g\|_1$. Let f be a density function and let $\epsilon > 0$ be given. Then there is some density function $g \in C^1[0,1]$ so that $\|f-g\|_1 \leq \epsilon/2$. Then we have

$$\begin{aligned}
\|T^n(f) - 1\|_1 &\leq \|T^n(f) - T^n(g)\|_1 + \|T^n(g) - 1\|_1 \\
&\leq \|T^n(f) - T^n(g)\|_1 + \|T^n(g) - 1\|_\infty \\
&\leq \|f-g\|_1 + \frac{3}{2}\|(T^n g)'\|_\infty \\
&\leq \|f-g\|_1 + \frac{3}{2^{n+1}}\|g'\|_\infty \leq \epsilon
\end{aligned}$$

for sufficiently large n. Thus $T^n f \to 1$ in $L^1[0,1]$ for any density f and so the only density function which satisfies (38) is $f(x) = 1$.

A.2 Proof of Lemma 1

Fix an initial value k_0 for capital, possibly, but not necessarily, such that $k_0 \in \left(0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$, and suppose, by contradiction, that the optimal saving/investment $y^* = k_{t+1}$ in each period t remains bounded inside the interval $\left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$. Hence, both definitions (8) and (9) imply that at each time $t \geq 0$ the probability of the shock r is constant—given by $p(y^*) \equiv 1-\delta$ or $p(y^*) \equiv \delta$ respectively—so that either the Bellman equation (11) or (13) fully represent problem (6). It is well known that the optimal policy solving either equation (11) or equation(13) is the same and is given by (15); such a policy generates trajectories $k_{t+1} = h(k_t, z_t) = \alpha\beta z_t k_t^\alpha$ having the deterministic trajectory generated by the lower map $k_{t+1} = \alpha\beta r k_t^\alpha$ as lower bound, so that $k_{t+1} = \alpha\beta z_t k_t^\alpha \geq \alpha\beta r k_t^\alpha$ for all $t \geq 0$. The lower bound trajectory generated by $\underline{k}_{t+1} = \alpha\beta r \underline{k}_t^\alpha$ converges to the (deterministic) fixed point $\lim_{t\to\infty} \alpha\beta r \underline{k}_t^\alpha = (\alpha\beta r)^{\frac{1}{1-\alpha}}$. As condition (16) is equivalent to

$$e^{-\frac{1-\delta-\gamma}{\varepsilon}} < (\alpha\beta r)^{\frac{1}{1-\alpha}},$$

we conclude that there exists a finite date $\tau \geq 0$ such that $k_\tau \geq \underline{k}_\tau \geq e^{-\frac{1-\delta-\gamma}{\varepsilon}}$, thus contradicting the assumption that k_t remains bounded inside the interval $\left[0, e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)$ for all $t \geq 0$.

A.3 Proof of Proposition 1

Under the assumption that $B - \varepsilon C \ln r > 0$ (we shall see that it holds at the end of the proof) the RHS in (17) is strictly concave in y, and the FOC with respect to y yields the unique solution

$$y^* = \frac{\beta(B - \varepsilon C \ln r)}{1 + \beta(B - \varepsilon C \ln r)} z k^\alpha, \tag{41}$$

Substituting y^* as in (41) into the RHS of (17) after some algebra yields

$$\begin{aligned}
V(k,z) &= A + B\ln k + C\ln z \\
&= \ln\left[zk^\alpha - \frac{\beta(B-\varepsilon C\ln r)}{1+\beta(B-\varepsilon C\ln r)}zk^\alpha\right] + \beta(B-\varepsilon C\ln r)\ln\left[\frac{\beta(B-\varepsilon C\ln r)}{1+\beta(B-\varepsilon C\ln r)}zk^\alpha\right] \\
&\quad + \beta(A+\gamma C\ln r) \\
&= \alpha\left[1+\beta(B-\varepsilon C\ln r)\right]\ln k + \left[1+\beta(B-\varepsilon C\ln r)\right]\ln z \\
&\quad + \beta(B-\varepsilon C\ln r)\ln\left[\beta(B-\varepsilon C\ln r)\right] - \left[1+\beta(B-\varepsilon C\ln r)\right]\ln\left[1+\beta(B-\varepsilon C\ln r)\right] \\
&\quad + \beta(A+\gamma C\ln r).
\end{aligned}$$

By equating all similar terms in both sides we find that a solution of the Bellman equation (17) is given by the constants A, B and C that satisfy

$$\begin{cases}
(1-\beta)A = \beta\gamma C\ln r + \beta(B-\varepsilon C\ln r)\ln\left[\beta(B-\varepsilon C\ln r)\right] \\
\qquad\qquad - \left[1+\beta(B-\varepsilon C\ln r)\right]\ln\left[1+\beta(B-\varepsilon C\ln r)\right] \\
B = \alpha\left[1+\beta(B-\varepsilon C\ln r)\right] \\
C = 1+\beta(B-\varepsilon C\ln r).
\end{cases}$$

From the second and third equations we see that $B = \alpha C$, so that, after substituting this in the third equation, we easily find the value of C as in (20), $C = \frac{1}{1-\beta(\alpha-\varepsilon\ln r)}$, which, when replaced into the second equation, yields the (crucial) value for B as in (19): $B = \frac{\alpha}{1-\beta(\alpha-\varepsilon\ln r)}$.

After cumbersome algebra the value of parameter A can be easily obtained; as $B = \alpha C = \frac{\alpha}{1-\beta(\alpha-\varepsilon\ln r)}$, we get

$$\beta(B-\varepsilon C\ln r) = \beta(\alpha C - \varepsilon C\ln r) = \beta(\alpha-\varepsilon\ln r)C = \frac{\beta(\alpha-\varepsilon\ln r)}{1-\beta(\alpha-\varepsilon\ln r)},$$

so that:

$$
\begin{aligned}
A = & \frac{1}{1-\beta}\left\{\frac{\beta(\alpha-\varepsilon\ln r)}{1-\beta(\alpha-\varepsilon\ln r)}\ln\left[\frac{\beta(\alpha-\varepsilon\ln r)}{1-\beta(\alpha-\varepsilon\ln r)}\right]\right. \\
& \left.-\left[1+\frac{\beta(\alpha-\varepsilon\ln r)}{1-\beta(\alpha-\varepsilon\ln r)}\right]\ln\left[1+\frac{\beta(\alpha-\varepsilon\ln r)}{1-\beta(\alpha-\varepsilon\ln r)}\right]+\frac{\beta\gamma\ln r}{1-\beta(\alpha-\varepsilon\ln r)}\right\} \\
= & \frac{[1-\beta(\alpha-\varepsilon\ln r)]\ln[1-\beta(\alpha-\varepsilon\ln r)]+\beta(\alpha-\varepsilon\ln r)\ln[\beta(\alpha-\varepsilon\ln r)]+\beta\gamma\ln r}{(1-\beta)[1-\beta(\alpha-\varepsilon\ln r)]},
\end{aligned}
$$

which is the expression in (18).

After replacing B and C as in (19) and in (20) respectively into (41) we easily obtain

$$
y^* = h(k,z) = \frac{\beta(B-\varepsilon C\ln r)}{1+\beta(B-\varepsilon C\ln r)}zk^\alpha = \beta(\alpha-\varepsilon\ln r)zk^\alpha,
$$

which confirms the expression in (21) for the optimal policy.

The solution in (21) is certainly interior under condition (16). In fact, on one hand it is straightforward to show that

$$
\varepsilon < -\frac{(1-\alpha)(1-\delta-\gamma)}{\ln(\alpha\beta r)} \qquad \Longleftrightarrow \qquad \alpha\beta r\left(e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)^\alpha > e^{-\frac{1-\delta-\gamma}{\varepsilon}},
$$

which implies that, as $-\varepsilon\ln r > 0$, for any $k \geq e^{-\frac{1-\delta-\gamma}{\varepsilon}}$,

$$
y^* = \beta(\alpha-\varepsilon\ln r)zk^\alpha > \alpha\beta zk^\alpha \geq \alpha\beta rk^\alpha \geq \alpha\beta r\left(e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)^\alpha > e^{-\frac{1-\delta-\gamma}{\varepsilon}},
$$

that is, $y^* > e^{-\frac{1-\delta-\gamma}{\varepsilon}}$. On the other hand, to prove that $y^* = \beta(\alpha-\varepsilon\ln r)zk^\alpha < zk^\alpha$ we show that condition (16) implies that $0 < \beta(\alpha-\varepsilon\ln r) < \alpha-\varepsilon\ln r < 1$. To this purpose note that, as $0 < \alpha\beta < 1$ and $0 < r < 1$, the following holds:

$$
\begin{aligned}
\ln(\alpha\beta)+\ln r = \ln(\alpha\beta r) < \ln r \quad & \Longleftrightarrow \quad -\frac{1-\alpha}{\ln r} > -\frac{1-\alpha}{\ln(\alpha\beta r)} \\
& \Longrightarrow \quad -\frac{1-\alpha}{\ln r} > -\frac{(1-\alpha)(1-\delta-\gamma)}{\ln(\alpha\beta r)} > \varepsilon,
\end{aligned}
$$

where the last inequality is condition (16); as the last two inequalities are equivalent to $\alpha - \varepsilon\ln r < 1$, we have just established that that $y^* < zk^\alpha$.

The property that $0 < \beta(\alpha-\varepsilon\ln r) < 1 \Longleftrightarrow 1-\beta(\alpha-\varepsilon\ln r) > 0$ also implies that both coefficients B and C are strictly positive; this establishes that the RHS in (17) is strictly concave.

Finally, it is a simple exercise to show that problem (6) satisfies all assumptions of Theorem 9.12 on p. 274 in Stokey and Lucas (1989): therefore, the function $V(k,z) = A + B\ln k + C\ln z$—with coefficients A, B and C defined in (18), (19) and (20) respectively—that solves the Bellman equation (17) is exactly the value function of problem (6), while the function $h(k_t^*, z_t) = \beta(\alpha-\varepsilon\ln r)z_t(k_t^*)^\alpha$ defined in (21) is exactly the optimal policy. We omit the details for brevity.

A.4 Proof of Proposition 2

Provided that $B+\varepsilon C\ln r > 0$, the RHS in (23) is strictly concave in y; hence, steps similar to those used in the proof of Proposition 1 easily yield the values A, B and C as in (25),

(26) and (27), together with the optimal policy as in (28) and the optimal consumption as in (29). Being the same exercise as in the previous proof of Proposition 1, also establishing that Theorem 9.12 on p. 274 in Stokey and Lucas (1989) holds is straightforward, so that the function $V(k,z) = A + B\ln k + C\ln z$—with coefficients A, B and C defined in (25), (26) and (27) respectively—that solves the Bellman equation (23) is exactly the value function of problem (6), while the function $h(k_t^*, z_t) = \beta(\alpha + \varepsilon \ln r) z_t (k_t^*)^\alpha$ defined in (28) is exactly the optimal policy.

We only need to establish that the unique solution in (28) is interior under condition (24). In fact, on one hand it is immediately shown that

$$e^{-\frac{(1-\alpha)(1-\delta-\gamma)}{\varepsilon}} - (\beta r \ln r)\varepsilon < \alpha\beta r \qquad \Longleftrightarrow \qquad \beta(\alpha + \varepsilon\ln r) r \left(e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)^\alpha > e^{-\frac{1-\delta-\gamma}{\varepsilon}},$$

which implies that, for any $k \geq e^{-\frac{1-\delta-\gamma}{\varepsilon}}$,

$$y^* = \beta(\alpha + \varepsilon\ln r) z k^\alpha \geq \beta(\alpha + \varepsilon\ln r) r k^\alpha \geq \beta(\alpha + \varepsilon\ln r) r \left(e^{-\frac{1-\delta-\gamma}{\varepsilon}}\right)^\alpha > e^{-\frac{1-\delta-\gamma}{\varepsilon}},$$

that is, $y^* > e^{-\frac{1-\delta-\gamma}{\varepsilon}}$. Note, in turn, that, as $e^{-\frac{1-\delta-\gamma}{\varepsilon}} > 0$, the last inequality also establishes that $\beta(\alpha + \varepsilon\ln r) > 0$. On the other hand, as $\varepsilon \ln r < 0$, $\beta(\alpha + \varepsilon\ln r) < 1$ definitely holds, so that $y^* = \beta(\alpha + \varepsilon\ln r) z k^\alpha < z k^\alpha$ as well.

Finally,

$$\beta(\alpha + \varepsilon\ln r) > 0 \quad \Longleftrightarrow \quad \frac{\beta\alpha}{1 - \beta(\alpha + \varepsilon\ln r)} + \frac{\beta\varepsilon\ln r}{1 - \beta(\alpha + \varepsilon\ln r)} = \beta(B + \varepsilon C \ln r) > 0,$$

which establishes that the RHS in (23) is strictly concave in y.

A.5 Proof of Proposition 3

The inverse transformation of (30) yields k as a function of x according to $k = (\theta_i r)^{\frac{1}{1-\alpha}} \left(r^{-\frac{1}{1-\alpha}}\right)^x$. Therefore, $\tilde{p}_i(x) = p(k) = p\left[(\theta_i r)^{\frac{1}{1-\alpha}} \left(r^{-\frac{1}{1-\alpha}}\right)^x\right]$, so that, for $i = 1$, according to (8), $\tilde{p}_1(x) = \gamma - \varepsilon\ln k = \gamma - \varepsilon \ln\left[(\theta_1 r)^{\frac{1}{1-\alpha}} \left(r^{-\frac{1}{1-\alpha}}\right)^x\right]$, which is equivalent to (32), while, for $i = 2$, according to (9), $\tilde{p}_2(x) = 1 - \gamma + \varepsilon\ln k = 1 - \gamma + \varepsilon\ln\left[(\theta_2 r)^{\frac{1}{1-\alpha}} \left(r^{-\frac{1}{1-\alpha}}\right)^x\right]$, which is equivalent to (33). As, under Assumptions A.1 and A.2, Propositions 1 and 2 establish that $k_t \in \left[e^{-\frac{1-\delta-\gamma}{\varepsilon}}, 1\right]$ for all $t \geq 0$, definitions (8) and (9) guarantee that $\tilde{p}_i(x) = p(k) = p\left[(\theta_i r)^{\frac{1}{1-\alpha}} \left(r^{-\frac{1}{1-\alpha}}\right)^x\right]$ satisfy $0 < \tilde{p}_i(x) < 1$ for all $x \in [0,1]$ and $i = 1, 2$. Finally, as $\frac{\varepsilon \ln r}{1-\alpha} < 0$, clearly $\tilde{p}_1(x)$ is decreasing, while, as $-\frac{\varepsilon\ln r}{1-\alpha} > 0$, $\tilde{p}_2(x)$ is increasing.

References

[1] Alvaredo F, Chancel L, Piketty T, Saez E, Zucman G. Global Inequality Dynamics: New Findings from WID.world. American Economic Review: Papers & Proceedings 107: 404—409 (2017).

[2] Barnsley MF. *Fractals Everywhere*, Academic Press, New York (1989).

[3] Barnsley MF, Demko S, Elton J, Geronimo J. Invariant measures for Markov processes arising from iterated function systems with state-dependent probabilities, Ann. Inst. H. Poincaré Prob. Stat. 24 367–394 (1988). Erratum: 25, 589–590 (1990).

[4] Bárány B. On iterated function systems with place-dependent probabilities, Proc. AMS, 143: 419–432 (2015).

[5] Banerjee AV, Duflo E. Inequality and Growth: What Can the Data Say?. Journal of Economic Growth 8: 267—299 (2003).

[6] Bethmann D. A closed-form solution of the Uzawa-Lucas model of endogenous growth. Journal of Economics 90: 87–107 (2007).

[7] Brock WA, Mirman LJ. Optimal Economic Growth and Uncertainty: the Discounted Case. Journal of Economic Theory 4: 479–513 (1972).

[8] Cozzi G, Privileggi F. The Fractal Nature of Inequality in a Fast Growing World: New Revision, University of Glasgow, Dept. of Economics Discussion Papers Series 2009-30, Business School - Economics, University of Glasgow (2009). https://www.gla.ac.uk/media/media_126649_en.pdf

[9] Hutchinson J, Fractals and self-similarity. Indiana Univ. J. Math. 30: 713–747 (1981).

[10] Elton J, An ergodic theorem for iterated maps, J. Erg. Theory Dyn. Sys. 7: 481–488 (1987).

[11] Kunze H, La Torre D, Mendivil F, Vrscay ER. Fractal-based Methods in Analysis, Springer, New York (2012).

[12] Kuznets S, Economic Growth and Income Inequality. American Economic Review: 45, 1–28 (1955).

[13] La Torre D, Marsiglio S, Privileggi F. Fractals and Self-similarity in Economics: the Case of a Two-sector Growth Model. Image Analysis & Stereology 30: 143–151 (2011).

[14] La Torre D, Marsiglio S, Mendivil F, Privileggi F. Self-similar Measures in Multi-sector Endogenous Growth Models. Chaos, Solitons and Fractals 79: 40–56 (2015).

[15] La Torre D, Maki E, Mendivil F, Vrscay ER. Iterated Function Systems with Place-Dependent Probabilities and the Inverse Problem of Measure Approximation using Moments. Fractals 26: 1850076 (2018a).

[16] La Torre D, Marsiglio S, Privileggi F. Fractal attractors in economic growth models with random pollution externalities, Chaos 28: 055916 (2018b).

[17] La Torre D, Marsiglio S, Mendivil F, Privileggi F. Fractal Attractors and Singular Invariant Measures in Two-Sector Growth Models with Random Factor Shares. Communications in Nonlinear Science and Numerical Simulation 58: 185–201 (2018c).

[18] La Torre D, Marsiglio S, Mendivil F, Privileggi F. A stochastic economic growth model with health capital and state-dependent probabilities. Chaos, Solitons & Fractals 129: 81–93 (2019)

[19] Mitra T, Montrucchio L, Privileggi F. The Nature of the Steady State in Models of Optimal Growth under Uncertainty. Econ Theor 23: 39–71 (2003).

[20] Mitra T, Privileggi F. Cantor Type Invariant Distributions in the Theory of Optimal Growth under Uncertainty. J Difference Equ Appl 10: 489–500 (2004).

[21] Mitra T, Privileggi F. Cantor Type Attractors in Stochastic Growth Models. Chaos, Solitons Fractals 29: 626–637 (2006).

[22] Mitra T, Privileggi F. On Lipschitz Continuity of the Iterated Function System in a Stochastic Optimal Growth Model. J Math Econ 45: 185–198 (2009).

[23] Montrucchio L, Privileggi F. Fractal Steady States in Stochastic Optimal Control Models. Ann Oper Res 88: 183–197 (1999).

[24] Olson LJ, Roy S. Theory of Stochastic Optimal Economic Growth, in Dana RA, Le Van C, Mitra T, Nishimura K (Eds.). "Handbook on Optimal Growth 1: Discrete Time", ch. 11, 297–336, Springer (2005)

[25] Piketty T. The Economics of Inequality, Harvard University Press (2015)

[26] Piketty T, Saez E. Inequality in the long run. Science 344: 838–844 (2014)

[27] Peres Y, Solomyak B. Absolute continuity of Bernoulli convolutions, a simple proof, Math. Res. Lett. 3: 231–239 (1996).

[28] Ramsey F. A mathematical theory of saving. Economic Journal 38: 543—559 (1928).

[29] Scheidel, W. The Great Leveler: Violence and the History of Inequality from the Stone Age to the Twenty-First Century. Princeton University Press, Princeton, NJ (2017)

[30] Shmerkin P. On the exceptional set for absolute continuity of Bernoulli convolutions. Geom. Func. Anal. 24: 946–958 (2014).

[31] Stenflo Ö. Uniqueness of invariant measures for place-dependent random iteration of functions, in Fractals in Multimedia, IMA Vol. Math. Appl. 132: 13–32 (2002).

[32] Stokey NL, Lucas RE. Recursive methods in economic dynamics. Harvard University Press, Cambridge, MA (1989).