Existence of the weak and strong core in a sharing model with arbitrary graph structures *

Jay Sethuraman†, Sonal Yadav‡

November 23, 2022

Abstract

We consider a model where agents are nodes on a graph and two agents are potential partners if they are connected by an edge in the graph. Agents have to be matched in pairs and have to contribute effort levels to complete a task of unit value. An allocation consists of pairs of agents and a sharing arrangement for each pair. Each agent has symmetric preferences around an ideal contribution level. We show that the strong core exists in an instance of the problem if and only if the optimal values of the integer matching, fractional matching and convering problems, coincide. The weak core exists if the optimal values of the integer and fractional matching problems coincide and always exists for the bipartite and complete graphs. A closely related paper is Nicolò et al. (2022). They provide an algorithm that generates a weak core allocation in the model with a complete graph and single-peaked preferences.

JEL classification: C78; D47; D71

Keywords: Matching; Strong Core; Weak Core

1 Introduction

We consider a model where agents are nodes in a graph. If two agents are connected by an edge, they are potential partners. Once a partnership consisting of a pair of agents is formed, they are required to perform a task of unit value. Each agent has an ideal contribution that she would like to make and preferences are symmetric. For example, agent i may optimally wish to contribute 0.3 with the contribution t_i preferred to t_i' if and only if t_i is closer to 0.3

*Sonal Yadav gratefully acknowledges financial support from the Jan Wallander and Tom Hedelius Foundation.

†Industrial Engineering and Operations Research, Columbia University, NY, USA.

‡Umeå University, Umeå, Sweden.

than t_i'. An *allocation* is a pairing (or matching) of all agents and a sharing arrangement for each pair. We investigate the existence and properties of strong and weak core allocations in this model.

We believe that situations we analyse arise naturally in several contexts. Managers may have to assign pairs of workers to tasks where each worker has to contribute effort. Job sharing in the software industry where programmers from different locations collaborate in order to execute projects, is fairly common. Roommates have to share household duties. In each case the compatibility of agents is specified by the graph structure. The requirement that allocations in these situations must belong to the core can be interpreted as requiring the absence of justified envy (see Roth (2002) and Kagel and Roth (2000)). The presence of justified envy is likely to cause disaffection among agents and disrupt the smooth functioning of institutions.

Our model is a variant of the well-known Shapley and Shubik (1971) model (henceforth referred to as the assignment model). In that model, the underlying graph is bipartite with buyers on one side and sellers on the other. In contrast, we consider a general graph structure - for example, we allow for the possibility that the underlying graph is complete so every agent is a potential partner for any other. Another significant departure from the assignment model is that in our setting, agents can be satiated by giving them their ideal contribution. Satiated agents will not be part of any coalition that blocks strongly. As a result, the strong and weak cores may differ. On the other hand, there is no distinction between the strong and the weak core in the assignment model. Moreover, the bipartite structure of the graph ensures the existence of the core. It is well-known that existence will fail if the graph is not assumed to be bipartite.

Fix a graph and a profile of preferences (i.e. a profile of ideal contributions). This induces a cost c_{ij} for every edge (i, j) in the graph. We construct three distinct linear programs from this data. In each case, the objective function is the cost associated with a matching. The three problems differ with respect to the class of matchings over which an optimum is sought. In the *Fractional Matching* problem (Program F), agents i and j can be matched with probability $x_{ij} \in [0, 1]$ and $\sum_j x_{ij} = 1$ for all i. In the *Integer Matching* problem (Program I), only deterministic matchings are considered, i.e. $x_{ij} \in \{0, 1\}$ and $\sum_j x_{ij} = 1$ for all i. Finally, in the *Covering* problem (Program P), $x_{ij} \in [0, 1]$ but $\sum_j x_{ij} \geq 1$ for all j. Let Z_I, Z_F and Z_P denote the optimal values of the three linear programs. It is clear from the nature of the constraints imposed in each of the problems that $Z_P \leq Z_F \leq Z_I$.

According to our results, the existence of the strong and weak cores depend crucially on the relationship between Z_I, Z_F and Z_P. The strong core exists if and only if $Z_I = Z_F = Z_P$. A sufficient condition for the existence of the weak core is $Z_I = Z_F$. Using these results, we show that the weak core exists in the bipartite and complete graphs for *all* profiles of preferences. We provide numerous examples that clarify the relationship between Z_I, Z_F and Z_P in the general case. Note that the existence of the strong and weak cores depends on *both* the structure of the graph and the preference profile. An open question is whether there are other graph structures where statements about existence can be made without reference to preferences (as in the bipartite and complete graph cases).

Our paper is closely related to Nicolò et al. (2022). They consider a very similar model where agents have to be matched in pairs and contribute effort in order to complete a task of unit value. They assume that all agents are compatible or equivalently, the underlying graph structure is complete. In this respect, their model is more specialized than ours. On the other hand, they assume that preferences of the agents are single-peaked with a unique peak. Our preferences are therefore included in the class of preferences they consider. They provide an algorithm called the SAM (Select-Allocate-Match) algorithm which explicitly computes an allocation in the weak core. They point out that the strong core may not exist but unlike us, do not investigate conditions under which it exists. However, a more fundamental difference between the two papers is that they employ completely different techniques to address the issue - we use ideas from linear programming (closer to the techniques in Shapley and Shubik (1971)) while their methods are more direct. We believe that our approach offers fresh insights into this interesting class of problems while proving several new results.

The rest of the paper is organized as follows. In Section 2, we introduce the model and basic definitions. Subsection 3.1 provides the linear programming formulation of the stable sharing problem. Subsections 3.2 and 3.3 provide the two main results on the strong and weak core respectively. Section 4 contains illustrative examples while Section 5 provides applications.

2 The Model

The set of agents is $N = \{1, \ldots, n\}$ where n is even. We consider a model where agents are nodes in a graph. If two agents are connected by an edge, they are potential partners. Once a partnership consisting of a pair of agents is formed, they are required to perform a task of unit value. We assume that no agent can remain on her own and each agent can have only one partner.

The set of possible partnerships is captured by the graph $G = (N, E)$ with N as the set of vertices. The edge-set E captures all feasible pairs of agents (partnerships) in the model. A pair of agents (i, j) is a feasible pair if $(i, j) \in E$. We refer to Graph G as the partnership graph. We assume that the partnership graph G admits at least one perfect matching.[1] Our objective is to identify who is paired with whom (matching), and how the unit value is shared by the matched agents.

An allocation (M, t) is a matching $M \subseteq E$ and a vector $t = (t_k)_{k \in N}$ where $t_k \in [0, 1]$ for all $k \in N$ and $t_i + t_j = 1$ for all $(i, j) \in M$. For a pair $(i, j) \in M$, we refer to (t_i, t_j)

[1] One special case is that G is a complete graph.

as the contribution vector (tuple) for the agents i, j who have been paired together in the allocation.

Each agent i has a preference ordering $\succsim_i$ over her contribution.[2] We assume $\succsim_i$ is *symmetric* single-peaked. The ordering $\succsim_i$ is single-peaked if there exists a unique contribution $p_i \in [0,1]$ such that for all $x, y \in [0,1]$, if $x < y < p_i$ or $x > y > p_i$ then $y \succ_i x$. The contribution p_i will be referred to as the peak of agent i in $\succsim_i$. A special instance of a single-peaked preference is a *symmetric* or *Euclidean* preference: $x \succsim_i y$ if and only if $|x - p_i| \leq |y - p_i|$. A preference profile $\succsim$ is an n-tuple of preferences $(\succsim_1, \ldots, \succsim_n)$.

The fundamental property that an allocation should satisfy is *stability*. Observe that in our model, "only" a feasible pair of agents can block an allocation.

Definition **1** *Fix a preference profile $\succsim$. Let (M,t) be an allocation and $i, j \in N$ be a feasible pair of agents with contributions t_i and t_j respectively in (M,t). Then the feasible pair (i,j) strongly blocks (M,t) at $\succsim$ if there exists a contribution vector $(\bar{t}_i, \bar{t}_j)$ such that $\bar{t}_i + \bar{t}_j = 1$, $\bar{t}_i \succ_i t_i$ and $\bar{t}_j \succ_j t_j$. An allocation belongs to the weak core if it is not strongly blocked by any feasible pair of agents.*

A more permissive notion of blocking is weak blocking where only one of the blocking agents is better-off while the other one no worse-off. An allocation that cannot be weakly blocked by any feasible pair of agents belongs to the strong core. It is easy to check that a strong core allocation is also a weak core allocation while the converse may not be true.

Definition **2** *Fix a preference profile $\succsim$. Let (M,t) be an allocation and $i, j \in N$ be a feasible pair of agents with contributions t_i and t_j respectively in (M,t). Then the feasible pair (i,j) weakly blocks (M,t) at $\succsim$ if there exists a contribution vector $(\bar{t}_i, \bar{t}_j)$ such that $\bar{t}_i + \bar{t}_j = 1$, $\bar{t}_i \succsim_i t_i$ and $\bar{t}_j \succsim_j t_j$ with either $\bar{t}_i \succ_i t_i$ or $\bar{t}_j \succ_j t_j$. An allocation belongs to the strong core if it is not weakly blocked by any feasible pair of agents.*

Observe that for any feasible pair of agents in our model, it is possible to compute the cost of a partnership between them using their peaks. Consider agents $i, j \in N$ with $(i,j) \in E$. We color the edge (i,j) red if $p_i + p_j > 1$, blue if $p_i + p_j < 1$ and black if $p_i + p_j = 1$. Define $c_{ij} = |p_i + p_j - 1|$ for all $(i,j) \in E$. The cost c_{ij} captures the net excess of a partnership between agents connected by a red edge. Similarly c_{ij} captures the net deficit of a partnership between agents connected by a blue edge. Also the cost of the partnership c_{ij} is zero for black edges. Note that $c_{ij} \in [0,1]$.

A natural approach is to follow the techniques in the analysis of the assignment model: computing the efficient[3] matching between the buyers and sellers and determining the shares

[2] The asymmetric and symmetric components of $\succsim_i$ are denoted by $\succ_i$ and $\sim_i$ respectively.

[3] The efficient matching maximises the sum of surplus generated.

in the surplus from the dual. In our model, an efficient matching is one that minimises the sum of costs incurred. The following example illustrates that this approach fails in our model. Consider a complete bipartite graph where the agents are partitioned into two sides.[4] We show that the efficient matching does not support a strong core allocation. In fact, the strong core is empty. The example highlights the reasons for the non-existence of strong core in our model.

Example **1** Let $N = \{1, 2, 3, 4, 5, 6\}$. The peaks of the agents are summarized in Table 1. Figure 1 shows the partnership graph, where every black edge has zero cost and every blue edge has cost 0.3.

p_1	p_2	p_3	p_4	p_5	p_6
0.1	0.4	0.4	0.9	0.9	0.6

Table 1: Peaks of agents in Example 1.

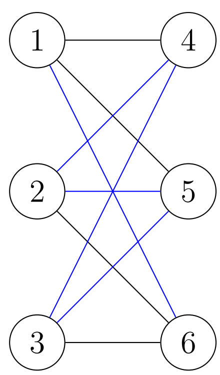

Figure 1: Graph in Example 1

We now compute the cost of the feasible partnerships (or edges) using the idea described above. The following edges have zero cost: $(1, 4)$, $(1, 5)$, $(2, 6)$ and $(3, 6)$. An edge between an agent in $\{2, 3\}$ and an agent in $\{4, 5\}$ has cost 0.3.[5] Any efficient matching must include two edges with zero cost and an edge with cost 0.3.

Here the efficient matching does not support a strong core allocation. In fact, the strong core is empty. To see this, observe that in any strong core allocation, each agent must be given their peak. Suppose not, then the agent not receiving her peak can (weakly) block using one of edges in $\{(1, 4), (1, 5), (2, 6), (3, 6)\}$.

[4] We use a bipartite graph to keep the setting as similar as possible to the Shapley-Shubik model.

[5] Costs of the other edges can be computed similarly. Edge $(1, 6)$ also has cost 0.3. However including the edge $(1, 6)$ in a matching will result in cost 0.9. This is because then the matching must include two edges, like $(2, 4)$ and $(3, 5)$, both with cost 0.3.

3 The Core

We first make some preliminary observations about core allocations.

Observation **1** In any core allocation (M, t), the following is true: (i) matched agents connected by a black edge have allocations equal to their peaks; (ii) matched agents connected by a red edge each have allocations (weakly) below their peaks; and (iii) matched agents connected by a blue edge each have allocations (weakly) exceeding their peaks. To see this, suppose not. Then there exist agents i and j who are matched and their contributions are on the "opposite" sides of their peak. Then (i, j) can block by transferring an infinitesimal amount from one to the other such that both agents are moved closer to their peaks.

Thus in any stable allocation, agents who are paired together must be given contributions on the same side of their peaks.

3.1 LP form of the model

Our goal is to provide a quick and efficient way to determine whether or not a given instance of the stable sharing problem admits an allocation in the core. It will be useful to recast the stable sharing problem into linear programming terminology. An allocation in our model consists of a matching of agents and a contribution vector for each pair.

We first consider the matching problem of agents: introduce real variables x_{ij} for all $(i, j) \in E$ (x_{ij} can be interpreted as the probability that agents i and j are paired together) and impose on them n constraints where $\sum_{j:(i,j)\in E} x_{ij} = 1$ for all $i \in N$. The objective is to obtain the matching with minimum cost (or weight). This is described by the integer programming formulation below.

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in E} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j:(i,j)\in E} x_{ij} = 1, \quad \forall i \in N, \\
& x_{ij} \in \{0, 1\}, \quad \forall (i, j) \in E.
\end{aligned}
$$

We refer to the minimum-weight matching problem as the integer program I. Let Z_I be the optimal cost of the integer program I. Since we assume that the graph G admits a perfect matching, there is at least one feasible solution to this integer program. Thus Z_I is finite.

Suppose we "relax" this integer programming problem by allowing (i) $x_{ij} \geq 0$ for all $(i, j) \in E$ and (ii) $\sum_{j:(i,j)\in E} x_{ij} = 1$ for all $i \in N$. These two features only enlarge the space of feasible solutions, so the optimal cost of the resulting linear program can only be lower. We write down this linear program F as

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in E} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j:(i,j)\in E} x_{ij} = 1, \quad \forall i \in N, \\
& x_{ij} \geq 0, \quad \forall (i,j) \in E.
\end{aligned}
$$

Let Z_F be the optimal cost of the linear program F. The linear program F is the minimum-weight fractional matching problem.

A further "relaxation" of the linear program F can be done by allowing (i) $x_{ij} \geq 0$ for all $(i,j) \in E$; and (ii) $\sum_{j:(i,j)\in E} x_{ij} \geq 1$ for all $i \in N$.[6] This enlarges the space of feasible solutions even further, so the optimal cost of the resulting linear programming problem can only be lower. We write down this linear program P as

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in E} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j:(i,j)\in E} x_{ij} \geq 1, \quad \forall i \in N, \\
& x_{ij} \geq 0, \quad \forall (i,j) \in E.
\end{aligned}
$$

Let Z_P be the optimal cost of the linear program P. The linear program P is the minimum-weight covering problem. A simple observation is that $Z_P \leq Z_F \leq Z_I$.

Let D be the dual of the linear program P. The dual of P has n non-negative real variables, π_i for all $i \in N$. The dual D is given by

$$
\begin{aligned}
\max \quad & \sum_{i\in N} \pi_i \\
\text{s.t.} \quad & \pi_i + \pi_j \;\leq\; c_{ij}, \quad \forall (i,j) \in E, \\
& \pi_i \geq 0, \quad \forall i \in N.
\end{aligned}
$$

Let Z_D be the optimal value of the dual D. By LP duality, $Z_D = Z_P$, and by our earlier discussion, $Z_D \leq Z_I$.

[6]Note that linear program F is also a relaxation of the integer program I.

The three programs play a central role in our analysis about the existence of the strong and weak core. Our first result shows that $Z_I = Z_F = Z_P$ is a necessary and sufficient condition for the existence of the strong core. The second result pertains to the weak core: $Z_I = Z_F$ is a sufficient condition for the existence of the weak core, but is not a necessary condition.

We provide some examples below to show that the structure of the graph and the preference profile "jointly" determine whether the equality $Z_I = Z_F = Z_F$ holds or not.[7]

EXAMPLE **2** Let $N = \{1, 2, 3, 4\}$. The peaks of the agents are summarized by Table 2. Figure 2 contains the partnership graph along with cost of each partnership.

p_1	p_2	p_3	p_4
0.9	0.2	0.2	0.2

Table 2: Peaks of agents in Example 2.

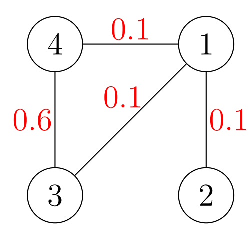

Figure 2: Graph in Example 2

The partnership graph has one perfect matching with pairs $(1, 2)$ and $(3, 4)$, with cost 0.7. So the optimal cost of the integer program is $Z_I = 0.7$.

Consider the minimum weight fractional matching problem F. Since agent 2 has only one edge in the graph, this edge $(1, 2)$ must have $x_{12} = 1$ in any feasible solution of F. This implies $x_{13} = x_{14} = 0$ as the sum of the x_{ij}'s for any node must be exactly one. Also $x_{34} = 1$. Thus $Z_F = 0.7 = Z_I$.

Finally we consider the minimum weight covering problem P. Note that $\sum_{j:(i,j)\in E} x_{ij} \geq 1$ for all nodes $i \in N$. A feasible solution of P is $x_{12} = x_{13} = x_{14} = 1$ and $x_{34} = 0$. Note that it is possible to assign probability 0 to the expensive edge $(3, 4)$. In fact the optimal value of the program P is attained at this feasible solution with $Z_P = 0.3$. Thus in this example, we observe that $Z_I = Z_F < Z_P$.

[7] In Section 5, we provide two applications: fix a graph structure and show that $Z_I = Z_F$ holds for all preference profiles.

The second part of the inequality is important even in the bipartite version of our problem. In assignment model, the second inequality plays no role. In our model, we require that agents must be paired while in assignment model, agents have the option of remaining unmatched.

3.2 Strong Core

We are now ready to prove the following theorem. The necessity part of the theorem shows that the efficient matching plays a critical role for the existence of the strong core. In fact, if a strong core allocation exists then it must be supported by the efficient matching.

Theorem **1** *Consider an arbitrary graph and assume that agents have symmetric preferences. The stable sharing problem has an element in the strong core if and only if $Z_I = Z_D$.*

Proof: We first prove the sufficiency result. Assume $Z_I = Z_D$. We will construct a strong core allocation using the optimal solutions to the primal and dual linear programs.

In particular, this implies that the minimum-cost fractional matching—one in which $x_{ij} \geq 0$, but $\sum_{j:(i,j)\in E} x_{ij} = 1$ for each $i \in V$—is exactly Z_I as well.

Consider the integer program I. Let M be an optimal integer matching. Define $x^*_{ij} = 1$ whenever $(i,j) \in M$ and zero otherwise. Let x^* be the characteristic vector of the given minimum-cost perfect matching M. Because $Z_I = Z_D$, we can view x^* as an optimal solution to the linear program P as well. Let π^* be an optimal solution to the dual D. Because x^* and π^* are optimal solutions to a primal-dual pair of linear programs, they must satisfy the complementary slackness conditions. In particular, $x^*_{ij} > 0 \implies \pi^*_i + \pi^*_j = c_{ij}$. Thus for any pair of agents (i,j) who are matched together, we know $\pi^*_i + \pi^*_j = c_{ij}$.

We next show how to construct a contribution vector t^* such that the allocation (M, t^*) is in the strong core of the given stable allocation problem. Consider the allocation t^*, computed based on the color of the matched edge in the following manner:

- If (i,j) is a black edge, let $t^*_i = p_i$ and $t^*_j = p_j$;
- If (i,j) is a red edge, let $t^*_i = p_i - \pi^*_i$ and $t^*_j = p_j - \pi^*_j$; and
- If (i,j) is a blue edge, let $t^*_i = p_i + \pi^*_i$ and $t^*_j = p_j + \pi^*_j$.

We claim that the allocation (M, t^*) is a strong core allocation. First we show that (M, t^*) is an allocation. We verify $t^*_i + t^*_j = 1$ for all edges $(i,j) \in M$. We know that $\pi^*_i + \pi^*_j = c_{ij}$ for all edges in M. We consider the three cases.[8]

(a) If (i,j) is a black edge, then $p_i + p_j = 1$. So $t^*_i + t^*_j = 1$.

[8]Note that the non-negativity of π^* is not used to show that (M, t^*) is an allocation.

(b) If (i,j) is a red edge, then $c_{ij} = p_i + p_j - 1$. So $t_i^* + t_j^* = p_i - \pi_i^* + p_j - \pi_j^* = 1 + c_{ij} - \pi_i^* - \pi_j^* = 1$.

(c) If (i,j) is a blue edge, then $c_{ij} = 1 - p_i - p_j$. So $t_i^* + t_j^* = p_i + \pi_i^* + p_j + \pi_j^* = 1 - c_{ij} + \pi_i^* + \pi_j^* = 1$.

We color each node the same color as the matched edge incident to it. We know $\pi_i^* \geq 0$ for all $i \in N$. Thus the black nodes receive their peak, the red nodes are (weakly) below their peak and the blue nodes are (weakly) above their peak.

Now we will prove that (M, t^*) is a strong core allocation, i.e. we show that no edge (or a pair of agents) can block the allocation (M, t^*). Clearly no edge of M can block this solution: the agents matched in M are on the "same" side of the peak, so there is no way for them to weakly block M.

Suppose $(i,j) \notin M$ blocks M via the contribution vector $(\tilde{t}_i, \tilde{t}_j)$. Without loss of generality, we can assume that i and j are not on opposite sides of their peaks.[9] Define $\tilde{\pi}_i = |\tilde{t}_i - p_i|$ and $\tilde{\pi}_j = |\tilde{t}_j - p_j|$. If (i,j) blocks M, then $\tilde{\pi}_i \leq \pi_i^*$, $\tilde{\pi}_j \leq \pi_j^*$, with at least one of these inequalities being strict. In particular, $\tilde{\pi}_i + \tilde{\pi}_j < \pi_i^* + \pi_j^* \leq c_{ij}$, where the last inequality follows from the dual feasibility of π^*.

Now, if $p_i + p_j \geq 1$, we must have $\tilde{t}_i \leq p_i$ and $\tilde{t}_j \leq p_j$. In this case,

$$\begin{aligned} \tilde{t}_i + \tilde{t}_j &= p_i - \tilde{\pi}_i + p_j - \tilde{\pi}_j \\ &= p_i + p_j - (\tilde{\pi}_i + \tilde{\pi}_j) \\ &> p_i + p_j - c_{ij} \\ &= 1, \end{aligned}$$

contradicting the requirement that $\tilde{t}_i + \tilde{t}_j = 1$.

If $p_i + p_j \leq 1$, we must have $\tilde{t}_i \geq p_i$ and $\tilde{t}_j \geq p_j$. In this case,

$$\begin{aligned} \tilde{t}_i + \tilde{t}_j &= p_i + \tilde{\pi}_i + p_j + \tilde{\pi}_j \\ &= p_i + p_j + (\tilde{\pi}_i + \tilde{\pi}_j) \\ &< p_i + p_j + c_{ij} \\ &= 1, \end{aligned}$$

contradicting the requirement that $\tilde{t}_i + \tilde{t}_j = 1$.

Thus we have shown that if $Z_I = Z_D$, the allocation (M, t^*) is in the strong core, i.e. the strong core is non empty.

[9] If $\tilde{t}_i < p_i$ and $\tilde{t}_j > p_j$, there is an even better contribution vector that they both prefer and can use to block.

To prove the necessity, we proceed similarly. Suppose we are given a strong core allocation (M, t) with matching M and contribution vector t. We will show that $Z_I = Z_D$.

Consider the solution x to the linear program P where we assign $x_{ij} = 1$ for each $(i, j) \in M$ and $x_{ij} = 0$ for each $(i, j) \notin M$. It is clear that x is feasible for the integer program I and for P.

We consider the following solution for the dual D: define $\pi_i = |p_i - t_i|$ for each $i \in N$. We will show that the constructed π is feasible for D. In particular, we will show $\pi_i + \pi_j = c_{ij}$ for all $(i, j) \in M$ and $\pi_i + \pi_j \leq c_{ij}$ for all $(i, j) \notin M$. By our assumption that (M, t) is a strong core allocation and Observation 1, we know the following holds for each $(i, j) \in M$: (a) if $p_i + p_j = 1$, $t_i = p_i$ and $t_j = p_j$; (b) if $p_i + p_j < 1$, $t_i \geq p_i$ and $t_j \geq p_j$; (c) if $p_i + p_j > 1$, $t_i \leq p_i$ and $t_j \leq p_j$. Thus, for each $(i, j) \in M$ with $p_i + p_j < 1$, we have

$$\pi_i + \pi_j \;=\; t_i - p_i + t_j - p_j \;=\; 1 - p_i - p_j \;=\; c_{ij}.$$

Similarly, for each $(i, j) \in M$ with $p_i + p_j > 1$, we have

$$\pi_i + \pi_j \;=\; p_i - t_i + p_j - t_j \;=\; p_i + p_j - 1 \;=\; c_{ij}.$$

Also, for each $(i, j) \in M$ with $p_i + p_j = 1$, we have $\pi_i \;=\; \pi_j \;=\; 0$, thus verifying

$$\pi_i + \pi_j \;=\; 0 \;=\; c_{ij}.$$

We now verify the dual feasibility of this solution for all pairs $(i, j) \notin M$. Suppose not. Fix an $(i, j) \in E \setminus M$ such that $\pi_i + \pi_j > c_{ij}$. Without loss of generality, suppose $\pi_i \geq \pi_j$. Define $\delta := \pi_i + \pi_j - c_{ij}$, and note that $\delta > 0$. We consider two cases.

(a) If $\pi_j \geq \delta/2$, let $\tilde{\pi}_i := \pi_i - \delta/2$ and $\tilde{\pi}_j = \pi_j - \delta/2$.

(b) If $\pi_j < \delta/2$, let $\tilde{\pi}_i := c_{ij}$ and $\tilde{\pi}_j = 0$.

In each case, we verify that $0 \leq \tilde{\pi}_i < \pi_i$ and $0 \leq \tilde{\pi}_j \leq \pi_j$. Furthermore, we verify that $\tilde{\pi}_i + \tilde{\pi}_j = c_{ij}$. This is immediate in case (a) as both $\pi_i, \pi_j \geq \delta/2$. In case (b), observe that

$$\pi_j < \delta/2 \iff 2\pi_j < \pi_i + \pi_j - c_{ij} \iff \pi_j + c_{ij} < \pi_i,$$

which implies $\tilde{\pi}_i = c_{ij} < \pi_i - \pi_j < \pi_i$.

Consider the following allocation, $(\tilde{t}_i, \tilde{t}_j)$, for agents i and j:

- If $p_i + p_j \geq 1$, let $\tilde{t}_i = p_i - \tilde{\pi}_i$ and $\tilde{t}_j = p_j - \tilde{\pi}_j$. Observe that

$$\tilde{t}_i + \tilde{t}_j \;=\; p_i + p_j - \tilde{\pi}_i - \tilde{\pi}_j \;=\; p_i + p_j - c_{ij} \;=\; 1.$$

- If $p_i + p_j < 1$, let $\tilde{t}_i = p_i + \tilde{\pi}_i$ and $\tilde{t}_j = p_j + \tilde{\pi}_j$. Observe that

$$\tilde{t}_i + \tilde{t}_j \;=\; p_i + p_j + \tilde{\pi}_i + \tilde{\pi}_j \;=\; p_i + p_j + c_{ij} \;=\; 1.$$

By construction, agent i strictly prefers and agent j weakly prefers $(\tilde{t}_i, \tilde{t}_j)$ to (t_i, t_j), contradicting the assumption that (M, t) is a strong core allocation. Thus, starting from a strong core allocation we have constructed a feasible integer solution to P and a feasible solution to D that satisfies the complementary slackness conditions. So $Z_I = Z_P = Z_D$. ■

Observation **2** Observe that if a strong core allocation exists, then it must be supoorted by an efficient matching. This follows from the necessity part of Theorem .

3.3 Weak Core

In this section, we provide a sufficient condition for the existence of the weak core. The condition requires that the optimal solution to the minimum-weight fractional matching problem is integral. Formally, we require $Z_I = Z_F$ where I is the minimum-weight integer program and F is the minimum-weight fractional matching problem. In particular, we show that the efficient matching supports a weak core allocation when $Z_I = Z_F$.

Before stating the theorem, we will define the dual $\bar{D}$ of the linear program F. The optimal solution of the dual will be used to construct the weak core allocation. The dual $\bar{D}$ of program F is,

$$\begin{aligned} \max \quad & \sum_{i \in N} \pi_i \\ \text{s.t.} \quad & \pi_i + \pi_j \ \leq \ c_{ij}, \quad \forall (i,j) \in E. \end{aligned}$$

Remark **1** It is important to note that the dual variables in $\bar{D}$ are unconstrained. This is because the primal problem F has equality constraints. This is in constrast to the dual of the minimum-weight covering problem P, where the dual variables are non-negative. It is important to note that non-negativity of the dual variables is essential for the construction of the strong core allocation when we prove the sufficiency result for the strong core (Theorem 2). However it is possible to construct a weak core allocation from the unconstrained dual variables in $\bar{D}$.

Let x^* be an optimal solution to F. The sufficient condition ensures that x^* is integral. By standard LP duality, we know that $\bar{D}$ has an optimal solution π^* such that:

(a) $\sum_{i,j} c_{ij} x^*_{ij} \ = \ \sum_{i \in N} \pi^*_i$ and

(b) $\pi^*_i + \pi^*_j = c_{ij}$ whenever $x^*_{ij} > 0$.

Theorem 2 establishes that the weak core exists when $Z_I = Z_F$. To prove this, we will use the optimal solution for $\bar{D}$, π^* to construct a weak core allocation.

Theorem 2 *Consider an arbitrary graph and assume that agents have symmetric preferences. If $Z_I = Z_F$, then the weak core is non-empty.*

Proof: We know $Z_I = Z_F$. Thus the optimal solution to F, x^* is integral. By standard LP duality, we know that $\bar{D}$ has an optimal solution π^* which satisfies the properties described above. In particular, $\pi_i^* + \pi_j^* = c_{ij}$ whenever $x_{ij}^* > 0$ and $\pi_i^* + \pi_j^* \le c_{ij}$ whenever $x_{ij}^* = 0$. We will use π^* to construct a weak core allocation. There are two cases: (I) $\pi_i^* \ge 0$ for all $i \in N$ and (II) $\pi_i^* < 0$ for some $i \in N$. We consider the two cases separately.

Case (I): $\pi_i^* \ge 0$ for all $i \in N$.

Consider the natural matching M where $(i, j) \in M$ whenever $x_{ij}^* > 0$ and $(i, j) \notin M$ whenever $x_{ij}^* = 0$. We will construct a contribution vector t^* such that the allocation (M, t^*) belongs to the weak core. We compute t^* based on the color of the matched edge in the following manner:

- If (i, j) is a black edge, let $t_i^* = p_i$ and $t_j^* = p_j$;
- If (i, j) is a red edge, let $t_i^* = p_i - \pi_i^*$ and $t_j^* = p_j - \pi_j^*$; and
- If (i, j) is a blue edge, let $t_i^* = p_i + \pi_i^*$ and $t_j^* = p_j + \pi_j^*$.

We show below that (M, t^*) is an allocation by verifying $t_i^* + t_j^* = 1$ for all edges $(i, j) \in M$. We know that $\pi_i^* + \pi_j^* = c_{ij}$ for all edges in M. There are three cases.[10]

(a) If (i, j) is a black edge, then $p_i + p_j = 1$. So $t_i^* + t_j^* = 1$.

(b) If (i, j) is a red edge, then $c_{ij} = p_i + p_j - 1$. So $t_i^* + t_j^* = p_i - \pi_i^* + p_j - \pi_j^* = 1 + c_{ij} - \pi_i^* - \pi_j^* = 1$.

(c) If (i, j) is a blue edge, then $c_{ij} = 1 - p_i - p_j$. So $t_i^* + t_j^* = p_i + \pi_i^* + p_j + \pi_j^* = 1 - c_{ij} + \pi_i^* + \pi_j^* = 1$.

We color each node the same color as the matched edge incident to it. We know $\pi_i^* \ge 0$ for all $i \in N$. Thus the black nodes receive their peak, the red nodes are (weakly) below their peak and the blue nodes are (weakly) above their peak.

We show below that (M, t^*) is a weak core allocation. Clearly no edge of M can block this solution: the agents matched in M are on the "same" side of the peak, so there is no way for them to weakly block M. Assume for contradiction that $(i, j) \notin M$ blocks (M, t^*) via the contribution vector $(\tilde{t}_i, \tilde{t}_j)$. Without loss of generality, we can assume that i and j are not on opposite sides of their peaks.[11] Define $\tilde{\pi}_i = |\tilde{t}_i - p_i|$ and $\tilde{\pi}_j = |\tilde{t}_j - p_j|$. If (i, j)

[10]Note that the non-negativity of π^* is not required to show that (M, t^*) is an allocation.

[11]If $\tilde{t}_i < p_i$ and $\tilde{t}_j > p_j$, there is an even better contribution vector that they both prefer.

blocks M, then $\tilde{\pi}_i < \pi_i^*$ and $\tilde{\pi}_j < \pi_j^*$. In particular, $\tilde{\pi}_i + \tilde{\pi}_j < \pi_i^* + \pi_j^* \leq c_{ij}$, where the last inequality follows from the dual feasibility of π^*.

Now, if $p_i + p_j \geq 1$, we must have $\tilde{t}_i \leq p_i$ and $\tilde{t}_j \leq p_j$. In this case,

$$\begin{aligned}\tilde{t}_i + \tilde{t}_j &= p_i - \tilde{\pi}_i + p_j - \tilde{\pi}_j \\ &= p_i + p_j - (\tilde{\pi}_i + \tilde{\pi}_j) \\ &> p_i + p_j - c_{ij} \\ &= 1,\end{aligned}$$

contradicting the requirement that $\tilde{t}_i + \tilde{t}_j = 1$.

If $p_i + p_j \leq 1$, we must have $\tilde{t}_i \geq p_i$ and $\tilde{t}_j \geq p_j$. In this case,

$$\begin{aligned}\tilde{t}_i + \tilde{t}_j &= p_i + \tilde{\pi}_i + p_j + \tilde{\pi}_j \\ &= p_i + p_j + (\tilde{\pi}_i + \tilde{\pi}_j) \\ &< p_i + p_j + c_{ij} \\ &= 1,\end{aligned}$$

contradicting the requirement that $\tilde{t}_i + \tilde{t}_j = 1$.

Case (II): There exists $i \in N$ such that $\pi_i^* < 0$.

Similar to Case (I), we consider the natural matching M. Let $\bar{I} = \{i \in N : \pi_i^* < 0\}$. Consider agent $i \in \bar{I}$ and let $j \in V$ be her partner in the matching M. By the dual feasibility constraints, we have $\pi_i^* + \pi_j^* = c_{ij}$ where $c_{ij} \geq 0$. Thus $\pi_j^* \geq 0$. This implies $j \notin \bar{I}$. Define $\bar{J} = \{j \in N : (i, j) \in M \text{ and } i \in \bar{I}\}$. Note that $\bar{I} \cap \bar{J} = \emptyset$. So we have partitioned the set of agents N into $\bar{I}$,$\bar{J}$ and $N \setminus [\bar{I} \cup \bar{J}]$ using the dual variable values.[12]

We construct a new set of non-negative dual variables, $\tilde{\pi}$ using the partition of N and π^* as follows:

- For any $i \in \bar{I}$, $j \in \bar{J}$ such that $(i, j) \in M$, define $\tilde{\pi}_i = 0$ and $\tilde{\pi}_j = c_{ij}$. Recall $\pi_i^* + \pi_j^* = c_{ij}$ where $\pi_i^* < 0$. So $\pi_j^* = c_{ij} - \pi_i^* > c_{ij} = \tilde{\pi}_j$.[13]
- For any $k \in N \setminus [\bar{I} \cup \bar{J}]$, define $\tilde{\pi}_k = \pi_k^*$.

Thus for any pair $(i, j) \in M$, we have

[12] All agents with negative dual variable values belong to $\bar{I}$. Their respective partners in M must have non-negative dual variable values and belong to $\bar{J}$. The remaining agents in $N \setminus [\bar{I} \cup \bar{J}]$ are matched to each other and have non-negative dual variable values.

[13] Since the non-negative dual variables in π^* are increased to 0, this means that all agents in $\bar{I}$ will receive their peak after the adjustment. Also the dual variables of their respective partners in $\bar{J}$ are decreased after the adjustment.

1. If $(i,j) \in N \setminus [\bar{I} \cup \bar{J}]$, then $\tilde{\pi}_i + \tilde{\pi}_j = \pi_i^* + \pi_j^* = c_{ij}$ where the latter equality comes from the dual feasibility conditions.

2. If $i \in \bar{I}$ and $j \in \bar{J}$, then $\tilde{\pi}_i + \tilde{\pi}_j = 0 + c_{ij} = c_{ij}$.

We will construct a contribution vector $\tilde{t}$ such that the allocation $(M, \tilde{t})$ belongs to the weak core. Consider the allocation $\tilde{t}$, computed based on the color of the matched edge in the following manner:

- If (i,j) is a black edge, let $\tilde{t}_i = p_i$ and $\tilde{t}_j = p_j$;
- If (i,j) is a red edge, let $\tilde{t}_i = p_i - \tilde{\pi}_i$ and $\tilde{t}_j = p_j - \tilde{\pi}_j$; and
- If (i,j) is a blue edge, let $\tilde{t}_i = p_i + \tilde{\pi}_i$ and $\tilde{t}_j = p_j + \tilde{\pi}_j$.

We claim that $(M, \tilde{t})$ is an allocation. We verify $\tilde{t}_i + \tilde{t}_j = 1$ for all edges $(i,j) \in M$. Consider an edge $(i,j) \in M$.

1. If $i, j \in N \setminus [\bar{I} \cup \bar{J}]$, we know $\tilde{\pi}_i + \tilde{\pi}_j = c_{ij}$. The argument for feasibility is similar to Case (I).

2. If $i \in \bar{I}$ and $j \in \bar{J}$, by construction $\tilde{\pi}_i + \tilde{\pi}_j = c_{ij}$. There are three subcases.

   - If (i,j) is a black edge, then $\tilde{t}_i + \tilde{t}_j = p_i + p_j = 1$.
   - If (i,j) is a red edge, we know $c_{ij} = p_i + p_j - 1$. Here $\tilde{t}_i + \tilde{t}_j = p_i - \tilde{\pi}_i + p_j - \tilde{\pi}_j = p_i + p_j - c_{ij} = 1 + c_{ij} - c_{ij} = 1$.
   - If (i,j) is a red edge, we know $c_{ij} = 1 - p_i - p_j$. We can argue similarly.

Finally we prove that $(M, \tilde{t})$ is a weak core allocation. Clearly no edge in M can block this allocation. The agents matched in M are on the "same" side of the peak, so there is no way for them to strongly block the allocation. Assume for contradiction that there exists an edge $(i,j) \notin M$ that strongly blocks the allocation. Since agents i and j strictly improve by blocking, we have $\tilde{t}_i \neq p_i$ and $\tilde{t}_j \neq p_j$. By construction, the contribution of any agent $k \in N$ is $\tilde{t}_k \in \{p_k, p_k - \tilde{\pi}_k, p_k + \tilde{\pi}_k\}$. So $\tilde{t}_k = p_k$ for all agents $k \in \bar{I}$.[14] Thus the agents in the blocking pair do not belong to $\bar{I}$, i.e. $i, j \notin \bar{I}$ and $i, j \in N \setminus \bar{I}$. There are four subcases to consider.

1. Agents $i, j \in N \setminus [\bar{I} \cup \bar{J}]$.

   By construction, $\tilde{\pi}_i = \pi_i^*$ and $\tilde{\pi}_j = \pi_j^*$. Thus $\tilde{\pi}_i + \tilde{\pi}_j = \pi_i^* + \pi_j^* \leq c_{ij}$.

[14] This is because $\tilde{\pi}_k = 0$ for such agents.

Suppose (i,j) blocks M via the contribution vector $(\hat{t}_i, \hat{t}_j)$. Without loss of generality, we may assume that i and j are not on opposite sides of their peaks, for if $\hat{t}_i < p_i$ and $\hat{t}_j > p_j$ there is an even better allocation that they both prefer. Define $\hat{\pi}_i = |\hat{t}_i - p_i|$ and $\hat{\pi}_j = |\hat{t}_j - p_j|$. If (i,j) blocks $(M, \tilde{t})$, then $\hat{\pi}_i < \tilde{\pi}_i$, $\hat{\pi}_j < \tilde{\pi}_j$. In particular, $\hat{\pi}_i + \hat{\pi}_j < \tilde{\pi}_i + \tilde{\pi}_j \leq c_{ij}$.

Now, if $p_i + p_j \geq 1$, we must have $\hat{t}_i \leq p_i$ and $\hat{t}_j \leq p_j$. In this case,

$$\begin{aligned}
\hat{t}_i + \hat{t}_j &= p_i - \hat{\pi}_i + p_j - \hat{\pi}_j \\
&= p_i + p_j - (\hat{\pi}_i + \hat{\pi}_j) \\
&> p_i + p_j - c_{ij} \\
&= 1,
\end{aligned}$$

contradicting the requirement that $\hat{t}_i + \hat{t}_j = 1$.

If $p_i + p_j \leq 1$, we must have $\hat{t}_i \geq p_i$ and $\hat{t}_j \geq p_j$. In this case,

$$\begin{aligned}
\hat{t}_i + \hat{t}_j &= p_i + \hat{\pi}_i + p_j + \hat{\pi}_j \\
&= p_i + p_j + (\hat{\pi}_i + \hat{\pi}_j) \\
&< p_i + p_j + c_{ij} \\
&= 1,
\end{aligned}$$

contradicting the requirement that $\tilde{t}_i + \tilde{t}_j = 1$.

2. Agents $i, j \in \bar{J}$.

   By construction, we know $\tilde{\pi}_i < \pi_i^*$ and $\tilde{\pi}_j < \pi_j^*$. So $\tilde{\pi}_i + \tilde{\pi}_j < \pi_i^* + \pi_j^*$. By the dual feasibility constraints, we know $\pi_i^* + \pi_j^* \leq c_{ij}$. Thus $\tilde{\pi}_i + \tilde{\pi}_j < c_{ij}$. Suppose (i,j) blocks $(M, \tilde{t})$ via the contribution vector $(\hat{t}_i, \hat{t}_j)$. We can argue like we did in Case 1.

3. Agents $i \in N \setminus [\bar{I} \cup \bar{J}]$ and $j \in \bar{J}$.

   Since $i \in V \setminus [\bar{I} \cup \bar{J}]$, we know $\tilde{\pi}_i = \pi_i^*$. Since $j \in \bar{J}$, we know $\tilde{\pi}_j < \pi_j^*$.

   By the dual feasibility constraints, we have $\pi_i^* + \pi_j^* \leq c_{ij}$. By the above facts, $\tilde{\pi}_i + \tilde{\pi}_j < \pi_i^* + \pi_j^*$. Thus $\tilde{\pi}_i + \tilde{\pi}_j < c_{ij}$. Suppose (i,j) blocks $(M, \tilde{t})$ via the contribution vector $(\hat{t}_i, \hat{t}_j)$. We can argue like we did in Case 1.

4. Agents $j \in N \setminus [\bar{I} \cup \bar{J}]$ and $i \in \bar{I}$.

   This case is symmetric to Case 3 above. Suppose (i,j) blocks $(M, \tilde{t})$ via the allocation $(\hat{t}_i, \hat{t}_j)$. We can argue like we did in Case 1.

We have shown that $(M, \tilde{t})$ belongs to the weak core and this completes the proof of theorem. ■

However this condition is not necessary, as shown by the following example.

Example **3** Let $N = \{1, \dots, 6\}$. Table 3 summarizes the peaks of the agents. Figure 3 provides the partnership graph and the cost of each partnership.

p_1	p_2	p_3	p_4	p_5	p_6
0.9	0.4	0.4	0.5	0.5	0.5

Table 3: Peaks of agents in Example 3.

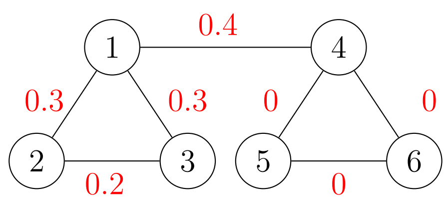

Figure 3: Graph in Example 3

There is a unique perfect matching $(1,4),(2,3),(5,6)$ with cost $0.4+0.2+0 = 0.6$. Thus $Z_I = 0.6$. The fractional solution that puts half on each of the triangles 123 and 456 and zero on the edge $(1,4)$ has cost $\frac{1}{2}(0.3+0.3+0.2) + \frac{1}{2}(0+0+0) = 0.4$. Thus $Z_I > Z_F$.

However a weak core allocation exists in this example: $(1,4)$ with $(0.9, 0.1)$, $(2,3)$ with $(0.5, 0.5)$ and $(5,6)$ with $(0.5, 0.5)$.

4 Examples

We present a sequence of examples and discuss the implications of the two theorems.

Example **4** Let $N = \{1, 2, \dots, 6\}$. The peaks of the agents are summarized in Table 4. Figure 4 contains the partnership graph along with the cost of each partnership.

The graph G has only one perfect matching: $\{(1,4),(2,3),(5,6)\}$ with cost $0.8+0.2+0.2 = 1.2$. Thus $Z_I = 1.2$.

Consider the feasible solution that puts a weight of $1/2$ on the edges of the two triangles, and zero on the edge $(1,4)$. The cost of this solution is 0.8. Furthermore, one can verify that this solution is optimal for the minimum cost fractional covering problem and hence

p_1	p_2	p_3	p_4	p_5	p_6
0.9	0.4	0.4	0.9	0.4	0.4

Table 4: Peaks of agents in Example 4.

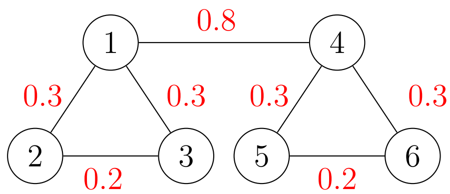

Figure 4: Graph with edge costs in Example 4

also the minimum cost fractional matching problem. (One way to verify this is to check that the solution $\pi_1 = \pi_4 = 0.2$, $\pi_2 = \pi_3 = \pi_5 = \pi_6 = 0.1$ is feasible for the dual of the minimum cost fractional covering problem and has the same cost as that of the primal.)

Thus, we have shown that $Z_I \neq Z_F = Z_P$. By Theorem 1, we know that the strong core is empty. Theorem 2 only provides a sufficient condition under which the weak core is nonempty, but that condition is violated in this example. As a result, a weak core allocation may or may not exist in this example. Indeed, we show next that the weak core is also empty.

Consider an arbitrary allocation (M, t) where the matching M is $(1,4), (2,3), (5,6)$. Suppose without loss of generality that $t_1 \leq t_4$. Then, $t_1 \in [0.1, 0.5]$, for otherwise $t_1 < 0.1$ and $t_4 > 0.9$, and this allocation is blocked by $t_1 = 0.1, t_4 = 0.9$ (leaving all the other t's unchanged). Now we can apply a similar reasoning to the pair $(2,3)$: supposing, without loss of generality, that $t_2 \leq t_3$, we can conclude that $t_3 \in [0.5, 0.6]$. However, agents 1 and 3 can block this allocation (M, t) by choosing $t_1' = 0.6, t_3' = 0.4$. Thus we can conclude that the weak core is empty.

Example **5** Let $N = \{1, 2, 3, 4, 5, 6\}$. The peaks of the agents are shown in Table 5. Figure 5 shows the partnership graph, where every red edge has zero cost and every blue edge has cost 0.3.

p_1	p_2	p_3	p_4	p_5	p_6
0.1	0.4	0.4	0.9	0.9	0.6

Table 5: Peaks of agents in Example 5.

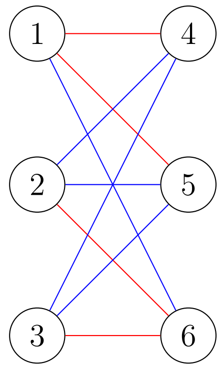

Figure 5: Graph in Example 5

We can verify that the edges $(1,4)$, $(1,5)$, $(2,6)$ and $(3,6)$, each have zero cost. Every other edge in the graph has a cost of 0.3.

Any perfect matching must include *at least* one edge with cost 0.3. Therefore any perfect matching that has exactly one edge with cost 0.3 (and there are many such matchings) is necessarily optimal. Moreover, because the graph is a bipartite graph, this is also the cost of an optimal fractional matching . Thus $Z_I = Z_F = 0.3$. Finally, observe that choosing all the edges with zero cost yields a valid cover, so $Z_P = 0$. Thus we have $Z_I = Z_F \neq Z_P$.

Theorem 1 implies that the strong core is empty. Because $Z_I = Z_F$, Theorem 2 implies that the weak core is non-empty: indeed, it is easy to construct a weak core allocation using a minimum-cost matching. For instance, the optimal matching $M = \{(1,4),(2,5),(3,6)\}$ can be supported as a weak core allocation with the respective contribution vectors being $\{(0.1,0.9),(0.5,0.5),(0.4,0.6)\}$.

Interestingly, a weak core allocation in this example can also be supported on a matching that is not optimal. For instance, consider the matching $M' = \{(1,6),(2,5),(3,4)\}$ with the respective contribution vectors $\{(0.4,0.6),(0.1,0.9),(0.1,0.9)\}$. Observe that all agents on the side $\{4,5,6\}$ receive their peaks, and so this allocation is in the weak core. A similar construction where all agents on the side $\{1,2,3\}$ receive their peaks also supports M' as a weak core allocation.

Observation **3** If (M,t) is an allocation in the weak core, then M is not necessarily a minimum-cost matching. This is illustrated in Example 5. In contrast, if (M,t) is an allocation in the strong core, then M is necessarily a minimum-cost matching.

Example **6** Let $N = \{1,\ldots,6\}$. Table 6 summarizes the agents' peaks. Figure 6 contains the partnership graph with cost of each partnership.

There are two matchings in G: $\{(1,4),(2,3),(5,6)\}$ and $\{(1,3),(2,5),(4,6)\}$ whose total costs are, respectively, 1 and 1.4. The first matching is the efficient one and $Z_I = 1$. To find

p_1	p_2	p_3	p_4	p_5	p_6
0.9	0.3	0.8	0.9	0.4	0.5

Table 6: Peaks of agents in Example 6.

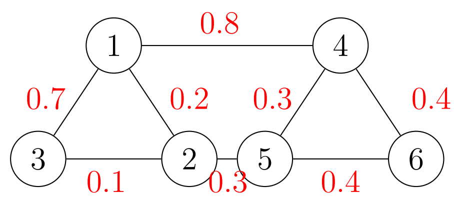

Figure 6: Graph in Example 6

a minimum-cost fractional matching, consider the feasible solution that puts a weight of 1/2 on all the edges of the two triangles [15], and zero everywhere else. This feasible solution has cost 0.9 and can be shown to be optimal. Thus $Z_F = 0.9$. Finally, the solution consisting of the edges $\{(1,2),(1,3),(4,5),(5,6)\}$ has cost 0.7 and can be shown to be a minimum-cost fractional cover. Thus $Z_P = 0.7$.

This is an instance for which $Z_I \neq Z_F \neq Z_P$. By Theorem 1, we know that the strong core is empty. Also, the sufficient condition of Theorem 2 is violated in this example, so we cannot conclude the existence of a weak core allocation may or may not exist in this example. In fact, the weak core is non-empty in this example. The efficient matching $(1,4),(2,3),(5,6)$ with contribution vectors $(0.3, 0.7)$, $(0.3, 0.7)$ and $(0.5, 0.5)$ belongs to the weak core: To see this, observe that for this allocation, neither the three edges in the matching nor any edge involving agents 2 or 6 can block the matching (as agents 2 and 6 receive their peaks); the only potential block pairs are $(1,3)$ and $(4,5)$. For the pair $(1,3)$, both agents receive an allocation below their peak and their allocations add up to 1, so there is no possibility of improvement; for the pair $(4,5)$, agent 4 needs to receive more than 0.7 to improve, which implies that agent 5 will receive an allocation *below* 0.3, which is worse than their current allocation.

Finally, in contrast to Example 5, the other perfect matching (with cost 1.4) cannot be supported by a weak core allocation.

The following examples show the existence of strong core allocations in different types of graphs.

Example **7** Let $N = \{1, \ldots, 6\}$. The peaks are summarized in Table 7. Figure 7 contains the partnership graph along with the cost of each partnership.

[15]These edges are $(1,2)$,$(1,3)$,$(2,3)$, $(4,5)$,$(4,6)$ and $(5,6)$.

p_1	p_2	p_3	p_4	p_5	p_6
0.9	0.2	0.6	0.3	0.4	0.5

Table 7: Peaks of agents in Example 7

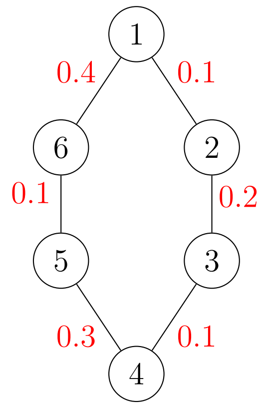

Figure 7: Graph in Example 7

The graph has two perfect matchings: $(1, 2), (3, 4), (5, 6)$ and $(1, 6), (2, 3), (4, 5)$. The first matching has cost 0.3 and the second matching has cost 0.9. Thus $Z_I = 0.3$. Now we examine the minimum cost fractional matching problem. Any fractional solution requires that the weights on the edges alternate between x and $1 - x$ where $0 \leq x \leq 1$. We will show that for any $x \in (0, 1)$, the cost of the solution will be higher than Z_I. Consider a feasible solution where the edge $(1, 2)$ has weight $x \in (0, 1)$. Then the edge $(2, 3)$ will have weight $1 - x$ as the sum of weights for node 2 is 1. Continuing in this manner, we get a sequence of weights alternating between x and $1 - x$, with the last edge $(1, 6)$ receiving a weight x. Thus cost of this feasible solution is $(x)(c_{12} + c_{34} + c_{56}) + (1 - x)(c_{23} + c_{25} + c_{16})$.[16] This cost of this solution is stricly less than Z_I iff $x > 1$. Thus the efficient matching $(1, 2), (3, 4), (5, 6)$ is also the optimal solution to the minimum weight fractional matching program. So $Z_F = Z_I$. Similarly we can argue that the efficient matching is the optimal solution to the minimum weight covering problem as well. We have shown $Z_I = Z_F = Z_P$. Theorem 1 implies that the strong core is non-empty and the efficient matching can be used to construct a strong core allocation.

The allocation $(1, 2)$ with $(0.9, 0.1)$, $(3, 4)$ with $(0.6, 04)$ and $(5, 6)$ with $(0.4, 0.6)$ belongs to the strong core. Agents 1, 3 and 5 receive their peaks. We can show that no weak blocking

[16]Observe that the value of any feasible solution to the fractional program is a weighted average of the cost of the two matchings in the graph, where x and $1 - x$ are the weights given to the efficient and inefficient matchings respectively.

is possible here.

Examples 8 and 9 below consider connected graphs with cliques.

EXAMPLE **8** Let $N = \{1, \dots, 6\}$. The peaks are summarized in Table 8. Figure 8 provides the partnership graph and the cost of each partnership.

p_1	p_2	p_3	p_4	p_5	p_6
0.5	0.4	0.4	0.5	0.4	0.4

Table 8: Peaks of agents in Example 8

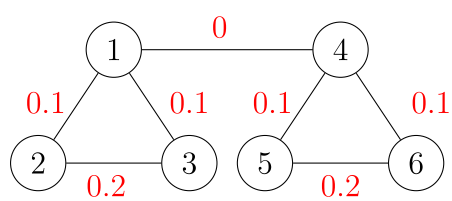

Figure 8: Graph in Example 8

A strong core allocation consists of the matching $(1,4)$, $(2,3)$ and $(5,6)$ where all pairs receive $(0.5, 0.5)$. The cost of the perfect matching is $0 + 0.2 + 0.2 = 0.4$.

EXAMPLE **9** Let $N = \{1, \dots, 8\}$. The peaks are summarized in Table 9. Figure 9 provides the partnership graph and the cost of each partnership.

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8
0.5	0.4	0.4	0.2	0.5	0.4	0.4	0.2

Table 9: Peaks of agents in Example 9

Since 1 and 5 have cost 0, they must receive their peaks in any strong core allocation. A strong core allocation is: $(1,5)$ with $(0.5, 0.5)$, $(3,4)$ with $(0.5, 0.5)$ and a similar matching/contribution vector for the other clique.

Agents 1 and 5 will not block as they both get their peaks. An agent from $\{3, 4\}$ will not block with 1 as they must give her $0.5 = p_1$ and then they have no change in their allocation. Agents 2 and 4 cannot block together as they both want to reduce their contributions from 0.5.

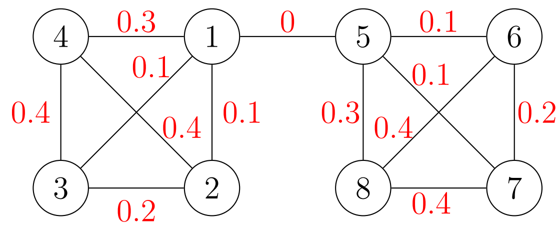

Figure 9: Graph in Example 9

5 Applications

This section provides two applications of Theorem 2. Our goal is to investigate whether some special graphs satisfy the sufficiency condition $Z_I = Z_F$, for all preference profiles. The bipartite graph is one such case and this is implied by well-known results. We show that complete graph also exhibits this property and hence always admit weak core allocations.

5.1 Complete graph

We show below that the sufficient condition for the existence of weak core holds for a complete graph, for any given configuration of peaks. We begin by proving a preliminary claim.

Claim **1** *Consider four agents i, j, k, ℓ, with their peaks satisfying $0 \le p_i \le p_j \le p_k \le p_\ell \le 1$. Then, the matching $\{(i,\ell),(j,k)\}$ is optimal. That is the cost of the matching $M := \{(i,\ell),(j,k)\}$ is no more than the cost of the other two matchings $M' := \{(i,j),(k,\ell)\}$ and $M'' := \{(i,k),(j,\ell)\}$.*

Proof: We enumerate all cases and verify the claim in each. Let $c(X)$ denote the cost of matching X. We start with two easy cases.

- $\mathbf{p_i + p_j \ge 1}$: In this case every pair of agents have a sum of peaks that is at least 1. The cost of any perfect matching is given by $p_i + p_j + p_k + p_\ell - 2$. Thus, $c(M) = c(M') = c(M'')$.
- $\mathbf{p_k + p_\ell \le 1}$: In this case every pair of agents have a sum of peaks that is at most 1. The cost of any perfect matching is given by $2 - p_i - p_j - p_k - p_\ell =: c(M) = c(M') = c(M'')$.

We may thus assume for the rest of the argument that $p_i + p_j < 1$ and that $p_k + p_\ell > 1$. Observe that, under these assumptions, the cost of M' is fixed and equals

$$c(M') \;=\; 1 - p_i - p_j + p_k + p_\ell - 1 \;=\; p_k + p_\ell - p_i - p_j.$$

We now look at some further subcases.

- $\mathbf{p_i} + \mathbf{p_k} \geq \mathbf{1}$: Here, every pair of agents, other than the pair $\{i, j\}$, has a sum of peaks that is at least 1, and so $c(M) = c(M'') = p_i + p_j + p_k + p_\ell - 2$. Now,

$$\begin{aligned} c(M') - c(M) &= (p_k + p_\ell - p_i - p_j) - (p_i + p_j + p_k + p_\ell - 2) \\ &= 2(1 - p_i - p_j), \end{aligned}$$

which is positive. Thus, $c(M) = c(M'') < c(M')$.

- $\mathbf{p_j} + \mathbf{p_\ell} \leq \mathbf{1}$: In this case every pair of agents, other than the pair $\{k, \ell\}$, has a sum of peaks that is at most 1, and so $c(M) = c(M'') = 2 - p_i - p_j - p_k - p_\ell$. Now,

$$\begin{aligned} c(M') - c(M) &= (p_k + p_\ell - p_i - p_j) - (2 - p_i - p_j - p_k - p_\ell) \\ &= 2(p_k + p_\ell - 1), \end{aligned}$$

which is positive. Thus, $c(M) = c(M'') < c(M')$.

For the rest of the argument, we may assume that $p_i + p_k < 1$ and that $p_j + p_\ell > 1$. Note that this fixes the cost of M'' as well:

$$c(M'') \;=\; 1 - p_i - p_k + p_j + p_\ell - 1 \;=\; p_j + p_\ell - p_i - p_k.$$

We note also that in all the cases considered so far $c(M) = c(M'') \leq c(M')$, so it is enough to compare M and M'' to complete the proof of the claim.

Of the six pairs of agents, the pairs $\{i, j\}$ and $\{i, k\}$ are assumed to have sum of peaks below 1; and the pairs $\{k, \ell\}$ and $\{j, \ell\}$ are assumed to have sum of peaks above 1. Each of the remaining pairs, $\{i, \ell\}$ and $\{j, k\}$, could have a sum of peaks above 1 or below 1, which naturally leads to the following four subcases:

- $\mathbf{p_i} + \mathbf{p_\ell} \geq \mathbf{1}$ and $\mathbf{p_j} + \mathbf{p_k} \geq \mathbf{1}$: In this case, $c(M) = p_i + p_\ell + p_j + p_k - 2$ and

$$\begin{aligned} c(M'') - c(M) &= (p_j + p_\ell - p_i - p_k) - (p_i + p_j + p_k + p_\ell - 2) \\ &= 2(1 - p_i - p_k), \end{aligned}$$

which is positive. Thus, $c(M) < c(M'')$.

- $\mathbf{p_i} + \mathbf{p_\ell} \geq \mathbf{1}$ and $\mathbf{p_j} + \mathbf{p_k} < \mathbf{1}$: In this case, $c(M) = p_i + p_\ell - 1 + 1 - p_j - p_k = p_i + p_\ell - p_j - p_k$.

$$\begin{aligned} c(M'') - c(M) &= (p_j + p_\ell - p_i - p_k) - (p_i + p_\ell - p_j - p_k) \\ &= 2(p_j - p_i), \end{aligned}$$

which is positive. Thus, $c(M) < c(M'')$.

- $\mathbf{p_i + p_\ell < 1}$ and $\mathbf{p_j + p_k \geq 1}$: In this case, $c(M) = 1 - p_i - p_\ell + p_j + p_k - 1 = p_j + p_k - p_i - p_\ell$.

$$\begin{aligned} c(M'') - c(M) &= (p_j + p_\ell - p_i - p_k) - (p_j + p_k - p_i - p_\ell) \\ &= 2(p_\ell - p_k), \end{aligned}$$

  which is positive. Thus, $c(M) < c(M'')$.

- $\mathbf{p_i + p_\ell < 1}$ and $\mathbf{p_j + p_k < 1}$: In this case, $c(M) = 2 - p_i + p_\ell - p_j - p_k$ and

$$\begin{aligned} c(M'') - c(M) &= (p_j + p_\ell - p_i - p_k) - (2 - p_i - p_j - p_k - p_\ell) \\ &= 2(1 - p_j - p_\ell), \end{aligned}$$

  which is positive. Thus, $c(M) < c(M'')$.

■

Claim **2** *Given* $2n$ *agents with* $0 \leq p_1 \leq p_2 \ldots \leq p_{2n-1} \leq p_{2n} \leq 1$*, the matching* $\{(1, 2n), (2, 2n-1), \ldots, (n, n+1)\}$ *is optimal.*

Proof: Given $2n$ agents with $0 \leq p_1 \leq p_2 \ldots \leq p_{2n-1} \leq p_{2n} \leq 1$, we can use Claim 1 iteratively to show that the matching $\{(1, 2n), (2, 2n-1), \ldots, (n, n+1)\}$ is optimal.

First, we argue that there is an optimal matching in which agents 1 and $2n$ are matched (*): suppose there is an optimal matching in which 1 is matched to j and $2n$ is matched to k. Because $p_1 \leq p_j, p_k \leq p_{2n}$, Claim 1 implies that the matching in which 1 is matched to $2n$ and j is matched to k is also optimal, establishing (*). The result now follows by applying the same observation to the smaller problem obtained by removing agents 1 and $2n$. ■

We will now show that there exists an optimal integer solution to the LP for a complete graph.

By well known results in the literature, we know that the extreme points of the LP formulation of the matching problem are half-integral: every component of each extreme point is in $\{0, 1/2, 1\}$. We now argue that in fact there is an optimal integer solution to the LP formulation as well. We do this by showing that the matching $\{(1, 2n), (2, 2n-1), \ldots, (n, n+1)\}$ is optimal for the LP as well.

Suppose x is an optimal solution to the LP. Because the entries are all $\{0, 1/2, 1\}$, we note that the fractional components of x will have to be a collection of odd cycles (and there should be an even number of them). We first argue that there is an optimal x in which every odd cycle, if any, has exactly three nodes. For otherwise, suppose there is an odd cycle involving nodes i, j, k, ℓ, m, and possibly some other nodes. Suppose i has the smallest peak among all nodes in this cycle, and suppose its neighbors are j and m, and suppose $p_j \leq p_m$. Suppose m's other neighbor in the cycle is ℓ. We can create a new solution x' that differs from x only on the pairs (i, m), (j, ℓ), (i, j) and (m, ℓ): we set $x'_{i,m} = 1$, $x'_{j,\ell} = 1/2$, $x'_{m,\ell} = 0$ and $x'_{i,j} = 0$. The values for x on these edges were originally $x_{i,m} = 1/2$, $x_{j,\ell} = 0$, $x_{m,\ell} = 1/2$ and $x_{i,j} = 1/2$. Observe that:

$$c(x') - c(x) \;=\; \frac{1}{2}(c_{i,m} + c_{j,\ell} - c_{i,j} - c_{m,\ell}) \;\leq\; 0,$$

where the last inequality follows from Claim 1 applied to the agents $\{i, j, m, \ell\}$. (We do not know where ℓ's peak falls....but regardless we know that the cost of $\{(i, m), (j, \ell)\}$ is at most the cost of $\{(i, j), (m, \ell)\}$.)

We have thus reduced the length of the cycle by 2, while ensuring that the cost of the cycle never increases. By applying this repeatedly we can eventually obtain an x in which every odd cycle has length 3.

If x has at least one odd cycle, then x must have an even number of odd cyles (because the total number of nodes is even). Pick two odd cycles, say, $\{i, j, k\}$ and $\{u, v, w\}$. Suppose $p_i \leq p_j \leq p_k$ and $p_u \leq p_v \leq p_w$, and suppose, without loss of generality, $p_i \leq p_u$. We shall exhibit an integer optimal solution on these six nodes.

We consider two cases.

Case 1: $p_k \geq p_w$

(a) Suppose further that $p_j \leq p_u$.

We claim that the cost of the matching $\mu := \{(i, k), (j, w), (u, v)\}$ is at most the cost of x:

$$\begin{aligned} c(\mu) - c(x) &= \frac{1}{2}\Big(c_{ik} + c_{uv} - c_{ij} - c_{uw} - c_{jk} - c_{vw} + c_{jw} + c_{jw}\Big) \\ &= \frac{1}{2}\Big\{(c_{ik} + c_{jw} - c_{iw} - c_{jk}) + (c_{uv} + c_{jw} - c_{vw} - c_{ju}) + (c_{iw} + c_{ju} - c_{ij} - c_{uw})\Big\} \\ &\leq 0, \end{aligned}$$

where the last inequality follows from applying Claim 1 repeatedly: when applied to the subset of agents $\{i, j, w, k\}$, the first term is seen to be non-positive; when applied to the subset of agents $\{j, u, v, w\}$, the second term is seen to be non-positive; and when applied to the subset of agents $\{i, j, u, w\}$, the final term is seen to be non-positive.

(b) Suppose $p_j \geq p_w$.

We claim that the cost of the matching $\mu' := \{(i,k),(v,w),(u,j)\}$ is at most the cost of x:

$$
\begin{aligned}
c(\mu') - c(x) &= \frac{1}{2}\Big(c_{ik} + c_{vw} - c_{ij} - c_{uw} - c_{jk} - c_{uv} + c_{uj} + c_{uj}\Big) \\
&= \frac{1}{2}\Big\{(c_{ik} + c_{uj} - c_{ij} - c_{uk}) + (c_{vw} + c_{uj} - c_{uw} - c_{vj}) + (c_{uk} + c_{vj} - c_{jk} - c_{uv})\Big\} \\
&\leq 0,
\end{aligned}
$$

where the last inequality follows from applying Claim 1 repeatedly: when applied to the subset of agents $\{i,u,j,k\}$, the first term is seen to be non-positive; when applied to the subset of agents $\{u,v,w,j\}$, the second term is seen to be non-positive; and when applied to the subset of agents $\{u,v,j,k\}$, the final term is seen to be non-positive.

(c) Finally, suppose $p_u \leq p_j \leq p_w$. We claim that the cost of the matching $\mu'' := \{(i,k),(u,w),(v,j)\}$ is at most the cost of x:

$$
\begin{aligned}
c(\mu'') - c(x) &= \frac{1}{2}\Big(c_{ik} + c_{uw} - c_{ij} - c_{vw} - c_{jk} - c_{uv} + c_{vj} + c_{vj}\Big) \\
&\leq \frac{1}{2}\Big(c_{iw} + c_{uk} - c_{ij} - c_{vw} - c_{jk} - c_{uv} + c_{vj} + c_{vj}\Big) \\
&= \frac{1}{2}\Big\{(c_{iw} + c_{vj} - c_{ij} - c_{vw}) + (c_{uk} + c_{vj} - c_{uv} + c_{jk})\Big\} \\
&\leq 0,
\end{aligned}
$$

where the first inequality follows from applying Claim 1 to the subset of agents $\{i,u,w,k\}$; and the final inequality follows from Claim 1 applied to the subsets of agents $\{i,j,v,w\}$ and $\{u,v,j,k\}$ respectively.

Case 2: $p_k < p_w$

Consider the solution x' obtained from x by setting

$$x'_{i,w} = 1;\;\; x'_{i,j} = x'_{i,k} = x'_{w,u} = x'_{w,v} = 0;\;\; x'_{j,u} = x'_{k,v} = \frac{1}{2},$$

and letting x' be the same as x for all other edges (including the edges (j,k) and (u,v), which both have values $1/2$. We claim that $c(x') \leq c(x)$. Note that

$$
\begin{aligned}
c(x') - c(x) &= \frac{1}{2}\Big(c_{iw} + c_{iw} - c_{ij} - c_{ik} + c_{uj} - c_{vw} - c_{uw} + c_{vk}\Big) \\
&= \frac{1}{2}\Big\{(c_{iw} + c_{uj} - c_{ij} - c_{uw}) + (c_{iw} + c_{vk} - c_{ik} - c_{vw}))\Big\} \\
&\leq 0,
\end{aligned}
$$

where the last inequality follows from applying Claim 1 repeatedly: when applied to the subset of agents $\{i, u, j, w\}$, the first term is seen to be non-positive; and when applied to the subset of agents $\{i, v, k, w\}$, the second term is seen to be non-positive.

Thus, we have exhibited a solution x' that is fractional with $c(x') \leq c(x)$. Morever, when restricted to the six nodes i, j, k, u, v, w, x' can be written as the average of the two matchings $\{(i,w),(j,k),(u,v)\}$ and $\{(i,w),(j,u),(k,v)\}$. One of these two matchings has cost no more than the cost of x' and is also integral (on these six nodes).

We have shown the following. Start from any extreme point solution. Standard results tell us that the fractional components are a collection of odd cycles. We first showed how one can convert each of them into cycles of length 3. We then showed how any pair of 3 cycles can be converted to an integer matching that is optimal. By repeatedly applying this process we can always arrive at an integer optimal solution.

References

Kagel, J. H. and A. E. Roth (2000): "The dynamics of reorganization in matching markets: A laboratory experiment motivated by a natural experiment," The Quarterly Journal of Economics, 115, 201–235.

Nicolò, A., P. Salmaso, A. Sen, and S. Yadav (2022): "Stable Sharing," .

Roth, A. E. (2002): "The economist as engineer: Game theory, experimentation, and computation as tools for design economics," Econometrica, 70, 1341–1378.

Shapley, L. S. and M. Shubik (1971): "The assignment game I: The core," International Journal of Game Theory, 1, 111–130.