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Selling to a naive agent with two rationales *

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Abstract

A seller is selling an object to an agent who uses two *rationales* to compare pairs of outcomes - (allocation probability, transfer) pairs. Each rationale is generated by quasilinear preference over the outcome space, and hence, can be represented by a valuation. However, the agent faces a budget constraint when making decisions using the first rationale. The agent compares any pair of outcomes using his pair of valuations in a lexicographic manner: first, he compares using the valuation corresponding to the first rationale; then, he compares using the valuation corresponding to the second rationale if and only if the first rationale cannot compare (due to budget constraint). We show that the optimal mechanism is either a posted price mechanism or a mechanism involving a pair of posted prices (a menu of three outcomes). In the latter case, the optimal mechanism involves randomization and *pools* types in the middle.

JEL Codes: D82, D40, D90

Keywords: optimal mechanism, posted-price mechanism, lexicographic choice, multiple rationales, budget constraint.

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1 Introduction

There is a vast literature in psychology and decision theory that intransitive preferences of agents can be explained by the fact that they are aggregation of multiple rationales (Tversky, 1969). As an example, consider a firm who is trying to acquire a resource. The firm (management) has a budget constraint but has access to shareholders or investors who can provide additional funds if needed - the contracting terms between the firm and the investors (shareholders) are exogenously given. There are various options (in terms of quantity of the resource and corresponding prices) in front of the firm. The firm first compares each pair of options using his own quasilinear preference (valuation) subject to budget constraint. If it cannot make the comparison due to the budget constraint, then the investors are consulted, who are not budget constrained. Further, the firm has to respect the quasilinear preference (valuation) of the investors for such options. For various reasons, the investors may have a different valuation than the firm - they may care about a different set of attributes of the resource than the firm. Such restrictions may lead to intransitive preference of the firm. Notice here that our firm is naive in the sense that it *only* goes to the investors when it cannot compare options due to budget constraint [1].

Generally, we are considering an agent whose intransitivity can be explained using two rationales, which he applies in a lexicographic manner to evaluate any pair of outcomes. In particular, our agent has two rationales R_1 and R_2 over the space of outcomes, which are (allocation probability, transfer) pairs. Each rationale evaluates outcomes using quasilinearity and can be represented by a valuation, but comparison using R_1 must satisfy a budget constraint. On the other hand, rationale R_2 is unconstrained, and hence, complete. The overall preference of the agent is derived in a lexicographic manner. For any pair of outcomes, if R_1 prefers one to the other then that becomes the preference of the agent. If R_1 cannot compare the two outcomes due to budget constraint, then the preference of R_2 becomes the preference of the agent. Because of the lexicographic nature of decision-making, the preference of the agent is intransitive (and no utility representation is possible). The type of an agent is the pair of values: the R_1 value and the R_2 value (budget is common knowledge).[2]

How should a seller sell an object to an agent which has such intransitive preferences?

[1]Later, in Section 4.2, we give more examples of how this simple framework can explain many settings of interest.

[2]We give partial results when budget is also private information of the agent.

In our model, a mechanism elicits values of both the rationales and offers an outcome. Incentive compatibility requires that truth-telling outcome is preferred to every other outcome in the range of the mechanism. Our main result shows that the *optimal* (expected revenue maximizing) mechanism is one of the two kinds of mechanisms.

- POST-1 MECHANISM. The first kind of mechanism consists of an optimally chosen reserve price less than the budget. If the value corresponding to the first rationale is less than this reserve price, then the object is not given. Else, the object is given with probability one. We show that such a mechanism is optimal if the budget is high enough. This is intuitive since with high budget, the first rationale can compare most of the outcomes and the seller can maximize her revenue by *only* targeting the first rationale.

- POST-2 MECHANISM. The second kind of mechanism consists of two optimally chosen reserve prices K_1 and K_2, both greater than the budget and $K_1 \leq K_2$. If value corresponding to the first rationale is less than K_1, then the object is not allocated. If both the values are greater than K_2, then the object is allocated with probability one. Else, the object is allocated with probability $\frac{B}{K_1}$ at a price equal to B, where B is the budget. Notice that a POST-2 mechanism *pools* types in the middle, where the object is allocated with a constant probability strictly less than one at a price equal to the budget. A sufficient condition on the budget, which ensures that the optimal mechanism is a POST-2 mechanism is that the budget is less than the optimal (unconstrained) monopoly reserve price for the first rationale. In such a case, the optimal mechanism necessarily involves randomization. This is unlike the standard setting, where the optimal mechanism is a deterministic mechanism - a posted-price mechanism (Mussa and Rosen, 1978; Myerson, 1981; Riley and Zeckhauser, 1983).

Thus, the optimal mechanism is simple since it can be described by a single parameter or a pair of parameters, and involves a menu of size two or three. The intransitivity complication of preferences does not complicate the menu of the optimal mechanism. Further, our result works for a rich class of priors (over values of the two rationales), which allows for correlation. The inclusion of the new class of mechanisms (using two pairs of posted prices) in the optimal menu is due to the naiveté of our agent. We stress here that our agent is *naive* in two ways: (a) his lexicographic decision-making using two rationales and (b) his ability to only compare pairs of outcomes. Indeed, our decision maker (buyer) can *only* make binary choices and we are silent about how he makes choices from larger sets. Our incentive constraints reflect this fact and we elaborate on this later.

It may be tempting to think that the budget constraint along the first rationale is the main driving force behind the randomized optimal mechanism result. This intuition is incorrect since in the standard model, with a single agent and public budget constraint, the optimal mechanism is a posted price mechanism (involving no randomization) - see Laffont and Robert (1996); Che and Gale (2000). The optimal mechanism in a standard model with budget constraint involves randomization only if the budget is private (Che and Gale, 2000) or if the number of agents is more than one and the budget is public (Laffont and Robert, 1996; Pai and Vohra, 2014).

The incentive constraints in our model are quite different from a standard model of mechanism design. This is because of the sequential nature of decision-making generating cyclic preference of the agent. Indeed, since no utility representation is possible, the incentive constraints are *ordinal* in nature. The fact that a POST-2 mechanism is incentive compatible is non-trivial. Compared to a standard multidimensional screening problem, where one runs into difficulty even in the two-dimensional case (Manelli and Vincent, 2007; Hart and Nisan, 2017), we still have tractability in our model because of the nature of decision-making and the incentive constraints.

We also consider an extension of our model where the budget information (along with values on both dimensions) is private. By restricting our attention to a reasonable class of mechanisms, we derive an optimal mechanism over this class of mechanisms - the projection of this optimal mechanism on the valuations space for each budget is (i) a POST-2 mechanism if the budget is low and (ii) a POST-1 mechanism if the budget is high. This shows some robustness of our main result.

We do an extensive literature review at the end in Section 6. Here, we point out that our investigation of such a behavioral agent (buyer) using two rationales in a mechanism design setting is inspired by a long list of papers which have considered such decision makers in other specific settings and provided axiomatic foundations for such decision making. Some examples include Rubinstein (1988) for choosing money lotteries; Tadenuma (2002); Houy and Tadenuma (2009) for studying problems in welfare economics; Manzini and Mariotti (2012); Apesteguia and Ballester (2013) for choice correspondences; Barak et al. (2013) in the context of law (and with references of such decision making in Talmud); Kohli and Jedidi (2007) in marketing.

2 The model

A seller is selling a single object to an agent who evaluates the object along two dimensions lexicographically. The two dimensions are indexed by $\{1,2\}$. The first dimension will be referred to as DIM_1 and the second dimension as DIM_2. In our firm-investors example, DIM_1 will be for the firm and DIM_2 will be for the investors. A consumption bundle is a pair (a,t), where $a \in [0,1]$ is the allocation probability and $t \in \mathbb{R}$ is the transfer - amount *paid* by the agent. The set of all consumption bundles is denoted by $Z \equiv [0,1] \times \mathbb{R}$. The agent evaluates the outcomes in Z along each dimension using **quasilinearity**. Hence, the rationale along each dimension can be represented using a valuation. The valuation for the object along DIM_i is v_i, where $i \in \{1,2\}$. We assume that $v_1, v_2 \in V \equiv [0,\beta]$ - all our results extend even if we allow for the fact $v_i \in [0,\beta_i]$ for each $i \in \{1,2\}$ and $\beta_1 \neq \beta_2$. The agent has a publicly observable budget $B \in \mathbb{R}_+$ for DIM_1. We assume $B \in (0,\beta)$. Since the budget is publicly observable, the only private information of the agent is his values along the two dimensions - we will come to the private budget case in Section 5.

Preference (rationale) along DIM_1, denoted by $\succeq_{v_1}$, is the following: $\forall\ (a,t),(a',t') \in Z$,

$$\big[(a,t) \succeq_{v_1} (a',t')\big] \quad \Leftrightarrow \quad \big[av_1 - t \geq a'v_1 - t' \text{ and } t \leq B\big].$$

Notice that $\succeq_{v_1}$ is incomplete. But whenever DIM_1 can compare two outcomes, it does so using quasilinearity.

Preference along DIM_2, denoted by $\succeq_{v_2}$, is the following: $\forall\ (a,t),(a',t') \in Z$,

$$\big[(a,t) \succeq_{v_2} (a',t')\big] \quad \Leftrightarrow \quad \big[av_2 - t \geq a'v_2 - t'\big].$$

Hence, $\succeq_{v_2}$ is complete and quasilinear.

We denote the preference of the agent with type $v \equiv (v_1,v_2)$ as $\succeq_v$. The preference $\succeq_v$ is a complete binary relation derived from $\succeq_{v_1}$ and $\succeq_{v_2}$ as follows. For every $(a,t),(a',t') \in Z$,

$$\big[(a,t) \succeq_v (a',t')\big] \Leftrightarrow$$

$$\text{either } \big[(a,t) \succeq_{v_1} (a',t')\big] \text{ or } \big[(a,t) \not\succeq_{v_1} (a',t'), (a',t') \not\succeq_{v_1} (a,t), (a,t) \succeq_{v_2} (a',t')\big].$$

As is expected, $\succeq_v$ is intransitive for some $v \equiv (v_1,v_2)$ - though we show below intransitivity of the strict part, even the symmetric component need not be transitive.

Lemma 1 **(Intransitive preference)** *For any type $v = (v_1, v_2)$ with $v_1, v_2 > 0$ and $v_1 \neq v_2$ there exist three outcomes $(a,t), (b,t'), (c,t'') \in Z$ such that*

$$(a,t) \succ_v (b,t') \succ_v (c,t'') \succ_v (a,t),$$

where $\succ_v$ is the strict part of the relation $\succeq_v$.

The proof of this lemma is given in Supplementary Appendix C.1 at the end. An important consequence of this lemma is that there is *no utility representation* of the preference of our agent.

We assume that joint distribution of $v \equiv (v_1, v_2)$ over $V \times V$ follows a distribution G with G_1 being the marginal for DIM_1 and G_2 being the marginal for DIM_2. Both G_1 and G_2 are assumed be differentiable functions with densities g_1 and g_2 respectively. Notice that we allow for values along both the dimensions to be correlated.

2.1 Some illustrations

Before getting into the formal definition of incentive compatibility and mechanisms, we explain using a simple example that the usual posted price mechanism need not be optimal any longer. For simplicity, consider a setting where values $v \equiv (v_1, v_2)$ are distributed in $[0,1] \times [0,1]$. Consider a budget $B > 0$. Suppose the seller uses a posted price mechanism with price $p > B$. We argue that such a posted price mechanism cannot be optimal. To see this, consider the menu to the agent in a posted price mechanism: $\{(1,p),(0,0)\}$. If the value of the agent is $v \equiv (v_1, v_2)$ such that $v_1 \leq p$ he will prefer $(0,0)$ to $(1,p)$. Also, if the value of the agent is $v \equiv (v_1, v_2)$ such that $v_2 \leq p$ and $v_1 \geq p$, then he will prefer $(0,0)$ to $(1,p)$. This is because even though $v_1 \geq p$, DIM_1 cannot compare $(1,p)$ and $(0,0)$ since $p > B$. So, the preference of DIM_2 becomes the preference of the agent. So, the only region where $(1,p)$ is preferred to $(0,0)$ is when $\min(v_1, v_2) \geq p$. This is shown in the left graph of Figure 1. [3]

Now, consider another mechanism with a menu of three outcomes: $\{(1,p), (\frac{B}{p}, B), (0,0)\}$. So, the new menu contains an outcome that involves randomization and a payment of B. Consider the agent with values $v \equiv (v_1, v_2)$. Using the same argument as before, we see that if $\min(v_1, v_2) \geq p$, then the agent prefers $(1,p)$ to the other two outcomes in the menu.

[3] In our firm and investors example, the firm will be able to buy the object at price $p > B$ if and only if both its and investors' values are higher than p.

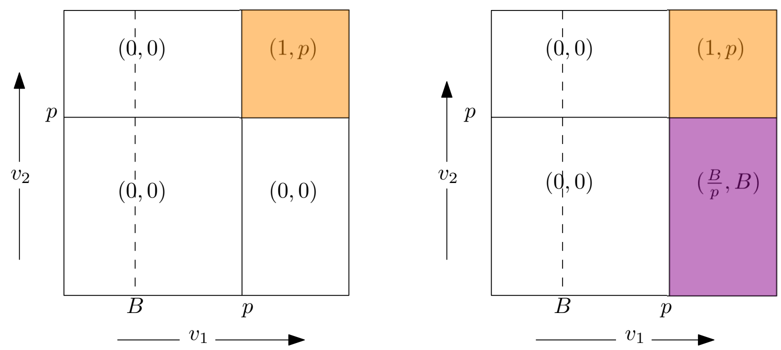

Figure 1: Non-optimality of posted prices

Similarly, if $v_1 \leq p$, then the agent prefers $(0,0)$ to the other two outcomes in the menu. However, if $v_1 \geq p$ but $v_2 \leq p$, then $v_1 - p \geq \frac{B}{p}(v_1 - p)$. But $p > B$ implies that the agent cannot compare $(1,p)$ and $(\frac{B}{p}, B)$ along $\textsc{dim}_1$. However, since $v_2 \leq p$, we see that $\frac{B}{p}(v_2 - p) \geq v_2 - p$. So, the agent prefers $(\frac{B}{p}, B)$ to $(1,p)$.[4] The agent also prefers $(\frac{B}{p}, B)$ to $(0,0)$ because he can compare these outcomes along $\textsc{dim}_1$, where $(\frac{B}{p}, B)$ is preferred to $(0,0)$. Hence, the agent prefers $(\frac{B}{p}, B)$ to the other outcomes in the menu. This is shown the right graph of Figure 1. This graph has an extra positive measure region where revenue of B can be earned by the seller at every profile. Hence, this mechanism generates strictly larger revenue than the posted price mechanism. As is apparent, the seller is able to exploit the lexicographic nature of decision-making of the buyer to extract more revenue than in a posted price mechanism. Our main result will show that it cannot exploit any more than this, i.e., such a mechanism will be optimal.

The above discussion shows that a posted price mechanism which posts a price above the budget cannot be optimal. Our main result will formalize this intuition - for low enough budgets, we will show that the optimal mechanism will involve randomization but we can be precise about the nature of the randomization. The optimal mechanism will be a posted price mechanism for "high enough" budgets. But for budgets below a certain threshold, it will be a mechanism involving an extra layer of pooling in the middle.

2.2 Layout of the paper

The rest of the paper is structured as follows. In Section 3, we introduce our notion of incentive compatibility and state our main results. The proofs of our main results are quite

[4]Again, in our firm and investors example, even though the firm likes $(1,p)$ to $(\frac{B}{p}, B)$, the investors shoot it down since $v_2 \leq p$ and the firm must respect their preference by preferring $(\frac{B}{p}, B)$ to $(1,p)$.

long. So, we have put them in Appendix A. We give a brief overview of the proofs in Section 3.4. Section 4 contains some discussions about the model and its motivations. Section 5 contains an extension where budget is also considered private information of the agent. The proofs of Section 5 is given in Appendix B. Supplementary Appendix C contains some missing proofs and discussions.

3 The optimal mechanism

3.1 Incentive compatibility

Since the preference of the agent is completely captured by $v \equiv (v_1, v_2)$, we will refer to v as the **type** of the agent - Section 5 discusses the private budget case, where the type will of the agent will be (v_1, v_2, B). A (direct) **mechanism** is a pair of maps: an allocation rule $f : V^2 \to [0,1]$ and a payment rule $p : V^2 \to \mathbb{R}$. For every $v \in V^2$, $f(v)$ denotes the allocation probability and $p(v)$ denotes the payment of the agent.

The restriction to such direct mechanisms is without loss of generality as a version of the revelation principle holds in our setting - see Section 4.1. Hence, we can discuss about incentive compatibility of direct mechanisms.

Definition 1 *A mechanism (f, p) is* **incentive compatible** *if for all $u, v \in V^2$,*

$$(f(u), p(u)) \ \succeq_u \ (f(v), p(v)).$$

Fix a mechanism (f, p) and let the range of the mechanism be

$$R^{f,p} := \{(a, t) : (f(v), p(v)) = (a, t) \text{ for some } v \in V^2\}.$$

In general, preferences over outcomes in $R^{f,p}$ may violate transitivity. However, our notion of incentive compatibility requires that at every type u, the outcome $(f(u), p(u))$ is preferred to any other outcome in $R^{f,p}$. This implies that the outcome chosen for every type is not involved in a cycle. This allows us to rule out Dutch book arguments (or money pump) using our notion of incentive compatibility. We discuss another notion of incentive compatibility and its relation to our notion later in Section 4.1.

Our notion of incentive compatibility can be broken down into two distinct cases. Fix $u, v \in V^2$. Then, there are two ways in which bundle $(f(u), p(u))$ can be (weakly) preferred over $(f(v), p(v))$ by an agent of type u.

1. First, DIM_1 prefers $(f(u), p(u))$ over $(f(v), p(v))$. This is possible if $p(u) \leq B$ and

$$u_1 f(u) - p(u) \geq u_1 f(v) - p(v).$$

2. Second, DIM_1 cannot compare $(f(u), p(u))$ and $(f(v), p(v))$, but DIM_2 prefers $(f(u), p(u))$ over $(f(v), p(v))$. This means $u_2 f(u) - p(u) \geq u_2 f(v) - p(v)$. Further, one of the following conditions must hold.

   (a) $u_1 f(u) - p(u) > u_1 f(v) - p(v)$ but $p(u) > B$.

   (b) $u_1 f(v) - p(v) > u_1 f(u) - p(u)$ but $p(v) > B$.

   (c) $u_1 f(v) - p(v) = u_1 f(u) - p(u)$ but $\min(p(u), p(v)) > B$.

Besides, incentive compatibility, we will impose a natural participation constraint. For this, we will assume that outside option of the agent is the consumption bundle $(0, 0)$, where he receives nothing and pays nothing.

Definition **2** *A mechanism (f, p) is* **individually rational** *if for all $v \in V^2$,*

$$(f(v), p(v)) \;\; \succeq_v \;\; (0, 0).$$

It is useful to note that the above individual rationality condition can be equivalently stated as follows. A mechanism (f, p) is individually rational if for all $v \in V^2$ (a) when $p(v) \leq B$, we have $v_1 f(v) - p(v) \geq 0$ and (b) when $p(v) > B$, we have $v_1 f(v) - p(v) \geq 0$ and $v_2 f(v) - p(v) \geq 0$. This leads us to the following characterization of individual rationality. Such characterizations are well known in standard settings and the result below shows that it extends to our model too.

Lemma **2** *Consider any incentive compatible mechanism (f, p). Then, (f, p) is individually rational if and only if $p(0, 0) \leq 0$.*

Proof: Suppose that $p(0, 0) \leq 0$. Consider any $u = (u_1, u_2) \in V^2$ such that $p(u) \leq B$, incentive compatibility implies that $(f(u), p(u)) \succeq_u (f(0, 0), p(0, 0))$, which implies $(f(u), p(u)) \succeq_{u_1} (f(0, 0), p(0, 0))$ since $p(u) \leq B$ and $p(0, 0) \leq 0 < B$. But this implies that $u_1 f(u) - p(u) \geq u_1 f(0, 0) - p(0, 0) \geq 0$, where the last inequality follows from the fact that $p(0, 0) \leq 0$. Hence, we have $(f(u), p(u)) \succeq_u (0, 0)$.

Similarly, consider any $v = (v_1, v_2) \in V^2$ such that $p(v) > B$, incentive compatibility implies $(f(v), p(v)) \succeq_{v_1} (f(0, 0), p(0, 0))$ and $(f(v), p(v)) \succeq_{v_2} (f(0, 0), p(0, 0))$ since $p(0, 0) \leq$

$B, p(v) > B$. This implies that $v_1 f(v) - p(v) \geq v_1 f(0,0) - p(0,0) \geq 0$. Similarly, we have $v_2 f(v) - p(v) \geq 0$. Hence, we have $(f(v), p(v)) \succeq_v (0,0)$.

For the other direction, consider the type $(0,0) \in V$. Individual rationality implies that $(f(0,0), p(0,0)) \succeq_{(0,0)} (0,0)$. This implies that $-p(0,0) \geq 0$. ■

3.2 New mechanisms

Incentive compatibility has different implications in our model because of the sequential nature of decision-making. There are some simple mechanisms that are incentive compatible and resemble similar mechanisms when the agent the agent uses only one rationale.

DEFINITION **3** *A mechanism (f, p) is a* POST-1 *mechanism if there exists a $K_1 \in [0, B]$ such that*

$$(f(v), p(v)) = \begin{cases} (0,0) & \text{if } v_1 \leq K_1 \\ (1, K_1) & \text{otherwise.} \end{cases}$$

A POST-1 mechanism is a mechanism where the object is allocated by only considering the value along DIM_1. So, it can be thought of as a posted price mechanism *for* DIM_1. This is because it posts a price K_1 which is less than the budget B, and hence, the agent can make a decision using his preference along DIM_1. So, if the value along DIM_1 is less than K_1, then the object is not allocated. Else, the object is allocated with probability 1. It is easy to see that such a mechanism is incentive compatible and individually rational.

We now introduce a new class of mechanisms that we call the POST-2 mechanisms. Unlike the POST-1 mechanism, the POST-2 mechanism considers the values of both the dimensions.

DEFINITION **4** *A mechanism (f, p) is a* POST-2 *mechanism if there exists a $K_1, K_2 \in [B, \beta]$ with $K_1 \leq K_2$, such that*

$$(f(v), p(v)) = \begin{cases} (0,0) & \text{if } v_1 \leq K_1 \\ (1, B + K_2(1 - \frac{B}{K_1})) & \text{if } \min(v_1, v_2) > K_2 \\ (\frac{B}{K_1}, B) & \text{otherwise} \end{cases}$$

The POST-2 mechanism has a pair of posted prices. The first posted price K_1 is for DIM_1. If the value along DIM_1 is below K_1, then the object is not sold. Else, the the object is sold with probability $\frac{B}{K_1}$ at per unit price of K_1, i.e., the total price paid equals K_1 times the probability of winning, which is $K_1 \times \frac{B}{K_1} = B$. The remaining probability $(1 - \frac{B}{K_1})$ is sold

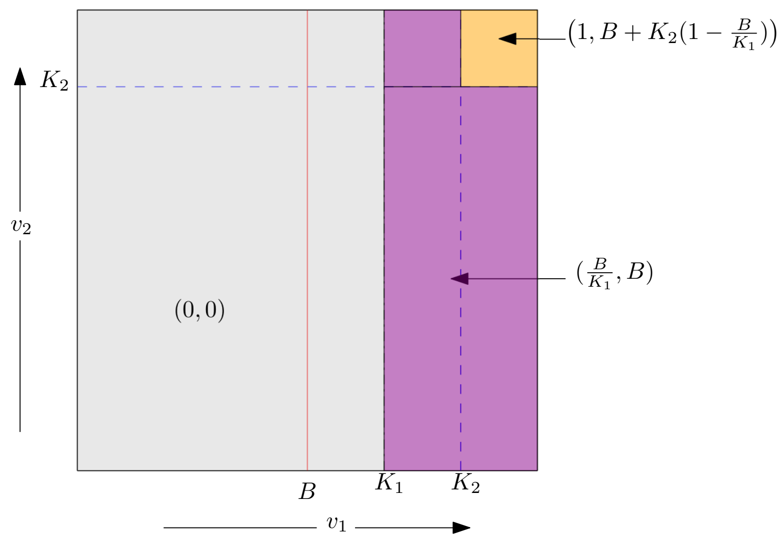

Figure 2: POST-2 mechanism

at per unit price K_2 if the value along both the dimensions exceed K_2. Figure 2 gives a graphical illustration of a POST-2 mechanism. We show below that a POST-2 mechanism is incentive compatible and individually rational.

PROPOSITION 1 *Every* POST-2 *mechanism is incentive compatible and individually rational.*

Though, we provide a formal proof of this result (and all subsequence omitted proofs) in the Appendix, we explain how the notion of incentive compatibility and the lexicographic decision-making make the result possible. There are three outcomes in the "menu" (range) of a POST-2 mechanism. The outcomes $(0, 0)$ and $(\frac{B}{K_1}, B)$ are outcomes which can be compared using DIM_1. On the other hand, outcome $(1, B + K_2(1 - \frac{B}{K_1}))$ has payment more than B. So, if a type $v \equiv (v_1, v_2)$ is assigned this outcome, incentive compatibility requires that $(1, B + K_2(1 - \frac{B}{K_1}))$ is preferred to $(0, 0)$ and $(\frac{B}{K_1}, B)$ in *both* DIM_1 and DIM_2. It is easy to verify that this is possible if $v_1, v_2 \geq K_2$ and $K_2 \geq K_1$. Similarly, the other incentive constraints can be shown to hold.

A POST-2 mechanism uses the behavioral nature of the agent by posting a pair of prices. There are other kinds of mechanisms that can be incentive compatible. Our main result below shows that the optimal mechanism can be either a POST-1 or a POST-2 mechanism.

3.3 Main results

The expected (ex-ante) revenue of a mechanism (f, p) is given by

$$\textsc{Rev}(f,p) = \int_{V^2} p(v) dG(v)$$

We say that a mechanism (f, p) is **optimal** if (a) (f, p) is incentive compatible and individually rational, and (b) $\textsc{Rev}(f, p) \geq \textsc{Rev}(f', p')$ for any other incentive compatible and individually rational mechanism (f', p').

For the optimality of our mechanisms, we will need a condition on the marginal distribution along $\textsc{dim}_1$. Define the function H_1 as follows:

$$H_1(x) = xG_1(x) \ \forall \ x \in [0, \beta].$$

Theorem **1** *Suppose H_1 is a strictly convex function. Then, either a* post-1 *or a* post-2 *mechanism is an optimal mechanism.*

Our results are slightly stronger than what Theorem 1 suggests. We prove that among all mechanisms which has a positive measure of types where the agent pays more than the budget, a post-2 mechanism is optimal. In the remaining class of mechanisms, a post-1 mechanism is optimal.

We can be more precise about the optimization programs that need to be solved to get the optimal mechanism in Theorem 1. In particular, we either need to solve a one-variable or a two-variable optimization program.

Proposition **2** *Suppose H_1 is strictly convex. Then, the expected revenue from the optimal mechanism is* $\max(R_1, R_2)$*, where*

$$R_1 = \max_{K_1 \in [0,B]} K_1(1 - G_1(K_1))$$

$$R_2 = \max_{K_2 \in [B,\beta], K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

The maximization expressions for R_1 and R_2 reflect the expected revenue from a post-1 and post-2 mechanism respectively.

If the budget B is high enough, then the post-1 mechanism becomes optimal - intuitively, the $\textsc{dim}_1$ rationale makes more decisions and screening along that dimension becomes optimal. It is more interesting to see how much restriction on budget we need to get post-2 mechanism to be optimal. Below, we derive such a sufficient condition on the budget.

Define the optimal monopoly reserve price as $\bar{K}$

$$\bar{K} := \arg\max_{r \in [0,\beta]} r(1 - G_1(r)).$$

If H_1 is a strictly convex function, $\bar{K}$ is uniquely defined since $x - xG_1(x)$ is a strictly concave function. The interpretation of $\bar{K}$ is that if the agent was *not* budget-constrained along DIM_1, then the optimal mechanism would have involved a posted-price of $\bar{K}$. Our other main result shows that if the budget constraint is less than $\bar{K}$, then the optimal mechanism is a POST-2 mechanism.

PROPOSITION **3** *Suppose H_1 is strictly convex and $B \leq \bar{K}$. Then, the optimal mechanism is a* POST-2 *mechanism. In particular, it is a solution to the following program.*

$$\max_{K_2 \in [B,\beta],\ K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

Proof: Since H_1 is strictly convex, $r(1 - G_1(r))$ is strictly increasing for all $r \leq \bar{K}$. Using $B \leq \bar{K}$, we get that $B(1 - G_1(B)) \geq r(1 - G_1(r))$ for all $r \leq B$. Hence, R_1 defined as the maximum possible revenue in a posted-price mechanism in our problem (Proposition 2) is

$$R_1 = \max_{K_1 \in [0,B]} K_1(1 - G_1(K_1)) = B(1 - G_1(B)).$$

But the POST-2 mechanism with $K_1 = K_2 = B$ generates a revenue of $B(1 - G_1(B))$. This proves the theorem. ■

The optimality of POST-2 mechanism is possible even for $B > \bar{K}$. Proposition 3 only gives a sufficient condition on the budget for optimality of a POST-2 mechanism. The exact optimal mechanism is difficult to describe in general. Section 3.5 works out the exact optimal mechanism for the uniform distribution prior.

3.4 Sketch of the proofs

We give an overview of the proof of Theorem 1 in this section. Fix a mechanism (f, p), and define the following partitioning of the type space:

$$V^+(f,p) := \{v : p(v) > B\}$$
$$V^-(f,p) = \{u : p(u) \leq B\}.$$

The proof considers two classes of mechanisms, those (f, p) where $V^+(f, p)$ has non-zero Lebesgue measure and those where $V^+(f, p)$ has zero Lebesgue measure. Define the following partitioning of the class of mechanisms:

$$M^+ := \{(f,p) : V^+(f,p) \text{ has positive Lebesgue measure}\}$$
$$M^- := \{(f,p) : V^+(f,p) \text{ has zero Lebesgue measure}\}.$$

The proof of Theorem 1 is completed by proving the following proposition.

PROPOSITION 4 *Suppose H_1 is strictly convex. Then, the following are true.*

1. *There exists a* POST-1 *mechanism $(f,p) \in M^-$ which is incentive compatible and individually rational such that for every incentive compatible and individually rational mechanism $(f',p') \in M^-$, we have*

$$\textsc{Rev}(f,p) \geq \textsc{Rev}(f',p').$$

2. *There exists a* POST-2 *mechanism $(f,p) \in M^+$ which is incentive compatible and individually rational such that for every incentive compatible and individually rational mechanism $(f',p') \in M^+$, we have*

$$\textsc{Rev}(f,p) \geq \textsc{Rev}(f',p').$$

The proof of (1) in Proposition 4 uses somewhat familiar ironing arguments. However, proof of (2) in Proposition 4 is quite different, and requires a lot of work to get to a simpler class of mechanisms where ironing can be applied. The proof proceeds by deriving some necessary conditions for incentive compatibility and reducing the space of mechanisms. It can be broken down into three steps.

1. STEP 1. The first step of the proof uses just incentive constraints to show that every incentive compatible mechanism has a simple form. In particular, there is a cutoff $K \geq B$ such that for all types v with $\min(v_1, v_2) > K$, the outcome of the mechanism is constant (with payment greater than the budget). This implication comes purely from the incentive constraints in the mechanism.

2. STEP 2. In the next step, we show that the *optimal* mechanism must belong to a class of simple mechanisms. In this class of mechanisms, there is a cutoff K (identified in Step 1), such that the outcome of the mechanism for types v with $\min(v_1, v_2) > K$ is one constant (where payment is greater than the budget) and for types v with $v_1 \geq K$ but $\min(v_1, v_2) \leq K$, it is another constant (where payment is equal to budget). For types v with $v_1 < K$, payment is not more than the budget.

3. Step 3. In this step, we further relax the class of above mechanisms. We show that it is without loss of generality to consider only those mechanisms where for all types u, v with $u_1 = v_1 < K$, the outcomes at u and v are the same. These steps allow us to apply standard ironing arguments and get to a POST-2 mechanism.

In summary, though the proof does not introduce new tools to deal with multidimensional mechanism design problems, it illustrates that multidimensional mechanism design problems may be tractable under certain behavioral assumptions.

3.5 Uniform distribution

In this section, we work out the exact optimal mechanism for the uniform distribution case. All the proofs of this section are given in Supplementary Appendix C.2.

We assume that $\beta = 1$ and G is the uniform distribution over $[0,1] \times [0,1]$. Call a POST-2 mechanism defined by posted prices (K_1^*, K_2^*) optimal POST-2 mechanism if it solves the optimization program in Proposition 2. Our result shows that for uniform distribution $K_1^* = K_2^*$.

Lemma **3** *Suppose $\beta = 1$ and G is the uniform distribution over $[0,1] \times [0,1]$. If (K_1^*, K_2^*) are values of (K_1, K_2) in the optimal* POST-2 *mechanism, then $K_1^* = K_2^*$.*

Further, the optimal POST-2 *mechanism must satisfy:*

1. *if $B \geq \frac{1}{2}(3-\sqrt{5})$, then $K_1^* = K_2^* = B$,*
2. *if $B < \frac{1}{2}(3-\sqrt{5})$, then $K_1^* = K_2^* = \frac{1}{3}\big(B + 2 - \sqrt{(B^2 + B + 1)}\big)$.*

Lemma 3 can be generalized to distributions beyond uniform distributions if enough assumptions on distributions are put. The details are skipped but available upon request. Using this lemma, we can provide a complete description of the optimal mechanism for the uniform distribution case.

Proposition **5** *Suppose $\beta = 1$ and G is the uniform distribution over $[0,1] \times [0,1]$. Then, the optimal mechanism is the following.*

1. *If $B > \frac{1}{2}$, then a* POST-1 *mechanism with $K_1 = \frac{1}{2}$ is optimal.*
2. *If $B \in [\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}]$, then a* POST-1 *mechanism with $K_1 = B$ is optimal. In this case, a* POST-2 *mechanism with $K_1 = K_2 = B$ is also optimal.*

3. *If* $B \in (0, \frac{1}{2}(3-\sqrt{5}))$*, then a* POST-2 *mechanism with*

$$K_1 = K_2 = \frac{1}{3}\big(B + 2 - \sqrt{(B^2 + B + 1)}\big)$$

is optimal.

Notice that as $B \to 0$, the optimal mechanism is a posted price mechanism with price $\frac{1}{3}$. So, in the limiting case when DIM_1 has zero budget to make decisions, the optimal mechanism is *not* a posted price mechanism with posted price $\frac{1}{2}$ - which is the optimal posted price in the standard model. To see why, consider the limiting case $B = 0$. Suppose the buyer uses a posted price mechanism with price p. Who are the types who will accept this price? This is shown in the left graph in Figure 1. All the types (v_1, v_2) such that $v_1 < p$ will choose outcome $(0, 0)$. All types (v_1, v_2) with $v_1 > p$ but $v_2 < p$ will also choose outcome $(0, 0)$ - this is because even though DIM_1 prefers $(1, p)$ over $(0, 0)$, it cannot make a decision because of budget constraint. Thus, the only types (v_1, v_2) which will prefer $(1, p)$ to $(0, 0)$ are those with $v_1 > p, v_2 > p$. Hence, the expected revenue from a posted price mechanism is $p(1-p)^2$, which is maximized at $\frac{1}{3}$. This argument establishes the optimal posted price mechanism. Proposition 5 shows that it is the optimal mechanism.

Finally, we show that the optimal mechanism revenue increases with B. Let $R^*(B)$ denote the revenue of the optimal mechanism described in Proposition 5 when the budget is B.

PROPOSITION **6** *Suppose* $\beta = 1$ *and* G *is the uniform distribution over* $[0, 1] \times [0, 1]$*. Then,* $B > B'$ *implies* $R^*(B) \geq R^*(B')$*.*

Figure 3 shows how the revenue increases (strictly) until some value of B is reached and then stays constant.

4 Discussions

In this section, we discuss some issues related to the revelation principle, our notion of incentive compatibility, and some motivating examples.

4.1 Notion of incentive compatibility

We show here a version of the revelation principle holds in our setting. To define an arbitrary mechanism, let M be a message space and $\mu : M \to Z$ be a mechanism. A strategy of the agent is a map $s : V \to M$. We say that mechanism μ **implements** the direct revelation mechanism (f, p) if there exists a strategy $s : V \to M$ such that

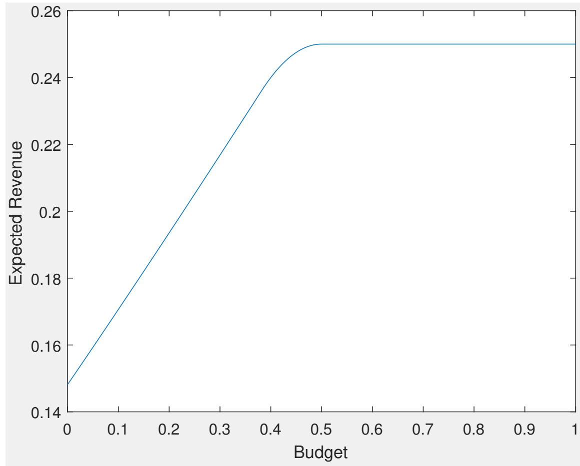

Figure 3: Revenue as a function of budget

- EQUILIBRIUM. $\mu(s(v)) \succeq_v \mu(m) \; \forall \; v \in V, \; \forall \; m \in M$.
- OUTCOME. $\mu(s(v)) = (f(v), p(v)) \; \forall \; v \in V$.

Suppose μ implements (f, p). Then, fix some $v, v' \in V$ and note that $(f(v), p(v)) = \mu(s(v)) \succeq_v \mu(s(v')) = (f(v'), p(v'))$, which proves incentive compatibility of (f, p). Hence, the revelation principle holds in this setting. It is well known that with behavioral agents, the revelation principle may not hold in general (de Clippel, 2014). There are at least two assumptions in our model which allows the revelation principle to work. The first is the completeness of our relation $\succeq_v$ (even though it may be intransitive). The second, and more important one, is the notion of incentive compatibility we use. We discuss this issue in detail next.

The primitives of our model involves how the agent chooses from pairs of outcomes. We are silent about how an agent chooses from a subset of alternatives. This is consistent with Tversky (1969) and most of the literature which works on binary choice models (Rubinstein, 1988; Tadenuma, 2002; Houy and Tadenuma, 2009). Our incentive constraints are appropriate for this binary choice model.

In Supplementary Appendix C.3, we consider a model where agents can choose from any subset of outcomes. We adapt a model of Manzini and Mariotti (2012) to our framework to consider choice correspondences. We then propose a notion of incentive compatibility which is appropriate for choice correspondences - we call it *choice incentive compatibility*. We argue that both the notions of incentive compatibility are independent. However, there are two main reasons why we use our existing notions of incentive compatibility instead of choice-incentive compatibility. First, to be able to use choice-incentive compatibility, we have to *assume* that the agent chooses from subsets of outcomes using some choice procedure. The current primitives of our model are much simpler - it just makes assumptions on how the agent chooses between pairs of outcomes. Importantly, our notion of incentive compatibility allows us tractability using minimal assumptions about deviations from rationality. Second, if the primitives of the model are choice correspondences, then a revelation principle need not hold - see de Clippel (2014). This implies that the space of mechanisms are more complex than the set of direct revelation mechanisms. In summary, it is not clear how an optimal mechanism will look like if we considered a model assuming certain choice behavior of agents over subsets of outcomes and choice-incentive compatibility as the notion of our incentive compatibility. We leave this issue for future research.

4.2 Other motivating examples

In this section, we discuss a couple of more examples that fit our story. In principle, any model where the agent cannot compare outcomes due to a budget constraint and consults an outside agent for such comparison can be fit into our model. The examples below make this point explicit for certain settings.

Consider a setting, where the seller is selling to a "delegate" (a manager in a firm or child in a family). The delegate is budget constrained. The delegate consults his boss-agent (board members of the firm or parent in a family) whenever he cannot compare outcomes. Whenever consulted, the preference of the boss-agent becomes the preference of the delegate.

Our model also fits a stylized setting of an agent with two selves. The agent compares outcomes using the two selves in a lexicographic manner: the first self (say, an impulsive self) compares outcomes whenever possible but there is threshold price (budget constraint) beyond which the comparison is done by the second self (contemplative self) [5]. Unlike traditional economic models of temptation and self-control (Gul and Pesendorfer, 2001; Fudenberg and Levine, 2006), our agent is naive and does not optimize between his two selves to form his preference.

[5] Imagine a situation where you compare products based on your underlying value but once the price hits a threshold, you start comparing them by doing some research (say, reading reviews of products etc.), in which case your comparison may look different.

Another motivation for our model is that the agent has two modes of payment - a mode of payment which is budget constrained (say, credit card) and a mode of payment which is unconstrained (say, direct debit or cheque). Whenever possible our agent makes payments using credit card. But if some pair of consumption bundles cannot be compared due to credit constraint on the card, the agent is more careful with the evaluation of these consumption bundles, resulting in a different value when he makes direct debit or check payments.

5 Private budgets: a partial result

In this section, we consider the scenario when budget is private information. This may be the case in various examples that we considered - the budget allocated to DIM_1 may not be observable if it is just a self control parameter of the agent. In such cases, the type space is three-dimensional. We only have a partial description of an optimal mechanism in this case.

We will assume that values on both dimensions and the budget lie in $[0, \beta]$. Thus, the type space is $W \equiv [0, \beta]^3$. We denote a type $(v, B) \equiv (v_1, v_2, B)$ to mean v_1 and v_2 are the values of DIM_1 and DIM_2 respectively and B is the budget. For any type $(v, B) \in W$, the preferences over the outcome space is same as the preferences of the type $v \in V$ with budget B in the public budget case. Since the outcome space is the same, this is well defined as before. For any type (v, B), we denote the corresponding preference as $\succeq_{(v,B)}$.

The seller has a prior Φ over the type space W. A (direct) **mechanism** is a pair of maps: an allocation rule $f : W \to [0, 1]$ and a payment rule $p : W \to \mathbb{R}$. The incentive compatibility and individual rationality constraints are as before.

Definition 5 *A mechanism (f, p) is* **incentive compatible** *if for all $(u, B), (v, B') \in W$,*

$$\big(f(u, B), p(u, B)\big) \;\succeq_{(u,B)}\; \big(f(v, B'), p(v, B')\big).$$

A mechanism (f, p) is **individually rational** *if for all $(v, B) \in W$,*

$$\big(f(v, B), p(v, B)\big) \;\succeq_{(v,B)}\; \big(0, 0\big).$$

We will only consider the following class of mechanisms in this section for our main result.

Definition 6 *A mechanism (f, p) is DIM_2* **non-trivial** *if there exists some budget $B \in [0, \beta]$ and $V' \subseteq [0, \beta]^2$ such that V' has non-zero Lebesgue measure in $[0, \beta]^2$ and*

$$p(v, B) > B \;\forall\; v \in V'.$$

A DIM$_2$ non-trivial mechanism rules out the possibility that at every budget B, the payment is not more than B at almost every valuation profile (given B). We only consider optimality in the class of DIM$_2$ non-trivial mechanisms. As before, the expected revenue of a mechanism (f, p) is

$$\textsc{Rev}(f,p) := \int_W p(v,B)d\Phi(v,B).$$

A DIM$_2$ non-trivial mechanism (f, p) is **partially optimal** if it is incentive compatible and individually rational and there is no other DIM$_2$ non-trivial mechanism (f', p') which is incentive compatible and individually rational and $\textsc{Rev}(f', p') > \textsc{Rev}(f, p)$.

We now introduce an analogue of the POST-2 mechanism in the private budget case.

DEFINITION **7** *A mechanism (f, p) is a* POST* *mechanism if there exists $K \in [0, \beta]$ such that*

$$(f(v,B),p(v,B)) = \begin{cases} \Big(1,K\Big) & \textit{if } \Big(\min(v_1,v_2) > K \textit{ and } B < K\Big) \\ & \textit{or } \Big(v_1 > K \textit{ and } B \geq K\Big) \\ (0,0) & \textit{if } v_1 \leq K \\ (\frac{B}{K}, B) & \textit{if } v_1 > K,\ v_2 \leq K \textit{ and } B < K \end{cases}$$

A pictorial description of a POST* mechanism is given in Figure 4. The similarity between

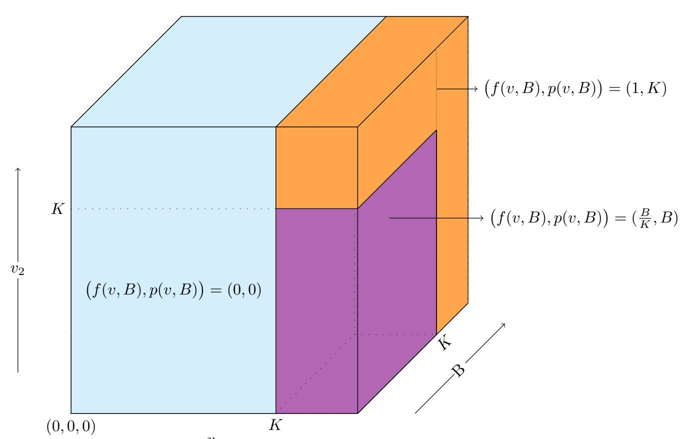

Figure 4: Illustration of a POST* mechanism

POST-2 and POST* is deceiving since POST-2 is defined for a fixed budget B but POST* is defined for all values of budget. As a result, the menu size of POST* is infinite - a separate outcome is chosen for every budget in the third case of the definition of POST* mechanism.

Notice that choice of $K \in [0, \beta]$ pins down a POST* mechanism. So, a POST* mechanism is defined by a single parameter. On the other hand, a POST-2 mechanism requires specification of two parameters. However, if we fix a POST* mechanism, defined by choosing K, and consider a budget $B < K$, then the projection of this POST* mechanism at B is a POST-2 mechanism with the two parameters of the POST-2 mechanism equal to K. Similarly, if we take $B > K$, then the projection of a POST* mechanism at B is a posted price mechanism.

We show below that every POST* mechanism is incentive compatible and individually rational.

PROPOSITION **7** *Every* POST* *mechanism is* DIM_2 *non-trivial, incentive compatible, and individually rational.*

The main result of this section establishes the partial optimality of POST* mechanism.

THEOREM **2** *A partially optimal mechanism is a* POST* *mechanism.*

We emphasize here that unlike Theorem 1, Theorem 2 does not require any distributional assumption. This is a consequence of the ironing required to arrive at the optimal mechanism in Theorem 1, and the absence of any ironing in the proof of Theorem 2 - see the respective proofs in Appendix. Intuitively, with private budgets, the set of incentive constraints become larger and the need for ironing reduces. We should also note here that if the lower support of budgets is positive (for simplicity, we have assumed it to be zero), Theorem 2 goes through with some minor changes, but it requires the distribution to satisfy the same condition as in Theorem 1. This is because, in that case, we need ironing to arrive at an optimal mechanism (very similar to Theorem 1). We skip these details for the interest of space but it is available upon request.

The derivation of an optimal mechanism without the DIM_2 non-triviality assumption for the private budget case seems difficult - even in the standard model, the private budget case is significantly complicated (Che and Gale, 2000). In Supplementary Appendix C.4, we state a sufficient condition on distributions (satisfied if values and budget are independently and uniformly distributed) that guarantee the optimality of a POST* mechanism.

6 Related literature

Our paper is related to a couple of strands of literature in mechanism design. We go over them in some detail.

Behavioral mechanism design. We discuss some literature in mechanism design which looks at specific models of behavioral agents and designing optimal contracts for selling to such agents. A very detailed survey with excellent examples can be found in Koszegi (2014). Our literature survey is limited in nature as we focus on models which are closer to ours.

A stream of papers investigate the optimal contract for a firm to a consumers in a two-period model, where the consumer has time inconsistent preferences. These papers differ in the way it treats inconsistent preferences and non-common priors between firm and consumers.

Eliaz and Spiegler (2006) consider a model where the type of the agent is his "cognitive" ability. In their model, there are two periods and the agent enjoys a valuation for an action in each period. In period 2, the agent's valuation may change to another value. Agents differ in their subjective assessment of the probability of that transition. So, in their model, the type is the subjective probability of the agent. They show how the optimal contract treats sophisticated and naive agents. While this paper allows agents to be time-inconsistent, in another paper, Eliaz and Spiegler (2008) study a similar model but do not allow time inconsistency. There, they allow the monopolist to have a separate belief about the change of state. They characterize the optimal contract and show the implications of non-common priors on the menu of optimal contract and ex-post efficiency. Grubb (2009) considers a two period model where a firm is selling a divisible good to consumers. The private type of the consumer is his demand in period 2. In period 1, the firm offers them a tariff which is accepted or rejected. If accepted, the consumers buy the quantity in period 2 once they realize their demand. The key innovation in his paper is again the lack of common prior between consumers and the firm - in particular, he shows that if the prior of the consumers is such that it *underestimates* the variance of the actual prior (for instance, if the consumer prior has the same mean as the firm, then consumer prior is a mean-preserving spread of the firm prior), then the optimal tariff of the firm must have three parts (with quantities offered at zero marginal cost).

de Clippel (2014) studies complete information implementation with behavioral agents - his main results extend Maskin's characterization (Maskin, 1999) to environments with behavioral agents. Esteban et al. (2007) consider a model where agents have temptation and self control preferences as in Gul and Pesendorfer (2001), and characterize the optimal contract - also see related work on self control preferences in DellaVigna and Malmendier (2004). There are several other papers who consider time inconsistent preferences and analyze the optimal contracting problem and our paper adds to this literature. Carbajal and Ely (2016) consider a model of optimal price discrimination when buyers have loss averse preferences with state dependent reference points. They characterize the optimal contract in their model.

MULTIDIMENSIONAL MECHANISM DESIGN. The type space of our agent is two-dimensional. It is well known that the problem of finding an optimal mechanism for selling multiple goods (even to a single buyer) is notorious. A long list of papers have shown the difficulties involved in extending the one-dimensional results in Mussa and Rosen (1978); Myerson (1981); Riley and Zeckhauser (1983) to multidimensional framework - see Armstrong (2000); Manelli and Vincent (2007) as examples. Even when the seller has just *two* objects and there is just one buyer with additive valuations (i.e., value for both the objects is sum of values of both the objects), the optimal mechanism is difficult to describe (Manelli and Vincent, 2007; Daskalakis et al., 2017; Hart and Nisan, 2017). This has inspired researchers to consider *approximately* optimal mechanisms (Chawla et al., 2007, 2010; Hart and Nisan, 2017) or additional robustness criteria for design (Carroll, 2017). Compared to these problems, our two-dimensional mechanism design problem becomes tractable because of the nature of incentive constraints, which in turn is a consequence of the preference of the agent.

MECHANISM DESIGN WITH BUDGET CONSTRAINTS. In our model, DIM_1 is budget constrained. We compare this with the literature in the standard model when there is a single object and the buyer(s) is budget constrained. The space of mechanisms is restricted to be such that payment is no more than the budget. This feasibility requirement on the mechanisms essentially translates to a violation of quasilinearity assumption of the buyer's preference for prices above the budget (utility assumed to be $-\infty$) but below the budget the utility is assumed to be quasilinear. This introduces additional complications for finding the optimal mechanism. Laffont and Robert (1996) show that an all-pay-auction with a suitable reserve price is an optimal mechanism for selling an object to multiple buyers who have publicly known budget constraints. When the budget is private information, the problem becomes even more complicated - see Che and Gale (2000) for a description of the optimal mechanism for the single buyer case and Pai and Vohra (2014) for a description of the optimal mechanism for the multiple buyers case. All these mechanisms involve randomization but the nature of randomization is quite different from ours. This is because the source of randomization in all these papers is either due to budget being private information (hence, part of the type, as in Che and Gale (2000); Pai and Vohra (2014)) or because of multiple agents with budget being common knowledge (as in Laffont and Robert (1996); Pai and Vohra (2014)). Indeed, with a single agent and public budget, the optimal mechanism in a standard single object allocation model is a posted price mechanism. This can be contrasted with our result where we get randomized optimal mechanism even with one buyer and budget being common knowledge. This shows that the lexicographic decision making using two rationales plays an important role in making a POST-2 mechanism optimal. Also, the set of menus in the optimal mechanism in the standard single object auction with budget constraint may have more than three outcomes. Further, the outcomes in the menu of these optimal mechanisms are not as simple as our POST-2 mechanism. Finally, like us, these papers assume that budget is exogenously determined by the agent. If the buyer can choose his budget constraint, then Baisa and Rabinovich (2016) shows that the optimal mechanism in a multiple buyers setting allocates the object efficiently whenever it is allocated - this is in contrast to the exogenous budget cases studied in Laffont and Robert (1996); Pai and Vohra (2014).

Burkett (2016) studies a principal-agent model where the agent is participating in an auction mechanism with a third-party. To decide on the report in the mechanism, the principal designs a contract with the agent. His main result shows that a simple budget-constraint contract is optimal for the principal. Hence, budget constraint comes via an optimization exercise of the principal in his problem. In contrast, our agent is naive and does not carry out any such optimization exercise across dimensions. We assume that the budget is exogenously assigned along DIM_1. Besides, in Burkett (2016), the third-party mechanism is *given* and his result specifies the optimal contract between the principal and the agent.

A Appendix: Omitted Proofs of Section 3

This section contains all omitted proofs of Section 3 - except for proofs of Section 3.5, which are given in the Supplementary Appendix C.2.

A.1 Proof of Proposition 1

Proof: Consider a POST-2 mechanism (f, p) defined by parameters K_1 and K_2 with $B \leq K_1 \leq K_2$. Since $p(0, 0) = 0$, Lemma 2 implies that (f, p) is individually rational if it is incentive compatible. We show incentive compatibility of (f, p). We will denote by $\bar{u} \rightarrow \tilde{u}$ the incentive constraint associated with type $\bar{u}$ when it cannot misreport $\tilde{u}$.

Consider types u, v, s taken from three different regions in Figure 2 with three different outcomes. In particular, u, v, s satisfy: $u_1 \leq K_1$, $\min(v_1, v_2) \leq K_2$ but $v_1 > K_1$, and $\min(s_1, s_2) > K_2$. Note that

$$(f(u), p(u)) = (0, 0),\ (f(v), p(v)) = (\frac{B}{K_1}, B),\ \text{and}\ (f(s), p(s)) = (1, B + K_2(1 - \frac{B}{K_1})).$$

We consider incentive compatibility of each of these types.

1. $u \rightarrow v, u \rightarrow s$. Note that since $u_1 \leq K_1$, we have $u_1 \frac{B}{K_1} - B \leq 0$. Hence, agent with type u weakly prefers $(0, 0)$ to $(\frac{B}{K_1}, B)$. Similarly,

$$\begin{aligned} u_1 - B - K_2\big(1 - \frac{B}{K_1}\big) &\leq K_1 - B - K_2 + \frac{K_2}{K_1}B \\ &= (K_2 - K_1)\Big(\frac{B}{K_1} - 1\Big) \leq 0, \end{aligned}$$

   where first inequality is due to $u_1 \leq K_1$ and the second is due to $K_2 \geq K_1$ and $B \leq K_1$. Hence, u prefers $(0, 0)$ to $(f(s), p(s))$.

2. $v \rightarrow u, v \rightarrow s$. For $v \rightarrow u$, we note that

$$v_1 \frac{B}{K_1} - B \geq 0$$

   This follows from the fact that $v_1 > K_1$. Hence, incentive constraint $v \rightarrow u$ holds as $p(v) = B$.

   For $v \rightarrow s$, we note that

$$\begin{aligned} \min(v_1, v_2) - B - K_2\Big(1 - \frac{B}{K_1}\Big) &\leq \min(v_1, v_2) - B - \min(v_1, v_2)\Big(1 - \frac{B}{K_1}\Big) \\ &= \frac{B}{K_1}\min(v_1, v_2) - B. \end{aligned}$$

If $\min(v_1, v_2) = v_1$, then we see that DIM_1 prefers $(f(v), p(v))$ to $(f(s), p(s))$. Else, $\min(v_1, v_2) = v_2$. In that case since $p(s) > B$, even if DIM_1 type prefers $(f(s), p(s))$ to $(f(v), p(v))$, he cannot compare. But DIM_2 prefers $(f(v), p(v))$ to $(f(s), p(s))$. Hence, incentive constraint $v \to s$ holds.

3. $s \to u, s \to v$. Note that for $x \in \{s_1, s_2\}$, we have

$$\begin{aligned} 0 \leq \frac{K_2}{K_1}B - B \leq \frac{B}{K_1}x - B \\ &= x - B - x\Big(1 - \frac{B}{K_1}\Big) \\ &\leq x - B - K_2\Big(1 - \frac{B}{K_1}\Big), \end{aligned}$$

where the inequalities follow from the fact that $\min(s_1, s_2) > K_2 \geq K_1 \geq B$. This shows that *both* the dimensions at s prefer $(f(s), p(s))$ to $(f(v), p(v))$ and $(f(u), p(u))$. Because $p(s) > B$, the incentive constraints $s \to v$ and $s \to u$ hold.

■

A.2 Proofs of Theorem 1 and Propositions 2 and 4

In this section, we provide the proof of the main results - Theorem 1 and Propositions 2 and 4. It is clear that Proposition 4 immediately implies Theorem 1. So, we first provide a proof of Proposition 4, followed by a proof of Proposition 2.

A.2.1 Preliminary Lemmas

We start off by proving a series of necessary conditions for incentive compatibility. The first lemma is a monotonicity condition of allocation rule: for every incentive compatible mechanism, type with higher payment implies higher allocation probability. Hence, the outcomes in the range of an incentive compatible mechanism are ordered in a natural sense.

LEMMA 4 *For any incentive compatible mechanism (f, p), if $p(u) < p(v)$ for any u, v, then $f(u) < f(v)$.*

Proof: Take any u, v such that $p(u) < p(v)$. Incentive compatibility implies that

$$(f(v), p(v)) \succeq_v (f(u), p(u)).$$

If $p(v) \leq B$, then we must use the incentive constraints along $\textsc{dim}_1$, which gives us

$$v_1 f(v) - p(v) \geq v_1 f(u) - p(u) > v_1 f(u) - p(v),$$

where the last inequality uses $p(v) > p(u)$. This implies $f(u) < f(v)$. If $p(v) > B$, then using the incentive constraint along $\textsc{dim}_2$, we have

$$v_2 f(v) - p(v) \geq v_2 f(u) - p(u) > v_2 f(u) - p(v),$$

where the last inequality uses $p(v) > p(u)$. This implies $f(u) < f(v)$. ■

Lemma **5** *For any incentive compatible mechanism (f, p), for all u, v*

1. *if $p(u), p(v) \leq B$ and $u_1 > v_1$, then $f(u) \geq f(v)$,*
2. *if $p(u), p(v) > B$ and $u_2 > v_2$, then $f(u) \geq f(v)$.*

Proof: Take any u, v. If $p(u), p(v) \leq B$, then adding the incentive constraints along $\textsc{dim}_1$ gives us the desired result and if $p(u), p(v) > B$, then adding the incentive constraints along $\textsc{dim}_2$ gives us the desired result. ■

Lemma **6** *For any incentive compatible mechanism (f, p), for all u, v the following holds:*

$$\Big[p(u) \leq B < p(v)\Big] \Rightarrow \Big[\min(v_1, v_2) \geq \min(u_1, u_2)\Big].$$

Proof: Since $p(u) \leq B < p(v)$, by Lemma 4, $f(v) > f(u)$. We consider the incentive constraint from v to u first. This gives us

$$v_2 f(v) - p(v) \geq v_2 f(u) - p(u). \tag{1}$$

$$v_1 f(v) - p(v) > v_1 f(u) - p(u). \tag{2}$$

Using $f(v) > f(u)$, and aggregating Inequalities 1 and 2 gives us

$$\min(v_1, v_2)\big(f(v) - f(u)\big) \geq p(v) - p(u). \tag{3}$$

Incentive compatibility from u to v implies one of the two conditions to holds:

Case 1. $\textsc{dim}_1$ prefers $(f(u), p(u))$ to $(f(v), p(v))$: this gives

$$u_1 f(u) - p(u) \geq u_1 f(v) - p(v) \text{ or } p(v) - p(u) \geq u_1(f(v) - f(u)).$$

Adding with Inequality 3, we get,

$$(\min(v_1, v_2) - u_1)(f(v) - f(u)) \geq 0.$$

Then, $f(v) > f(u)$ implies that $\min(v_1, v_2) \geq u_1$.

CASE 2. $\textsc{dim}_1$ does not prefer $(f(u), p(u))$ to $(f(v), p(v))$ but budget has a bite - so, $\textsc{dim}_2$ prefers $(f(u), p(u))$ to $(f(v), p(v))$: this gives

$$u_2 f(u) - p(u) \geq u_2 f(v) - p(v). \tag{4}$$

Adding Inequalities (4) and (3), we get $(\min(v_1, v_2) - u_2)(f(v) - f(u)) \geq 0$. Since $f(v) > f(u)$, we get $\min(v_1, v_2) \geq u_2$.

Combining both the cases, $\min(v_1, v_2) \geq \min(u_1, u_2)$. ■

Now, fix a mechanism (f, p), and define

$$V^+(f, p) := \{v : p(v) > B\}$$
$$V^-(f, p) = \{u : p(u) \leq B\}.$$

LEMMA **7** *Fix an incentive compatible mechanism* (f, p)*. If* $V^+(f, p)$ *and* $V^-(f, p)$ *are non-empty, then the following holds:*

$$\inf_{v \in V^+(f,p)} \min(v_1, v_2) = \sup_{u \in V^-(f,p)} \min(u_1, u_2).$$

Proof: Since $V^+(f, p)$ is non-empty and $\min(v_1, v_2) \geq 0$, we have that $\inf_{v \in V^+(f,p)} \min(v_1, v_2)$ is a non-negative real number - we denote it as $\underline{v}$. By Lemma 6, $\sup_{u \in V^-(f,p)} \min(u_1, u_2)$ is also a non-negative real number as it is bounded above - we denote this as $\bar{v}$.

First, we show that $\underline{v} \geq \bar{v}$. If not, then $\underline{v} < \bar{v}$. Then, there is some v such that $\underline{v} < \min(v_1, v_2) < \bar{v}$. By definition of $\underline{v}$, there is a v' such that $\min(v'_1, v'_2)$ is arbitrarily close to $\underline{v}$ and $p(v') > B$. Since $\min(v'_1, v'_2) < \min(v_1, v_2)$, Lemma 6 gives us $p(v) > B$. Similarly, by definition of $\bar{v}$, there is a u' such that $\min(u'_1, u'_2)$ is arbitrarily close to $\bar{v}$ and $p(u') \leq B$. Since $\min(u'_1, u'_2) > \min(v_1, v_2)$, Lemma 6 gives us $p(v) \leq B$, giving us the desired contradiction.

Next, we show that $\underline{v} = \bar{v}$. If not, $\underline{v} > \bar{v}$. But this is not possible since for any v with $\underline{v} > \min(v_1, v_2) > \bar{v}$, we will have both $p(v) \leq B$ and $p(v) > B$, giving us a contradiction. ■

For any mechanism (f, p), we will denote by $K_{(f,p)}$ the following:

$$K_{(f,p)} := \inf_{v \in V^+(f,p)} \min(v_1, v_2) = \sup_{u \in V^-(f,p)} \min(u_1, u_2). \tag{5}$$

By Lemma 7, this is well-defined if $V^+(f, p)$ and $V^-(f, p)$ is non-empty.

Lemma 8 *If (f, p) is an incentive compatible and individual rational mechanism, then $V^-(f, p)$ is non-empty.*

Proof: Lemma 2 ensures that $(0, 0) \in V^-(f, p)$ if (f, p) is incentive compatible and individually rational. ■

Define the following partitioning of the class of mechanisms:

$$\begin{aligned} M^+ &:= \{(f, p) : V^+(f, p) \text{ has positive Lebesgue measure}\} \\ M^- &:= \{(f, p) : V^+(f, p) \text{ has zero Lebesgue measure}\}. \end{aligned}$$

We now prove a series of Lemmas for M^+ class of mechanisms.

A.2.2 Lemmas for M^+

The following lemma shows that $K_{(f,p)}$ is well defined if $(f, p) \in M^+$.

Lemma 9 *Suppose (f, p) is an incentive compatible and individually rational mechanism.*

1. *If $V^+(f, p)$ is non-empty, then $K_{(f,p)}$ defined in Equation (5) exists and satisfies: for all $v \in V$,*

$$\Big[\min(v_1, v_2) > K_{(f,p)}\Big] \Rightarrow \Big[p(v) > B\Big],$$
$$\Big[\min(v_1, v_2) < K_{(f,p)}\Big] \Rightarrow \Big[p(v) \leq B\Big].$$

2. *If $(f, p) \in M^+$, then $\beta > K_{(f,p)} > B$.*

Proof: The first part follows from Lemma 7, Lemma 8, and the definition of M^+.

For the second part, we first argue that $K_{(f,p)} \geq B$. Suppose $K_{(f,p)} < B$. Then, for some v with $K_{(f,p)} < \min(v_1, v_2) \leq B$, we have $p(v) > B$. But this violates individual rationality.

Now, assume for contradiction $K_{(f,p)} = B$. In that case, fix some $\epsilon \in (0, 1)$ and positive integer k, and consider the type $v^{k,\epsilon} \equiv (B + \epsilon^k, B + \epsilon^k)$. By (1), we know that $p(v^{k,\epsilon}) > B$. By individual rationality,

$$(B + \epsilon^k) f(v^{k,\epsilon}) \geq p(v^{k,\epsilon}) > B.$$

This gives us $f(v^{k,\epsilon}) > \frac{B}{B+\epsilon^k}$. Since $B+\epsilon > B+\epsilon^k$ for all $k > 1$, by (1) of Lemma 5, we have $f(v^{1,\epsilon}) \geq f(v^{k,\epsilon}) > \frac{B}{B+\epsilon^k}$. As $\frac{B}{B+\epsilon^k}$ can be made arbitrarily close to 1, we conclude that $f(v^{1,\epsilon}) = 1$ - notice that $v^{1,\epsilon} \equiv (B+\epsilon, B+\epsilon)$ and the claim holds for *all* $\epsilon \in (0,1)$. By Lemma 4, for all $\epsilon, \epsilon' \in (0,1)$, since $f(v^{1,\epsilon}) = f(v^{1,\epsilon'}) = 1$, we get that $p(v^{1,\epsilon}) = p(v^{1,\epsilon'})$. Denote $p(v^{1,\epsilon}) = B+\delta$, where $\epsilon \in (0,1)$. By definition, $\delta > 0$. Now, individual rationality requires that for *every* $\epsilon \in (0,1)$,

$$(B+\epsilon)f(v^{1,\epsilon}) - p(v^{1,\epsilon}) = (B+\epsilon) - (B+\delta) \geq 0.$$

But this will mean $\epsilon > \delta$ for all $\epsilon \in (0,1)$. Since $\delta > 0$ is fixed, this is a contradiction.

Finally, we know that $(f,p) \in M^+$ implies $V^+(f,p)$ has positive Lebesgue measure. If $\beta = K_{(f,p)}$, then by (1), we know that $V^+(f,p)$ has zero Lebesgue measure, which is a contradiction. ■

Next, we show a useful inequality involving $K_{(f,p)}$ for any $(f,p) \in M^+$.

LEMMA **10** *Suppose (f,p) is an incentive compatible and individually rational mechanism. If $(f,p) \in M^+$, then for all types $u \in V$ with $B < p(u)$, we must have*

$$K_{(f,p)}f(K_{(f,p)},0) - p(K_{(f,p)},0) \geq K_{(f,p)}f(u) - p(u).$$

Proof: First, consider two types $v \equiv (K_{(f,p)},0)$ and $v' \equiv (K_{(f,p)}, K_{(f,p)} - \epsilon)$, where $\epsilon > 0$ such that $K_{(f,p)} - \epsilon > 0$. Notice that $\min(v_1,v_2) < K_{(f,p)}$ and $\min(v'_1,v'_2) < K_{(f,p)}$. Hence, by Lemma 9, $p(v) \leq B$ and $p(v') \leq B$. As a result incentive constraints $v \to v'$ and $v' \to v$ imply that

$$K_{(f,p)}f(v) - p(v) \geq K_{(f,p)}f(v') - p(v')$$
$$K_{(f,p)}f(v') - p(v') \geq K_{(f,p)}f(v) - p(v).$$

This gives us

$$K_{(f,p)}f(K_{(f,p)},0) - p(K_{(f,p)},0) = K_{(f,p)}f(K_{(f,p)}, K_{(f,p)} - \epsilon) - p(K_{(f,p)}, K_{(f,p)} - \epsilon). \quad (6)$$

Now, assume for contradiction that for some u with $p(u) > B$ we have

$$K_{(f,p)}f(K_{(f,p)},0) - p(K_{(f,p)},0) < K_{(f,p)}f(u) - p(u).$$

We can choose an $\epsilon > 0$ but arbitrarily close to zero such that

$$K_{(f,p)}f(K_{(f,p)},0) - p(K_{(f,p)},0) < \big(K_{(f,p)} - \epsilon\big)f(u) - p(u).$$

Using Equation 6, we get,

$$K_{(f,p)}f(K_{(f,p)},K_{(f,p)}-\epsilon)-p(K_{(f,p)},K_{(f,p)}-\epsilon)<\big(K_{(f,p)}-\epsilon\big)f(u)-p(u).$$

But then

$$\begin{aligned}(K_{(f,p)}-\epsilon)f(K_{(f,p)},K_{(f,p)}-\epsilon)-p(K_{(f,p)},K_{(f,p)}-\epsilon)\\ <K_{(f,p)}f(K_{(f,p)},K_{(f,p)}-\epsilon)-p(K_{(f,p)},K_{(f,p)}-\epsilon)\\ <\big(K_{(f,p)}-\epsilon\big)f(u)-p(u)<K_{(f,p)}f(u)-p(u).\end{aligned}$$

Hence, the incentive constraint $(K_{(f,p)},K_{(f,p)}-\epsilon)\to u$ does not hold - a contradiction. ■

Lemma **11** *Suppose $(f,p)\in M^+$ is an incentive compatible and individually rational mechanism. Then, for any $\gamma\in(K_{(f,p)},\beta]$, the following limits exist:*

$$\begin{aligned}\lim_{\delta\to 0^+}f(K_{(f,p)}+\delta,\gamma)&=A_{(f,p),\gamma}\\ \lim_{\delta\to 0^+}p(K_{(f,p)}+\delta,\gamma)&=P_{(f,p),\gamma}.\end{aligned}$$

Further, the following equations hold:

$$K_{(f,p)}A_{(f,p),\gamma}-P_{(f,p),\gamma}=K_{(f,p)}f(K_{(f,p)},0)-p(K_{(f,p)},0) \tag{7}$$

$$\gamma A_{(f,p),\gamma}-P_{(f,p),\gamma}=\gamma f(\beta,\gamma)-p(\beta,\gamma). \tag{8}$$

Proof: Fix any $\gamma\in(K_{(f,p)},\beta]$ and any $\delta>0$ such that $K_{(f,p)}+\delta\leq\beta$ - by Lemma 9, such $\delta>0$ exists. Consider two types $v\equiv(K_{(f,p)}+\delta,\gamma)$ and $v'\equiv(\beta,\gamma)$. By Lemma 9, $p(v),p(v')>B$. The pair of incentive constraints between v and v' gives us

$$\begin{aligned}\gamma f(v)-p(v)&\geq\gamma f(v')-p(v')\\ \gamma f(v')-p(v')&\geq\gamma f(v)-p(v).\end{aligned}$$

Combining these and using the definition of v', we get

$$\gamma f(v)-p(v)=\gamma f(\beta,\gamma)-p(\beta,\gamma). \tag{9}$$

Now, consider $v''\equiv(K_{(f,p)},0)$. By Lemma 9, $p(v'')\leq B$. But $p(v)>B$ implies that incentive constraint $v\to v''$ must imply

$$\begin{aligned}(K_{(f,p)}+\delta)f(v)-p(v)&\geq(K_{(f,p)}+\delta)f(v'')-p(v'')\\ &\geq K_{(f,p)}f(v)-p(v)+\delta f(v''),\end{aligned}$$

where the second inequality comes from Lemma 10 and the fact that $p(v) > B$. Using Equation 9, we replace $p(v)$ in the previous equation to get,

$$\begin{aligned}(K_{(f,p)} + \delta)f(v) &\geq (K_{(f,p)} + \delta)f(v'') - p(v'') + \gamma f(v) - \gamma f(\beta,\gamma) + p(\beta,\gamma)\\ &\geq K_{(f,p)}f(v) + \delta f(v'')\end{aligned}$$

Rearranging terms, we get

$$\begin{aligned}\Big[\gamma - K_{(f,p)}\Big]f(v) &\leq \Big[\gamma f(\beta,\gamma) - p(\beta,\gamma)\Big] - \Big[K_{(f,p)}f(v'') - p(v'')\Big]\\ &\leq \Big[\gamma - K_{(f,p)}\Big]f(v) + \delta\Big[f(v'') - f(v)\Big]\end{aligned}$$

Since $v'' \equiv (K_{(f,p)}, 0)$ is independent of δ and $v \equiv (K_{(f,p)} + \delta, \gamma)$, we get that

$$\Big[\gamma - K_{(f,p)}\Big]\lim_{\delta\to 0^+} f(K_{(f,p)} + \delta, \gamma) = \Big[\gamma f(\beta,\gamma) - p(\beta,\gamma)\Big] - \Big[K_{(f,p)}f(K_{(f,p)},0) - p(K_{(f,p)},0)\Big].$$

This gives us the desired expression for $A_{(f,p),\gamma}$. Using Equation 9, we also get the desired expression for $P_{(f,p),\gamma}$.

Then, it is routine to check that Equations (7) and (8) hold. ■

LEMMA **12** *Suppose $(f,p) \in M^+$ is an incentive compatible and individually rational mechanism. For every $\delta \in (0, \beta - K_{(f,p)}]$ and $\gamma \in (K_{(f,p)}, \beta]$, the following is true:*

1. $f(K_{(f,p)} + \delta, \gamma) \geq A_{(f,p),\gamma}$,
2. $p(K_{(f,p)} + \delta, \gamma) \geq P_{(f,p),\gamma}$.

Proof: Fix any $\delta \in (0, \beta - K_{(f,p)}]$ and $\gamma \in (K_{(f,p)}, \beta]$ and let $v \equiv (K_{(f,p)} + \delta, \gamma)$. By Lemma 9, we know that $p(v) > B$. Then Lemma 10 applies and we must have,

$$K_{(f,p)}f(K_{(f,p)},0) - p(K_{(f,p)},0) \geq K_{(f,p)}f(v) - p(v).$$

Equation 7 then directly implies,

$$K_{(f,p)}A_{(f,p),\gamma} - P_{(f,p),\gamma} \geq K_{(f,p)}f(v) - p(v).$$

Combining Equations (8) and (9) yields,

$$\gamma A_{(f,p),\gamma} - P_{(f,p),\gamma} = \gamma f(v) - p(v).$$

Combining the above two expressions gives us

$$K_{(f,p)}\Big(A_{(f,p),\gamma} - f(v)\Big) \geq P_{(f,p),\gamma} - p(v) = \gamma\Big(A_{(f,p),\gamma} - f(v)\Big).$$

Since $\gamma > K_{(f,p)}$, we get $A_{(f,p),\gamma} \leq f(v)$, which further implies $P_{(f,p),\gamma} \leq p(v)$. This gives us the desired results. ■

Lemma **13** *Suppose $(f,p) \in M^+$ is an incentive compatible and individually rational mechanism. For every $\gamma_1, \gamma_2 \in (K_{(f,p)}, \beta]$,*

$$A_{(f,p),\gamma_1} = A_{(f,p),\gamma_2}$$
$$P_{(f,p),\gamma_1} = P_{(f,p),\gamma_2}.$$

Proof: Fix any $\gamma_1, \gamma_2 \in (K_{(f,p)}, \beta]$. First, we note that Equation 7 implies

$$K_{(f,p)} A_{(f,p),\gamma_1} - P_{(f,p),\gamma_1} = K_{(f,p)} A_{(f,p),\gamma_2} - P_{(f,p),\gamma_2}. \tag{10}$$

Assume for contradiction that $A_{(f,p),\gamma_1} < A_{(f,p),\gamma_2}$, which implies that $P_{(f,p),\gamma_1} < P_{(f,p),\gamma_2}$. Then Equation 10 combined with the fact that $K_{(f,p)} < \gamma_1$ implies

$$\gamma_1 A_{(f,p),\gamma_1} - P_{(f,p),\gamma_1} < \gamma_1 A_{(f,p),\gamma_2} - P_{(f,p),\gamma_2}.$$

Let $\Delta > 0$ be defined by the equation

$$\Delta = \big[\gamma_1\big(A_{(f,p),\gamma_2} - A_{(f,p),\gamma_1}\big)\big] - \big[P_{(f,p),\gamma_2} - P_{(f,p),\gamma_1}\big]. \tag{11}$$

Fix some $\delta > 0$ be such that the following inequality holds

$$p(K_{(f,p)} + \delta, \gamma_2) - P_{(f,p),\gamma_2} < \Delta.$$

Existence of such a δ is guaranteed by the definition of $P_{(f,p),\gamma_2}$. Lemma 12 implies that

$$0 \leq \gamma_1\big(f(K_{(f,p)} + \delta, \gamma_2) - A_{(f,p),\gamma_2}\big).$$

Adding above two inequalities we arrive at

$$\gamma_1 A_{(f,p),\gamma_2} - P_{(f,p),\gamma_2} < \Delta + \gamma_1 f(K_{(f,p)} + \delta, \gamma_2) - p(K_{(f,p)} + \delta, \gamma_2).$$

Substituting Δ from Equation 11 we get

$$\gamma_1 A_{(f,p),\gamma_1} - P_{(f,p),\gamma_1} < \gamma_1 f(K_{(f,p)} + \delta, \gamma_2) - p(K_{(f,p)} + \delta, \gamma_2).$$

Combining this with Equation 8 we get

$$\gamma_1 f(\beta, \gamma_1) - p(\beta, \gamma_1) < \gamma_1 f(K_{(f,p)} + \delta, \gamma_2) - p(K_{(f,p)} + \delta, \gamma_2).$$

By Lemma 9, we know that $p(\beta, \gamma_1) > B$ and $p(K_{(f,p)} + \delta, \gamma_2) > B$. Then, the above inequality implies that the incentive constraint $(\beta, \gamma_1) \to (K_{(f,p)} + \delta, \gamma_2)$ does not hold, which is a contradiction. ■

In light of Lemma 13, for every incentive compatible and individually rational mechanism (f, p) in M^+ we denote $A_{(f,p),\gamma}$ and $P_{(f,p),\gamma}$ defined in the Lemma 11 by $A_{(f,p)}$ and $P_{(f,p)}$, i.e., we drop the subscript γ.

A.2.3 A structure lemma for M^+ mechanisms

The following lemma identifies an important structure of incentive compatible and individually rational mechanisms in M^+.

LEMMA **14** *Suppose $(f, p) \in M^+$ is an incentive compatible and individually rational mechanism. Then the following are true.*

1. *$p(u) = P_{(f,p)}$ and $f(u) = A_{(f,p)}$, for all u with $u_2 \in (K_{(f,p)}, \beta)$ and $u_1 > K_{(f,p)}$.*
2. *$P_{(f,p)} > B$.*
3. *$A_{(f,p)} > f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\Big[B - p(K_{(f,p)}, 0)\Big]$.*

Proof: PROOF OF (1). Consider a type $(K_{(f,p)} + \delta, \beta)$ for some $\delta > 0$ but close to zero. By Lemma 9, we know that $p(K_{(f,p)} + \delta, \beta) > B$. Now, choose any u with $u_2 \in (K_{(f,p)}, \beta)$ and $u_1 > K_{(f,p)}$. By Lemma 9, we have $p(u) > B$. By Lemma 5, we get $f(K_{(f,p)} + \delta, \beta) \geq f(u)$. Now, the incentive constraint $u \to (K_{(f,p)} + \delta, \beta)$ implies

$$\begin{aligned} u_2 f(u) - p(u) &\geq u_2 f(K_{(f,p)} + \delta, \beta) - p(K_{(f,p)} + \delta, \beta) \\ \Rightarrow p(K_{(f,p)} + \delta, \beta) - p(u) &\geq u_2\Big[f(K_{(f,p)} + \delta, \beta) - f(u)\Big] \geq 0. \end{aligned}$$

Since this holds for all $\delta > 0$ but arbitrarily close to zero,

$$P_{(f,p)} = \lim_{\delta \to 0^+} p(K_{(f,p)} + \delta, \beta) \geq p(u).$$

Now, applying Lemmas 12 and 13, we have

$$P_{(f,p)} \leq p(u).$$

The above two inequalities give us $p(u) = P_{(f,p)}$. Then, using Equations (8) and (9) give us $f(u) = A_{(f,p)}$.

PROOF OF (2). By Lemma 9, for all u with $u_2 \in (K_{(f,p)}, \beta)$ and $u_1 > K_{(f,p)}$, we have $p(u) > B$. By (1), the result then follows.

Proof of (3). Assume for contradiction that

$$A_{(f,p)} \leq f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\Big[B - p(K_{(f,p)}, 0)\Big].$$
$$\Leftrightarrow K_{(f,p)}A_{(f,p)} - B \leq K_{(f,p)}f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0).$$

Using the expression of $A_{(f,p)}$ and $P_{(f,p)}$ in Lemma 11, we get that

$$K_{(f,p)}f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) = K_{(f,p)}A_{(f,p)} - P_{(f,p)}.$$

Substituting this above, we get $P_{(f,p)} \leq B$. This contradicts (2) above. ■

Lemma 14 shows that how certain regions in the type space look like for any incentive compatible and individually rational mechanism (f, p). This is shown in Figure 5.

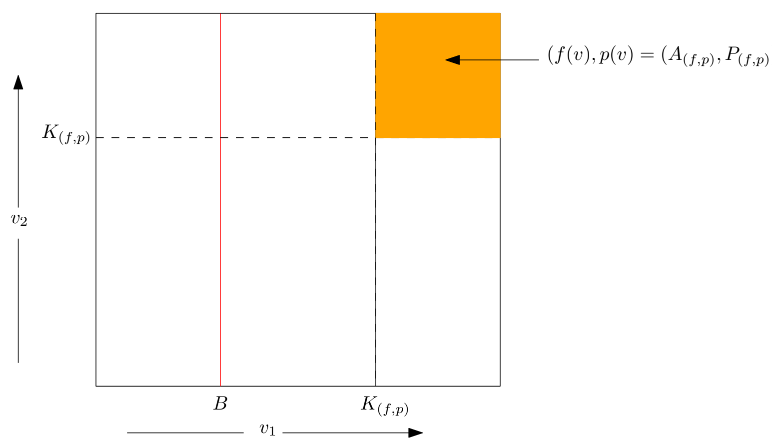

Figure 5: Implication of Lemma 14

Notice that Lemma 14 is silent about the outcome of the mechanism for types v with $v_1 > K_{(f,p)}$ and $v_2 = \beta$.

A.2.4 Reduction of space of M^+ mechanisms: implications of optimality

The next lemma shows that it is without loss of generality to make the outcomes for those types also $(A_{(f,p)}, P_{(f,p)})$.

Lemma **15** *Suppose $(f, p) \in M^+$ is an incentive compatible and individually rational mechanism. Then, there is another incentive compatible and individually rational mechanism (f', p') such that*

$$(f'(v), p'(v)) = \begin{cases} (A_{(f,p)}, P_{(f,p)}) & \textit{if } v_1 > K_{(f,p)} \textit{ and } v_2 = \beta \\ (f(v), p(v)) & \textit{otherwise.} \end{cases}$$

and

$$p'(v) \geq p(v) \textit{ for almost all } v.$$

Proof: By Lemma 14, the only difference between the mechanisms (f', p') and (f, p) is at v with $v_1 > K_{(f,p)}$ and $v_2 = \beta$ with $\beta > K_{(f,p)}$ (see (2) in Lemma 9). Also, such a modification changes the outcome at these types to $(A_{(f,p)}, P_{(f,p)})$ which is already in the menu of outcomes in the original mechanism (f, p). Hence, the only possibility of a manipulation in (f', p') is for type (v_1, β) with $v_1 > K_{(f,p)}$ to report another type v' to get $(f(v'), p(v')) \neq (A_{(f,p)}, P_{(f,p)})$. This manipulation is possible if $p(v') \leq B$ and

$$v_1 f(v') - p(v') > v_1 A_{(f,p)} - P_{(f,p)}$$

or $p(v') > B$ and

$$\beta f(v') - p(v') > \beta A_{(f,p)} - P_{(f,p)}.$$

Now, consider a type u such that $u_1 = v_1$ and $u_2 = \beta - \epsilon$ for small enough $\epsilon > 0$. Note that $(f(u), p(u)) = (f'(u), p'(u)) = (A_{(f,p)}, P_{(f,p)})$ by Lemma 14. Since $\epsilon > 0$ is small enough, this implies that one of the above constraints must hold for type u too, which further implies that type u can manipulate the mechanism (f, p). This is a contradiction.

Since $p'(0, 0) = p(0, 0) = 0$, individual rationality follows from Lemma 2. Since (f', p') is a modification of (f, p) at measure zero profiles, $p'(v) \geq p(v)$ for almost all v. ■

Lemma 15 has a straightforward implication - we can assume without loss of generality that the top (and right) boundary of the upper rectangle in Figure 5 is assigned outcome $(A_{(f,p)}, P_{(f,p)})$. This simplifies our analysis. Using Lemmas 14 and 15, we assume that every incentive compatible and individually rational mechanism $(f, p) \in M^+$ has the feature that for all v with $\min(v_1, v_2) > K_{(f,p)}$, we have $((f(v), p(v)) = (A_{(f,p)}, K_{(f,p)})$.

Next, we will look at a subclass of mechanisms which fixes some other regions of the type space. Further, we will show that such a restriction is also without loss of generality for optimal mechanisms. To show this property, we consider an arbitrary incentive compatible and individually rational mechanism $(f, p) \in M^+$. We then construct a new incentive compatible and individually rational mechanism which generates more expected revenue and has the property we require. The new mechanism, which we denote as (f', p') is defined as follows.

$$(f'(v), p'(v)) = \begin{cases} (f(v), p(v)) & \text{if } v_1 < K_{(f,p)} \text{ or } \min(v_1, v_2) > K_{(f,p)} \\ \Big(f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big), B\Big) & \text{otherwise} \end{cases}$$

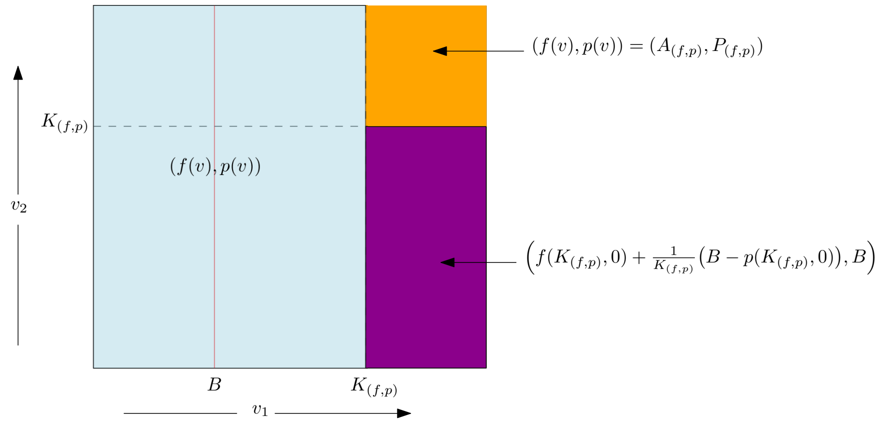

Figure 6: New mechanism

The new mechanism is shown in Figure 6. The rectangle at the top-right corner of the type space (excluding the lower boundaries) continues to have the outcome $(A_{(f,p)}, P_{(f,p)})$ - by Lemma 14, this is the same outcome as in the original mechanism (f, p). The outcomes in the big white rectangle to the left (but excluding the right boundary) is left unchanged. Note that $v_1 < K_{(f,p)}$ implies $p'(v) = p(v) \leq B$ by Lemma 9 in this region. The outcomes along the vertical line corresponding to $K_{(f,p)}$ value in $\textsc{dim}_1$ and the outcomes for all types such that $v_1 > K_{(f,p)}$ and $v_2 \leq K_{(f,p)}$ is assigned value

$$\Big(f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big), B\Big)$$

We prove the following.

Lemma **16** *If $(f, p) \in M^+$ is an incentive compatible and individually rational mechanism, then the mechanism (f', p') is incentive compatible, individually rational, and*

$$p'(v) \geq p(v) \textit{ for almost all } v.$$

Proof: As stated earlier, we assume $(f, p) \in M^+$ is an incentive compatible and individually rational mechanism such that $(f(v), p(v)) = (A_{(f,p)}, P_{(f,p)})$ for all v with $\min(v_1, v_2) > K_{(f,p)}$. Since $p(0, 0) = p'(0, 0)$ and (f, p) is individually rational, Lemma 2 implies that (f', p') is also individually rational if we can show that (f', p') is incentive compatible. First, we establish that $p'(v) \geq p(v)$ for **almost** all $v \in V$. To see this, first observe that $p(v)$ and $p'(v)$ may be unequal *only* when v belongs to the following set of types:

$$\tilde{V} := \{v : v_1 \geq K_{(f,p)} \text{ and } \min(v_1, v_2) \leq K_{(f,p)}\}.$$

Now, consider the set of types $\bar{V} := \{v : \big(v_1 > K_{(f,p)}, v_2 \leq K_{(f,p)}\big) \text{ or } v_1 = K_{(f,p)}\}$. For each $v \in \bar{V}$, we have $p'(v) = B$ and $p(v) \leq B$ (due to Lemma 9). The set of types $\tilde{V} \setminus \bar{V}$ forms a set of measure zero. So, for almost all v, we have $p'(v) \geq p(v)$.

For incentive compatibility, we consider a partition of the type space as follows:

$$\begin{aligned}
V^1 &:= \{v : \min(v_1, v_2) > K_{(f,p)}\} \\
V^2 &:= \{v : v_1 < K_{(f,p)}\} \\
V^3 &:= (V \times V) \setminus (V^1 \cup V^2).
\end{aligned}$$

For any $v, v' \in V^1 \cup V^2$, we have $(f'(v), p'(v)) = (f(v), p(v))$ and $(f'(v'), p'(v')) = (f(v'), p(v'))$. Since (f, p) is incentive compatible, the incentive constraints $v \to v'$ and $v' \to v$ hold. For any $v, v' \in V^3$, we have $(f'(v), p'(v)) = (f'(v'), p'(v'))$. Hence, the incentive constraints $v \to v'$ and $v' \to v$ hold.

Hence, we pick $u \in V^1, s \in V^2, t \in V^3$, and verify the incentive constraints

$$s \to t, t \to s, t \to u, u \to t.$$

1. $s \to t$. Note that $p(K_{(f,p)}, 0) \leq B$ and since $p(s) \leq B$, incentive constraint $s \to (K_{(f,p)}, 0)$ in (f, p) implies that

$$\begin{aligned}
s_1 f(s) - p(s) &\geq s_1 f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) \\
&\geq s_1 f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) - \Big[B - p(K_{(f,p)}, 0)\Big]\Big(1 - \frac{s_1}{K_{(f,p)}}\Big),
\end{aligned}$$

   where the inequality follows because $p(K_{(f,p)}, 0) \leq B$ and $s_1 < K_{(f,p)}$. Using $f(s) = f'(s)$, $p(s) = p'(s)$, and a slight rearrangement of RHS of the above inequality gives us

$$\begin{aligned}
s_1 f'(s) - p'(s) &\geq s_1 \Big[f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big)\Big] - B \\
&= s_1 f'(t) - p'(t).
\end{aligned}$$

   Hence, the incentive constraint $s \to t$ holds for (f', p').

2. $t \to s$. Since $p(s) \leq B$, incentive constraint $(K_{(f,p)}, 0) \to s$ in (f, p) implies that

$$\begin{aligned}
K_{(f,p)} f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) &\geq K_{(f,p)} f(s) - p(s) \\
\Rightarrow K_{(f,p)} \Big[f(K_{(f,p)}, 0) + \frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big)\Big] - B &\geq K_{(f,p)} f(s) - p(s) \\
\Rightarrow K_{(f,p)} f'(t) - p'(t) &\geq K_{(f,p)} f'(s) - p'(s).
\end{aligned}$$

This implies that

$$K_{(f,p)}\Big[f'(t) - f'(s)\Big] \geq p'(t) - p'(s).$$

But $p'(t) = B \geq p'(s) = p(s)$ implies that $f'(t) \geq f'(s)$. Using the fact that $t_1 \geq K_{(f,p)}$, we get

$$t_1\Big[f'(t) - f'(s)\Big] \geq p'(t) - p'(s),$$

Since $p'(t) = B$ and $p'(s) \leq B$, this is the desired incentive constraint $t \to s$ in (f', p').

3. $t \to u$, $u \to t$. By Lemma 11, we know that

$$\begin{aligned} &K_{(f,p)}f(K_{(f,p)}, 0) - p(K_{(f,p)}, 0) = K_{(f,p)}A_{(f,p)} - P_{(f,p)} \\ \Leftrightarrow K_{(f,p)}\Big[f(K_{(f,p)}, 0) - &\frac{1}{K_{(f,p)}}\big(B - p(K_{(f,p)}, 0)\big)\Big] - B = K_{(f,p)}A_{(f,p)} - P_{(f,p)}. \end{aligned}$$

Hence, we get

$$K_{(f,p)}\Big[f'(u) - f'(t)\Big] = p'(u) - p'(t). \tag{12}$$

Using Lemma 14, $p'(u) = p(u) = P_{(f,p)} > p'(t) = B$. Hence, Equation 12 implies that $f'(u) > f'(t)$. Using $\min(u_1, u_2) > K_{(f,p)}$, we get

$$\begin{aligned} u_1 f'(u) - p'(u) &\geq u_1 f'(t) - p'(t) \\ u_2 f'(u) - p'(u) &\geq u_2 f'(t) - p'(t). \end{aligned}$$

Hence, the incentive constraint $u \to t$ holds in (f', p').

Similarly, we now use the fact that $\min(t_1, t_2) \leq K_{(f,p)}$. If $\min(t_1, t_2) = t_1$, then using Equation 12, we get

$$t_1 f'(t) - p'(t) \geq t_1 f'(u) - p'(u).$$

Else, $\min(t_1, t_2) = t_2$, in which case again, we get

$$t_2 f'(t) - p'(t) \geq t_2 f'(u) - p'(u).$$

So, one of the above constraints must hold. Since $p'(t) = B$ and $p'(u) > B$, this ensures that the incentive constraint $t \to u$ holds in (f', p').

■

A.2.5 Ironing Lemmas

The final Lemma before we start ironing, further simplifies the class of mechanisms that we need to consider for optimal mechanism design.

LEMMA **17** *Suppose $(f,p) \in M^+$ is an incentive compatible and individually rational mechanism. Then, there exists another mechanism $(\hat{f},\hat{p})$ such that*

1. *$(\hat{f}(v),\hat{p}(v)) = (f(v),p(v))$ for all v with $v_1 \geq K_{(f,p)}$,*
2. *$(\hat{f}(v),\hat{p}(v)) = (\hat{f}(u),\hat{p}(u))$ for all u,v with $u_1 = v_1 < K_{(f,p)}$,*
3. *$\hat{p}(u) \geq p(u)$ for all u,*
4. *$\hat{p}(0,0) = p(0,0)$,*
5. *incentive constraints $u \to v$ for every u,v with $\hat{p}(u),\hat{p}(v) \leq B$ hold in $(\hat{f},\hat{p})$.*

Proof: Consider an incentive compatible and individually rational mechanism (f,p), and let $K_{(f,p)}$ be as defined in Lemma 9. We complete the proof in two steps.

STEP 1. In this step, we show some implications of incentive constraints $u \to v$, where $u_1, v_1 < K_{(f,p)}$. Consider any $(u_1,u_2),(u_1,u_2')$ such that $u_1 < K_{(f,p)}$. Then, by Lemma 9, we have $p(u_1,u_2) \leq B$ and $p(u_1,u_2') \leq B$. Hence, the relevant pair of incentive constraints give us:

$$u_1 f(u_1,u_2) - p(u_1,u_2) \geq u_1 f(u_1,u_2') - p(u_1,u_2')$$
$$u_1 f(u_1,u_2') - p(u_1,u_2') \geq u_1 f(u_1,u_2) - p(u_1,u_2).$$

This gives us

$$u_1 f(u_1,u_2) - p(u_1,u_2) = u_1 f(u_1,u_2') - p(u_1,u_2'). \tag{13}$$

Also, notice that Equation 13 implies that for all $u_2 \in [0,\beta]$,

$$p(0,u_2) = p(0,0) \tag{14}$$

Finally, since only DIM_1 incentive constraints are relevant in this region, revenue equivalence formula implies that for every $u_1 < K_{(f,p)}$ and $u_2, u_2' \in [0,\beta]$, we have

$$u_1 f(u_1,u_2) - p(u_1,u_2) = \int_0^{u_1} f(x,u_2)dx - p(0,u_1) = \int_0^{u_1} f(x,u_2)dx - p(0,0)$$

$$u_1 f(u_1, u_2') - p(u_1, u_2') = \int_0^{u_1} f(x, u_2')dx - p(0, u_1) = \int_0^{u_1} f(x, u_2')dx - p(0,0)$$

Using Equation 13, we get

$$\int_0^{u_1} f(x, u_2)dx = \int_0^{u_1} f(x, u_2')dx.$$

Hence, we can write for every $u_1 < K_{(f,p)}$ and every $u_2 \in [0, \beta]$,

$$u_1 f(u_1, u_2) - p(u_1, u_2) = \int_0^{u_1} f(x, 0)dx - p(0,0). \tag{15}$$

Notice that the RHS of the above equation is independent of u_2. Denoting the RHS of the above equation as $\mathcal{U}^{(f,p)}(u_1)$, we see that

$$u_1 \sup_{u_2 \in [0,\beta]} f(u_1, u_2) = \sup_{u_2 \in [0,\beta]} p(u_1, u_2) + \mathcal{U}^{(f,p)}(u_1). \tag{16}$$

Notice that f and p are bounded from above (p is bounded from above because $p(u_1, u_2) \leq B$ for each $u_2 \in [0, \beta]$ due to Lemma 9). As a result, the supremums in the above equation exist. We denote this supremums as follows:

$$\alpha(u_1) := \sup_{u_2 \in [0,\beta]} f(u_1, u_2) \; \forall \; u_1 < K_{(f,p)} \tag{17}$$

$$\pi(u_1) := \sup_{u_2 \in [0,\beta]} p(u_1, u_2) \; \forall \; u_1 < K_{(f,p)}. \tag{18}$$

We use these to define our new mechanism in the next step.

Step 2. Now, we define the following mechanism $(\hat{f}, \hat{p})$. For every v with $v_1 \geq K_{(f,p)}$, we have $(\hat{f}(v), \hat{p}(v)) = (f(v), p(v))$. For all v with $v_1 < K_{(f,p)}$, we define

$$\hat{f}(v) := \alpha(v_1); \hat{p}(v) := \pi(v_1).$$

By definition of $\hat{p}$, it is clear that $\hat{p}(v) \geq p(v)$ for all v. Also, Equation 14 ensures that $\hat{p}(0,0) = \pi(0) = p(0,0)$. Hence, (1), (2), (3), (4) hold for $(\hat{f}, \hat{p})$.

For (5), assume for contradiction that the incentive constraint $u \to v$ in $(\hat{f}, \hat{p})$ does not hold for some u, v with $\hat{p}(u), \hat{p}(v) \leq B$. So, the violation of incentive constraint must happen along $\textsc{dim}_1$. Note that by definition of $\hat{p}$, we must have $p(u) \leq B$ and $p(v) \leq B$. Also, incentive constraints cannot be violated if $u_1, v_1 \geq K_{(f,p)}$ since (f, p) is incentive compatible and $(\hat{f}(u), \hat{p}(u)) = (f(u), p(u))$ and $(\hat{f}(v), \hat{p}(v)) = (f(v), p(v))$. The other possibilities are analyzed below.

CASE 1. $u_1, v_1 < K_{(f,p)}$. In that case, we must have

$$u_1\alpha(u_1) - \pi(u_1) < u_1\alpha(v_1) - \pi(v_1) = (u_1 - v_1)\alpha(v_1) + v_1\alpha(v_1) - \pi(v_1).$$

Using Equation (16), we get that

$$\mathcal{U}^f(u_1) < \mathcal{U}^f(v_1) + (u_1 - v_1)\alpha(v_1).$$

By definition, there exists, $y \in [0, \beta]$ such that $\alpha(v_1)$ is arbitrarily close to $f(v_1, y)$. Using Equation (15) gives us

$$u_1 f(u_1, y) - p(u_1, y) < v_1 f(v_1, y) - p(v_1, y) + (u_1 - v_1)f(v_1, y) = u_1 f(v_1, y) - p(v_1, y).$$

This contradicts incentive compatibility of (f, p).

CASE 2. $u_1 < K_{(f,p)}$ and $v_1 \geq K_{(f,p)}$. In that case, we must have

$$u_1\alpha(u_1) - \pi(u_1) < u_1 f(v) - p(v).$$

But using Equations (15) and (16), we see that there is some y such that

$$u_1 f(u_1, y) - p(u_1, y) < u_1 f(v) - p(v)$$

which contradicts incentive compatibility of (f, p).

CASE 3. $u_1 \geq K_{(f,p)}$ and $v_1 < K_{(f,p)}$. In that case, we must have

$$u_1 f(u) - p(u) < u_1\alpha(v_1) - \pi(v_1) = (u_1 - v_1)\alpha(v_1) + \mathcal{U}^f(v_1).$$

Now, pick y such that $\alpha(v_1)$ is arbitrarily close to $f(v_1, y)$. By Equations (15) and (16), we get

$$u_1 f(u) - p(u) < (u_1 - v_1)f(v_1, y) + v_1 f(v_1, y) - p(v_1, y) = u_1 f(v_1, y) - p(v_1, y).$$

This contradicts incentive compatibility of (f, p) and completes the proof. ■

DEFINITION 8 *We call a mechanism* (f, p) **simple** *if there exists* $K, A, \hat{A}, P$ *with* $K \in (0, B)$, $P \in (B, \beta]$, $A, \hat{A} \in [0, 1]$, $A > \hat{A}$ *such that*

1. $p(0, 0) \leq 0$.

2. $K(A - \hat{A}) = P - B$ *with* $KA - P \geq 0$.

3. $(f(v), p(v)) = (A, P)$ *for all* v *with* $\min(v_1, v_2) > K$,

4. $p(v) \leq B$ *for all* v *with* $v_1 < K$.

5. $(f(v), p(v)) = (\hat{A}, B)$ *for all* v *with* $\min(v_1, v_2) \leq K$ *and* $v_1 \geq K$.

6. $(f(v), p(v)) = (f(v'), p(v'))$ *for all* v, v' *with* $v_1 = v_1' < K$.

7. *incentive constraints* $v \to v'$ *hold for all types with* $p(v), p(v') \leq B$.

Based on Lemmas 16 and 17, the following is a simple corollary.

Corollary **1** *If* (f, p) *is an optimal mechanism in* M^+, *then there is a simple mechanism* $(\hat{f}, \hat{p})$ *such that*

$$\text{Rev}(f, p) \leq \text{Rev}(\hat{f}, \hat{p}).$$

Proof: Suppose (f, p) is an optimal mechanism in M^+, then Lemma 16 says that there is another incentive compatible and individually rational mechanism (f', p') such that $\text{Rev}(f', p') \geq \text{Rev}(f, p)$. Using $K = K_{(f,p)}$, Lemma 17 shows that (f', p') satisfies all the properties of a simple mechanism. ■

Because of property (6), for any simple mechanism (f, p), we denote the allocation probability at any type v with $v_1 < K$ as simply $\alpha^f(v_1)$ and the payment as $\pi^p(v_1)$. We also denote by $\alpha^f(K) \equiv \hat{A}$ and $\pi^p(K) \equiv B$, where $\hat{A}$ is the parameter specified in the simple mechanism (f, p).

Lemma **18** *Suppose* (f, p) *is a simple mechanism with parameters* $(K, A, \hat{A}, P)$. *Then, the revenue from* (f, p) *is*

$$\begin{aligned}\text{Rev}(f, p) = {} & G_1(K)\Big[B - K\alpha^f(K)\Big] + \int_0^K h(x)\alpha^f(x)dx \\ & + B(1 - G_1(K)) + K(A - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K, K)),\end{aligned}$$

where $h(x) = xg_1(x) + G_1(x)$ *for all* $x \in [0, K]$.

Proof: Fix a simple mechanism with parameters $(K, A, \hat{A}, P)$. We divide the proof into two parts, where we compute revenue from two disjoint regions of the type space.

Region 1. Here, we consider all v such that $v_1 \le K$. By properties (4) and (5) of the simple mechanism, payments in this region of type space is not more than B and by property (7), all the incentive constraints in this region hold. Using standard Myersonian techniques, it is easy to see that

$$\alpha^f(v_1) \ge \alpha^f(v_1') \qquad \forall\ v_1' < v_1 \le K \tag{19}$$

$$\pi^p(v_1) = \pi^p(0) + v_1\alpha^f(v_1) - \int_0^{v_1} \alpha^f(x)dx \qquad \forall\ v_1 \le K \tag{20}$$

Hence, the expected payment from this region is

$$\begin{aligned}
\int_0^K \pi^p(v_1)g_1(v_1)dv_1 &= \int_0^K \pi^p(0)g_1(v_1)dv_1 + \int_0^K v_1\alpha^f(v_1)g_1(v_1)dv_1 - \int_0^K \big(\int_0^{v_1} \alpha^f(x)dx\big)g_1(v_1)dv_1 \\
&= G_1(K)\pi^p(0) + \int_0^K v_1\alpha^f(v_1)g_1(v_1)dv_1 - \int_0^K \big((G_1(K) - G_1(v_1)\big)\alpha^f(v_1)dv_1 \\
&= G_1(K)\big[\pi^p(0) - \int_0^K \alpha^f(x)dx\big] + \int_0^K h(x)\alpha^f(x)dx \\
&= G_1(K)\big[\pi^p(K) - K\alpha^f(K)\big] + \int_0^K h(x)\alpha^f(x)dx \\
&= G_1(K)\big[B - K\alpha^f(K)\big] + \int_0^K h(x)\alpha^f(x)dx,
\end{aligned}$$

where the last but one equality follows from Equation 20 at $v_1 = K$ and the last equality follows from the fact $\pi^p(K) = B$.

Region 2. Finally, we consider all v such that $v_1 > K$. By definition, the expected revenue from this region is

$$\begin{aligned}
&B(1 - G_1(K)) + (P - B)(1 - G_1(K) - G_2(K) + G(K,K)) = \\
&\qquad B(1 - G_1(K)) + K(A - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K,K)),
\end{aligned}$$

where the equality follows from property (2) of simple mechanism.

Putting together the revenues from both the regions, we get the desired expression of the expected revenue from the simple mechanism. ■

We now prove that for every simple mechanism, there is a post-2 mechanism that generates as much expected revenue.

Lemma **19** *For every simple mechanism (f,p), there is a* post-2 *mechanism $(\bar{f},\bar{p})$ such that*

$$\textsc{Rev}(\bar{f},\bar{p}) \ge \textsc{Rev}(f,p).$$

Proof: Suppose (f, p) is a simple mechanism with parameters $(K, A, \hat{A}, P)$. Now, by property (5) of the simple mechanism, Equation 20 along with property (1) imply that

$$\pi^f(K) = B \leq K\alpha^f(K) - \int_0^K \alpha^f(x)dx. \tag{21}$$

Now, define a POST-2 mechanism by parameters:

$$K_1 := \frac{B}{\hat{A}} = \frac{B}{\alpha^f(K)}, \quad K_2 := K.$$

By property (1) of simple mechanism, we get that $K_1 = \frac{B}{\alpha^f(K)} \leq K_2 = K$. Also, $K_1 > B$. This means that the new mechanism is a well-defined POST-2 mechanism. Denote this mechanism as (f', p').

It is also easily verified that it is a simple mechanism: the parameters are

$$K' := K_2 = K; A' = 1; \hat{A}' := \hat{A} = \alpha^f(K); P' := B + K_2(1 - \frac{B}{K_1}) = B + K(1 - \alpha^f(K)),$$

and also note that every POST-2 mechanism is incentive compatible (Proposition 1). Note here that $\alpha^{f'}(K) = \alpha^f(K)$. Also, $\alpha^{f'}(x) = 0$ for all $x \leq K_1$ and $\alpha^{f'}(x) = \frac{B}{K_1} = \alpha^f(K)$ for all $x \in (K_1, K]$. Using these observations and Lemma 18,

$$
\begin{aligned}
&\textsc{Rev}(f', p') - \textsc{Rev}(f, p) \\
&= \Bigg(G_1(K)\Big[B - K\alpha^f(K)\Big] + \int_0^K h(x)\alpha^{f'}(x)dx + B(1 - G_1(K)) + \\
&K(1 - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K, K)) \Bigg) \\
&- \Bigg(G_1(K)\Big[B - K\alpha^f(K)\Big] + \int_0^K h(x)\alpha^f(x)dx + B(1 - G_1(K)) + \\
&K(A - \alpha^f(K))(1 - G_1(K) - G_2(K) + G(K, K)) \Bigg) \\
&\geq \int_0^K h(x)\alpha^{f'}(x)dx - \int_0^K h(x)\alpha^f(x)dx \\
&\geq \int_{K_1}^K h(x)\big(\alpha^f(K) - \alpha^f(x)\big)dx - \int_0^{K_1} h(x)\alpha^f(x)dx. \\
&\geq (K - K_1)h(K_1)\alpha^f(K) - h(K_1)\int_{K_1}^K \alpha^f(x)dx - h(K_1)\int_0^{K_1} \alpha^f(x)dx \\
&\text{(using } h \text{ and } \alpha \text{ to be increasing functions)} \\
&= (K - K_1)h(K_1)\alpha^f(K) - h(K_1)\int_0^K \alpha^f(x)dx \\
&\geq h(K_1)(K - K_1)\alpha^f(K) - h(K_1)(K - K_1)\alpha^f(K) \\
&\text{(using Equation (21) and definition of } K_1) \\
&= 0.
\end{aligned}
$$

■

A.2.6 Proof of Proposition 4

The proof of (2) in Proposition 4 now follows from Corollary 1 and Lemma 19. Proof of (1) in Proposition 4 is given below.

This requires to show that the optimal mechanism in M^- is a POST-1 mechanism. Every mechanism $(f, p) \in M^-$ satisfies the property that types satisfying $p(v) > B$ have zero measure. We first argue that it is without loss of generality to assume that $p(v) \leq B$ for *all* v. To see this, note that by (1) in Lemma 9 and the fact that $V^+(f, p)$ has zero measure, it must be that $K_{(f,p)} = \beta$. Let $\pi^p(\beta) := \sup_{v_2 < \beta} p(\beta, v_2)$ and $\alpha^f(\beta) := \sup_{v_2 < \beta} f(\beta, v_2)$. Observe that

$\alpha^p(\beta) \leq B$. Hence, we consider the following mechanism (f', p'): $(f'(v), p'(v)) = (f(v), p(v))$ if $v \notin V^+(f,p)$ and $(f'(v), p'(v)) = (\alpha^f(\beta), \pi^p(\beta))$ otherwise. By construction, the expected revenue of (f', p') is the same as (f, p) and $p'(v) \leq B$ for all v. Further, (f', p') is incentive compatible (we only need to worry about incentive constraints of types $v \in V^+(f,p)$, and they hold because for all v, $p'(v) \leq B$ implies we only need to check incentive constraints along $\textsc{dim}_1$, which holds due to an argument similar to that in Lemma 17(5)). Individual rationality of (f', p') follows from Lemma 2.

Now, we state an analogue of Lemma 17 for M^- class of mechanisms - the proof of this lemma is identical to that of Lemma 17, and is skipped.

Lemma **20** *Suppose $(f,p) \in M^-$ is an incentive compatible and individually rational mechanism. Then, there exists another mechanism $(\hat{f}, \hat{p})$ such that*

1. *$(\hat{f}(v), \hat{p}(v)) = (\hat{f}(u), \hat{p}(u))$ for all u, v with $u_1 = v_1$,*
2. *$\hat{p}(u) \geq p(u)$ for all u,*
3. *$\hat{p}(0,0) = p(0,0)$,*
4. *$(\hat{f}, \hat{p})$ is incentive compatible and individually rational.*

Using Lemma 20, we only focus on mechanisms satisfying the properties stated in Lemma 20. Let (f,p) be such a mechanism and define α^f and π^p as before, i.e., $\alpha^f(v_1) = f(v_1, v_2)$ and $\pi^p(v_1) = p(v_1, v_2)$ for all v with $v_1 < \beta$.

Hence, the expected revenue from a mechanism (f,p) given in Lemma 20 is given by

$$\begin{aligned}
\textsc{Rev}(f,p) &= p(0,0) + \int_0^\beta u_1 \alpha^f(u_1) g_1(u_1) du_1 - \int_0^\beta \Big(\int_0^{u_1} \alpha^f(x) dx \Big) g_1(u_1) du_1 \\
&= p(0,0) + \int_0^\beta x \alpha^f(x) g_1(x) dx - \int_0^\beta (1 - G_1(x)) \alpha^f(x) dx \\
&= p(0,0) + \int_0^\beta \Big[h(x) - 1 \Big] \alpha^f(x) dx.
\end{aligned}$$

We now construct another posted-price mechanism (f', p') that generates no less revenue than (f,p). The posted-price mechanism (f', p') is defined as follows. Let $K_1 := \frac{\pi^f(\beta)}{\alpha^f(\beta)}$. For all v with $v_1 \leq K_1$, we set

$$f'(v) = 0, p'(v) = 0$$

and for all v with $v_1 > K_1$, we set

$$f'(v) = \alpha^f(\beta), \quad p'(v) = K_1 \alpha^f(\beta) = \pi^p(\beta).$$

It is not difficult to see that (f', p') is individually rational and incentive compatible. The expected revenue from (f', p') is given by

$$\textsc{Rev}(f', p') = K_1 \alpha^f(\beta)(1 - G_1(K_1))$$

Now, note that

$$\alpha^f(\beta) \int_{K_1}^{\beta} \Big[h(x) - 1\Big] dx = \alpha^f(\beta)\Big(K_1 - K_1 G_1(K_1)\Big) = \textsc{Rev}(f', p').$$

So, we get

$$\begin{aligned}
\textsc{Rev}(f', p') - \textsc{Rev}(f, p) &= \left(\alpha^f(\beta) \int_{K_1}^{\beta} \Big[h(x) - 1\Big] dx\right) - \left(p(0,0) + \int_0^{\beta} \Big[h(x) - 1\Big] \alpha^f(x) dx\right) \\
&= \alpha^f(\beta) \int_{K_1}^{\beta} h(x) dx - \int_0^{\beta} h(x) \alpha^f(x) dx + \int_0^{\beta} \alpha^f(x) dx - (\beta - K_1)\alpha^f(\beta) - p(0,0) \\
&= \alpha^f(\beta) \int_{K_1}^{\beta} h(x) dx - \int_0^{\beta} h(x) \alpha^f(x) dx + \int_0^{\beta} \alpha^f(x) dx - \beta\alpha^f(\beta) - \pi^p(\beta) - p(0,0) \\
&\text{(Using definition of } K_1\text{)} \\
&= \alpha^f(\beta) \int_{K_1}^{\beta} h(x) dx - \int_0^{\beta} h(x) \alpha^f(x) dx \\
&\text{(Using revenue equivalence formula (Equation 20) at } \beta\text{)} \\
&= \int_{K_1}^{\beta} \Big[\alpha^f(\beta) - \alpha^f(x)\Big] h(x) dx - \int_0^{K_1} \alpha^f(x) h(x) dx \\
&\geq h(K_1) \int_{K_1}^{\beta} \Big[\alpha^f(\beta) - \alpha^f(x)\Big] dx - h(K_1) \int_0^{K_1} \alpha^f(x) dx \\
&\text{(since } h \text{ is increasing and } \alpha \text{ is non-decreasing)} \\
&= h(K_1)(\beta - K_1)\alpha^f(\beta) - h(K_1) \int_0^{\beta} \alpha^f(x) dx \\
&\geq h(K_1)(\beta - K_1)\alpha^f(\beta) - h(K_1)(\beta - K_1)\alpha^f(\beta) \\
&\text{(Using revenue equivalence formula (Equation 20) at } \beta \text{ and } p(0,0) \leq 0\text{)} \\
&= 0.
\end{aligned}$$

Hence, every optimal mechanism in M^- is a posted-price mechanism described in (f', p'). It is characterized by a posted-price K_1 and an allocation probability α if the value along

DIM_1 is above the posted price. The optimization program can be written as follows.

$$\max_{K_1,\alpha} K_1\alpha(1 - G_1(K_1))$$
$$\text{subject to}$$
$$K_1\alpha \leq B$$
$$\alpha \in [0, 1].$$

We argue that the optimal solution to this program must have $\alpha = 1$. To see this, let K^* be the unique solution to the following optimization

$$\max_{K_1 \in [0,B]} K_1(1 - G_1(K_1)).$$

The fact that this optimization program has a unique solution follows from the fact that $x - xG_1(x)$ is strictly concave (since $xG_1(x)$ is strictly convex). Hence, the revenue from the solution when $\alpha = 1$ is $K^*(1 - G_1(K^*))$. Now, suppose the optimal solution has $\hat{K}$ and $\hat{\alpha}$. Note that the $\hat{K}\hat{\alpha} \leq B$. So, define $\tilde{K} = \hat{K}\hat{\alpha} \leq B$. By definition,

$$\begin{aligned} K^*(1 - G_1(K^*)) &\geq \tilde{K}(1 - G_1(\tilde{K})) \\ &= \hat{K}\hat{\alpha}(1 - G_1(\hat{K}\hat{\alpha})) \\ &\geq \hat{K}\hat{\alpha}(1 - G_1(\hat{K})), \end{aligned}$$

where the final inequality used the fact that $G_1(\hat{K}\hat{\alpha}) \leq G_1(\hat{K})$. This implies that the optimal solution must have $\alpha = 1$ and K_1 must be the unique solution to $K_1(1 - G_1(K_1))$ with the constraint $K_1 \in [0, B]$. Hence, the optimal solution in M^- must be a posted price mechanism, where the posted price is a unique solution to

$$\max_{K_1 \in [0,B]} K_1(1 - G_1(K_1)).$$

A.2.7 Proof of Proposition 2

We now combine the optimal solutions in M^+ and M^- as follows. The optimal in M^- is a solution to

$$\max_{K_1 \in [0,B]} K_1(1 - G_1(K_1)).$$

The optimal in M^+ is a solution to

$$\max_{K_2 \in (B,\beta), K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

Notice that the optimization for M^+ does not admit $K_2 = B$. But if $K_2 = B$ and $K_1 \in [B, K]$, we must have $K_1 = B$ and then the objective function value reduces to $B(1 - G_1(B))$. This is the same objective function value of the program for M^- when $K_1 = B$. Similarly, if $K_2 = \beta$ is allowed in the optimization for M^+, we see that the objective function is maximized at $K_1 = B$ giving a value of $B(1 - G_1(B))$ to the objective function. Again, this is the same objective function value of the program for M^- when $K_1 = B$.

Summarizing these findings, we get that the expected revenue from the optimal mechanism is $\max(R_1, R_2)$, where

$$R_1 = \max_{K_1 \in [0,B]} K_1(1 - G_1(K_1))$$

$$R_2 = \max_{K_2 \in [B,\beta], K_1 \in [B,K_2]} B\Big[1 - G_1(K_1)\Big] + K_2\big(1 - \frac{B}{K_1}\big)\Big[1 - G_1(K_2) - G_2(K_2) + G(K_2, K_2)\Big].$$

This proves Proposition 2.

B Appendix: Proofs of Section 5

This appendix contains all omitted proofs of Section 5.

B.1 Proof of Proposition 7

We establish a stronger result. We show that a larger class mechanisms, which includes the POST* mechanism, is incentive compatible.

DEFINITION **9** *A mechanism (f, p) is a generalized* POST* *(*G-POST**) mechanism if there exists* $K, P \in (0, \beta]$ *and* $A \in [0, 1]$ *such that*

$$0 \leq A - \frac{P}{K} \leq 1 - \frac{B}{K}$$

and for all $(v, B) \in W$

$$(f(v,B), p(v,B)) = \begin{cases} (A - \frac{P}{K}, 0) & \text{if } v_1 \leq K \\ \Big(A, P\Big) & \text{if } \{\min(v_1, v_2) > K \text{ and } B < P\} \\ & \text{or } \{v_1 > K \text{ and } B \geq P\} \\ (A - \frac{P-B}{K}, B) & \text{if } v_1 > K,\ v_2 \leq K \text{ and } B < P \end{cases}$$

Note that if we put $A = 1, P = K$, we get a POST* mechanism. We prove the following proposition, which implies Proposition 7.

PROPOSITION **8** *Every* G-POST* *mechanism is* DIM_2 *non-trivial, incentive compatible, and individually rational.*

Proof: It is clear that a G-POST* mechanism is DIM_2 non-trivial. Individual rationality will follow from Lemma 2 once we show incentive compatibility. So, we show incentive compatibility below.

Fix a G-POST* mechanism (f, p) defined by parameters K, P, A. Partition the type space W into three regions:

$$W^1 := \{(u, B) : u_1 \leq K\},$$
$$W^2 := \{(u, B) : \min(u_1, u_2) > K, B < P\} \cup \{(u, B) : u_1 > K, B \geq P\},$$
$$W^3 := \{(u, B) : u_1 > K, u_2 \leq K, B < P\}.$$

By definition, we have $(f(u, B), p(u, B)) = (f(u', B'), p(u', B'))$ if $(u, B), (u', B') \in W^1$ or $(u, B), (u', B') \in W^2$. Now, pick $(u, B), (u', B') \in W^3$ with $B < B'$. Notice that

$$K\Big[f(u, B) - f(u', B')\Big] = p(u, B) - p(u', B') = B - B' < 0.$$

This gives us $f(u,B) < f(u',B')$. Since, $u_1' > K$, we get

$$u_1'\Big[f(u,B) - f(u',B')\Big] < p(u,B) - p(u',B'),$$

which implies that incentive constraint $(u',B') \to (u,B)$ holds for (f,p). Similarly, using $u_2 \le K$, we notice that

$$u_2\Big[f(u,B) - f(u',B')\Big] \ge p(u,B) - p(u',B').$$

Using $p(u',B') = B' > B$, the above inequality implies that incentive constraint $(u,B) \to (u',B')$ also holds for (f,p).

We now show incentive constraints hold across each pair of types in W^1, W^2, W^3. For this, pick $(u,B) \in W^1, (u',B') \in W^2, (u'',B'') \in W^3$. By definition, we have

$$Kf(u,B) - p(u,B) = Kf(u',B') - p(u',B') = Kf(u'',B'') - p(u'',B'') = KA - P. \quad (22)$$

Now, we consider three cases.

Case 1. $(u,B) \to (u',B')$ and $(u',B') \to (u,B)$. Using Equation (22), we get

$$K\Big[f(u,B) - f(u',B')\Big] = p(u,B) - p(u',B') = -P < 0.$$

Using $u_1 < K$, we get

$$u_1 f(u,B) - p(u,B) \ge u_1 f(u',B') - p(u',B').$$

This is enough for incentive constraint $(u,B) \to (u',B')$ since $p(u,B) = 0$.

Similarly, using $u_1' > K$ implies

$$u_1' f(u',B') - p(u',B') \ge u_1' f(u,B) - p(u,B). \quad (23)$$

This is enough for incentive constraint $(u',B') \to (u,B)$ if $p(u',B') = P \le B'$. Else, $p(u',B') = P > B'$, which also means $\min(u_1', u_2') > K$. But this means, we also have

$$u_2' f(u',B') - p(u',B') \ge u_2' f(u,B) - p(u,B). \quad (24)$$

Inequalities (23) and (24) ensure that incentive constraint $(u',B') \to (u,B)$ holds.

Case 2. $(u',B') \to (u'',B'')$ and $(u'',B'') \to (u',B')$. Using Equation (22) and $B'' < P$, we get

$$K\Big[f(u',B') - f(u'',B'')\Big] = p(u',B') - p(u'',B'') = P - B'' > 0.$$

Since $u_2'' \le K$, we get

$$u_2'' f(u'', B'') - p(u'', B'') \ge u_2'' f(u', B') - p(u', B').$$

This is enough for incentive constraint $(u'', B'') \to (u', B')$ to hold since $p'(u', B') = P > B''$.

Similarly, using $u_1' > K$ implies

$$u_1' f(u', B') - p(u', B') > u_1' f(u'', B'') - p(u'', B''). \tag{25}$$

This is enough for incentive constraint $(u', B') \to (u'', B'')$ if $p(u', B') = P \le B'$. Else, $p(u', B') = K > B'$, which also means $\min(u_1', u_2') > K$. But this means, we also have

$$u_2' f(u', B') - p(u', B') > u_2' f(u'', B'') - p(u'', B''). \tag{26}$$

Inequalities (25) and (26) ensure that incentive constraint $(u', B') \to (u'', B'')$ holds.

Case 3. $(u, B) \to (u'', B'')$ and $(u'', B'') \to (u, B)$. Using Equation (22), we get

$$K\Big[f(u, B) - f(u'', B'')\Big] = p(u, B) - p(u'', B'') = 0 - B'' \le 0.$$

Using $u_1 \le K$, we get

$$u_1 f(u, B) - p(u, B) \ge u_1 f(u'', B'') - p(u'', B'').$$

This is enough for incentive constraint $(u, B) \to (u'', B'')$ since $p(u, B) = 0$. Also, since $u_1'' > K$, we get

$$u_1'' f(u'', B'') - p(u'', B'') \ge u_1'' f(u, B) - p(u, B).$$

This is enough for incentive constraint $(u'', B'') \to (u, B)$ since $p(u'', B'') = B''$. ∎

B.2 Proof of Theorem 2

We give the proof of Theorem 2. We start by giving some preparatory lemmas.

B.2.1 Preparatory Lemmas

Fix a DIM_2 non-trivial mechanism (f, p). Let

$$B^+_{(f,p)} := \{B : \{v \in V : p(v, B) > B\} \text{ has non-zero measure}\}.$$

By DIM_2 non-triviality $B^+_{(f,p)}$ is non-empty. This means for any $B \in B^+_{(f,p)}$, we observe that $V^+(f, p)$ defined in the public budget case has non-zero measure and hence (f, p) restricted to B belongs to M^+. We can then directly state equivalent of lemmas from the public budget case for any $B \in B^+_{(f,p)}$.

Lemma **21** *Suppose (f,p) is an incentive compatible and individually rational mechanism satisfying* DIM_2 *non-triviality. Then, for any $B \in B^+_{(f,p)}$, there exists $P_{(f,p),B}$, $A_{(f,p),B}$ and $K_{(f,p),B}$ such that the following are true.*

1. *$p(u,B) = P_{(f,p),B}$ and $f(u,B) = A_{(f,p),B}$, for all u with $u_2 \in (K_{(f,p),B}, \beta)$ and $u_1 > K_{(f,p),B}$.*
2. *$A_{(f,p),B} > f(K_{(f,p),B}, 0, B) + \frac{1}{K_{(f,p),B}}\Big[B - p(K_{(f,p),B}, 0, B)\Big]$.*
3. *$\beta A_{(f,p),B} - P_{(f,p),B} = \beta f(u,B) - p(u,B)$ for all u with $u_2 = \beta$ and $u_1 > K_{(f,p),B}$.*
4. *$K_{(f,p),B}A_{(f,p),B} - P_{(f,p),B} = K_{(f,p),B} f(K_{(f,p),B}, 0, B) - p(K_{(f,p),B}, 0, B)$.*

Proof: Fix any $B \in B^+_{(f,p)}$. Define $K_{(f,p),B}$ as in Lemma 7 and $P_{(f,p),B}$, $A_{(f,p),B}$ as in Lemma 11. Then it is easy to see that the first two statements are direct equivalent statements from Lemma 14. (3) follows by combining Lemma 13 with Equations 8 and 9. Combining Equation 7 with Lemma 13 we get (4). ■

Lemma **22** *Suppose (f,p) is an incentive compatible and individually rational mechanism satisfying* DIM_2 *non-triviality. Then, there exists $P_{(f,p)}$, $A_{(f,p)}$ and $K_{(f,p)}$ such that the following hold.*

1. *$p(u,B) = P_{(f,p)}$, $f(u,B) = A_{(f,p)}$ for all $(u,B) \in W$ with*
$$u_1 > K_{(f,p)},\ u_2 \in (K_{(f,p)}, \beta) \text{ and } B < P_{(f,p)}.$$
2. *If $B < P_{(f,p)}$, then $B \in B^+_{(f,p)}$.*
3. *$p(u,B) \leq B$ for all $(u,B) \in W$ with $(u_1,u_2) \neq (\beta,\beta)$ and $B \geq P_{(f,p)}$.*
4. *$p(u,B) = P_{(f,p)}$ and $f(u,B) = A_{(f,p)}$ for all $(u,B) \in W$ with $B \geq P_{(f,p)}$, $u_1 \in (K_{(f,p)}, \beta)$, and $u_2 < \beta$*
5. *$K_{(f,p)}A_{(f,p)} - P_{(f,p)} = K_{(f,p)} f(K_{(f,p)}, 0, B) - p(K_{(f,p)}, 0, B)$ for all $B < P_{(f,p)}$.*
6. *$p(u,B) \leq p(K_{(f,p)}, 0, B')$ for all $(u,B) \in W$ with $u_1 < K_{(f,p)}$ and for all $B' < P_{(f,p)}$.*
7. *$p(u,B) \leq 0$ for all $(u,B) \in W$ with $u_1 < K_{(f,p)}$.*

Proof: PROOFS OF (1) AND (2). Fix an incentive compatible and individually rational mechanism (f,p) and pick any $\acute{B} \in B^+_{(f,p)}$. From Lemma 21, we know that there exist $K_{(f,p),\acute{B}}$, $P_{(f,p),\acute{B}}$, and $A_{(f,p),\acute{B}}$ such that $p(u,\acute{B}) = P_{(f,p),\acute{B}} > \acute{B}$ and $f(u,\acute{B}) = A_{(f,p),\acute{B}}$, for all $u \in V$ with $u_2 \in (K_{(f,p),\acute{B}}, \beta)$ and $u_1 > K_{(f,p),\acute{B}}$. We do the proof in two steps.

STEP 1. Consider an outcome (a,t) in the range of the mechanism. First, consider the case when $t < P_{(f,p)}$. Analogous to Lemma 4, it can be shown that incentive compatibility of (f,p) implies that $a < A_{(f,p),\acute{B}}$. Now, consider any type of the form $(v,\acute{B})$ where $v_1 = v_2 = x \in (K_{(f,p),\acute{B}}, \beta)$. Such a v exists since $K_{(f,p),\acute{B}} < \beta$. Lemma 21 implies that $(f(v,\acute{B})), p(v,\acute{B})) = (A_{(f,p),\acute{B}}, P_{(f,p),\acute{B}})$. Incentive compatibility from $(v,\acute{B})$ to any type with the outcome (a,t) gives us:

$$xA_{(f,p),\acute{B}} - P_{(f,p),\acute{B}} \geq xa - t.$$

Since this is true for *all* $x \in (K_{(f,p),\acute{B}}, \beta)$ and noting that $t < P_{(f,p),\acute{B}}$ and $a < A_{(f,p),\acute{B}}$ we conclude that

$$xA_{(f,p),\acute{B}} - P_{(f,p),\acute{B}} > xa - t \text{ for all } x \in (K_{(f,p),\acute{B}}, \beta). \tag{27}$$

If $t > P_{(f,p)}$, a similar reasoning establishes that Inequality (27) continues to hold (the only adjustment we need to do is that a will be strictly greater than $A_{(f,p)}$).

STEP 2. Pick any budget B' with $B' \neq \acute{B}$ but $B' < P_{(f,p),\acute{B}}$. Further, pick any type (u,B') with $u_1 > K_{(f,p),\acute{B}}$ and $u_2 \in (K_{(f,p),\acute{B}}, \beta)$. We will argue that $(f(u,B'), p(u,B')) = (A_{(f,p),\acute{B}}, P_{(f,p),\acute{B}})$. Assume for contradiction, $(f(u,B'), p(u,B')) = (a,t)$ for some $(a,t) \neq (A_{(f,p),\acute{B}}, P_{(f,p),\acute{B}})$. Since Inequality (27) holds for $x = u_2$, incentive compatibility implies that $t \leq B'$ and

$$u_1 a - t \geq u_1 A_{(f,p),\acute{B}} - P_{(f,p),\acute{B}}.$$

But $B' < P_{(f,p),\acute{B}}$ implies that $t < P_{(f,p),\acute{B}}$, and hence, $a < A_{(f,p),\acute{B}}$. So, for any $x \in (K_{(f,p),\acute{B}}, \beta)$ with $x < u_1$, we must have

$$xa - t > xA_{(f,p),\acute{B}} - P_{(f,p),\acute{B}},$$

which is a contradiction to Inequality (27).

So, we conclude that for all $u_1 > K_{(f,p),\acute{B}}$ and $u_2 \in (K_{(f,p),\acute{B}}, \beta)$, we have $(f(u,B'), p(u,B')) = (A_{(f,p),\acute{B}}, P_{(f,p),\acute{B}})$. Further, this ensures that $B' \in B^+_{(f,p)}$. Hence, we have shown that for any

$\acute{B} \in B^+_{(f,p)}$ and any $B' < P_{(f,p),\acute{B}}$, we have

$$B' \in B^+_{(f,p)}. \tag{28}$$

Now, Lemma 21 implies that for every (u, B') with $u_1 > K_{(f,p),B'}$ and $u_2 \in (K_{(f,p),B'}, \beta)$, we have $p(u, B') = P_{(f,p),B'}$, we get that $P_{(f,p),B'} = P_{(f,p),\acute{B}}$. Consequently, $A_{(f,p),B'} = A_{(f,p),\acute{B}}$. Clearly, $K_{(f,p),B'} \leq K_{(f,p),\acute{B}}$. But since $P_{(f,p),B'} = P_{(f,p),\acute{B}}$ and the choice of $B', \acute{B}$ is arbitrary, we could swap their positions to conclude $K_{(f,p),\acute{B}} = K_{(f,p),B'}$.

We can now define $P_{(f,p)} := P_{(f,p),\acute{B}}$, $A_{(f,p)} := A_{(f,p),\acute{B}}$, and $K_{(f,p)} := K_{(f,p),\acute{B}}$. This concludes proof of (1).

For (2), by $\textsc{dim}_2$ non-triviality, $B^+_{(f,p)}$ is non-empty, and using the conclusion in (1) along with the set inclusion in (28), we get that for all $B < P_{(f,p)}$, we have $B \in B^+_{(f,p)}$.

From this step, using Inequality (27), we can write that for all outcomes $(a, t) \neq (A_{(f,p)}, P_{(f,p)})$ in the mechanism, we must have

$$xA_{(f,p)} - P_{(f,p)} > xa - t \qquad \forall\, x \in (K_{(f,p)}, \beta). \tag{29}$$

This obviously implies that if $a > A_{(f,p)}$, then

$$xA_{(f,p)} - P_{(f,p)} > xa - t \qquad \forall\, x < \beta. \tag{30}$$

Proof of (3) and (4). Fix any type (u, B) such that $B > P_{(f,p)}$, and $(u_1, u_2) \neq (\beta, \beta)$. Assume for contradiction that $p(u, B) > B$ - this implies that $f(u, B) > A_{(f,p)}$. Since $p(u, B) > B > P_{(f,p)}$ and $f(u, B) > A_{(f,p)}$, the following inequalities must hold for incentive compatibility

$$u_1 f(u, B) - p(u, B) \geq u_1 A_{(f,p)} - P_{(f,p)}$$
$$u_2 f(u, B) - p(u, B) \geq u_2 A_{(f,p)} - P_{(f,p)}$$

This contradicts Inequality (30) for $x = u_1$ or $x = u_2$ (note that $f(u, B) > A_{(f,p)}$). This proves (2).

Fix any (u, B) such that $B \geq P_{(f,p)}$, $u_1 \in (K_{(f,p)}, \beta)$, and $u_2 < \beta$. From (2) above, we have $p(u, B) \leq B$. Substituting $x = u_1$ in Inequality (29), we notice that for every other outcome (a, t) in the range of the mechanism, we have

$$u_1 A_{(f,p)} - P_{(f,p)} > u_1 a - t.$$

Hence, the agent prefers $(A_{(f,p)}, P_{(f,p)})$ to any other outcome (a, t) in the range of the mechanism along $\textsc{dim}_1$. By incentive compatibility $(f(u, B), p(u, B)) = (A_{(f,p)}, P_{(f,p)})$. This proves (3).

Proof of (5). By (1), we know that every $B < P_{(f,p)}$ belongs to $B^+_{(f,p)}$. Then, (4) in Lemma 21 gives the result.

Proof of (6). Fix any $(u, B) \in W$ such that $u_1 < K_{(f,p)}$. Since $u_1 < K_{(f,p)}$, Lemma 9 implies that $p(u, B) \leq B$.

Substituting $x = K_{(f,p)}$ and $(a, t) = \big(f(u, B), p(u, B)\big)$, Inequality (29) implies

$$K_{(f,p)} A_{(f,p)} - P_{(f,p)} \geq K_{(f,p)} f(u, B) - p(u, B)$$

Now pick $B' < P_{(f,p)}$ and use (4) above to get

$$K_{(f,p)} f(K_{(f,p)}, 0, B') - p(K_{(f,p)}, 0, B') \geq K_{(f,p)} f(u, B) - p(u, B). \tag{31}$$

Now, assume for contradiction that $p(u, B) > p(K_{(f,p)}, 0, B')$. Since, $p(u, B) \leq B$ we have $p(K_{(f,p)}, 0, B') < B$. Then incentive constraint $(u, B) \to (K_{(f,p)}, 0, B')$ implies that

$$u_1 f(u, B) - p(u, B) \geq u_1 f(K_{(f,p)}, 0, B') - p(K_{(f,p)}, 0, B'). \tag{32}$$

Adding Inequalities (31) and (32), and using $u_1 < K_{(f,p)}$, we get $f(u, B) \leq f(K_{(f,p)}, 0, B')$. But this implies that $p(u, B) \leq p(K_{(f,p)}, 0, B')$, which is contradiction.

Proof of (7). This is a corollary to (5) above. Set $B' = 0$ and the result follows since $p(K_{(f,p)}, 0, 0) \leq 0$ from Lemma 9. ■

Figure 7 gives a pictorial description of an incentive compatible and individually rational mechanism as implied by Lemma 22.

B.2.2 Optimality of $\textsc{post}^*$

We now complete the proof of Theorem 2 by using the preparatory lemmas. For every incentive compatible, individually rational, and $\textsc{dim}_2$ non-trivial mechanism (f, p), we first construct a new $\textsc{g-post}^*$ mechanism (f', p') in the following way.

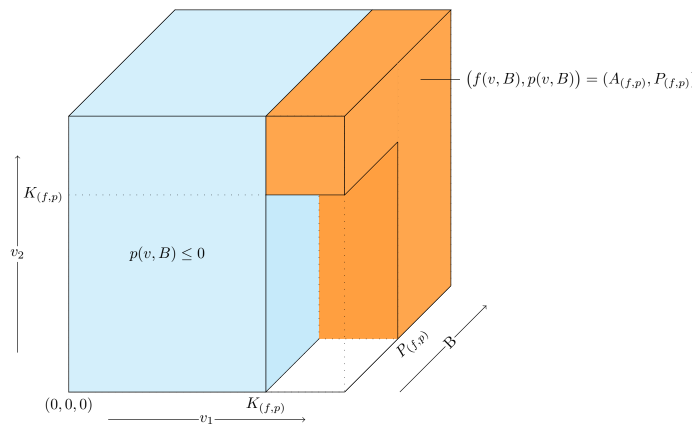

Figure 7: Structure of incentive compatible and individually rational mechanism

$$(f'(v,B),p'(v,B)) = \begin{cases} (A_{(f,p)},P_{(f,p)}) & \text{if } \Big(\min(v_1,v_2) > K_{(f,p)} \text{ and } B < P_{(f,p)}\Big) \\ & \text{or } \Big(v_1 > K_{(f,p)} \text{ and } B \geq P_{(f,p)}\Big) \\ \big(A_{(f,p)} - \frac{1}{K_{(f,p)}}P_{(f,p)}, 0\big) & \text{if } v_1 \leq K_{(f,p)} \\ \big(A_{(f,p)} - \frac{1}{K_{(f,p)}}(P_{(f,p)} - B), B\big) & \text{if } v_1 > K_{(f,p)},\ v_2 \leq K_{(f,p)} \text{ and } B < P_{(f,p)} \end{cases}$$

The new mechanism (f',p') is shown in Figure 8. It is easy to verify that $f'(v,B) \in [0,1]$ for all $(v,B) \in W$. To see this, assume for contradiction that $A_{(f,p)} - \frac{1}{K_{(f,p)}}(P_{(f,p)} - B) > 1$ when $B < P_{(f,p)}$. Then, we get $K_{(f,p)}A_{(f,p)} - P_{(f,p)} > K_{(f,p)} - B$, which is a contradiction since $A_{(f,p)} \in [0,1]$ and $B < P_{(f,p)}$. This shows that $A_{(f,p)} - \frac{1}{K_{(f,p)}}(P_{(f,p)} - B) \leq 1$, which also implies that $A_{(f,p)} - \frac{1}{K_{(f,p)}}P_{(f,p)} \leq 1$. Finally, $A_{(f,p)} - \frac{1}{K_{(f,p)}}P_{(f,p)} \geq 0$ follows from (5) in Lemma 22 and individual rationality of (f,p).

Lemma **23** *If (f,p) is an incentive compatible, individually rational,* DIM$_2$ *non-trivial mechanism, then the* G-POST* *mechanism (f',p') is a* DIM$_2$ *non-trivial, incentive compatible, individually rational, and*

$$p'(v,B) \geq p(v,B) \textit{ for almost all } (v,B) \in W.$$

Proof: Since (f',p') is a G-POST* mechanism, Proposition 8 implies that (f',p') is a DIM$_2$ non-trivial, incentive compatible, individually rational. We establish that $p'(v,B) \geq p(v,B)$ for **almost** all $(v,B) \in W$. To see this, consider the following three cases.

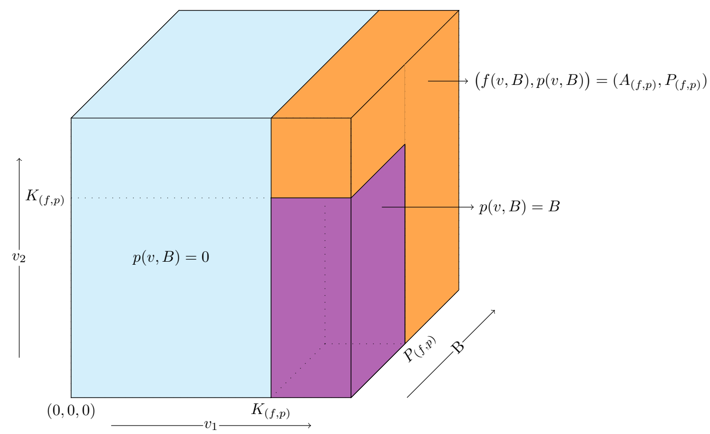

Figure 8: Mechanism (f', p')

- Case 1. Consider $(v, B) \in W$ such that $\{\min(v_1, v_2) > K_{(f,p)}$ and $B < P_{(f,p)},\ v_2 \neq \beta\}$ or $\{v_1 \in (K_{(f,p)}, \beta)$ and $B \geq P_{(f,p)},\ v_2 \neq \beta\}$. By (1) and (4) in Lemma 22,

$$p'(v) = P_{(f,p)} = p(v).$$

- Case 2. Consider $(v, B) \in W$ such that $v_1 < K_{(f,p)}$. By (7) in Lemma 22, we have $p'(v, B) = 0 \geq p(v, B)$.

- Case 3. Finally, consider $(v, B) \in W$ such that $v_2 < K_{(f,p)},\ v_1 > K_{(f,p)}$ and $B < P_{(f,p)}$. By (2) in Lemma 22, we get that $B \in B^{+}_{(f,p)}$. Then, since $\min(v_1, v_2) < K_{(f,p)}$, by the definition of $K_{(f,p)}$, we get $p(v, B) \leq B = p'(v, B)$, which concludes this case.

Denote by W' the set of type profiles covered in the above three cases. It is easy to see (for instance, refer to Figure 8) that $W \setminus W'$ has zero Lebesgue measure. So, for almost all (v, B), we have $p'(v, B) \geq p(v, B)$. ■

The proof of Theorem 2 is completed by the following lemma.

Lemma **24** *For every* G-POST* *mechanism* (f, p)*, there is a* POST* *mechanism* (f', p') *such that*

$$p'(v, B) \geq p(v, B)\ \forall\ (v, B) \in W.$$

Proof: Take any G-POST* mechanism (f, p) defined by parameters A, P, K. Consider the POST* mechanism (f', p') defined by parameter K. By definition of G-POST* mechanism (f, p), we know that $K \geq P$. Now, consider the following cases:

- $p'(v, B) = p(v, B) = 0$ for all (v, B) if $v_1 \leq K$.
- $p'(v, B) = p(v, B) = B$ for all (v, B) if $v_1 > K$, $v_2 \leq K$ and $B < P$.
- $p'(v, B) = K \geq P = p(v, B)$ for all (v, B) if $\{\min(v_1, v_2) > K$ and $B < K\}$ or $\{v_1 > K$ and $B \geq K\}$
- $p'(v, B) = K \geq P = p(v, B)$ for all (v, B) if $v_1 > K$, $v_2 \leq K$ and $P \leq B < K$.

This concludes the proof. ■

Lemma 24 thus establishes that a POST* mechanism is a partially optimal mechanism, which concludes the proof of Theorem 2.

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C Supplementary Appendix

C.1 Proof of Lemma 1

Proof: We consider two cases where $v_1 < v_2$ and then $v_1 > v_2$. The proof is by construction of three outcomes as stated above.

Case 1. Fix any $v = (v_1, v_2)$ such that $0 < v_1 < v_2$. Consider three outcomes

$$(a,t) := (\frac{1}{2}, B),\ (b,t') := (1, B + \frac{3v_1}{8} + \frac{v_2}{8}),\ \text{and } (c,t'') = (\frac{3}{4} - \frac{v_1}{8v_2}, B + \frac{v_1}{8}).$$

First,

$$v_1 a - t = \frac{1}{2}v_1 - B = v_1 - B - \frac{v_1}{2} > v_1 - B - (\frac{3v_1}{8} + \frac{v_2}{8}) = v_1 b - t',$$

where the inequality is true because $v_1 < v_2$. Combining this with $t \leq B$ gives us

$$(a,t) \succ_v (b,t').$$

Second,

$$\begin{aligned} v_2 b - t' &= v_2 - B - (\frac{3v_1}{8} + \frac{v_2}{8}) = v_2 - B - (\frac{v_1}{4} + \frac{v_1 + v_2}{8}) \\ &> v_2 - B - (\frac{v_1}{4} + \frac{v_2}{4}) = v_2(\frac{3}{4} - \frac{v_1}{8v_2}) - B - \frac{v_1}{8} \\ &= v_2 c - t''. \end{aligned}$$

where the inequality is true because $v_1 < v_2$. Combining this with the fact that $t', t'' > B$, we have

$$(b,t') \succ_v (c,t'').$$

Third,

$$v_1 c - t'' = v_1(\frac{3}{4} - \frac{v_1}{8v_2}) - B - \frac{v_1}{8} > \frac{3}{4}v_1 - B - \frac{v_1}{4} = \frac{1}{2}v_1 - B = v_1 a - t,$$

where the inequality is true because $v_1 < v_2$. Hence, $(a,t) \not\succeq_{v_1} (c,t'')$.

But since $t'' > B$, we need to compare the outcomes with respect to v_2. For that, notice

$$v_2 c - t'' = v_2(\frac{3}{4} - \frac{v_1}{8v_2}) - B - \frac{v_1}{8} = v_2(\frac{3}{4} - \frac{v_1}{4v_2}) - B > \frac{1}{2}v_2 - B,$$

where the inequality is due to $v_1 < v_2$. This implies that $(c,t'') \succ_v (a,t)$.

CASE 2. Fix any $v = (v_1, v_2)$ such that $v_1 > v_2$. Set $K = \max(2, \left\lceil \frac{v_2}{B} \right\rceil)$, where we use the notation that $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Consider three outcomes

$$(a,t) := (1 - \frac{2}{K}, B - \frac{v_2}{K}),\ (b,t') := (1, B + \frac{v_2(3 - \frac{v_2}{v_1})}{2K}),\ \text{and } (c,t'') := (1 - \frac{7 - 3(\frac{v_2}{v_1})}{4K}, B).$$

The value of K set above ensures that all the consumption bundles are feasible.
First,

$$(v_1 b - t') - (v_1 a - t) = \frac{1}{K}(2v_1 - v_2) - \frac{1}{2K}\frac{v_2}{v_1}(3v_1 - v_2) \geq \frac{1}{K}(2v_1 - v_2) - \frac{1}{2K}(3v_1 - v_2) > 0,$$

where the inequalities are true because $v_1 > v_2$. Since $t' > B$ we have $(b,t') \not\succeq_{v_1} (a,t)$. We need to check the outcomes with respect to the agent's type, that is v_2. For that, notice

$$(v_2 a - t) - (v_2 b - t') = \frac{v_2}{v_1}\Big(\frac{3v_1 - v_2}{2K}\Big) - \frac{v_2}{K} > 0.$$

The inequality is true because $v_1 > v_2$. From above discussions, we have

$$(a,t) \succ_v (b,t').$$

Second,

$$(v_2 b - t') - (v_2 c - t'') = \frac{1}{4K}\Big(\big(7 - 3\frac{v_2}{v_1}\big) - \big(6 - 2\frac{v_2}{v_1}\big)\Big) = \frac{1}{4K}(1 - \frac{v_2}{v_1}) > 0,$$

where the inequality is due to $v_1 > v_2$. Also, notice that from above we derive $t' - t'' < v_2(b - c) < v_1(b - c)$ which implies $v_1 b - t' > v_1 c - t''$. Combining the above two results with the fact that $t' > B$, we conclude that

$$(b,t') \succ_v (c,t'').$$

Third,

$$(v_1 c - t'') - (v_1 a - t) = \frac{1}{K}(2v_1 - v_2) - \frac{1}{4K}\big(7v_1 - 3v_2\big) = \frac{1}{4K}(v_1 - v_2) > 0.$$

The inequality is because $v_1 > v_2$. Noticing that $t'' \leq B$, we have $(c,t'') \succ_v (a,t)$. ■

C.2 Proofs for the uniform distribution case

In this section, we give the proofs of Lemma 3 and Proposition 5.

C.2.1 Proof of Lemma 3

Proof: Suppose (K_1^*, K_2^*) are values of (K_1, K_2) in the optimal POST-2 mechanism. By definition $K_1^* \leq K_2^*$. Using the uniform distribution of G, we see that (K_1^*, K_2^*) are optimal solutions to the following optimization problem:

$$\max_{K_2 \in [B,1],\ K_1 \in [B, K_2]} B\Big[1 - K_1\Big] + \big(1 - \frac{B}{K_1}\big) K_2 (1 - K_2)^2. \tag{33}$$

We consider the following optimization problem, where we fix the value of K_1^* and maximize over all K_2:

$$\max_{K_2 \in [0,1]} B\Big[1 - K_1^*\Big] + \big(1 - \frac{B}{K_1^*}\big) K_2 (1 - K_2)^2.$$

Notice that the objective function is strictly concave in K_2, and the unique maximum occurs when $K_2 = \frac{1}{3}$.

Now, assume for contradiction $K_1^* < K_2^*$. We consider two cases and reach a contradiction in both the cases.

CASE 1. Suppose $K_1^* \geq \frac{1}{3}$. Then, $K_2^* > \frac{1}{3}$. But $K_2 = K_1^*$ and K_1^* defines a feasible POST-2 mechanism, and generates more revenue. This is a contradiction.

CASE 2. Suppose $K_1^* < \frac{1}{3}$. Since $K_2^* \geq K_1^*$, we see that $K_2 = \frac{1}{3}$ and K_1^* defines a feasible POST-2 mechanism and generates more revenue. Hence, K_2^* must be equal to $\frac{1}{3}$. Now, fixing the value of K_2 at $\frac{1}{3}$, we optimize the Expression (33) with relaxed constraints on K_1:

$$\max_{K_1 \in [0,1]} B\Big[1 - K_1\Big] + \big(1 - \frac{B}{K_1}\big) \frac{4}{27}.$$

This objective function is strictly concave with a unique maxima at $K_1 = \frac{2}{3\sqrt{3}} > \frac{1}{3}$. Hence, the objective function of the Expression in (33) is higher at $K_1 = \frac{1}{3} = K_2^*$ than at (K_1^*, K_2^*) with $K_1^* < \frac{1}{3}$. Further, $K_1 = K_2 = \frac{1}{3}$ is a POST-2 mechanism since (K_1^*, K_2^*) with $K_2^* = \frac{1}{3}$ is a POST-2 mechanism. This is a contradiction.

Using this, we can conclude that the optimal POST-2 mechanism is a solution to the following single-variable constrained optimization problem.

$$\max_{K \in [B,1]} B\big(1 - K\big) + \big(K - B\big)(1 - K)^2. \tag{34}$$

We denote $J(K) := B\big(1-K\big) + \big(K-B\big)(1-K)^2$ for all K. Notice that

$$\begin{aligned} J'(K) &= 3K^2 - K(2B+4) + (B+1) \\ J''(K) &= 6K - (2B+4). \end{aligned}$$

Note that

$$J'(B) = B^2 - 3B + 1 = \Big(B - \frac{3-\sqrt{5}}{2}\Big)\Big(B - \frac{3+\sqrt{5}}{2}\Big).$$

Hence, $J'(B) \le 0$ if and only if $B \ge \frac{1}{2}\big(3-\sqrt{5}\big)$.

Notice that $J''(K) = 0$ for $K = \frac{1}{3}(B+2)$. Hence, $J'(K)$ is decreasing in $[B, \frac{1}{3}(B+2)]$ and increasing in $[\frac{1}{3}(B+2), 1]$. Also, $J'(1) = -B < 0$. Hence, if $J'(B) \le 0$, we must have $J'(K) < 0$ for all $K \in (B, 1]$.

Proof of (1). This implies that for $B \ge \frac{1}{2}\big(3-\sqrt{5}\big)$, we have $J'(K) < 0$ for all $K \in (B, 1]$. This implies that J is decreasing in $[B, 1]$, and hence, the optimal solution of Optimization (34) must have $K = B$. Then, the first part implies that the optimal post-2 mechanism must have $K_1^* = K_2^* = B$.

Proof of (2). If $B < \frac{1}{2}\big(3-\sqrt{5}\big)$, then $J'(B) > 0$ and $J'(K) = 0$ at a unique point

$$K = \frac{1}{3}\big(B + 2 - \sqrt{(B^2+B+1)}\big).$$

Denote this point of inflection as $\tilde{K}$. Notice that $J'(K) < 0$ for all $K > \tilde{K}$, and, hence, J is decreasing after $\tilde{K}$. Further, $\tilde{K} < \frac{1}{3}(B+2)$ and $J''(K) < 0$ for all $K < \tilde{K}$. This means J is strictly concave from B to $\frac{1}{3}(B+2)$. Combining these observations, we conclude that $K = \tilde{K}$ solves the Optimization in (34). The first part implies that the optimal post-2 mechanism must have

$$K_1^* = K_2^* = \frac{1}{3}\big(B + 2 - \sqrt{(B^2+B+1)}\big),$$

if $B < \frac{1}{2}\big(3-\sqrt{5}\big)$. ■

C.2.2 Proof of Proposition 5

Proof: To do the proof, we first compute the optimal post-1 mechanism, which is the solution to the following optimization program:

$$\max_{K_1 \in [0,B]} K_1(1-K_1). \tag{35}$$

It is clear the optimal POST-1 mechanism is $K_1 = \frac{1}{2}$ if $B > \frac{1}{2}$ and $K_1 = B$ if $B \leq \frac{1}{2}$. Now, we consider the three cases separately.

CASE 1 - $B > \frac{1}{2}$. Optimal POST-1 mechanism generates a revenue of $\frac{1}{4}$. By Lemma 3, optimal POST-2 mechanism generates a revenue of $B(1-B)$, which is less than $\frac{1}{4}$. Hence, the optimal mechanism is a POST-1 mechanism with $K_1 = \frac{1}{2}$.

CASE 2 - $B \in [\frac{1}{2}(3-\sqrt{5}), \frac{1}{2}]$. In this case, both the optimal POST-1 mechanism and the optimal POST-2 mechanism (due to Lemma 3) generates a revenue of $B(1-B)$. Hence, the optimal POST-1 mechanism with $K_1 = B$ is optimal.

CASE 3 - $B \in (0, \frac{1}{2}(3-\sqrt{5}))$. In this case, the optimal POST-1 mechanism generates a revenue of $B(1-B)$, which is also the revenue generates by a POST-2 mechanism with $K_1 = K_2 = B$. But the optimal POST-2 is unique and has $K_1 = K_2 = \frac{1}{3}\big(B + 2 - \sqrt{(B^2+B+1)}\big)$ due to Lemma 3. Hence, the result follows. ■

C.2.3 Proof of Proposition 6

Proof: By Proposition 5, the revenue clearly increases with budget in the region $B \geq \frac{1}{2}\big(3-\sqrt{5}\big)$ since the mechanism is a POST-1 mechanism. We show that $R^*(B) > R^*(B')$ if $\frac{1}{2}\big(3-\sqrt{5}\big) > B > B'$. This will conclude the proof since

$$\lim_{B \to \frac{1}{2}\left(3-\sqrt{5}\right)} R^*(B) = R^*(\frac{1}{2}\big(3-\sqrt{5}\big)).$$

To show this, we use Proposition 5 to note that if $\frac{1}{2}\big(3-\sqrt{5}\big) > B > B'$, then the optimal mechanism at B and B' are POST-2 mechanisms. Hence, we just show the monotonicity of revenue of an optimal POST-2 mechanism as a function of budget when $B \in (0, \frac{1}{2}(3-\sqrt{5}))$.

Denote the optimal value of K_1 and K_2 (both are equal by Proposition 5) in the optimal POST-2 mechanism as:

$$K(B) := \frac{1}{3}\big(B + 2 - \sqrt{(B^2+B+1)}\big) = \frac{1}{3}\big(B + 2 - q(B)\big),$$

where $q(B) := \sqrt{(B^2+B+1)}$. Notice that

$$K(B) - \frac{1}{2} = \frac{1}{3}\Big(\big(B + \frac{1}{2}\big) - \sqrt{B^2+B+1}\Big) < 0.$$

Hence, $K(B) < \frac{1}{2}$.

Further, notice that

$$K'(B) = \frac{1}{3}\big(1 - (B + \frac{1}{2})\frac{1}{q(B)}\big) = \frac{1}{3}\left(1 - \sqrt{\frac{(B^2 + B + \frac{1}{4})}{(B^2 + B + 1)}}\right) > 0.$$

Our first claim is that $K'(B) < K(B)$. To see this,

$$\begin{aligned}
3\big[K(B) - K'(B)\big] &= B + 1 + (B + \frac{1}{2})\frac{1}{q(B)} - q(B) \\
&= (B+1) + \frac{1}{q(B)}\big[B + \frac{1}{2} - B^2 - B - 1\big] \\
&= (B+1) - \frac{1}{q(B)}\big(B^2 + \frac{1}{2}\big) \\
&= \frac{1}{q(B)}\Big((B+1)\sqrt{B^2 + B + 1} - (B^2 + \frac{1}{2})\Big) \\
&\geq \frac{1}{q(B)}\Big(\sqrt{(B^2+1)}\sqrt{(B^2+1)} - (B^2 + \frac{1}{2})\Big) \\
&= \frac{1}{2} \\
&> 0.
\end{aligned}$$

Now, revenue in the optimal POST-2 mechanism for these values of B is given by

$$R(B) := B\big(1 - K(B)\big) + \big(K(B) - B\big)\big(1 - K(B)\big)^2.$$

Now, we differentiate $R(B)$ with respect to B and observe:

$$\begin{aligned}
R'(B) &= \big(1 - K(B)\big) - BK'(B) + \big(K'(B) - 1\big)\big(1 - K(B)\big)^2 - 2\big(1 - K(B)\big)K'(B)\big(K(B) - B\big) \\
&= K(B)\big(1 - K(B)\big) + K'(B)\Big[-B + \big(1 - K(B)\big)^2 - 2\big(1 - K(B)\big)\big(K(B) - B\big)\Big] \\
&> K'(B)\Big[1 - K(B) - B + \big(1 - K(B)\big)^2 - 2\big(1 - K(B)\big)\big(K(B) - B\big)\Big] \\
&\quad \text{(using } K'(B) > 0 \text{ and } K'(B) < K(B)) \\
&= K'(B)\Big[\big(1 - K(B)\big)\big(2 - 3K(B)\big) + B\big(1 - 2K(B)\big)\Big] \\
&> 0 \\
&\quad \text{(using } K(B) < \frac{1}{2} \text{ and } K'(B) > 0).
\end{aligned}$$

This concludes the proof. ∎

C.3 An alternate notion of incentive compatibility

In this section, we adapt the choice correspondence procedure defined in Manzini and Mariotti (2012) to propose an extension of our binary choice model. We then propose an appropriate notion of incentive compatibility for this model and show its relation to our notion of incentive compatibility.

Consider an agent of type $v \equiv (v_1, v_2)$. For any subset of outcomes $S \subseteq Z$, define

$$M^1(S; v_1) := \{(a,t) \in S : av_1 - t \geq a'v_1 - t' \ \forall \ (a', t') \in S \text{ and } t \leq B\}$$

and define

$$M^2(S; v_2) := \{(a,t) \in S : av_2 - t \geq a'v_2 - t' \ \forall \ (a', t') \in S\}.$$

Using $M^1(S; v_1)$ and $M^2(S; v_2)$, we can now define a choice correspondence $C^v : 2^Z \to 2^Z$ with $\emptyset \neq C^v(S) \subseteq S$ for each $S \subseteq Z$ as follows:

$$C^v(S) = \begin{cases} M^1(S; v_1) & \text{if } M^1(S; v_1) \neq \emptyset \\ M^2(S; v_2) & \text{otherwise} \end{cases}$$

Intuitively, the agent tries to choose from S using v_1 first, i.e., if the maximal elements according to DIM_1 satisfy budget constraint, then they are chosen. Otherwise, the maximal elements according to DIM_2 are chosen. This is a plausible extension of our binary choice model to accommodate choice from arbitrary subsets.

If we *assume* that our agent makes choices using such choice correspondences (or some other choice correspondence "consistent" with type v), then a familiar notion of incentive compatibility for choice correspondences can be applied. In particular, we say that (f, p) is **choice-incentive compatible** if for every v,

$$(f(v), p(v)) \in C^v(R^{f,p}),$$

where $R^{f,p}$ is the range of the mechanism (f, p). This definition can be extended to arbitrary mechanisms $\mu : M \to Z$ defined on message space M. Notice that our definition requires that

$$(f(v), p(v)) \succeq_v (a, t) \ \forall \ (a, t) \in R^{f,p}.$$

If the agent makes choices using C^v for each type v, we show that choice-incentive compatibility and incentive compatibility are independent conditions. We give two examples below to illustrate this.

EXAMPLE **1**

To see this, consider a type space with three types $V := \{v, v', v''\}$, where

$$v = (1, 1.2), v' = (0, 0), v'' = (1, 1).$$

Assume $B = 0.5$ and consider the following mechanism (f, p) defined on this type space.

$$(f(v), p(v)) := (1, 0.6), \quad (f(v'), p(v')) := (0.81, 0.4), \quad (f(v''), p(v'')) = (0.924, 0.51).$$

We can check that

$$M^1(R^{f,p}; v_1) = \emptyset, M^1(R^{f,p}; v'_1) = \{(f(v'), p(v'))\}, M^1(R^{f,p}; v''_1) = \emptyset$$
$$M^2(R^{f,p}; v_2) = \{(f(v), p(v))\}, M^2(R^{f,p}; v'_2) = \{(f(v'), p(v'))\}, M^2(R^{f,p}; v''_2) = \{(f(v''), p(v''))\}.$$

Hence, we get

$$C^v(R^{f,p}) = \{(f(v), p(v))\}, C^{v'}(R^{f,p}) = \{(f(v'), p(v'))\}, C^{v''}(R^{f,p}) = \{(f(v''), p(v''))\}.$$

Hence, (f, p) is choice-incentive compatible. But it can also be checked that

$$(f(v), p(v)) = (1, 0.6) \not\succeq_v (0.81, 0.4).$$

Hence, (f, p) is not incentive compatible.

EXAMPLE **2**

Now, consider another type space $V' = \{u, u', u''\}$, where

$$u = (3, 2), \; u' = (0, 0), \; \text{and } u'' = (2.5, 2.5).$$

As before, assume $B = 0.5$. Now, consider the following mechanism (f', p') defined on the type space V'.

$$(f'(u), p'(u)) := (0.99, 0.49), \quad (f'(u'), p'(u')) := (0.989, 0.487), \quad (f'(u''), p'(u'')) = (1, 0.51).$$

Now, the following binary relations can be verified.

$$(0.99, 0.49) \succeq_u (0.989, 0.487), (0.99, 0.49) \succeq_u (1, 0.51).$$
$$(0.989, 0.487) \succeq_{u'} (0.99, 0.49), (0.989, 0.487) \succeq_{u'} (1, 0.51).$$
$$(1, 0.51) \succeq_{u''} (0.99, 0.49), (1, 0.51) \succeq_{u''} (0.989, 0.487).$$

This shows that (f', p') is incentive compatible. But notice that

$$M^1(R^{f',p'}; u_1) = \emptyset, M^2(R^{f',p'}; u_2) = \{(0.989, 0.487)\}.$$

Hence, $(f'(u), p'(u)) = (0.99, 0.49) \notin C^u(R^{f',p'})$. This shows that (f', p') is not choice-incentive compatible.

C.4 A sufficient condition for optimality of POST*

In this section, we will identify some restrictions on the distribution that ensures that POST* is an *optimal* mechanism for the private budgets case. We summarize our assumptions below.

DEFINITION **10** *We say distribution* Φ *satisfies* **Assumption A** *if*

- *Values and budget are distributed independently, i.e., there exists a prior* G *over* $V \equiv [0,\beta] \times [0,\beta]$ *and a prior* Π *over* $[0,\beta]$ *such that* $\Phi(v,B) = G(v)\Pi(B)$ *for all* (v,B).
- *Marginal* G_1 *satisfies the property that* $H_1(x) := xG_1(x)\ \forall\ x$ *is strictly convex.*
- *Finally, define* $\bar{K}$ *as before:* $\bar{K} := \arg\max_{r\in[0,\beta]} r(1-G_1(r))$ *- this is well defined because* H_1 *is strictly convex. Then, the following must hold:*

$$[1-G(\bar{K},\beta)-G(\beta,\bar{K})+G(\bar{K},\bar{K})]\int_0^{\bar{K}}(\bar{K}-B)d\Pi(B) \geq \int_0^{\bar{K}} B[G_1(\bar{K})-G_1(B)]d\Pi(B)$$

If G is the uniform distribution over $[0,1]\times[0,1]$ and Π is uniform over $[0,1]$, then the resulting distribution satisfies Assumption A.

PROPOSITION **9** *If* Φ *satisfies Assumption A, then a* POST* *mechanism is optimal.*

Proof: Fix any B in $(0,\beta)$ and consider the optimal POST-1 mechanism in M^- derived in Proposition 4. We use this mechanism for each B (using the expression in Proposition 2) to define a new mechanism (f',v') for the private budget case - for $B \in \{0,\beta\}$, we use the limiting mechanisms of the POST-1 mechanism suggested in Proposition 2.

$$(f'(v),p'(v)) = \begin{cases} (1,B) & \text{if } v_1 > B \text{ and } B < \bar{K} \\ (1,\bar{K}) & \text{if } v_1 > \bar{K} \text{ and } B \geq \bar{K} \\ (0,0) & \text{otherwise.} \end{cases}$$

Of course, this mechanism is not incentive compatible in the private budget case - when $v_1 > B > 0$, the agent has an incentive to report a budget equal to zero get the outcome $(1,0)$. But notice that the expected revenue of the optimal mechanism in the class of incentive compatible and individually rational mechanisms that are not DIM$_2$ non-trivial cannot exceed the expected revenue of (f',p').

Now, consider the POST* mechanism by setting $K = \bar{K}$:

$$(f^*(v),p^*(v)) = \begin{cases} (1,\bar{K}) & \text{if } \{v_1 > \bar{K} \text{ and } B \geq \bar{K}\} \text{ or } \{v_1, v_2 > \bar{K} \text{ and } B < \bar{K}\} \\ (\frac{B}{\bar{K}}, B) & \text{if } v_1 > \bar{K},\ v_2 \leq \bar{K}, \text{ and } B < \bar{K} \\ (0,0) & \text{otherwise} \end{cases}$$

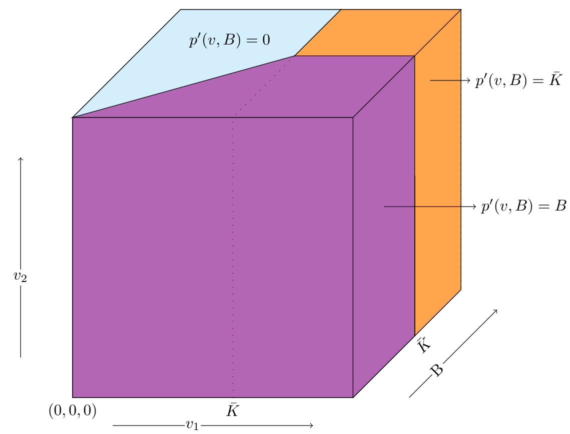

Figure 9: Upper bound

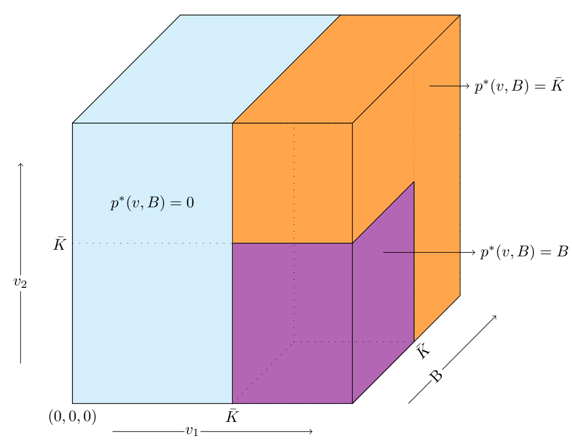

Figure 10: Lower bound

The two mechanisms are shown in Figures 9 and 10 below.

We argue that POST* generates weakly greater expected revenue that (f', p') under Assumption A. Hence, the optimal mechanism must be a POST* mechanism by Theorem 2.

Note that (f', p') and (f^*, p^*) yield the same revenue for the following types:

$$(v, B) \text{ such that } B \geq \bar{K}$$
$$(v, B) \text{ such that } v_1 > \bar{K},\ v_2 \leq \bar{K}, \text{ and } B < \bar{K}$$
$$(v, B) \text{ such that } v_1 \leq B, \text{ and } B < \bar{K}$$

So, we ignore these types and focus on rest of the types.

- for any type (v, B) such that $v_1, v_2 > \bar{K}$ and $B < \bar{K}$, revenue from (f^*, p^*) is $\bar{K}$ whereas revenue from (f', p') is B; so the difference in revenue is $\bar{K} - B$.
- for any type (v, B) such that $v_1 \in (B, \bar{K}]$ and $B < \bar{K}$, revenue from (f^*, p^*) is 0 whereas revenue from (f', p') is B; so the difference in revenue is B.

Then the condition for revenue from (f^*, p^*) to be more than that of (f', p') is:

$$[1 - G(\bar{K}, \beta) - G(\beta, \bar{K}) + G(\bar{K}, \bar{K})] \int_0^{\bar{K}} (\bar{K} - B) d\Pi(B) \geq \int_0^{\bar{K}} B[G_1(\bar{K}) - G_1(B)] d\Pi(B)$$

This holds because of Assumption A. ■