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Pareto efficient combinatorial auctions: dichotomous preferences without quasilinearity.

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Pareto efficient combinatorial auctions: dichotomous preferences without quasilinearity *

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Abstract

We consider a combinatorial auction model where preferences of agents over bundles of objects and transfers need not be quasilinear. We show the salience of dichotomous preferences in this model: an agent with dichotomous preference partitions the set of bundles of objects as *acceptable* and *unacceptable*, and at the same transfer level, she is indifferent between bundles in each class but strictly prefers acceptable to unacceptable bundles. We show that there is no Pareto efficient, dominant strategy incentive compatible (DSIC), individually rational (IR) mechanism satisfying no subsidy if the domain of preferences includes *all* dichotomous preferences. However, a generalization of the VCG mechanism is Pareto efficient, DSIC, IR and satisfies no subsidy if the domain of preferences is the set of all *positive income effect* dichotomous preferences. We show tightness of this result: adding a non-dichotomous preference (satisfying some natural properties) to such a domain of preferences brings back the impossibility result.

JEL Codes: D82, D90

Keywords: combinatorial auctions; non-quasilinear preferences; dichotomous preferences; single-minded bidders

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1 Introduction

The Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973) occupies a central role in mechanism design theory (specially, with private values). It satisfies two fundamental desiderata: it is dominant strategy incentive compatible (DSIC) and Pareto efficient. The focus of the current paper is on combinatorial auctions, where multiple objects are sold to agents simultaneously, who may buy any bundle of objects. For such combinatorial auction models, the VCG mechanism and its indirect implementations (like ascending price auctions) have been popular. In such models, the VCG mechanism is also individually rational (IR) and satisfies no subsidy (i.e., does not subsidize any agent).

Unfortunately, these desirable properties of the VCG mechanism critically rely on the fact that agents have quasilinear preferences. While analytically convenient and a good approximation of actual preferences when transfers involved are low, quasilinearity is a debatable assumption in practice. For instance, consider an agent participating in a combinatorial auction (i.e., simultaneous auction of multiple objects where agents may be allocated more than one object) for spectrum licenses, where agents often borrow from various investors at non-negligible interest rates. Such borrowing naturally leads to preferences which are non-quasilinear. Further, income effects are ubiquitous in settings with non-negligible transfers. [1]

This has initiated a small literature in mechanism design theory (discussed later in this section and again in Section 5), where the quasilinearity assumption is relaxed to allow any *classical* preference of the agent over consumption bundles: (bundle of objects, transfer) pairs [2]. The main research question addressed in this literature is the following:

> *In combinatorial auction models, if agents have classical preferences, is it possible to construct a "desirable" mechanism - a mechanism which inherits the DSIC, Pareto efficiency, IR, and no subsidy properties of the VCG mechanism?*

This paper contributes to this literature by showing the salience of a particular class of preferences, which we call *dichotomous*. If an agent has a dichotomous preference, she partitions the set of bundles of objects into "acceptable" and "unacceptable". If the transfer amount for all the bundles of objects are the same, then an agent is indifferent between

[1] Designing mechanisms without the quasilinearity assumption saves us from a form of robustness critique since we do not have to rely on specific assumptions about the functional form of utility functions of agents.

[2] Classical preferences assume mild continuity and monotonicity (in money and bundles of objects) properties of preferences.

her acceptable bundles of objects; she is also indifferent between unacceptable bundles of objects; but she prefers every acceptable bundle to every unacceptable bundle.

We show that if the domain of preferences contains *all* dichotomous classical preferences, there is no desirable mechanism. However, we show that a natural generalization of the VCG mechanism to classical preferences, which we call the *generalized VCG* (GVCG) mechanism, is desirable if the domain contains *only positive income effect* dichotomous preferences. Further, the GVCG mechanism is the *unique* desirable mechanism in any domain of positive income effect dichotomous preferences if it contains the quasilinear dichotomous preferences. However, this positive result is tight: we get back impossibility in any domain containing quasilinear dichotomous preferences and at least one more positive income effect non-dichotomous preference (satisfying an extra reasonable condition). As a corollary, we discover new type spaces where a desirable mechanism does not exist. The tightness result also hints that classical preference domains that admit a desirable mechanism cannot contain the set of dichotomous preferences.

Though the dichotomous preference appears restrictive, it is natural in many settings. For instance, consider a setting where firms are trying to procure bundles of resources for their production (such resources can be workers, as in the Kelso and Crawford (1982) model). For every firm, there are certain bundles (acceptable ones) which will satisfy the requirements for production and the firm is indifferent between those bundles - for instance, the firm may be looking to build a team of workers having certain skills and as long as the team has those skills, it is acceptable to the firm. Other examples include firms (data providers) buying paths on (data) networks (Babaioff et al., 2006) - a firm is interested in sending data from node x to node y on a directed graph whose edges are up for sale, and as long as a bundle of edges contain a path from x to y, it is acceptable to the firm. One may also consider the dichotomous preference restriction as a behavioral assumption, where the agent does not consider computing values for each of the exponential number of bundles but classifies the bundles as acceptable and unacceptable. Even in quasilinear setting, the dichotomous restriction poses interesting combinatorial challenges for computing the VCG outcome. This has led to a large literature in computer science for looking at *approximately desirable* VCG-style mechanisms (Babaioff et al., 2005, 2006; Lehmann et al., 2002; Ledyard, 2007; Milgrom and Segal, 2014). Also related is the literature in matching and social choice theory (models without transfers), where dichotomous preferences have been widely studied (Bogomolnaia and Moulin, 2004; Bogomolnaia et al., 2005; Bade, 2015).

Connecting the literature to our results. We briefly connect our results to some relevant results from the literature - a thorough literature survey is given in Section 5. As discussed earlier, classical preferences imply that willingness to pay for a bundle of objects depends on the transfer level. Thus, it is not clear what the counterpart of "valuation" of a bundle of objects is in this setting. Our generalized VCG is defined by treating the willingness to pay at *zero* transfer as the "valuation" of a bundle and then defining the VCG outcome with respect to these valuations. We are not the first one to take this approach.

Saitoh and Serizawa (2008) was the first paper to define the generalized VCG mechanism using this approach for single object auction [3]. They show that the generalized VCG mechanism is desirable even if preferences have *negative income effect*. This is in contrast to our results - we get impossibility with negative income effect preferences but the generalized VCG mechanism is desirable with positive income effect.

When we go from single object to multiple object combinatorial auctions, the generalized VCG may fail to be DSIC. For instance, Demange and Gale (1985) consider a combinatorial auction model where multiple objects are sold but each agent demands at most one object - note that objects are heterogeneous in this model. In this model, the generalized VCG is no longer DSIC. However, Demange and Gale (1985) propose a different mechanism (based on the idea of market-clearing prices), which is desirable.

When agents can demand more than one object in a combinatorial auction model with multiple heterogeneous objects, Kazumura and Serizawa (2016) show that a desirable mechanism may not exist - this result requires certain richness of the domain of preferences which is violated by our dichotomous preference model. Similarly, Baisa (2017) shows that in the homogeneous objects sale case, if agents demand multiple units, then a desirable mechanism may not exist - he requires slightly different axioms than our desirability axioms [4].

These results point to a conjecture that when agents demand multiple objects in a combinatorial auction model, a desirable mechanism may not exist. Since ours is a combinatorial auction model where agents can consume multiple objects, our impossibility result with dichotomous preferences complement these results. However, what is surprising is that we

[3] They consider a slightly more general model where they allow for multiple *homogeneous* units of the same object to be sold but agents can only consume one unit.

[4] The impossibility result in Baisa (2017) requires there to be at least three agents, and he shows the existence of a desirable mechanism with two agents.

recover the desirability of the generalized VCG mechanism with positive income effect dichotomous preferences. This shows that *not all* multi-demand combinatorial auction models without quasilinearity are impossibility domains.

The rest of the paper is organized as follows. We formally describe our combinatorial auctions and the dichotomous preference restriction in Section 2 with a motivating example. Section 3 describes all our results - the impossibility result in Section 3.1 and the possibility result with a characterization in Section 3.2. Section 4 discusses tightness of our positive results. A detailed literature review is given in Section 5. All proofs are in the Appendix at the end. A Supplementary Appendix at the end provides a missing proof from the text and an additional tightness result.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ be the set of agents and M be a set of m objects. Let $\mathcal{B}$ be the set of all subsets of M. We will refer to elements in $\mathcal{B}$ as **bundles** (of objects). A seller (or a planner) is selling/allocating bundles from $\mathcal{B}$ to agents in N using transfers. We introduce the notion of classical preferences and type spaces corresponding to them below.

2.1 Classical Preferences

Each agent has preference over possible *outcomes* - pairs of the form (A, t), where $A \in \mathcal{B}$ is a bundle and $t \in \mathbb{R}$ is a transfer amount paid by the agent. Let $\mathcal{Z} = \mathcal{B} \times \mathbb{R}$ denote the set of all outcomes. A preference R_i of agent i over $\mathcal{Z}$ is a complete transitive preference relation with strict part denoted by P_i and indifference part denoted by I_i.

We restrict attention to the following class of preferences over outcomes.

Definition 1 *Preference ordering R_i of agent i over $\mathcal{Z}$ is* **classical** *if it satisfies*

1. **Monotonicity**. *For each $A, A' \in \mathcal{B}$ with $A' \subseteq A$ and for each $t, t' \in \mathbb{R}$ with $t' > t$,*

$$(A, t) \ P_i \ (A, t')$$
$$(A, t) \ R_i \ (A', t).$$

2. **Continuity**. *For each $Z \in \mathcal{Z}$, the upper contour set $\{Z' \in \mathcal{Z} : Z' \ R_i \ Z\}$ and the lower contour set $\{Z' \in \mathcal{Z} : Z \ R_i \ Z'\}$ are closed.*

3. **Finiteness**. *For each $t \in \mathbb{R}$ and for each $A, A' \in \mathcal{B}$, there exist $t', t'' \in \mathbb{R}$ such that $(A', t')\ R_i\ (A, t)$ and $(A, t)\ R_i\ (A', t'')$.*

The monotonicity conditions mentioned above are quite natural. The continuity and finiteness are technical conditions needed to ensure nice structure of the indifference vectors. A quasilinear preference is always classical, where *indifference vectors* are "parallel". Notice that the monotonicity condition requires a free-disposal property: at a fixed transfer level, every bundle is weakly preferred to every other bundle which is a subset of it. All our results continue to hold even if we relax this free-disposal property to require that at a fixed transfer level, every bundle be weakly preferred to the empty bundle *only*.

Given a classical preference R_i, the **willingness to pay (WP)** of agent i at t for bundle A is defined as solution x to the following equation:

$$(A, t + x)\ I_i\ (\emptyset, t).$$

We denote this solution as $WP(A, t; R_i)$. The following fact is immediate from monotonicity, continuity, and finiteness - a proof can be found in Kazumura et al. (2017).

FACT **1** *For every classical preference R_i, for every $A \in \mathcal{B}$ and for every $t \in \mathbb{R}$, $WP(A, t; R_i)$ is a non-negative real number.*

For quasilinear preference, $WP(A, t; R_i)$ is independent of t and represents the valuation for bundle A.

Another way to represent a classical preference is by a collection of indifference vectors. Fix a classical preference R_i. Then, by definition, for every $t \in \mathbb{R}$ and for every $A \in \mathcal{B}$, agent i with classical preference R_i will be indifferent between the following outcomes:

$$(\emptyset, t)\ I_i\ (A, t + WP(A, t; R_i)).$$

Figure 1 shows a representation of classical preference for three objects $\{a, b, c\}$. The horizontal lines correspond to transfer levels for each of the bundles. Hence, these lines are the set of all outcomes Z - the space between these eight lines have no meaning and are kept only for ease of illustration. As we go to the right along any of these lines, the outcomes become worse since the transfer (payment made by the agent) increases. Figure 1 shows eight points, each corresponding to a unique bundle and a transfer level for that bundle. These points are joined to show that the agent is indifferent between these outcomes for a classical preference. Classical preference implies that all the points to the left of this indifference vector are better than these outcomes and all the points to the right of this indifference vector are worse than these outcomes. Indeed, every classical preference can be represented by a collection of an infinite number of such indifference vectors.

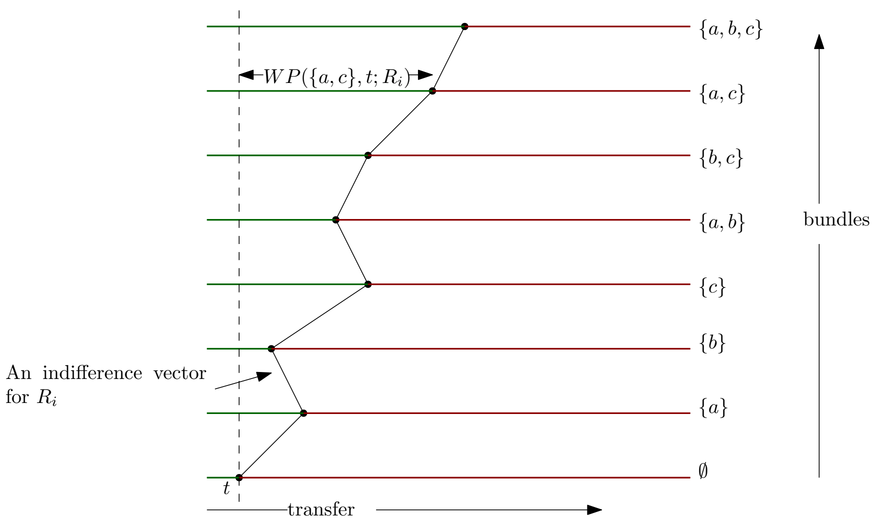


Figure 1: Representation of classical preferences

2.2 Domains and mechanisms

A *bundle allocation* is an ordered sequence of objects $(A_1, \ldots, A_n)$ such that for each $A_i, A_j \in \mathcal{B}$, we have $A_i \cap A_j = \emptyset$ - note that A_i can be equal to $\emptyset$ for any i in an object allocation. Let $\mathcal{X}$ denote the set of all bundle allocations.

An outcome profile $((A_1, t_1), \ldots, (A_n, t_n))$ is a collection of n outcomes such that $(A_1, \ldots, A_n)$ is the bundle allocation and t_i denotes the payment made by agent i. An outcome profile $((A_1, t_1), \ldots, (A_n, t_n))$ is **Pareto efficient** at $R \equiv (R_1, \ldots, R_n)$, if there does not exist another outcome profile $((A'_1, t'_1), \ldots, (A'_n, t'_n))$ such that

1. for each $i \in N, (A'_i, t'_i)\ R_i\ (A_i, t_i)$,

2. $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$,

with one of the inequalities strictly satisfied. The first relation says that each agent i prefers (A'_i, t'_i) to (A_i, t_i). The second relation requires that the seller is not spending money to make everyone better off - without this, we can always improve any outcome profile by subsidizing the agents.

A **domain or type space** is any subset of classical preferences. A typical domain of preferences will be denoted by $\mathcal{T}$. A **mechanism** is a pair $(f, \mathbf{p})$, where $f : \mathcal{T}^n \to \mathcal{X}$ and $\mathbf{p} \equiv (p_1, \ldots, p_n)$ with each $p_i : \mathcal{T}^n \to \mathbb{R}$. Here, f is the bundle allocation rule and p_i is the transfer rule of agent i. We denote the bundle allocated to agent i at type profile R by $f_i(R)$ in the bundle allocation rule f.

We require the following properties from a mechanism, which we term desirable.

DEFINITION **2 (Desirable mechanisms)** *A mechanism* $(f, \mathbf{p})$ *is* **desirable** *if*

1. *it is* **dominant strategy incentive compatible (DSIC)***: for all* $i \in N$*, for all* $R_{-i} \in \mathcal{T}^{n-1}$*, and for all* $R_i, R'_i \in \mathcal{T}$*, we have*

$$\Big(f_i(R), p_i(R)\Big) \; R_i \; \Big(f_i(R'_i, R_{-i}), p_i(R'_i, R_{-i})\Big).$$

2. *it is* **Pareto efficient***:* $\Big((f_1(R), p_1(R)), \ldots, (f_n(R), p_n(R))\Big)$ *is Pareto efficient at* R *for all* $R \in \mathcal{T}^n$*.*

3. *it is* **individually rational***: for all* $R \in \mathcal{T}^n$ *and for all* $i \in N$*,*

$$\Big(f_i(R), p_i(R)\Big) \; R_i \; (\emptyset, 0).$$

4. *satisfies* **no subsidy***: for all* $R \in \mathcal{T}^n$ *and for all* $i \in N$

$$p_i(R) \geq 0.$$

We will explore domains where a desirable mechanism exists.

2.3 A motivating example

We give an example to illustrate why existence of a desirable mechanism is difficult with classical preferences in our model. The example illustrates the intuition for some of our results. For this example, we will consider a simple setting with three agents $N = \{1, 2, 3\}$ and two objects $M = \{a, b\}$. We will investigate a particular preference profile in this setting. In this preference profile, agents 2 and 3 have identical preference: $R_2 = R_3 = R_0$. In particular, R_0 will satisfy

$$WP(\{a,b\}, t; R_0) = WP(\{a\}, t; R_0) = WP(\{b\}, t; R_0) = 2 + 3t,$$

for $t > -\frac{1}{2}$. We are silent about the willingness to pay below $-\frac{1}{2}$, but it can be taken to be 0.5. We will only consider transfers $t > -\frac{1}{2}$ for this example. In other words, an agent with preference R_0 treats all non-empty bundles the same way. Further,

$$(\{a,b\}, 2+4t)\ I_0\ (\{b\}, 2+4t)\ I_0\ (\{a\}, 2+4t)\ I_0\ (\emptyset, t).$$

Hence, as t increases, bundle $\{a\}$ (or $\{b\}$ or $\{a,b\}$) will require more transfer to be indifferent to $(\emptyset, t)$. We term this *negative income effect.*

Agent 1 has *quasilinear* preference with a value of 3.9 for bundle $\{a,b\}$; value zero (or, arbitrarily close to zero) for bundle $\{a\}$ and bundle $\{b\}$, and value of bundle $\emptyset$ is normalized to zero. We denote this preference as R_1.

Suppose $(f, \mathbf{p})$ is a desirable mechanism defined on a (rich enough) type space $\mathcal{T}^n$ containing $R \equiv (R_1, R_2 = R_0, R_3 = R_0)$. Notice that value of $\{a,b\}$ for agent 1 is 3.9 but $WP(\{a\}, 0; R_2) + WP(\{b\}, 0; R_3) = 4$. Hence, a consequence of Pareto efficiency, individual rationality, and no subsidy is that $f_1(R) = \emptyset$ - individual rationality and no subsidy imply that agents who are not allocated any object pay zero, and hence, any outcome where agent 1 is given both the objects can be Pareto improved. Then, without loss of generality, agent 2 gets bundle $\{a\}$ and agent 3 gets bundle $\{b\}$ - this follows from Pareto efficiency.

Next, we can pin down the payments of agents at R. Since agent 1 gets $\emptyset$, her payment must be zero by IR and no subsidy. Now, pretend as if agents 2 and 3 have quasilinear preference with valuations equal to their willingness to pay at zero transfer. Then, the VCG mechanism would charge them their externalities, which is equal to 1.9 for both the agents. If the type space $\mathcal{T}$ is sufficiently rich (in a sense, we make precise later), DSIC will still require that $p_2(R) = p_3(R) = 1.9$ - the precise argument is flushed out in the proof of Theorem 1.

But this is a problem since the following outcome vector Pareto dominates the outcome of the mechanism at R:

$$z_1 := (\{a,b\}, 3.9), \quad z_2 := (\emptyset, -0.025), \quad z_3 := (\emptyset, -0.025).$$

To see why, note that (a) sum of transfers in z is $3.85 > p_2(R) + p_3(R) = 3.8$; (b) agent 1 is indifferent between z_1 and $(\emptyset, 0)$; (c) agents 2 and 3 are also indifferent between their outcomes in the mechanism and z since $(\emptyset, -0.025)\ I_0\ (\{a\}, 1.9)$ (because $WP(\{a\}, t; R_0) = 2 + 3t$ for all t).

If the preference R_0 was quasilinear with values on all non-empty bundles equal to 2, then agents 2 and 3 would have strictly preferred $(\{a\}, 1.9)$ (or $(\{b\}, 1, 9)$ outcome) to $(\emptyset, -0.025)$. The negative income effect of R_0 destroys this property, and we get the desired impossibility. It is important to note that R_1 having high value on $\{a, b\}$ and (almost) zero value on all other bundles played a crucial role in determining payments of agents, and hence, in the impossibility. Indeed, the preference R_0 is called a *unit demand* preference in the literature (since agents are only interested singleton bundles), and it is well known that if the domain contains *only* unit demand preferences, a desirable mechanism exists (Demange and Gale, 1985). This motivates our study of preferences of the form R_1, which we call dichotomous preference, and the implications of various income effects.

2.4 Dichotomous preferences

We turn our focus on a subset of classical preferences which we call dichotomous. The dichotomous preferences can be described by: (a) a collection of bundles, which we call the *acceptable* bundles, and (b) a willingness to pay function, which only depends on the transfer level. Formally, it is defined as follows.

DEFINITION **3** *A classical preference R_i of agent i is* **dichotomous** *if there exists a non-empty set of bundles $\emptyset \neq \mathcal{S}_i \subseteq (\mathcal{B} \setminus \{\emptyset\})$ and a willingness to pay (WP) map $w_i : \mathbb{R} \to \mathbb{R}_{++}$ such that for every $t \in \mathbb{R}$,*

$$WP(A, t; R_i) = \begin{cases} w_i(t) & \forall\ A \in \mathcal{S}_i \\ 0 & \forall\ A \in \mathcal{B} \setminus \mathcal{S}_i. \end{cases}$$

In this case, we refer to $\mathcal{S}_i$ as the collection of **acceptable** *bundles.*

The interpretation of the dichotomous preference is that, given same price (transfer) for all the bundles, the agent is indifferent between the bundles in $\mathcal{S}_i$. Similarly, it is indifferent between the bundles in $\mathcal{B} \setminus \mathcal{S}_i$, but it strictly prefers a bundle in $\mathcal{S}_i$ to a bundle outside it. Hence, a dichotomous preference can be succinctly represented by a pair $(w_i, \mathcal{S}_i)$, where $w_i : \mathbb{R} \to \mathbb{R}_{++}$ is a WP map and $\emptyset \neq \mathcal{S}_i \subseteq (\mathcal{B} \setminus \{\emptyset\})$ is the set of acceptable bundles.

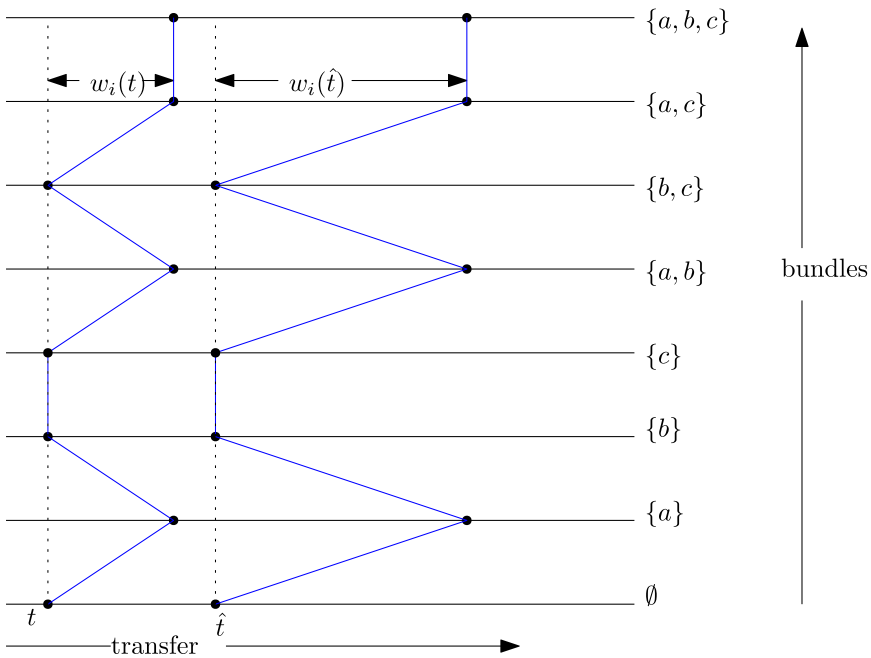
By our monotonicity requirement (free-disposal) of classical preference, for every $S, T \in \mathcal{B}$, we have

$$\Big[S \subseteq T, S \in \mathcal{S}_i\Big] \Rightarrow \Big[T \in \mathcal{S}_i\Big].$$

Hence, a dichotomous preference can be described by w_i and a *minimal* set of bundles $\mathcal{S}_i^{min}$ such that

$$\mathcal{S}_i := \{T \in \mathcal{B} : S \subseteq T \text{ for some } S \in \mathcal{S}_i^{min}\}.$$

Figure 2 shows a dichotomous preference - it only shows two indifference vectors of this preference. The figure makes it clear that the bundles $\{a\}, \{c\}, \{a, c\}, \{a, b\}$ and $\{a, b, c\}$ are acceptable but others are not.



Two indifference vectors corresponding to a dichotomous classical preference

Acceptable bundles: $\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}$.

Figure 2: A dichotomous preference

We will denote the domain of **all** dichotomous preferences as $\mathcal{D}$ - each preference in $\mathcal{D}$ for agent i is described by a w_i map and a collection of minimal bundles $\mathcal{S}_i^{min}$. A **dichotomous domain** is any subset of dichotomous preferences.

3 The results

We describe our main results in this section.

3.1 An impossibility result

We start with our main negative result: if the domain consists of *all* dichotomous preferences, then there is no desirable mechanism. This generalizes the intuition we demonstrated in the example in Section 2.3.

Theorem 1 **(Impossibility)** *Suppose $\mathcal{T} \supseteq \mathcal{D}$ (i.e., the domain contains* all *dichotomous preferences), $n \geq 3$, and $m \geq 2$. Then, no desirable mechanism can be defined on $\mathcal{T}^n$.*

As discussed in the introduction, Theorem 1 adds to a small list of papers that have established such negative results in other combinatorial auction problems. Notice that the domain $\mathcal{T}$ may contain preferences that are not dichotomous or it may be equal to $\mathcal{D}$, the set of all dichotomous preferences. We make two remarks about the result.

Remark. The impossibility result in Theorem 1 is slightly stronger than stated. The proof, given in the Appendix, reveals that not *all* dichotomous preferences need to be present in the domain. The proof only uses existence of preferences where each agent's *minimal* acceptable bundle ($\mathcal{S}^{min}$) contains a unique bundle - such preferences are called *single-minded* preferences in the algorithmic game theory literature (Babaioff et al., 2005, 2006; Lehmann et al., 2002). This means that as long as the domain contains all single-minded preferences the impossibility in Theorem 1 holds.

Remark. The conditions $m \geq 2$ and $n \geq 3$ are both necessary: if $m = 1$, we know that a desirable mechanism exists (Saitoh and Serizawa, 2008); if $n = 2$, the mechanism that we propose next is desirable.

Definition 4 *The* **generalized Vickrey-Clarke-Groves (GVCG) mechanism**, *denoted as $(f^{vcg}, \mathbf{p}^{vcg})$, is defined as follows: for every profile of preferences R,*

$$f^{vcg}(R) \in \arg\max_{A \in \mathcal{X}} \sum_{i \in N} WP(A_i, 0; R_i)$$

$$p_i^{vcg}(R) = \max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, 0; R_j) - \sum_{j \neq i} WP(f_j^{vcg}(R), 0; R_j).$$

The GVCG is a natural generalization of the VCG mechanism to our setting without quasilinearity - in fact, the current definition does not use anything about dichotomous preferences. Theorem 1 implies that the GVCG mechanism is not desirable. We show that the GVCG mechanism is DSIC and individually rational in *any* dichotomous preference domain. This shows the limits of the impossibility result in Theorem 1.

PROPOSITION **1** *The GVCG mechanism is DSIC, individually rational, and satisfies no subsidy on any dichotomous domain.*

3.2 Positive income effect and possibility

Proposition 1 and Theorem 1 point out that the GVCG is not Pareto efficient in the entire dichotomous domain. A closer look at the proof of Theorem 1 (and the example in Section 2.3) reveals that the impossibility is driven by a particular kind of dichotomous preferences: the ones where the willingness to pay of an agent increases with payment. We term such preferences *negative income effect* - a restatement of Theorem 1 will say that as long as the domain of preferences contain the entire set of negative income effect preferences, no desirable mechanism can exist.

A standard definition of positive income effect will say that as income rises, a preferred bundle becomes "more preferred". We do not model income explicitly, but our preferences implicitly account for income. So, if payment decreases from t to t', the income level of the agent increases implicitly. As a result, she is willing to pay more for his acceptable bundles at t' than at t. Thus, positive income effect captures a reasonable (and standard) restriction on preferences of the agents.

DEFINITION **5** *A dichotomous preference $R_i \equiv (w_i, \mathcal{S}_i)$ satisfies* **positive income effect** *if for all $t > t'$, we have $w_i(t) \leq w_i(t')$.*

A dichotomous domain of preferences $\mathcal{T}$ satisfies positive income effect if every preference in $\mathcal{T}$ satisfies positive income effect.

As an illustration, the indifference vectors shown in Figure 2 cannot be part of a dichotomous preference satisfying positive income effect - we see that $\hat{t} > t$ but $w_i(\hat{t}) > w_i(t)$. The example in Section 2.3 also violated positive income effect. A quasilinear preference (where $w_i(t) = w_i(t')$ for all t, t') always satisfies positive income effect, and the GVCG mechanism is known to be a desirable mechanism in this domain. Our next result says that the impossibility in Theorem 1 is overturned in any domain of dichotomous preferences satisfying positive income effect.

THEOREM **2 (Possibility)** *The GVCG mechanism is desirable on any dichotomous domain satisfying positive income effect.*

Theorem 2 can be interpreted to be a generalization of the well-known result that the VCG mechanism is desirable in the quasilinear domain. Indeed, we know that if the domain of preferences is the set of *all* quasilinear preferences, then standard revenue equivalence result (which holds in quasilinear domains) implies that the VCG mechanism is the *only* desirable mechanism. Though we do not have a revenue equivalence result, we show below a similar uniqueness result of the GVCG mechanism. For this, we first remind ourselves the definition of a quasilinear preference. A dichotomous preference $(w_i, \mathcal{S}_i)$ is **quasilinear** if for every $t, t' \in \mathbb{R}$, we have $w_i(t) = w_i(t')$. We denote by $\mathcal{D}^{QL}$ the set of **all** quasilinear preferences which is a subset of $\mathcal{D}$ - in other words, $\mathcal{D}^{QL}$ is the set of all dichotomous quasilinear preferences. This leads to a characterization of the GVCG mechanism.

THEOREM **3 (Uniqueness)** *Suppose the domain of preferences $\mathcal{T}$ is a dichotomous domain satisfying positive income effect and contains $\mathcal{D}^{QL}$. Let $(f, \mathbf{p})$ be a mechanism defined on $\mathcal{T}^n$. Then, the following statements are equivalent.*

1. $(f, \mathbf{p})$ *is a desirable mechanism.*
2. $(f, \mathbf{p})$ *is the generalized VCG mechanism.*

4 Tightness of results

In this section, we investigate if the positive results in the previous sections continue to hold if the domain includes (positive income effect) non-dichotomous preferences. In particular, we investigate the consequences of adding a non-dichotomous preference satisfying (a) positive income effect and (b) decreasing marginal willingness to pay. Both these conditions are natural properties to impose on preferences. Our results below can be summarized as follows: if we take the set of *all* quasilinear dichotomous preferences and add *any* non-dichotomous preference satisfying the above two conditions, then no desirable mechanism can exist in such a type space. As corollaries, we uncover new type spaces where no desirable mechanism can exist with non-quasilinear preferences, and establish the role of dichotomous preferences in such type spaces.

4.1 Heterogenous objects

In this section, we consider a preference where an agent can demand multiple heterogeneous objects. We require that at least two objects are heterogeneous in the following sense.

DEFINITION **6** *A preference R_i satisfies* **heterogenous** *demand if there exists $a, b \in M$,*

$$WP(\{a\}, 0; R_0) \neq WP(\{b\}, 0; R_0).$$

Heterogeneous demand requires that for *some* pair of objects, the WP at 0 must be different for them. If objects are not the same (i.e., not homogeneous), then we should expect this condition to hold - see Section 4.2 for results relating to the homogeneous goods case.

Besides the heterogeneous demand, we will impose two natural conditions on preferences. The first condition is a mild form of substitutability condition.

DEFINITION **7** *A preference R_i satisfies* **weak decreasing marginal WP** *if for every $a, b \in M$,*

$$WP(\{a\}, 0; R_i) + WP(\{b\}, 0; R_i) > WP(\{a, b\}, 0; R_i).$$

Weak decreasing marginal WP requires a minimal degree of submodularity - the marginal increase in WP (at 0) by adding $\{a\}$ to $\{b\}$ is less than adding $\{a\}$ to $\emptyset$. Notice that we require this substitutability requirement *only* for bundles of size two. Hence, larger bundles may exhibit complementarity or substitutability. Because of free disposal, for every $a, b \in M$, we have

$$WP(\{a, b\}, 0; R_i) \geq \max(WP(\{a\}, 0; R_i), WP(\{b\}, 0; R_i)).$$

Hence, weak decreasing marginal WP implies that $WP(\{a\}, 0; R_i) > 0$ and $WP(\{b\}, 0; R_i)) > 0$, i.e., each object is *desirable* in a weak sense (getting an object in M is preferred to getting nothing at transfer 0).

We point out that unit demand preferences (studied in (Demange and Gale, 1985; Morimoto and Serizawa, 2015)) satisfy weak decreasing marginal WP. A preference R_i is called a **unit demand** preference if for every S,

$$WP(S, t; R_i) = \max_{a \in S} WP(\{a\}, t; R_i) \ \forall \ t \in \mathbb{R}_+.$$

If R_i is a unit demand preference and objects are *desirable*, then it satisfies weak decreasing marginal WP. To see this, call every object $a \in M$ **desirable** if $WP(\{a\}, 0; R_i) > 0$ at every R_i. If objects are desirable, then for every $a, b \in M$, we see that

$$WP(\{a\}, 0; R_i) + WP(\{b\}, 0; R_i) > \max_{x \in \{a,b\}} WP(\{x\}, 0; R_i) = WP(\{a, b\}, 0; R_i).$$

Besides the weak decreasing marginal WP condition, we will also be requiring strict positive income effect, but *only* for singleton bundles.

Definition 8 *A classical preference R_i satisfies* **strict positive income effect** *if for every $a, b \in M$ and for every t, t' with $t' > t$, the following holds for every $\delta > 0$:*

$$\Big[(\{b\}, t')\ I_i\ (\{a\}, t)\Big] \ \Rightarrow\ \Big[(\{b\}, t' - \delta)\ P_i\ (\{a\}, t - \delta)\Big].$$

This definition of strict positive income effect requires that if two objects are indifferent then decreasing their prices by the same amount makes the higher priced object better. This is a generalization of the definition of positive income effect we had introduced for dichotomous preferences in Definition 5, but only restricted to singleton bundles. [5] This means that for larger bundles, we do not require positive income effect to hold.

We are ready to state the main tightness result with heterogeneous objects.

Theorem 4 *Suppose $n \geq 4, m \geq 2$. Let R_0 be a heterogeneous demand preference satisfying strict positive income effect and weak decreasing marginal WP. Consider any domain $\mathcal{T}$ containing $\mathcal{D}^{QL} \cup \{R_0\}$. Then, no desirable mechanism can be defined on $\mathcal{T}^n$.*

Unlike the negative result in Theorem 1, Theorem 4 does not require existence of negative income effect dichotomous preferences. It just requires presence of quasilinear dichotomous preferences along with at least one heterogeneous demand preference satisfying some reasonable conditions. This negative result parallels a result of Kazumura and Serizawa (2016) who show that adding *any* multi-demand preference to a class of *rich* unit demand preference gives rise to a similar impossibility. While they show impossibility with multiple object demand preferences, our impossibility is driven by existence of dichotomous preferences. We now spell out an exact implication of Theorem 4 in a corollary below.

Let $\mathcal{D}^+$ be the set of all *positive income effect* dichotomous preferences (note that $\mathcal{D}^{QL} \subsetneq \mathcal{D}^+$) and $\mathcal{U}^+$ be the set of all heterogeneous unit demand preferences satisfying positive income effect (as argued earlier, unit demand preferences satisfy weak decreasing marginal WP). Then, the following corollary is immediate from Theorem 4.

Corollary 1 *Suppose $\mathcal{T} = \mathcal{D}^+ \cup \mathcal{U}^+$. Then, no desirable mechanism can be defined on $\mathcal{T}^n$.*

[5] An alternate definition along the lines of Definition 5 using willingness to pay map is also possible.

Theorem 3 shows that the GVCG mechanism is the unique desirable mechanism on $\mathcal{D}^+$. Similarly, Demange and Gale (1985) have shown that a desirable mechanism exists in $\mathcal{U}^+$ - this mechanism is called the *minimum Walrasian equilibrium price mechanism* and collapses to the VCG mechanism if preferences are quasilinear. Corollary 1 says that we lose these possibility results if we consider the unions of these two type spaces.

4.2 Homogeneous objects

In this section, we will assume that objects offered for sale are *homogeneous*. If objects are homogeneous, then at every preference R_i, for any $S, T \in \mathcal{B}$ with $|S| = |T|$, we have $(S, t)\ I_i\ (T, t)\ \forall\ t \in \mathbb{R}$. So, only the size of the bundle matters in preferences. For sake of compact notation, instead of calling elements of M objects, we will refer to them as m units (of a *single* object). Hence, set of outcomes for any agent is $\{0, 1, \ldots, m\} \times \mathbb{R}$. A consumption bundle is a pair (k, t), where $k \in \{0, 1, \ldots, m\}$ is the number of units of the object and t is the transfer amount. We call this model the **multi-unit** model.

Willingness to pay definition can also be adapted. Now, $WP(k, t; R_i)$ is the willingness to pay of agent i with preference R_i for k units at transfer t, i.e., $WP(k, t; R_i)$ is the transfer amount that makes the agent indifferent between outcome $(k, WP(k, t; R_i) + t)$ and $(0, t)$ at preference R_i. A mechanism $(f, \mathbf{p})$ will assign the number of units and a transfer amount to each agent, and hence, $f_i(R)$ will denote the number of units assigned to agent i at preference profile R.

A dichotomous preference R_i will be characterized by $\mathcal{S}_i^{min}$, which will be completely characterized by a single positive number indicating the number of acceptable units to agent i, and a value for those many units. Notice that our earlier results on dichotomous preferences continue to hold with homogeneous objects.

We will need a preference which has multi-unit demand in the following sense.

Definition 9 *A preference R_i in the multi-unit model is a* **multi-unit demand** *preference if for every $j \in \{0, 1, \ldots, m-1\}$ and for every $t \in \mathbb{R}$,*

$$(j + 1, t)\ P_i\ (j, t).$$

Next, just like in the heterogeneous demand model, we are going to impose strict positive income effect, but on arbitrary number of units (recall, the positive income effect definition earlier only required the relation to hold for singleton bundles).

Definition **10** *A preference R_i in the multi-unit model satisfies* **strict positive income effect** *if for every $j, k \in \{0, 1, \ldots, m\}$ and every $t, t' \in \mathbb{R}$ with $t' > t$ and $(j, t')\ I_i\ (k, t)$, we have*

$$(j, t' - \delta)\ P_i\ (k, t - \delta)\ \forall\ \delta \in \mathbb{R}_{++}.$$

Finally, we are going to impose a mild restriction on willingness to pay at 0 transfer. It requires that the WP at 0 for 1 unit is greater than the difference between WP at 0 for 3 units and 2 units.

Definition **11** *A preference R_i in the multi-unit model satisfies* **weak decreasing marginal WP** *if*

$$WP(1, 0; R_i) + WP(2, 0; R_i) > WP(3, 0; R_i).$$

A more general version of this requirement would be to have $WP(j+1, t; R_i) - WP(j, t; R_i) > WP(k + 1, t; R_i) - WP(k, t; R_i)$ for all t and for all j, k with $j < k$. Our weak decreasing marginal WP condition is only for $j = 1, k = 2$, and $t = 0$. With these conditions, we are now ready to state the main result of this section. Let $\mathcal{D}_H^{QL}$ denote the set of all quasilinear dichotomous preferences in the homogeneous objects model.

Theorem **5** *Suppose objects are homogeneous and $n \geq 4, m \geq 3$. Let R_0 be a multi-unit demand preference satisfying strict positive income effect and weak decreasing marginal WP. Consider any domain $\mathcal{T}$ containing $\mathcal{D}_H^{QL} \cup \{R_0\}$. Then, no desirable mechanism can be defined on $\mathcal{T}^n$.*

The proof of Theorem 5 is almost identical to the proof of Theorem 4 and is given in the Supplementary Appendix. The main idea in the proof of Theorem 4 was to have two agents with identical preferences but they were given two different (heterogeneous according to their preferences) objects. However, the payments could not be constructed to make incentive constraints and efficiency compatible. The proof of Theorem 5 follows a similar approach. However, to assign two "different objects" in the homogeneous object case, we need to assign them different number of units - in particular, one agent 1 unit and the other one 2 units. This means we need at least three units for the proof to work. We do not know if a desirable mechanism can always be constructed for $m = 2$.

Unlike Theorem 4, Theorem 5 does not hold if preference R_0 satisfies unit demand (the proof requires multi-unit demand). Indeed, unit demand preferences in the homogeneous good case is a dichotomous preference (with 1 unit as acceptable bundle). Hence, the GVCG mechanism remains desirable in $\mathcal{D}_H^{QL} \cup \{R_0\}$ if R_0 satisfies unit demand.

Just like Theorem 4, Theorem 5 applies to many type spaces in the homogeneous objects model. In particular, consider the set of *all* multi-unit demand preference satisfying positive income effect with weak decreasing marginal WP, and denote it as $\mathcal{H}^+$. Similarly, let $\mathcal{D}_H^+$ denote the set of all dichotomous preferences satisfying positive income effect in the homogeneous objects model. Then, a corollary of Theorem 5 is the following.

Corollary **2** *No desirable mechanism exists in* $\mathcal{T}^n$ *if* $\mathcal{T} = \mathcal{H}^+ \cup \mathcal{D}_H^+$.

Theorem 5 parallels impossibility results in Baisa (2016), who established two impossibility results in the homogeneous objects model. Baisa (2016) assumes that preferences are formed by a parameter θ_i, which is the *type* of agent i. Depending on whether θ_i is "single-dimensional" or "multi-dimensional", Baisa (2016) shows incompatibility of DSIC, Pareto efficiency, individual rationality with some other properties if the type space contains enough variety of homogeneous multi-unit demand preferences. Hence, Baisa (2016) requires certain richness in the set of homogeneous multi-unit demand preference, whereas our result requires inclusion of quasilinear dichotomous preferences and just *one* multi-unit demand preference satisfying strict positive income effect and weak decreasing marginal WP.

5 Related Literature

The quasilinearity assumption is at the heart of mechanism design literature with transfers. Our formulation of classical preferences was studied in the context of single object auction by Saitoh and Serizawa (2008), who proposed the generalized VCG mechanism and axiomatized it for that setting - also see axiomatizations in Sakai (2008, 2013). As discussed, Demange and Gale (1985) had shown that a mechanism different from the generalized VCG mechanism is desirable when multiple heterogeneous objects are sold to agents with unit demand - see characterizations of this mechanism in Morimoto and Serizawa (2015), Zhou and Serizawa (2018) and Kazumura et al. (2018). However, impossibility results for the existence of a desirable mechanism were shown (a) by Kazumura and Serizawa (2016) for multi-object auctions with multi-demand agents and (b) by Baisa (2017) for multiple homogeneous object model with multi-demand agents. Baisa (2016) considers non-quasilinear preferences with randomization in a single object auction environment. He proposes a randomized mechanism and establishes strategic properties of this mechanism. Dastidar (2015) considers a model where agents have same utility function but models income explicitly to allow for different incomes. He considers equilibria of standard auctions. Social choice problems with transfers are studied with particular form of non-quasilinear preferences in Ma et al. (2016,

2018) - these papers establish dictatorship results in this setting with non-quasilinear preferences. Samuelson and Noldeke (2018) discuss an implementation duality without quasilinear preferences and apply it to matching and adverse selection problems.

A growing literature in mechanism design with quasilinear preferences studies problems where agents are budget constrained. The modeling of budget constraint is such that if an agent has to pay more than budget, then his utility is minus infinity - this introduces non-quasilinear utility functions but it does not fit our model because of the hard budget constraint. For the single object auction with such budget-constrained agents, Lavi and May (2012) establish that no desirable mechanism can exist - see a multi-unit extension of this result in Dobzinski et al. (2012). For combinatorial auctions with a particular kind of dichotomous (called single-minded agents) and quasilinear preferences, Le (2018) shows that these impossibilities with budget-constrained agents can be overcome in a *generic* sense - he defines a "truncated" VCG mechanism and shows that it is desirable *almost everywhere*.

There is a literature in algorithmic mechanism design on combinatorial auctions with quasilinear but a particular dichotomous preference - this literature terms such dichotomous preferences "single-minded" preferences, where there is a unique bundle such that all supersets of that bundle constitute the acceptable bundles. Apart from practical significance, the problem is of interest because computing a VCG outcome in this problem is computationally challenging but various "approximately" desirable mechanisms can be constructed (Babaioff et al., 2005, 2006; Lehmann et al., 2002). Rastegari et al. (2011) show that in this model, the revenue from the VCG mechanism (and any DSIC mechanism) may not satisfy monotonicity, i.e., adding an agent may *decrease* revenue. Our paper adds to this literature by illustrating the implications of non-quasilinear preferences.

6 Conclusion

Our results highlight some issues with general combinatorial auctions with and without the dichotomous restrictions. Theorem 1 suggests that in a general combinatorial auction model, if the domain consists of *all* dichotomous classical preferences, then no desirable mechanism exists. Further, if we only allow for positive income effect preferences, and consider a combinatorial auction domain that includes *all* quasilinear dichotomous preferences, then restriction of any desirable mechanism to the dichotomous domain must be the generalized VCG mechanism - this follows from Theorem 3. A natural future research question is to characterize the positive income effect combinatorial auction domains where a desirable mechanism may exist. Our tightness results in Theorem 4 and Theorem 5 point that such domains must necessarily exclude dichotomous preferences.

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A Proofs

A.1 Proof of Theorem 1

Proof: We start by providing two useful lemmas.

Lemma **1** *Suppose $(f, \mathbf{p})$ is an individually rational mechanism satisfying no subsidy. Then for every agent $i \in N$ and every $R \in \mathcal{T}^n$, we have $p_i(R) = 0$ if $f_i(R) \notin \mathcal{S}_i$.*

Proof: Suppose R is a profile such that $f_i(R) \notin \mathcal{S}_i$ for agent i. By individual rationality, $(f_i(R), p_i(R))\ R_i\ (\emptyset, 0)$. But $f_i(R) \notin \mathcal{S}_i$ implies that $(\emptyset, p_i(R))\ I_i\ (f_i(R), p_i(R))\ R_i\ (\emptyset, 0)$. Hence, $p_i(R) \leq 0$. But no subsidy implies that $p_i(R) = 0$. ■

Lemma **2** *Suppose $(f, \mathbf{p})$ is an individually rational mechanism satisfying no subsidy. Then for every agent $i \in N$ and every $R \in \mathcal{T}^n$, we have $0 \leq p_i(R) \leq WP(f_i(R), 0; R_i)$.*

Proof: If $f_i(R) \notin \mathcal{S}_i$, then the claim follows from Lemma 1. Suppose $f_i(R) \in \mathcal{S}_i$. By individual rationality, $(f_i(R), p_i(R))\ R_i\ (\emptyset, 0)\ I_i\ (f_i(R), WP(f_i(R), 0; R_i))$. This implies that $p_i(R) \leq WP(f_i(R), 0; R_i)$. No subsidy implies that $p_i(R) \geq 0$. ■

Consider any three non-empty bundles S, S_1, S_2 such that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Consider a profile of dichotomous preferences $R^* \in \mathcal{D}^n$ as follows. Since all the agents have dichotomous preferences, to describe any agent i's preference, we describe the *minimal* acceptable bundles $\mathcal{S}_i^{min}$ (i.e., the set of acceptable bundles $\mathcal{S}_i$ are derived by taking supersets of each element in $\mathcal{S}_i^{min}$) and the willingness to pay map w_i. Preference R_1^* of agent 1 is quasilinear:

$$\mathcal{S}_1^{min} = \{S\}, w_1(t) = 3.9\ \forall\ t \in \mathbb{R}.$$

Preference R_2^* of agent 2 is:

$$\mathcal{S}_2^{min} = \{S_1\}, w_2(t) = (2-t) - ((2-t)^3 - 8)^{\frac{1}{3}}\ \forall\ t \in \mathbb{R}.$$

Preference R_3^* of agent 3 is:

$$\mathcal{S}_3^{min} = \{S_2\}, w_3(t) = (2-t) - ((2-t)^3 - 8)^{\frac{1}{3}}\ \forall\ t \in \mathbb{R}.$$

Note that a utility function representing such a preference is $u^*(S, t) = 8 + (2-t)^3$ if S is acceptable and $u^*(S, t) = (2-t)^3$ if S is not acceptable.

Preference R_i^* of each agent $i \notin \{1, 2, 3\}$ is quasilinear:

$$\mathcal{S}_i^{min} = \{S\}, w_i(t) = \epsilon\ \forall\ t \in \mathbb{R},$$

where $\epsilon > 0$ but very close to zero.

Assume for contradiction that there exists a DSIC, Pareto efficient, individually rational mechanism $(f, \mathbf{p})$ satisfying no subsidy. We now do the proof in several steps.

STEP 1. In this step, we show that at every preference profile R with $R_i = R_i^*$ for all $i \notin \{2, 3\}$, we must have $S \not\subseteq f_i(R)$ if $i \notin \{1, 2, 3\}$. We know that $\mathcal{S}_i^{min} = \{S\}$ for all $i \notin \{2, 3\}$. Assume for contradiction $S \subseteq f_k(R)$ for some $k \notin \{1, 2, 3\}$. Then, $S \not\subseteq f_1(R)$. By Lemma 1, $p_1(R) = 0$. Consider the following outcome:

$$Z_1 = (S, \epsilon), Z_k = (\emptyset, p_k(R) - \epsilon), Z_j = (f_j(R), p_j(R)) \; \forall \; j \notin \{1, k\}.$$

Since preferences of agent 1 and agent k are quasilinear (note that $R_1 = R_1^*$ and $R_k = R_k^*$) and ϵ is very close to zero, we have

$$Z_1 \; P_1 \; (f_1(R), p_1(R) = 0), Z_k \; I_k \; (f_k(R), p_k(R)), Z_j \; I_j \; (f_j(R), p_j(R)) \; \forall \; j \notin \{1, k\}.$$

Also, the sum of transfers in the outcome vector $Z \equiv (Z_1, \ldots, Z_n)$ is $\sum_{i \in N} p_i(R)$. This contradicts Pareto efficiency of $(f, \mathbf{p})$.

STEP 2. Fix a preference $\hat{R}_2$ of agent 2 such that $\hat{\mathcal{S}}_2^{min} = \{S_1\}$ and $\hat{w}_2(0) > 1.9$. We show that at preference profile $\hat{R} = (\hat{R}_2, R_{-2}^*)$, $S \not\subseteq f_1(\hat{R})$. Suppose $S \subseteq f_1(\hat{R})$. Then, $S_1 \not\subseteq f_2(\hat{R})$ and $S_2 \not\subseteq f_3(\hat{R})$. By Lemma 1, $p_2(\hat{R}) = 0, p_3(\hat{R}) = 0$. Consider a new outcome vector:

$$Z_1 = (\emptyset, p_1(\hat{R}) - 3.9), Z_2 = (S_1, \hat{w}_2(0)), Z_3 = (S_2, w_3(0)), Z_j = (f_j(\hat{R}), p_j(\hat{R})) \; \forall \; j \notin \{1, 2, 3\}.$$

By quasilinearity of R_1^*, we get $Z_1 \; I_1^* \; (f_1(\hat{R}), p_1(\hat{R}))$. By definition,

$$Z_2 \; \hat{I}_2 \; (\emptyset, 0) \; \hat{I}_2 \; (f_2(\hat{R}), p_2(\hat{R})).$$

Similarly, $Z_3 \; I_3^* \; (f_3(\hat{R}), p_3(\hat{R}))$. Further, the sum of transfers in the outcome vector Z is

$$p_1(\hat{R}) - 3.9 + \hat{w}_2(0) + w_3(0) + \sum_{j \notin \{1,2,3\}} p_j(\hat{R}) > \sum_{j \in N} p_j(\hat{R}),$$

where the inequality used the fact that $p_2(\hat{R}) = p_3(\hat{R}) = 0$ and $\hat{w}_2(0) > 1.9, w_3(0) = 2$. This contradicts Pareto efficiency of $(f, \mathbf{p})$.

STEP 3. Fix any quasilinear preference $\hat{R}_2$ of agent 2 such that $\hat{\mathcal{S}}_2^{min} = \{S_1\}$ and $\hat{w}_2(t) = 1.9 - \delta$, where $\delta \in (0, 1.9)$. We show that at preference profile $\hat{R} = (\hat{R}_2, R_{-2}^*)$, we must have

$S \subseteq f_1(\hat{R})$. If not, then by Step 1 and by Pareto efficiency, $S_1 \subseteq f_2(\hat{R})$ and $S_2 \subseteq f_3(\hat{R})$. Now, consider the following outcome Z':

$$Z'_1 = (S, 3.9),\ Z'_2 = (\emptyset, p_2(\hat{R}) - (1.9 - \frac{\delta}{2})),\ Z'_3 = (\emptyset, p_3(\hat{R}) - 2),$$

$$Z'_j = (f_j(\hat{R}), p_j(\hat{R}))\ \forall\ j \notin \{1, 2, 3\}.$$

Note that by Lemma 1, $p_1(\hat{R}) = 0$. Hence, using quasilinearity of R^*_1, we get $(f_1(\hat{R}), p_1(\hat{R}) = 0)\ I^*_1\ (S, 3.9)$. Similarly, by quasilinearity of $\hat{R}_2$, we get $Z'_2\ \hat{P}_2\ (f_2(\hat{R}), p_2(\hat{R}))$. Also, the sum of transfers in outcome Z' is

$$3.9 + p_2(\hat{R}) - (1.9 - \frac{\delta}{2}) + p_3(\hat{R}) - 2 + \sum_{j \notin \{1,2,3\}} p_j(\hat{R}) = \sum_{i \in N} p_i(\hat{R}) + \frac{\delta}{2} > \sum_{i \in N} p_i(\hat{R}),$$

where we used the fact that $p_1(\hat{R}) = 0$.

We now prove that $(\emptyset, p_3(\hat{R}) - 2)\ R^*_3\ (f_3(\hat{R}), p_3(\hat{R}))$. For this, let $t = p_3(\hat{R}) - 2$. By Lemma 2, we have $p_3(\hat{R}) \leq 2$. By no subsidy, $p_3(\hat{R}) \geq 0$. So, $2 - t = 4 - p_3(\hat{R}) \in [2, 4]$. Now, observe the following:

$$\begin{aligned} t + w_3(t) &= 2 - \big((2-t)^3 - 8\big)^{\frac{1}{3}} \\ &\leq 2 - \big((2-t) - 2\big) \\ &= 2 + t \\ &= p_3(\hat{R}), \end{aligned}$$

where the inequality used the fact that $(2 - t) \geq 2$ and $(2 - t)^3 - 2^3 \geq \big((2 - t) - 2\big)^3$.

Using this, we now observe that (still using $t := p_3(\hat{R}) - 2$ below),

$$(\emptyset, p_3(\hat{R}) - 2)\ I^*_3\ (f_3(\hat{R}), t + w_3(t))\ R^*_3\ (f_3(\hat{R}), p_3(\hat{R})).$$

Hence, we get a contradiction to Pareto efficiency.

Step 4. In this step, we show that at preference profile R^*,

$$S_1 \subseteq f_2(R^*), S_2 \subseteq f_3(R^*),$$

and

$$p_2(R^*) = p_3(R^*) = 1.9.$$

Since $w_2(0) = 2$ in preference R^*_2, by Step 2, $S \not\subseteq f_1(R^*)$. By Step 1, $S \not\subseteq f_i(R^*)$ for all $i \notin \{1, 2, 3\}$. By Pareto efficiency, it must be

$$S_1 \subseteq f_2(R^*), S_2 \subseteq f_3(R^*).$$

Now, assume for contradiction $p_2(R^*) > 1.9$. Fix a preference $\hat{R}_2$ of agent 2 such that $\hat{\mathcal{S}}_2^{min} = \{S_1\}$ and $p_2(R^*) > \hat{w}_2(0) > 1.9$. By Step 2, $S_1 \subseteq f_2(\hat{R}_2, R^*_{-2})$. By DSIC, $p_2(R^*) = p_2(\hat{R}_2, R^*_{-2})$. Hence, $p_2(\hat{R}_2, R^*_{-2}) > \hat{w}_2(0)$. This is a contradiction to Lemma 2.

Finally, assume for contradiction $p_2(R^*) < 1.9$. Then, consider any quasilinear preference $\hat{R}_2$ of agent 2 such that $\hat{\mathcal{S}}_2^{min} = \{S_1\}$ and $p_2(R^*) < \hat{w}_2(0) < 1.9$. By Step 3, $S_1 \nsubseteq f_2(\hat{R}_2, R^*_{-2})$ and by Lemma 1, $p_2(\hat{R}_2, R^*_{-2}) = 0$. But by reporting R^*_2, agent 2 gets S_1 at a transfer less than $\hat{w}_2(0)$. By quasilinearity of $\hat{R}_2$ and the fact that $S_1 \nsubseteq f_2(\hat{R}_2, R^*_{-2})$, she prefers this outcome to outcome $(f_2(\hat{R}_2, R^*_{-2}), 0)$, which is a contradiction to DSIC.

This concludes the proof that $p_2(R^*) = 1.9$. A similar argument establishes (with Steps 2 and 3 applied to agent 3) that $p_3(R^*) = 1.9$.

Step 5. We now complete the proof. By Step 4, we know that the outcome at preference profile R^* satisfies:

$$S \nsubseteq f_1(R^*), S_1 \subseteq f_2(R^*), S_2 \subseteq f_3(R^*),$$

$$p_1(R^*) = 0,\ p_2(R^*) = p_3(R^*) = 1.9.$$

Now, consider the following outcome: $Z'_j = (f_j(R^*), p_j(R^*))$ for all $j \notin \{1, 2, 3\}$ and

$$Z'_1 = (S, 3.9), Z'_2 = (\{\emptyset\}, -0.05),\ Z'_3 = (\{\emptyset\}, -0.05).$$

Notice that the sum of transfers in this outcome is $\sum_{i\in N} p_i(R^*)$. Agent 1 is indifferent between Z'_1 and $(f_1(R^*), p_1(R^*))$. For agents 2 and 3, verify that

$$(-0.05) + w_2(-0.05) = (-0.05) + w_3(-0.05) < 1.9.$$

Hence, for $i \in \{2, 3\}$, we have

$$Z'_i = (\emptyset, -0.05)\ I^*_i\ (f_i(R^*), w_i(-0.05) - 0.05)\ P^*_i\ (f_i(R^*), 1.9).$$

This contradicts Pareto efficiency, giving us a contradiction. ■

A.2 Proof of Proposition 1

Proof: Fix a dichotomous domain $\mathcal{T}$. We start with the proof of individual rationality. Fix agent i and a profile of preferences $R \in \mathcal{T}^n$. By the definition of maximum,

$$\max_{A\in\mathcal{X}} \sum_{j\in N} WP(A_j, 0; R_j) \geq \max_{A\in\mathcal{X}} \sum_{j\neq i} WP(A_j, 0; R_j)$$
$$\Rightarrow \sum_{j\in N} WP(f_j^{vcg}(R), 0; R_j) \geq \max_{A\in\mathcal{X}} \sum_{j\neq i} WP(A_j, 0; R_j)$$
$$\Rightarrow WP(f_i^{vcg}(R), 0; R_i) \geq \max_{A\in\mathcal{X}} \sum_{j\neq i} WP(A_j, 0; R_j) - \sum_{j\neq i} WP(f_j^{vcg}(R), 0; R_j) = p_i^{vcg}(R).$$

But this implies that

$$(f_i^{vcg}(R), p_i^{vcg}(R)) \; R_i \; (f_i^{vcg}(R), WP(f_i^{vcg}(R), 0; R_i)) \; I_i \; (\emptyset, 0).$$

This shows that the GVCG mechanism is individually rational.

No subsidy follows because,

$$p_i^{vcg}(R) = \max_{A\in\mathcal{X}} \sum_{j\neq i} WP(A_j, 0; R_j) - \sum_{j\neq i} WP(f_j^{vcg}(R), 0; R_j) \geq 0,$$

where the last inequality follows from the definition of maximum.

Now, we prove that the GVCG is DSIC. Fix agent $i \in N$, $R_{-i} \in \mathcal{T}^{n-1}$, and $R_i, R_i' \in \mathcal{T}$. Let $A \equiv f^{vcg}(R_i, R_{-i})$ and $A' \equiv f^{vcg}(R_i', R_{-i})$. We start with a simple lemma.

Lemma **3** *If A_i, A_i' belongs to the acceptable bundle set at R_i, then $p_i^{vcg}(R_i, R_{-i}) \leq p_i^{vcg}(R_i', R_{-i})$.*

Proof: Note that

$$\begin{aligned}
p_i^{vcg}(R_i, R_{-i}) - p_i^{vcg}(R_i', R_{-i}) &= \big[\max_{\hat{A}\in\mathcal{X}} \sum_{j\neq i} WP(\hat{A}_j, 0; R_j) - \sum_{j\neq i} WP(A_j, 0; R_j)\big] \\
&\quad - \big[\max_{\hat{A}\in\mathcal{X}} \sum_{j\neq i} WP(\hat{A}_j, 0; R_j) - \sum_{j\neq i} WP(A_j', 0; R_j).\big] \\
&= \sum_{j\neq i} WP(A_j', 0; R_j) - \sum_{j\neq i} WP(A_j, 0; R_j). \\
&= WP(A_i', 0; R_i) + \sum_{j\neq i} WP(A_j', 0; R_j) \\
&\quad - WP(A_i, 0; R_i) - \sum_{j\neq i} WP(A_j, 0; R_j). \\
&= \sum_{j\in N} WP(A_j', 0; R_j) - \sum_{j\in N} WP(A_j, 0; R_j) \\
&\leq 0,
\end{aligned}$$

where the third equality follows from the fact that A_i, A'_i belong to the acceptable bundle set at R_i and the last inequality follows from the definition of A. ■

Now, we complete the proof of the proposition. Let $\mathcal{S}_i$ be the acceptable bundle set of agent i according to R_i. We consider two cases.

CASE 1. $A_i \in \mathcal{S}_i$. If $A'_i \in \mathcal{S}_i$, then Lemma 3 implies that

$$(A_i, p_i^{vcg}(R_i, R_{-i}))\ I_i\ (A'_i, p_i^{vcg}(R_i, R_{-i}))\ R_i\ (A'_i, p_i^{vcg}(R'_i, R_{-i}).$$

If $A'_i \notin \mathcal{S}_i$, then Lemma 1 implies that $p_i^{vcg}(R'_i, R_{-i}) = 0$. But, then individual rationality implies that

$$(A_i, p_i^{vcg}(R_i, R_{-i}))\ R_i\ (\emptyset, 0)\ I_i\ (A'_i, 0).$$

CASE 2. $A_i \notin \mathcal{S}_i$. By Lemma 1, $p_i^{vcg}(R_i, R_{-i}) = 0$. Now, note that since $A_i \notin \mathcal{S}_i$, we have $WP(A_i, 0; R_i) = 0$, and hence,

$$\sum_{j \in N} WP(A_j, 0; R_j) = \max_{\hat{A} \in \mathcal{X}} \sum_{j \neq i} WP(\hat{A}_j, 0; R_j).$$

This implies that

$$\sum_{j \in N} WP(A'_j, 0; R_j) \leq \sum_{j \in N} WP(A_j, 0; R_j) = \max_{\hat{A} \in \mathcal{X}} \sum_{j \neq i} WP(\hat{A}_j, 0; R_j),$$

where the first inequality followed from the definition of A. This implies that

$$WP(A'_i, 0; R_i) \leq \max_{\hat{A} \in \mathcal{X}} \sum_{j \neq i} WP(\hat{A}_j, 0; R_j) - \sum_{j \neq i} WP(A'_j, 0; R_j) = p_i^{vcg}(R'_i, R_{-i}).$$

This further implies that

$$(A_i, p_i^{vcg}(R_i, R_{-i}))\ I_i\ (\emptyset, 0)\ I_i\ (A'_i, WP(A'_i, 0; R_i))\ R_i\ (A'_i, p_i^{vcg}(R'_i, R_{-i})).$$

Hence, in both cases, we see that agent i prefers his outcome $(A_i, p_i^{vcg}(R_i, R_{-i}))$ in the GVCG mechanism to the outcome obtained by reporting R'_i. This concludes the proof. ■

A.3 Proof of Theorem 2

Proof: By Proposition 1, the GVCG mechanism is DSIC, individually rational, and satisfies no subsidy. Now, we prove Pareto efficiency. Let $\mathcal{T}$ be a dichotomous domain satisfying positive income effect. Assume for contradiction that there exists a profile $R \in \mathcal{T}^n$ such that $(f^{vcg}(R), \mathbf{p}^{vcg}(R))$ is not Pareto efficient. As before, let $(\mathcal{S}_i, w_i)$ denote the dichotomous preference R_i of any agent i. Let $f^{vcg}(R) \equiv A$ and $\mathbf{p}^{vcg}(R) \equiv (p_1, \ldots, p_n)$. Then there exists, an outcome profile $((A'_1, p'_1), \ldots, (A'_n, p'_n))$ which Pareto dominates $((A_1, p_1), \ldots, (A_n, p_n))$.

We consider various cases to derive relationship between p_i and p'_i for each $i \in N$.

CASE 1. Pick $i \in N$ such that $A_i, A'_i \in \mathcal{S}_i$ or $A_i, A'_i \notin \mathcal{S}_i$. Dichotomous preference implies that $(A'_i, p'_i)\ I_i\ (A_i, p'_i)$. But $(A'_i, p'_i)\ R_i\ (A_i, p_i)$ implies that $(A_i, p'_i)\ R_i\ (A_i, p_i)$. Hence, we get

$$p_i \geq p'_i \ \forall\ i \text{ such that } A_i, A'_i \in \mathcal{S}_i \text{ or } A_i, A'_i \notin \mathcal{S}_i. \tag{1}$$

CASE 2. Pick $i \in N$ such that $A_i \notin \mathcal{S}_i$ but $A'_i \in \mathcal{S}_i$. This implies that $p_i = 0$ (by Lemma 1). Hence, $(A'_i, p'_i)\ R_i\ (A_i, p_i)\ I_i\ (A_i, 0)\ I_i\ (\emptyset, 0)\ I_i\ (A'_i, w_i(0))$. Thus,

$$w_i(0) + p_i \geq p'_i \ \forall\ i \text{ such that } A_i \notin \mathcal{S}_i, A'_i \in \mathcal{S}_i. \tag{2}$$

CASE 3. Pick $i \in N$ such that $A_i \in \mathcal{S}_i$ but $A'_i \notin \mathcal{S}_i$. Since $A'_i \notin \mathcal{S}_i$, we can write $(A'_i, p'_i)\ I_i\ (\emptyset, p'_i)\ I_i\ (A_i, p'_i + w_i(p'_i))$. But $(A'_i, p'_i)\ R_i\ (A_i, p_i)$ implies that

$$p_i \geq p'_i + w_i(p'_i).$$

Also, $(\emptyset, p'_i)\ I_i\ (A'_i, p'_i)\ R_i\ (A_i, p_i)\ R_i\ (\emptyset, 0)$, where the last inequality is due to individual rationality of the GVCG mechanism. Hence, $p'_i \leq 0$. But then, positive income effect implies that $w_i(p'_i) \geq w_i(0)$. This gives us

$$p_i \geq p'_i + w_i(0) \ \forall\ i \text{ such that } A_i \in \mathcal{S}_i, A'_i \notin \mathcal{S}_i. \tag{3}$$

By summing over Inequalities 1, 2, and 3, we get

$$\begin{aligned}
\sum_{i \in N} p_i &\geq \sum_{i \in N} p'_i + \sum_{i: A_i \in \mathcal{S}_i, A'_i \notin \mathcal{S}_i} w_i(0) - \sum_{i: A_i \notin \mathcal{S}_i, A'_i \in \mathcal{S}_i} w_i(0). \\
&= \sum_{i \in N} p'_i + \sum_{i: A_i \in \mathcal{S}_i, A'_i \notin \mathcal{S}_i} w_i(0) + \sum_{i: A_i, A'_i \in \mathcal{S}_i} w_i(0) - \sum_{i: A_i, A'_i \in \mathcal{S}_i} w_i(0) - \sum_{i: A_i \notin \mathcal{S}_i, A'_i \in \mathcal{S}_i} w_i(0). \\
&= \sum_{i \in N} p'_i + \sum_{i \in N} WP(A_i, 0; R_i) - \sum_{i \in N} WP(A'_i, 0; R_i) \\
&\geq \sum_{i \in N} p'_i,
\end{aligned}$$

where the inequality follows from the definition of the GVCG mechanism. Also, note that the inequality above is strict if any of the Inequalities 1, 2, and 3 is strict. This contradicts the fact that the outcome $((A'_1, p'_1), \ldots, (A'_n, p'_n))$ Pareto dominates $((A_1, p_1), \ldots, (A_n, p_n))$. ■

A.4 Proof of Theorem 3

Proof: Let $(f, \mathbf{p})$ be a Pareto efficient, DSIC, IR mechanism satisfying no subsidy. The proof proceeds in two steps. We assume without loss of generality that at every preference profile R, if an agent $i \in N$ is assigned an acceptable bundle $f_i(R)$, then $f_i(R)$ is a *minimal* acceptable bundle at R_i, i.e., there does not exist another acceptable bundle $S_i \subsetneq f_i(R)$ at R_i [6]. We now proceed with the proof in two Steps.

Allocation is GVCG allocation. In this step, we argue that f must satisfy:

$$f(R) \in \arg\max_{A \in \mathcal{X}} \sum_{i \in N} WP(A_i, 0; R_i) \; \forall \; R \in \mathcal{T}^n$$

Assume for contradiction that for some $R \in \mathcal{T}^n$, we have

$$\sum_{i \in N} WP(f_i(R), 0; R_i) < \max_{A \in \mathcal{X}} \sum_{i \in N} WP(A_i, 0; R_i).$$

Before proceeding with the rest of the proof, we fix a generalized VCG mechanism (f^{vcg}, p^{vcg}) and introduce a notation. For every R', denote by

$$N_{0+}(R') := \Big\{ i \in N : \big[(f_i^{vcg}(R'), p_i^{vcg}(R')) \; I_i' \; (\emptyset, 0)\big] \text{ and } \big[(f_i(R'), p_i(R')) \; P_i' \; (\emptyset, 0)\big] \Big\}.$$

We now construct a sequence of preference profiles, starting with preference profile R, as follows. Let $R^0 := R$. Also, we will maintain a sequence of subsets of agents, which is initialized as $B^0 := \emptyset$. We will denote the preference profile constructed in step t of the sequence as R^t and the willingness to pay map at preference R_i^t as w_i^t for each $i \in N$.

S1. If $N_{0+}(R^t) \setminus B^t = \emptyset$, then stop. Else, go to the next step.

S2. Choose $k^t \in N_{0+}(R^t) \setminus B^t$ and consider $R_{k^t}^{t+1}$ to be a quasilinear dichotomous preference with valuation $w_{k^t}^{t+1}(0) \in (p_{k^t}(R^t), w_{k^t}^t(0))$ and a unique minimal acceptable bundle $f_{k^t}(R^t)$ - such a quasilinear preference exists because $\mathcal{T} \supseteq \mathcal{D}^{QL}$. Let $R_j^{t+1} = R_j^t$ for all $j \neq k^t$.

[6] This is without loss of generality for the following reason. For every Pareto efficient, DSIC, IR mechanism $(f, \mathbf{p})$ satisfying no subsidy, we can construct another mechanism $(f', \mathbf{p}')$ such that: for all R and for all $i \in N$, $f_i'(R) \subseteq f_i(R)$ and $f_i'(R)$ is a minimal acceptable bundle at R_i whenever $f_i(R)$ is an acceptable bundle at R_i and $f_i'(R) = f_i(R)$ otherwise. Further, $\mathbf{p}' = \mathbf{p}$. It is routine to verify that $(f', \mathbf{p}')$ is DSIC, IR, Pareto efficient and satisfies no subsidy. Finally, by construction, if $(f', \mathbf{p}')$ is a generalized VCG mechanism, then $(f, \mathbf{p})$ is also a generalized VCG mechanism.

S3. Set $B^{t+1} := B^t \cup \{k^t\}$ and $t := t+1$. Repeat from Step S1.

Because of finiteness of number of agents, this process will terminate finitely in some $T < \infty$ steps. We establish some claims about the preference profiles generated in this procedure.

Claim **1** *For every $t \in \{0, \ldots, T-1\}$, $f_{k^t}(R^{t+1}) = f_{k^t}(R^t)$ and $p_{k^t}(R^{t+1}) = p_{k^t}(R^t)$.*

Proof: Fix t and assume for contradiction $f_{k^t}(R^{t+1}) \neq f_{k^t}(R^t)$. Since $f_{k^t}(R^t)$ is the unique minimal acceptable bundle at $R^{t+1}_{k^t}$ and f only assigns a minimal acceptable bundle whenever it assigns acceptable bundles, it must be that $f_{k^t}(R^{t+1})$ is not an acceptable bundle at $R^{t+1}_{k^t}$. Then, by Lemma 1, we get $p_{k^t}(R^{t+1}) = 0$. Since $w^{t+1}_{k^t}(0) > p_{k^t}(R^t)$ and $f_{k^t}(R^t)$ is an acceptable bundle at $R^{t+1}_{k^t}$, we get

$$(f_{k^t}(R^t), p_{k^t}(R^t)) \; P^{t+1}_{k^t} \; (\emptyset, 0) \; I^{t+1}_{k^t}(f_{k^t}(R^{t+1}), p_{k^t}(R^{t+1})).$$

This contradicts DSIC. Finally, if $f_{k^t}(R^{t+1}) = f_{k^t}(R^t)$, we must have $p_{k^t}(R^{t+1}) = p_{k^t}(R^t)$ due to DSIC since acceptable bundle at $R^{t+1}_{k^t}$ is $f_{k^t}(R^t)$ and $f_{k^t}(R^t)$ is also an acceptable bundle at $R^t_{k^t}$. ■

The next claim establishes a useful inequality.

Claim **2** *For every $t \in \{0, \ldots, T\}$, the following holds:*

$$w^t_{k^t}(0) + \max_{A \in \mathcal{X}, A_{k^t} = f_{k^t}(R^t)} \sum_{j \neq k^t} WP_j(A_j, 0; R^t_j) \leq \max_{A \in \mathcal{X}} \sum_{j \neq k^t} WP_j(A_j, 0; R^t_j).$$

Proof: Pick some $t \in \{0, \ldots, T\}$ and suppose the above inequality does not hold. We complete the proof in two steps.

Step 1. In this step, we argue that $f^{vcg}_{k^t}$ must be an acceptable bundle for agent k^t at preference R^t. If this is not true, then we must have

$$\begin{aligned}
\sum_{j \in N} WP(f^{vcg}_j(R^t), 0; R^t_j) &= \sum_{j \neq k^t} WP(f^{vcg}_j(R^t), 0; R^t_j) \\
&\leq \max_{A \in \mathcal{X}} \sum_{j \neq k^t} WP(A_j, 0; R^t_j) \\
&< w^t_{k^t}(0) + \max_{A \in \mathcal{X}, A_{k^t} = f_{k^t}(R^t)} \sum_{j \neq k^t} WP(A_j, 0; R^t_j) \\
&= WP(f_{k^t}(R^t), 0; R^t_{k^t}) + \max_{A \in \mathcal{X}, A_{k^t} = f_{k^t}(R^t)} \sum_{j \neq k^t} WP(A_j, 0; R^t_j),
\end{aligned}$$

where the last inequality follows from our assumption that the claimed inequality does not hold and the last equality follows from the fact that $f_{k^t}(R^t)$ is an acceptable bundle of agent k^t at $R^t_{k^t}$. But, then the resulting inequality contradicts the definition of f^{vcg}.

STEP 2. We complete the proof in this step. Notice that the payment of agent k^t in (f^{vcg}, p^{vcg}) is defined as follows.

$$\begin{aligned}
p^{vcg}_{k^t}(R^t) &= \max_{A\in\mathcal{X}} \sum_{j\neq k^t} WP(A_j, 0; R^t_j) - \sum_{j\neq k^t} WP(f^{vcg}_j(R^t), 0; R^t_j) \\
&< w^t_{k^t}(0) + \max_{A\in\mathcal{X}, A_{k^t}=f_{k^t}(R^t)} \sum_{j\neq k^t} WP(A_j, 0; R^t_j) - \sum_{j\neq k^t} WP(f^{vcg}_j(R^t), 0; R^t_j) \\
&= w^t_{k^t}(0) + \max_{A\in\mathcal{X}, A_{k^t}=f_{k^t}(R^t)} \sum_{j\neq k^t} WP(A_j, 0; R^t_j) \\
&- \sum_{j\in N} WP(f^{vcg}_j(R^t), 0; R^t_j) + WP(f^{vcg}_{k^t}(R^t), 0; R^t_{k^t}) \\
&= w^t_{k^t}(0) + \max_{A\in\mathcal{X}, A_{k^t}=f_{k^t}(R^t)} \sum_{j\in N} WP(A_j, 0; R^t_j) - \sum_{j\in N} WP(f^{vcg}_j(R^t), 0; R^t_j) \\
&\leq w^t_{k^t}(0),
\end{aligned}$$

where the strict inequality followed from our assumption and the last equality follows from the fact both $f_{k^t}(R^t)$ and $f^{vcg}_{k^t}(R^t)$ are acceptable bundles for agent k^t at $R^t_{k^t}$ (Step 1). But, this implies that

$$(f^{vcg}_{k^t}(R^t), p^{vcg}_{k^t}(R^t)) \; P^t_{k^t} \; (f^{vcg}_{k^t}(R^t), w^t_{k^t}(0)) \; I^t_{k^t} \; (\emptyset, 0).$$

This is a contradiction to the fact that $k^t \in N_{0+}(R^t)$. This completes the proof. ■

We now establish an important claim regarding an inequality satisfied by the sequence of preferences generated.

CLAIM **3** *For every* $t \in \{0, \ldots, T\}$,

$$\sum_{j\in N} WP(f_j(R^t), 0; R^t_j) < \sum_{j\in N} WP(f^{vcg}_j(R^t), 0; R^t_j).$$

Proof: The inequality holds for $t = 0$ by assumption. We now use induction. Suppose the inequality holds for $t \in \{0, \ldots, \tau - 1\}$. We show that it holds for τ. To see this, denote $k \equiv k^{\tau-1}$. By Claim 1, we know that $f_k(R^{\tau-1}) = f_k(R^\tau)$. Further, by definition, $f_k(R^\tau)$

belongs to the acceptable bundle of k at R_k^τ and $R_k^{\tau-1}$. Now, observe the following:

$$\begin{aligned}
\sum_{j\in N} WP(f_j(R^\tau),0;R_j^\tau) &= w_k^\tau(0) + \sum_{j\neq k} WP(f_j(R^\tau),0;R_j^\tau) \qquad \text{(follows from definition of } k\text{)}\\
&\leq w_k^\tau(0) + \max_{A\in\mathcal{X}:A_k=f_k(R^{\tau-1})=f_k(R^\tau)} \sum_{j\neq k} WP(A_j,0;R_j^\tau)\\
&= w_k^\tau(0) + \max_{A\in\mathcal{X}:A_k=f_k(R^{\tau-1})=f_k(R^\tau)} \sum_{j\neq k} WP(A_j,0;R_j^{\tau-1})\\
&\text{(using the fact that } R_j^\tau = R_j^{\tau-1} \text{ for all } j\neq k\text{)}\\
&\leq w_k^\tau(0) - w_k^{\tau-1}(0) + \max_{A\in\mathcal{X}} \sum_{j\neq k} WP(A_j,0;R_j^{\tau-1}) \qquad \text{(using Claim 2)}\\
&< \max_{A\in\mathcal{X}} \sum_{j\neq k} WP(A_j,0;R_j^{\tau-1}) \qquad \text{(using the fact that } w_k^\tau(0) < w_k^{\tau-1}(0)\text{)}\\
&= \max_{A\in\mathcal{X}} \sum_{j\neq k} WP(A_j,0;R_j^{\tau})\\
&\leq \max_{A\in\mathcal{X}} \sum_{j\in N} WP(A_j,0;R_j^{\tau}).
\end{aligned}$$

■

We now complete our claim the allocation is the same as in a GVCG mechanism. Let $R^T \equiv R'$. Let $f^{vcg}(R') = A^{vcg}$ and $f(R') = A'$. Partition the set of agents as follows.

$$\begin{aligned}
N_{++} &:= \{i : WP_i(A_i^{vcg},0;R_i') = WP(A_i',0;R_i') > 0\}\\
N_{+-} &:= \{i : WP_i(A_i^{vcg},0;R_i') > 0, WP(A_i',0;R_i') = 0\}\\
N_{-+} &:= \{i : WP_i(A_i^{vcg},0;R_i') = 0, WP(A_i',0;R_i') > 0\}\\
N_{--} &:= \{i : WP_i(A_i^{vcg},0;R_i') = WP(A_i',0;R_i') = 0\}.
\end{aligned}$$

Now, consider the following consumption bundle Z:

$$Z_i := \begin{cases} (A_i^{vcg}, p_i(R')) & \text{if } i \in N_{++}\cup N_{--}\\ (A_i^{vcg}, p_i(R') - WP(A_i',0;R_i')) & \text{if } i\in N_{-+}\\ (A_i^{vcg}, WP(A_i^{vcg},0;R_i')) & \text{if } i \in N_{+-}\end{cases}$$

Notice that for each $i \in N_{++}\cup N_{--}$, we have $Z_i = (A_i^{vcg}, p_i(R'))\ I_i'\ (A_i', p_i(R'))$. For each $i \in N_{+-}$, we know that $WP(A_i',0;R_i') = 0$ - this implies that A_i' is not an acceptable bundle at R_i'. Hence, for all $i \in N_{+-}$, we have $Z_i = (A_i^{vcg}, WP(A_i^{vcg},0;R_i'))\ I_i'\ (\emptyset,0)\ I_i'\ (A_i', p_i(R'))$, where the last relation follows from Lemma 1. Finally, for all $i\in N_{-+}$, $WP_i(A_i^{vcg},0;R_i') = 0$ implies that $(A_i^{vcg}, p_i^{vcg}(R'))\ I_i'\ (\emptyset,0)$. Then, for every $i\in N_{-+}$, either we have $(A_i', p_i(R'))\ I_i'\ (\emptyset,0)$ or we have $i \in B^T$ (i.e., R'_i is a quasilinear preference). In the first case, $p_i(R') = WP(A'_i, 0; R'_i)$ implies

$$(A_i^{vcg}, p_i(R') - WP(A'_i, 0; R'_i)) \; I'_i \; (A_i^{vcg}, 0) \; I'_i \; (\emptyset, 0) \; I'_i \; (A'_i, p_i(R')).$$

In the second case, quasilinearity of R'_i implies $(A_i^{vcg}, p_i(R') - WP(A'_i, 0; R'_i)) \; I'_i \; (A'_i, p_i(R'))$. This completes the argument that $Z_i \; R'_i \; (A'_i, p_i(R'))$ for every $i \in N$.

Now, observe the sum of transfers across all agents in Z is:

$$\begin{aligned}
&\sum_{i \notin N_{+-}} p_i(R') - \sum_{i \in N_{-+}} WP(A'_i, 0; R'_i) + \sum_{i \in N_{+-}} WP(A_i^{vcg}, 0; R'_i) \\
&= \sum_{i \in N} p_i(R') - \sum_{i \in N_{-+}} WP(A'_i, 0; R'_i) + \sum_{i \in N_{+-}} WP(A_i^{vcg}, 0; R'_i) \\
&\text{(since } A'_i \text{ is not acceptable, Lemma 1 implies } p_i(R') = 0 \text{ for all } i \in N_{+-}) \\
&= \sum_{i \in N} p_i(R') + \sum_{i \in N} WP(A_i^{vcg}, 0; R'_i) - \sum_{i \in N} WP(A'_i, 0; R'_i) \\
&> \sum_{i \in N} p_i(R'),
\end{aligned}$$

where the last inequality follows from Claim 3.

Hence, Z Pareto dominates the outcome $(f(R'), p(R'))$, contradicting Pareto efficiency. We now proceed to the next step to show that the payment in $(f, \mathbf{p})$ must also coincide with the generalized VCG outcome.

Payment is GVCG payment. Fix a preference profile R. We now know that

$$f(R) \in \arg\max_{A \in \mathcal{X}} \sum_{i \in N} WP(A_i, 0; R_i).$$

By Lemma 1, for every $i \in N$, if $f_i(R) = f_i^{vcg}(R)$ is not acceptable for agent i, then $p_i(R) = p_i^{vcg}(R) = 0$ - here, we assume, without loss of generality, that $f(R') = f^{vcg}(R')$ for all R' [7]. We now consider two cases.

Case 1. Assume for contradiction that there exists $i \in N$ such that $f_i(R)$ is an acceptable bundle of agent i and

$$p_i(R) > \max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, 0; R_j) - \sum_{j \neq i} WP(f_j(R), 0; R_j). \tag{4}$$

[7] Depending on how we break ties for choosing a maximum in the maximization of sum of willingness to pay, we have a different generalized VCG mechanism. This assumption ensures that we pick the generalized VCG mechanism that breaks the ties the same way as f.

Now consider R'_i with the set of acceptable bundles the same in R_i and R'_i but $WP(f_i(R), 0; R'_i) < p_i(R)$ but arbitrarily close to $p_i(R)$. Let $A' \equiv f(R'_i, R_{-i})$. We argue that A'_i is an acceptable bundle (at R'_i). If not, then

$$\max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, 0; R_j) \geq \sum_{j \neq i} WP(A'_j, 0; R_j) = WP(A'_i, 0; R'_i) + \sum_{j \neq i} WP(A'_j, 0, R_j),$$

where we used the fact that A'_i is not an acceptable bundle for i. But then, by construction of R'_i and Inequality (4), we get

$$WP(f_i(R), 0; R'_i) + \sum_{j \neq i} WP(f_j(R), 0; R_j) > \max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, 0; R_j) \geq WP(A'_i, 0; R'_i) + \sum_{j \neq i} WP(A'_j, 0, R_j),$$

which is a contradiction to our earlier step that f is the same allocation as in the GVCG mechanism. Hence, A'_i is an acceptable bundle at R'_i. But, then $p_i(R) = p_i(R'_i, R_{-i})$ by DSIC (since $f_i(R)$ is also an acceptable bundle at R_i and the set of acceptable bundles at R_i and R'_i are the same). Since $WP(A'_i, 0; R'_i) < p_i(R) = p_i(R'_i, R_{-i})$, we get a contradiction to individual rationality.

Case 2. Assume for contradiction that there exists $i \in N$ such that $f_i(R)$ is an acceptable bundle of agent i and

$$p_i(R) < p_i^{vcg}(R) = \max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, 0; R_j) - \sum_{j \neq i} WP(f_j(R), 0; R_j).$$

Pick R'_i such that the set of acceptable bundles at R'_i and R_i are the same but $WP(f_i(R), 0; R'_i) \in (p_i(R), p_i^{vcg}(R))$. Notice that if $f_i(R'_i, R_{-i})$ is not an acceptable bundle at R'_i, then his payment is zero (Lemma 1). In that case, $WP(f_i(R), 0; R'_i) > p_i(R)$ implies that

$$(f_i(R), p_i(R)) \ P'_i \ (\emptyset, 0) \ I'_i \ (f_i(R'_i, R_{-i}), p_i(R'_i, R_{-i})),$$

contradicting DSIC. Hence, $f_i(R'_i, R_{-i}) = f_i^{vcg}(R'_i, R_{-i})$ is an acceptable bundle at R'_i. This implies that $f_i^{vcg}(R'_i, R_{-i})$ is an acceptable bundle at R'_i. Since the generalized VCG is DSIC, we get that $p_i^{vcg}(R) = p_i^{vcg}(R'_i, R_{-i})$. But $WP(f_i^{vcg}(R'_i, R_{-i}), 0; R'_i) < p_i^{vcg}(R) = p_i^{vcg}(R'_i, R_{-i})$ is a contradiction to IR of the generalized VCG. This completes the proof. ■

A.5 Proof of Theorem 4

Proof: Assume for contradiction that $(f, \mathbf{p})$ is a desirable mechanism on $\mathcal{T}^n$. By heterogeneous demand, there exist objects a and b such that $0 < WP(a, 0; R_0) < WP(b, 0; R_0)$. Consider a preference profile $R \in \mathcal{T}^n$ as follows:

1. Agent 1 has quasilinear dichotomous preference with $\mathcal{S}_i^{min} = \{\{a,b\}\}$ and value $w_1(0)$ that satisfies

$$WP(\{a,b\},0;R_0) < w_1(0) < WP(\{a\},0;R_0) + WP(\{b\},0;R_0). \tag{5}$$

2. $R_i = R_0$ for all $i \in \{2,3\}$.

3. If $m > 2$, agent 4 has quasilinear dichotomous preference with acceptable bundle $M \setminus \{a,b\}$ and value *very high*. If $m = 2$, agent 4 has quasilinear dichotomous preference with acceptable bundle M and value equals to ϵ, which is very close to zero.

4. For all $i > 4$, let R_i be a quasilinear dichotomous preference with $\mathcal{S}_i^{min} = \{M\}$ and value equals to ϵ, which is very close to zero.

We begin by a useful claim.

Claim 4 *Pick $k \in \{2,3\}$ and $x \in \{a,b\}$. Let R' be a preference profile such that $R'_i = R_i$ for all $i \neq k$. Suppose R'_k is such that*

$$WP(\{x\},0;R'_k) + WP(\{a,b\} \setminus \{x\},0;R_0) > w_1(0) > WP(\{a,b\},0;R'_k). \tag{6}$$

Then, the following are true:

1. $f_1(R') = \emptyset$
2. $f_2(R') \cup f_3(R') = \{a,b\}$
3. $f_2(R') \neq \emptyset$ *and* $f_3(R') \neq \emptyset$.

Proof: It is without loss of generality (due to Pareto efficiency) that $f_i(R') = \emptyset$ or $f_i(R') \in \mathcal{S}_i^{min}$ for all i who has dichotomous preference. Since ϵ is very close to zero, Pareto efficiency implies that (a) if $m = 2$, $f_i(R') = \emptyset$ for all $i > 3$; and (b) if $m > 2$, since agent 4 has very high value for $M \setminus \{a,b\}$, $f_4(R') = M \setminus \{a,b\}$ and $f_i(R') = \emptyset$ for all $i > 4$. Hence, agents $1, 2,$ and 3 will be allocated $\{a,b\}$ at R'. Denote $y \equiv \{a,b\} \setminus \{x\}$ and $\ell \equiv \{2,3\} \setminus \{k\}$.

Proof of (1) and (2). Assume for contradiction $f_1(R') \neq \emptyset$. Pareto efficiency implies that $f_1(R') = \{a,b\}$ and $f_2(R') = f_3(R') = \emptyset$. Lemma 1 implies that $p_2(R') = p_3(R') = 0$. Then, consider the following outcome:

$$z_1 := \Big(\emptyset, p_1(R') - w_1(0)\Big),\ z_k := \Big(\{x\}, WP(\{x\},0;R'_k)\Big),\ z_\ell := \Big(\{y\}, WP(\{y\},0;R'_\ell)\Big),$$

$$z_i := \Big(f_i(R'), p_i(R')\Big) \ \forall\, i > 3.$$

By definition of willingness to pay, $z_i \ I_i \ (\emptyset, 0) \equiv \Big(f_i(R'), p_i(R')\Big)$ for all $i \in \{2, 3\}$. Since agent 1 has quasilinear preferences, he is also indifferent between z_1 and $\Big(\{a, b\}, p_1(R')\Big) \equiv \Big(f_1(R'), p_1(R')\Big)$. Thus, the difference in total payment between the outcome z and the payment in $(f, \mathbf{p})$ at R' is

$$WP(\{x\}, 0; R'_k) + WP(\{y\}, 0; R'_\ell) - w_1(0) = WP(\{x\}, 0; R'_k) + WP(\{y\}, 0; R_0) - w_1(0) > 0,$$

where the inequality follows from Inequality (6). This is a contradiction to Pareto efficiency of $(f, \mathbf{p})$. Hence, $f_1(R) = \emptyset$. By Pareto efficiency, $f_2(R') \cup f_3(R') = \{a, b\}$.

Proof of (3). Now, we show that $f_2(R') \neq \emptyset$ and $f_3(R') \neq \emptyset$. Suppose $f_3(R') = \emptyset$. Then, $f_2(R') = \{a, b\}$ and Lemma 1 implies that $p_3(R') = 0$. We first argue that $p_2(R') = WP(\{a, b\}, 0; R'_2)$. To see this, consider a quasilinear dichotomous preference $\tilde{R}_2$ with acceptable bundle $\{a, b\}$ and value equal to $WP(\{a, b\}, 0; R'_2)$. Notice that $w_1(0) > WP(\{a, b\}, 0; R'_2)$ - if $k = 2$, then this is true by Inequality (6) and if $\ell = 2$, then $R'_\ell = R_0$ satisfies $w_1(0) > WP(\{a, b\}, 0; R_0)$ by Inequality (5). Since agents 1 and 2 have the same acceptable bundle at $(\tilde{R}_2, R'_{-2})$ but $w_1(0) > WP(\{a, b\}, 0; R'_2)$, this implies that (due to Pareto efficiency), $f_2(\tilde{R}_2, R'_{-2}) = \emptyset$ and $p_2(\tilde{R}_2, R'_{-2}) = 0$ (Lemma 1). By DSIC, $(\emptyset, 0) \ \tilde{R}_2 \ (\{a, b\}, p_2(R'))$. This implies that $WP(\{a, b\}, 0; R'_2) \leq p_2(R')$. IR of agent 2 at R' implies $WP(\{a, b\}, 0; R'_2) = p_2(R')$.

Next, consider the following outcome

$$z'_k := (\{x\}, WP(\{x\}, 0; R'_k), z'_\ell := (\{y\}, WP(\{y\}, 0; R'_\ell), z'_i := (f_i(R'), p_i(R')) \ \forall\, i \notin \{2, 3\}.$$

By definition, for every agent i, $z'_i \ I'_i \ (f_i(R'), p_i(R'))$. The difference between the sum of transfers of agents in z' and $(f, \mathbf{p})$ at R is:

$$\begin{aligned} WP(\{x\}, 0; R'_k) + WP(\{y\}, 0; R'_\ell) - p_2(R') &= WP(\{x\}, 0; R'_k) + WP(\{y\}, 0; R_0) - WP(\{a, b\}, 0; R'_2) \\ &> w_1(0) - WP(\{a, b\}, 0; R'_2) \\ &> 0, \end{aligned}$$

where the first inequality follows from Inequality (6) and the second inequality follows from Inequality (6) if $k = 2$ and from Inequality (5) if $\ell = 2$. This contradicts Pareto efficiency of $(f, \mathbf{p})$. A similar proof shows that $f_2(R') \neq \emptyset$. ■

Now, pick any $k \in \{2,3\}$ and set $R'_k = R_0$ in Claim 4. By Inequality (5), Inequality (6) holds for R_0. As a result, we get that $f_2(R) \neq \emptyset, f_3(R) \neq \emptyset$, and $f_2(R) \cup f_3(R) = \{a,b\}$. Hence, without loss of generality, assume that $f_2(R) = \{a\}$ and $f_3(R) = \{b\}$. [8] We now complete the proof in two steps.

STEP 1. We argue that $p_2(R) = w_1(0) - WP(\{b\}, 0; R_0)$ and $p_3(R) = w_1(0) - WP(\{a\}, 0; R_0)$. Suppose $p_2(R) > w_1(0) - WP(\{b\}, 0; R_0)$. Then, consider the quasilinear dichotomous preference R_2^Q such that the minimum acceptable bundle of agent 2 is $\{a\}$ and his value v satisfies

$$w_1(0) - WP(\{b\}, 0; R_0) < v < p_2(R). \tag{7}$$

Now, note that by IR of agent 2 at R, we have

$$p_2(R) \leq WP(\{a\}, 0; R_0) \leq WP(\{a,b\}, 0; R_0) < w_1(0),$$

where the strict inequality followed from Inequality (5). Hence, $v < w_1(0)$ and $w_1(0) < v + WP(\{b\}, 0; R_0)$ by Inequality (7). Hence, choosing $k = 2$, $x = a$ and $R'_k = R_2^Q$, we can apply Claim 4 to conclude that $f_2(R_2^Q, R_{-2}) \cup f_3(R_2^Q, R_{-2}) = \{a,b\}$ and $f_2(R_2^Q, R_{-2}) \neq \emptyset$, $f_3(R_2^Q, R_{-2}) \neq \emptyset$. Since R_2^Q is a dichotomous preference with acceptable bundle $\{a\}$, Pareto efficiency implies that $f_2(R_2^Q) = \{a\} = f_2(R)$. By DSIC, $p_2(R) = p_2(R_2^Q, R_{-2})$. But Inequality (7) gives $v < p_2(R) = p_2(R_2^Q, R_{-2})$, and this contradicts individual rationality.

Next, suppose $p_2(R) < w_1(0) - WP(\{b\}, 0; R_0)$. Then, consider the quasilinear dichotomous preference $\hat{R}_2^Q$ such that the minimal acceptable bundle of agent 2 is $\{a\}$ and his value $\hat{v}$ satisfies

$$p_2(R) < \hat{v} < w_1(0) - WP(\{b\}, 0; R_0). \tag{8}$$

Now, consider the preference profile $\hat{R}$ such that $\hat{R}_2 = \hat{R}_2^Q$ and $\hat{R}_i = R_i$ for all $i \neq 2$. We first argue that $f_2(\hat{R}) = \emptyset$. Suppose not, then by Pareto efficiency, $f_2(\hat{R}) = \{a\}$. By Pareto efficiency, we have $f_3(\hat{R}) = \{b\}$ and $f_1(\hat{R}) = \emptyset$. By Lemma 1, $p_1(\hat{R}) = 0$. We argue that $p_3(\hat{R}) = WP(\{b\}, 0; R_0)$. To see this, consider a profile $\hat{R}'$ where $\hat{R}'_i = \hat{R}_i$ for all $i \neq 3$ and $\hat{R}'_3$ is a quasilinear dichotomous preferences with minimum acceptable bundle $\{b\}$ and value equal to $WP(\{b\}, 0; R_0)$ - notice that every agent in $\hat{R}'$ has quasilinear preference. As a result,

[8] Since we have assumed $WP(\{b\}, 0; R_0) > WP(\{a\}, 0; R_0)$, this may appear to be with loss of generality. However, if we have $f_2(R) = \{b\}$ and $f_3(R) = \{a\}$, then we will swap 2 and 3 in the entire argument following this.

Theorem 3 implies that the outcome of $(f, \mathbf{p})$ at $\hat{R}'$ must coincide with the GVCG mechanism. But $w_1(0) > \hat{v} + WP(\{b\}, 0; R_0)$ implies that $f_1(\hat{R}') = \{a, b\}$ and $f_2(\hat{R}') = f_3(\hat{R}') = \emptyset$. Then, DSIC implies that (incentive constraint of agent 3 from $\hat{R}'$ to $\hat{R}$) $0 \geq WP(\{b\}, 0; R_0) - p_3(\hat{R})$. By individual rationality of agent 3 at $\hat{R}$ we get, $p_3(\hat{R}) \leq WP(\{b\}, 0; R_0)$, and combining these we get $p_3(\hat{R}) = WP(\{b\}, 0; R_0)$.

Now, consider the following allocation vector $\hat{z}$:

$$\hat{z}_1 := \Big(\{a, b\}, w_1(0)\Big), \hat{z}_2 := \Big(\emptyset, p_2(\hat{R}) - \hat{v}\Big), \hat{z}_3 := \Big(\emptyset, 0\Big),$$

$$\hat{z}_i := \Big(f_i(\hat{R}), p_i(\hat{R})\Big) \; \forall \; i > 3.$$

By definition of $w_1(0)$, we get that $\hat{z}_1 \; \hat{I}_1 \; (\emptyset, 0)$. Also, since $\hat{R}_2$ is quasilinear with value $\hat{v}$, we get $(\emptyset, p_2(\hat{R}) - \hat{v}) \; \hat{I}_2 \; (\{a\}, p_2(\hat{R}))$. For agent 3, notice that $R_3 = R_0$ and by the definition of willingness to pay, we get $(\emptyset, 0) \; \hat{I}_3 \; \Big(\{b\}, WP(\{b\}, 0; R_0)\Big)$. For $i > 3$, each agent i gets the same outcome in $\hat{z}$ and $(f, \mathbf{p})$. Finally, the sum of transfers of agents $1, 2$, and 3 (transfers of other agents remain unchanged) in $\hat{z}$ is

$$w_1(0) + p_2(\hat{R}) - \hat{v} > p_2(\hat{R}) + p_3(\hat{R}),$$

where the strict inequality follows from Inequality (8) and the fact that $p_3(\hat{R}) = WP(\{b\}, 0; R_0)$. This contradicts the fact that $(f, \mathbf{p})$ is Pareto efficient.

Hence, we must have $f_2(\hat{R}) = \emptyset$. By Lemma 1, we have $p_2(\hat{R}) = 0$. But since $v > p_2(R)$, we get $(\{a\}, p_2(R)) \; \hat{P}_2 \; (\emptyset, 0)$. Hence, $(f_2(R), p_2(R)) \; \hat{P}_2 \; (f_2(\hat{R}), p_2(\hat{R}))$. This contradicts DSIC.

An identical argument establishes that $p_3(R) = w_1(0) - WP(\{a\}, 0; R_0)$.

Step 2. In this step, we show that agent 2 can manipulate at R, thus contradicting DSIC and completing the proof. Consider a quasilinear dichotomous preference $\bar{R}_2^Q$ where the minimum acceptable bundle of agent 2 is $\{b\}$ (note that $f_2(R) = \{a\}$) and his value $\bar{v}$ is $WP(\{b\}, 0; R_0)$. Consider the preference profile $\bar{R}$ where $\bar{R}_2 = \bar{R}_2^Q$ and $\bar{R}_i = R_i$ for all $i \neq 2$. Notice that if we let $k = 2$, $x = b$, and $R'_k = \bar{R}_2^Q$, Inequality (6) holds, and hence, Claim 4 implies that $f_2(\bar{R}) \neq \emptyset$ and $f_3(\bar{R}) \neq \emptyset$ but $f_2(\bar{R}) \cup f_3(\bar{R}) = \{a, b\}$. Hence, Pareto efficiency implies that $f_2(\bar{R}) = \{b\}$ and $f_3(\bar{R}) = \{a\}$. Then, we can mimic the argument in Step 1 to conclude that

$$p_2(\bar{R}) = w_1(0) - WP(\{a\}, 0; R_0).$$

Now, by the definition of willingness to pay,

$$\Big(\{b\}, WP(\{b\}, 0; R_0)\Big) \ I_0 \ \Big(\{a\}, WP(\{a\}, 0; R_0)\Big)$$

and by our assumption, $WP(\{b\}, 0; R_0) > WP(\{a\}, 0; R_0)$. By subtracting $WP(\{a\}, 0; R_0) + WP(\{b\}, 0; R_0) - w_1(0)$ (which is positive by Inequality (5)) from transfers on both sides, and using the fact that R_0 satisfies strict positive income effect, we get

$$\Big(\{b\}, w_1(0) - WP(\{a\}, 0; R_0)\Big) \ P_0 \ \Big(\{a\}, w_1(0) - WP(\{b\}, 0; R_0)\Big).$$

Hence, $(f_2(\bar{R}), p_2(\bar{R})) \ P_2 \ (f_2(R), p_2(R))$. This contradicts DSIC. ■

B Supplementary Appendix

B.1 Proof of Theorem 5

We provide the proof of Theorem 5. The proof is similar to the proof of Theorem 4 with slight changes.

Proof: Assume for contradiction that $(f, \mathbf{p})$ is a desirable mechanism on $\mathcal{T}^n$. Consider the preference profile $R \in \mathcal{T}^n$ as follows:

1. R_1: Agent 1 has quasilinear dichotomous preference with $\mathcal{S}_1^{min} = 3$ and value $w_1(0)$ satisfies [9]

$$WP(3, 0; R_0) < w_1(0) < WP(1, 0; R_0) + WP(2, 0; R_0). \tag{9}$$

   By weak decreasing marginal WP of R_0, Inequality 9 can be satisfied.

2. R_2, R_3: Agents 2 and 3 have homogeneous multi-demand preference R_0 satisfying strong positive income effect and decreasing marginal WP.

3. R_4: Agent 4 has quasilinear dichotomous preference with $\mathcal{S}_4^{min} = m - 3$ if $m > 3$ and $\mathcal{S}_4^{min} = m$ if $m = 3$. Value of her acceptable bundle is *very* high if $m > 3$ and is $\epsilon > 0$ but arbitrarily close to zero if $m = 3$.

4. $R_i, i > 4$: Agent $i > 4$ has quasilinear dichotomous preference with $\mathcal{S}_i^{min} = m$ and value $\epsilon > 0$ but arbitrarily close to zero.

Claim **5** *Pick $k \in \{2, 3\}$ and $x \in \{1, 2\}$. Let R' be a preference profile such that $R'_i = R_i$ for all $i \neq k$. Suppose R'_k is such that*

$$WP(x, 0; R'_k) + WP((3 - x), 0; R_0) > w_1(0) > WP(3, 0; R'_k). \tag{10}$$

Then, the following are true:

1. $f_1(R') = 0$
2. $f_2(R') + f_3(R') = 3$
3. $f_2(R') \neq 0$ *and* $f_3(R') \neq 0$.

[9] Note that since this preference is quasilinear $w_1(0) = w_1(t)$ for all t.

Proof: It is without loss of generality (due to Pareto efficiency) that $f_i(R') = 0$ or $f_i(R') = \mathcal{S}_i^{min}$ for all i with dichotomous preferences. Since ϵ is very close to zero, Pareto efficiency implies that (a) if $m = 3$, $f_i(R') = 0$ for all $i > 3$; and (b) if $m > 3$, since agent 4 has very high value, $f_i(R') = 0$ for all $i > 4$ and $f_4(R') = m - 3$ units. As a result, 3 units would be allocated among agents 1, 2 and 3 in $(f, \mathbf{p})$ at R.

Proof of (1) and (2). We argue that $f_1(R') = 0$. Suppose not, then Pareto efficiency implies that $f_1(R') = 3$ and $f_2(R') = f_3(R') = 0$. Lemma 1 implies that $p_2(R') = p_3(R') = 0$. Then, consider the following outcome:

$$z_1 := \Big(0, p_1(R') - w_1(0)\Big),\ z_2 := \Big(1, WP(1, 0; R'_2)\Big),\ z_3 := \Big(2, WP(2, 0; R'_3)\Big),$$

$$z_i := \Big(f_i(R'), p_i(R')\Big)\ \forall\ i > 3.$$

By definition of willingness to pay, $z_i\ I_i\ (0, 0) \equiv \Big(f_i(R'), p_i(R')\Big)$ for all $i \in \{2, 3\}$. Since agent 1 has quasilinear preferences, he is also indifferent between z_1 and $\Big(f_1(R'), p_1(R')\Big)$. Thus, the difference in total payment between the outcome z and the payment in the $(f, \mathbf{p})$ is $WP(1, 0; R'_2) + WP(2, 0; R'_3) - w_1(0) > 0$, where the inequality follows from Inequality (9). This is a contradiction to Pareto efficiency of $(f, \mathbf{p})$. Hence, $f_1(R') = 0$. By pareto efficiency, $f_2(R') + f_3(R') = 3$.

Proof of (3). Suppose for contradiction that $f_3(R') = 0$. This implies $f_2(R') = 3$ and by lemma 1, $p_3(R') = 0$. We first show that $p_2(R') = WP(3, 0; R'_2)$. Consider preference profile $(\tilde{R}_2, R'_{-2})$, where $\tilde{R}_2$ is quasilinear dichotomous preference with acceptable bundle of units 3 and value equal to $WP(3, 0; R'_2)$. Since agents 1 and 2 have the same acceptable bundle but agent 2 has lower willingness to pay (since $WP(3, 0; R'_2) < w_1(0)$ - if $k = 2$, it follows from Inequality 10, otherwise it follows from that fact that $R'_2 = R_0$ and Inequality 9), it is not Pareto efficient to allocate the acceptable bundle to agent 2.[10] Thus, $f_2(\tilde{R}_2, R'_{-2}) = 0$ and $p_2(\tilde{R}_2, R'_{-2}) = 0$. By DSIC, $(0, 0)\ \tilde{R}_2\ (3, p_2(R'))$. Hence, $p_2(R') \geq WP(3, 0; R'_2)$. By IR at R', we get $p_2(R') \leq WP(3, 0; R'_2)$. Hence, $p_2(R') = WP(3, 0; R'_2)$.

Now, consider the following outcome:

$$\hat{z}_1 := \big(3, w_1(0)\big),\ \hat{z}_2 := \big(0, 0\big),\ \ \hat{z}_3 := \big(0, 0\big)$$

[10] Suppose not. The outcome $\tilde{z}$ defined as: $\tilde{z}_1 := (3, w_1(0))$, $\tilde{z}_2 := (0, p_2(\tilde{R}) - WP(3, 0; R'_2))$, $\tilde{z}_i := (\tilde{R}_2, f_i(R'_{-2}), p_i(\tilde{R}_2, R'_{-2}))\ \forall i > 2$, Pareto dominates $(f(\tilde{R}_2, R'_{-2}), \mathbf{p}(\tilde{R}_2, R'_{-2}))$.

$$\hat{z}_i := (f_i(R'), p_i(R')),\ \forall i > 3.$$

All agents are indifferent between $\big(f(R'), p(R')\big)$ and $\hat{z}$. The difference between sum of transfer payments in $\hat{z}$ and in $(f, \mathbf{p})$ is $w_1(0) - WP(3, 0; R'_2) > 0$. This is a contradiction to Pareto efficiency.

Similarly, we can show that $f_2(R') \neq 0$. ■

Now, pick any $k \in \{2, 3\}$ and set $R'_k = R_0$ in Claim 5. By Inequality (9), Inequality (10) holds for R_0. As a result, we get that $f_2(R) \neq 0, f_3(R) \neq 0$, and $f_2(R) + f_3(R) = 3$. Hence, without loss of generality, assume that $f_2(R) = 1$ and $f_3(R) = 2$.

Arguing in the same manner as we did in Step 1 of Theorem 4, we would have [11]

$$p_2(R) = w_1(0) - WP(2, 0; R_0) \text{ and } p_3(R) = w_1(0) - WP(1, 0; R_0).$$

Now, we show that agent 2 can manipulate at R, thus contradicting DSIC and completing the proof. Consider a quasilinear dichotomous preference $\bar{R}^Q_2$ where the minimum acceptable bundle of agent 2 is 2 units ($= f_3(R)$) and his value $\bar{v}$ is $WP(2, 0; R_0)$. Consider the preference profile $\bar{R}$ where $\bar{R}_2 = \bar{R}^Q_2$ and $\bar{R}_i = R_i$ for all $i \neq 2$.

For $k = 2$, $x = 2$ and $R'_k = \bar{R}^Q_2$, Inequality 10 holds, and hence, Claim 5 implies $f_2(\bar{R}) \neq 0$. By Pareto efficiency, $f_2(\bar{R}) = 2$.

Then, again, we can mimic the argument in Step 1 of Theorem 4 to conclude that

$$p_2(\bar{R}) = w_1(0) - WP(1, 0; R_0).$$

Now, by the definition of willingness to pay,

$$\Big(2, WP(2, 0; R_0)\Big)\ I_0\ \Big(1, WP(1, 0; R_0)\Big)$$

and by our assumption, $WP(2, 0; R_0) > WP(1, 0; R_0)$. By subtracting $WP(1, 0; R_0) + WP(2, 0; R_0) - w_1(0)$ (which is positive by Inequality 9) from transfers on both sides, and using the fact that R_0 satisfies strong positive income effect, we get

$$\Big(2, w_1(0) - WP(1, 0; R_0)\Big)\ P_0\ \Big(1, w_1(0) - WP(2, 0; R_0)\Big).$$

Hence, $(f_2(\bar{R}), p_2(\bar{R}))\ P_2\ (f_2(R), p_2(R))$. This contradicts DSIC. ■

[11] The proof is identical - we replace $\emptyset$ by 0, $\{a\}$ by 1, $\{b\}$ by 2, and $\{a, b\}$ by 3 in the proof of Step 1 of Theorem 4.

B.2 Another robustness in heterogeneous goods case

We now consider the case where objects are heterogeneous and agents demand more than one object. Formally, we consider a model of heterogenous objects with multi-demand preferences.

Definition **12** *A classical preference R_i satisfies* **multi-demand** *if for every S,T with $S \subsetneq T$, we have*

$$(T,t)\ P_i\ (S,t) \quad \forall\ t \in \mathbb{R}.$$

Besides the multi-demand property, we impose two properties of multi-demand preference that we consider. To define them, we need some notation. At any classical preference R_i, for every $S,T \in \mathcal{B}$ and every $t \in \mathbb{R}$, define $V^{R_i}(S,(T,t))$ to be the transfer such that

$$(S, V^{R_i}(S,(T,t)))\ I_i\ (T,t).$$

By the assumptions of classical preferences, $V^{R_i}(S,(T,t))$ is a well-defined real number.

Definition **13** *A multi-demand classical preference R_i is* **flexible** *if it satisfies the following two conditions for every partition M_1, M_2 of M:*

- **A1.** *For every $t \in \mathbb{R}$,*

$$t < V^{R_i}(M_1,(M,t)) + V^{R_i}(M_2,(M,t)).$$

- **A2.** *Either $(M, WP(M_2, 0; R_i))\ P_i\ (M_1, 0)$ or $(M, WP(M_1, 0; R_i))\ P_i\ (M_2, 0)$.*

Out of the two assumptions of flexibility, **A1** requires some sort of *submodularity*. On the other hand, **A2** requires some leeway in constructing the submodular preferences. Figure 3 gives a pictorial description of flexible preferences by drawing indifference vectors.

With the help of multi-demand flexible preference, we can state the main result of this section.

Theorem **6** *Let R_0 be a multi-demand flexible classical preference. Consider any domain $\mathcal{T}$ containing $\mathcal{D}^{QL} \cup \{R_0\}$. Then, no desirable mechanism can be defined on $\mathcal{T}^n$.*

Proof: Assume for contradiction that $(f, \mathbf{p})$ is a desirable mechanism on $\mathcal{T}^n$. Consider a preference profile $R \in \mathcal{T}^n$ as follows:

1. $R_i = R_0$ for all $i \in \{1, 2\}$.

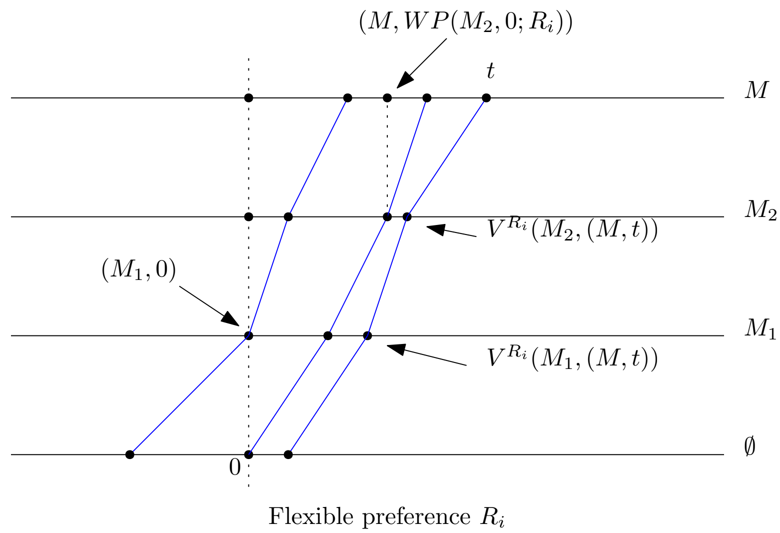

Figure 3: Illustration of flexible a preference

2. for all $i > 2$, let R_i be a quasilinear dichotomous preference with $\mathcal{S}_i^{min} = \{M\}$ and value equal to ϵ, where ϵ is very close to zero.

A straightforward consequence of Pareto efficiency is that $f_i(R) = \emptyset$ for all $i > 2$. By Lemma 1, $p_i(R) = 0$ for all $i > 2$. Hence, Pareto efficiency implies that $f_1(R) \cup f_2(R) = M$. We now do the proof in several steps.

Step 1. In the first step of the proof, we show that $f_1(R) \neq \emptyset$ and $f_2(R) \neq \emptyset$. Suppose $f_1(R) = \emptyset$. Then, $f_2(R) = M$ and Lemma 1 implies that $p_1(R) = 0$. Then, consider the following outcome

$$z_1 := (M_1, V^{R_0}(M_1, (M, p_2(R)))), z_2 := (M_2, V^{R_0}(M_2, (M, p_2(R)))), z_i := (f_i(R), p_i(R)) \,\forall\, i > 2.$$

By definition, $(M, p_2(R))\ I_0\ z_1\ I_0\ z_2$. Hence, $z_2\ I_2\ (f_2(R), p_2(R))$. By IR of agent 2, $(M, p_2(R))\ R_0\ (\emptyset, 0)$. Hence, $z_1\ R_1\ (f_1(R), p_1(R))$. By assumption **A1**,

$$p_2(R) < V^{R_0}(M_1, (M, p_2(R))) + V^{R_0}(M_2, (M, p_2(R))).$$

But the LHS of the above inequality is the sum of transfers at preference profile R in mechanism $(f, \mathbf{p})$ and the RHS is the sum of transfers in the outcome vector z. This contradicts Pareto efficiency of $(f, \mathbf{p})$. A similar proof shows that $f_2(R) \neq \emptyset$.

Step 2. From Step 1 and Pareto efficiency, we conclude that $f_1(R) = M_1^*, f_2(R) = M_2^*$, where $M_1^* \cup M_2^* = M$, $M_1^* \cap M_2^* = \emptyset$, and $M_1^*, M_2^* \neq \emptyset$. By assumption **A2**, either