Asymptotics and Simulation of Heavy-Tailed Processes

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Overview

Lecture 1 Heavy-Tailed Heuristics

- The One Big Jump Heuristics/The Heaviest Tail Wins/Breiman's Lemma
- Applying the Heavy-Tailed Heuristics: Lévy Processes/Stochastic Integrals/Linear Processes

Lecture 2 Regular Variation – the Technical Framework

- Regular Variation and Weak Convergence
- Convergence in the space $\mathbf{M}_0$
- The Quality of Heavy-Tailed Asymptotics

Lecture 3 Efficient Simulation of Heavy-Tailed Processes

- Introduction to Rare-Event Simulation
- Importance Sampling in a Heavy-Tailed Setting
- Markov Chain Monte Carlo in Rare-Event Simulation

Outline

The Objective

Heavy-tailed asymptotics and simulation are used to

- approximate the probability of extreme events,
- gain understanding of the underlying mechanism that is most likely to lead to extreme events. For example,
  - cumulative build-up,
  - shocks transferred through a system,
  - passage through bottleneck states,
  - etc. . .

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The Regular Variation Framework

We consider a regular variation framework where $\{X_n\}$ is a sequence of random variables, A_n a sequence of events, and the probabilities $\{p_n\}$ where

$$p_n = \mathbb{P}\{X_n \in A_n\},$$

form a regularly varying sequence; for any $\lambda > 0$,

$$\lim_{n \to \infty} \frac{p_{[\lambda n]}}{p_n} = \lambda^{-\alpha},$$

for some $\alpha \geq 0$ called the *index of regular variation*.

Outline

The One Big Jump Heuristic

- Let $\{Z_k\}$ be independent and identically distributed (iid) random variables.
- Suppose $\mathbb{P}\{Z_1 > n\}$ is regularly varying.
- Then, for each fixed $k \geq 1$,

$$\mathbb{P}\{Z_1 + \cdots + Z_k > n\} \sim k\mathbb{P}\{Z_1 > n\}, \quad \text{as } n \to \infty.$$

Notation: $a_n \sim b_n$ *if* $a_n/b_n \to 1$.

Proof

- Suppose for simplicity that $Z_1 \geq 0$.
- Lower bound (inclusion/exclusion):

$$\mathbb{P}\{S_k > n\} \geq \mathbb{P}\{\cup_{k=1}^{n} Z_k > n\} \geq k\mathbb{P}\{Z_1 > n\} - k(k-1)\mathbb{P}\{Z_1 > n\}^2.$$

- Upper bound: $(k = 2)$, $\epsilon > 0$ arbitrary,

$$\begin{aligned}
\mathbb{P}\{S_2 > n\} &= 2\mathbb{P}\{Z_1 + Z_2 > n, Z_2 \leq \epsilon n\} \\
&\quad + \mathbb{P}\{Z_1 + Z_2 > n, Z_1 > \epsilon n, Z_2 > \epsilon n\} \\
&\leq 2\mathbb{P}\{Z_1 > (1-\epsilon)n\} + \mathbb{P}\{Z_1 > n\epsilon\}^2 \\
&\sim 2(1-\epsilon)^{-\alpha}\mathbb{P}\{Z_1 > n\}.
\end{aligned}$$

The One Big Jump Heuristic

A General Version

- Let $\{Z_k\}$ be independent and identically distributed (iid) random variables.
- Suppose $\mathbb{P}\{Z_1 > n\}$ is regularly varying and put $S_n = Z_1 + \cdots + Z_n$.
- Then, there is uniform convergence:

$$\lim_{n\to\infty} \sup_{x \geq \lambda_n} \left| \frac{\mathbb{P}\{S_n > x\}}{n\mathbb{P}\{Z_1 > x\}} - 1 \right| = 0,$$

  for $\lambda_n \to \infty$ sufficiently fast.[1]
- Ex: ($\alpha > 2$): $\lambda_n = a\sqrt{n \log n}$, $a > \sqrt{\alpha - 2}$,
  ($\alpha = 2$): $\lambda_n / \sqrt{n^{1+\gamma}} \to \infty$, $\gamma > 0$,
  ($\alpha < 2$): $S_n/\lambda_n \to 0$, in probability.

[1] These conditions are called the Nagaev conditions. General conditions under supexponentiality are given in [3].

Outline

The Heaviest Tail Wins

Let Y and Z be random variables. Suppose $\mathbb{P}\{Z > n\}$ is regularly varying with index $-\alpha$ and $\mathbb{P}\{Y > n\} = o(\mathbb{P}\{Z > n\})$. Then[2]

$$\mathbb{P}\{Z + Y > n\} \sim \mathbb{P}\{Z > n\}.$$

[2] See [1].

Proof

Suppose for simplicity that Z and Y are non-negative. For arbitrary $\epsilon \in (0,1)$,

$$\begin{aligned}\mathbb{P}\{Z+Y>n\} &= \mathbb{P}\{Z+Y>n, Z>(1-\epsilon)n\}+\mathbb{P}\{Z+Y>n, Z\leq(1-\epsilon)n\}\\ &\leq \mathbb{P}\{Z>(1-\epsilon)n\}+\mathbb{P}\{Y>\epsilon n\}\\ &= \mathbb{P}\{Z>(1-\epsilon)n\}+o(\mathbb{P}\{Z>\epsilon n)\})\\ &\sim (1-\epsilon)^{-\alpha}\mathbb{P}\{Z>n\}.\end{aligned}$$

The reverse inequality is trivial when Y is non-negative.

Outline

Breiman's Lemma[3]

Let Y and Z be independent random variables with Y non-negative. Suppose $\mathbb{P}\{Z > n\}$ be regularly varying with index $-\alpha$ and $E[Y^{\alpha+\epsilon}] < \infty$ for some $\epsilon > 0$.
Then

$$\mathbb{P}\{YZ > n\} \sim E[Y^{\alpha}]\mathbb{P}\{Z > n\}.$$

[3] see [2]

Proof

Suppose for simplicity that Y is bounded by m. Then, by conditioning on Y,

$$\begin{aligned}\mathbb{P}\{YZ > n\} &= \int_0^m \mathbb{P}\{Z > n/y\}\mathbb{P}\{Y \in dy\} \\ &\sim \int_0^m y^\alpha \mathbb{P}\{Z > n\}\mathbb{P}\{Y \in dy\} \\ &= E[Y^\alpha]\mathbb{P}\{Z > n\}.\end{aligned}$$

Outline

Random Sums

- Let $\{Z_k\}$ be iid with $\mathbb{P}\{Z_1 > n\}$ regularly varying with index $-\alpha$.
- Let N be the random number of terms (N has sufficiently light tails, e.x. exponential), independent of $\{Z_k\}$.
- Determine the asymptotic decay of $\mathbb{P}\{S_N > n\}$, where $S_N = \sum_{k=1}^{N} Z_k$.

Random Sums

- Heuristic: Think of N as "not very large". Then, $S_N = \sum_{k=1}^N Z_k$ is large, most likely because precisely one of the Z_k's is large, so expect
$$\mathbb{P}\{S_N > n\} \sim \text{const}\, \mathbb{P}\{Z_1 > n\}.$$
- What is the constant?
- By conditioning on N:

$$\begin{aligned}\mathbb{P}\{S_N > n\} &= \sum_{k=1}^{\infty} \mathbb{P}\{N = k\}\mathbb{P}\{Z_1 + \cdots + Z_k > n\} \\ &\sim \sum_{k=1}^{\infty} \mathbb{P}\{N = k\} k \mathbb{P}\{Z_1 > n\} \\ &= E[N]\, \mathbb{P}\{Z_1 > n\}.\end{aligned}$$

See e.g. [7] for more details.

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Outline

Infinitely Divisible Laws

- X has an infinitely divisible law if, for each n, there are iid random variables $Y_{1,n}, \ldots, Y_{n,n}$ such that

$$X \stackrel{d}{=} Y_{1,n} + \cdots + Y_{n,n}.$$

- The Lévy-Itô decomposition states that X can be represented in law as a sum of three independent parts

$$X \stackrel{d}{=} \mu + \sum_{k=1}^{N} Z_k + \text{ small jumps } + \text{ Gaussian.}$$

where $Z_k \geq 1$ is distributed according to $\nu(\cdot)/\nu(1,\infty)$, ν is the Lévy measure, and N has a Poisson distribution with mean $\nu(1,\infty)$.

Infinitely Divisible Laws

- Suppose $\nu(n,\infty)$ is regularly varying with index $-\alpha$, so $\mathbb{P}\{Z_1 > n\}$ is regularly varying with index $-\alpha$.
- Then

$$X \stackrel{d}{=} \mu + \sum_{k=1}^{N} Z_k + \underbrace{\text{small jumps}}_{\text{light tails}} + \underbrace{\text{Gaussian}}_{\text{light tails}}.$$

- The random sum $\sum_{k=1}^{N} Z_k$ satisfies

$$\mathbb{P}\Big\{\sum_{k=1}^{N} Z_k > n\Big\} \sim E[N]\,\mathbb{P}\{Z_1 > n\} = \nu(1,\infty)\frac{\nu(n,\infty)}{\nu(1,\infty)} = \nu(n,\infty).$$

- The **heaviest tail wins** argument implies that

$$\mathbb{P}\{X > n\} \sim \nu(n,\infty).$$

The reverse implication also holds, even under subexponentiality, see [4].

Outline

Stochastic Integrals[4]

- Consider a Lévy process X with regularly varying Lévy measure ν (index α).
- Let Y be an adapted process with lighter tails than ν:

$$E[\sup_{t\in[0,1]} Y_t^{\alpha+\epsilon}] < \infty, \qquad \text{some } \epsilon > 0.$$

- Consider tail probabilities $\mathbb{P}\{\int_0^1 Y_t dX_t > n\}$

[4]This part is based on [5]

Stochastic Integrals

Decomposing the Lévy process as

$$X_t \stackrel{\text{d}}{=} \sum_{k=1}^{N_t} \Delta X_{\tau_k} + \underbrace{S_t}_{\text{small jumps}} + \underbrace{W_t}_{\text{Gaussian}},$$

the stochastic integral becomes

$$\int_0^1 Y_t dX_t \stackrel{\text{d}}{=} \sum_{k=1}^{N_1} Y_{\tau_k} \Delta X_{\tau_k} + \underbrace{\int_0^1 Y_t dS_t}_{\text{light tails}} + \underbrace{\int_0^1 Y_t dW_t}_{\text{light tails}},$$

the **heaviest tail wins** argument tells us that $\sum_{k=1}^{N_t} Y_{\tau_k} \Delta X_{\tau_k}$ is the most important term (if it has heavy tails).

Stochastic Integrals

Studying $\sum_{k=1}^{N_t} Y_{\tau_k}\Delta X_{\tau_k}$

- Each term $Y_{\tau_k}\Delta X_{\tau_k}$ is a product of independent rv's so **Breiman's Lemma** implies that each term satisfies

$$\mathbb{P}\{Y_{\tau_k}\Delta X_{\tau_k} > n\} \sim E[Y_{\tau_k}^{\alpha}]\mathbb{P}\{\Delta X_{\tau_k} > n\}.$$

- The terms $Y_{\tau_k}\Delta X_{\tau_k}, k = 1, 2, \dots$ are not independent, but the **one big jump heuristic** tells us that, most likely, only one ΔX_{τ_k} is large, so one expects that

$$\mathbb{P}\{\int_0^1 Y_t dX_t > n\} \sim \mathbb{P}\{\sum_{k=1}^{N_1} Y_{\tau_k}\Delta X_{\tau_k} > n\} \sim E[N_1]\mathbb{P}\{Y_{\tau}\Delta X_{\tau} > n\}$$
$$\sim E[Y_{\tau}^{\alpha}]E[N_1]\mathbb{P}\{\Delta X_{\tau} > n\} = E[Y_{\tau}^{\alpha}]\nu(n,\infty),$$

where τ is the time of the big jump.

Outline

Moving Average Processes

- MA(2) process: Let $\{Z_k\}$ be iid regularly varying (α) and A_0, A_1 constants. Put

$$X_k = A_0 Z_k + A_1 Z_{k-1}, \qquad k \geq 1.$$

- The **one big jump heuristic** implies that

$$\mathbb{P}\{X_1 > n\} \sim \mathbb{P}\{A_0 Z_1 > n\} + \mathbb{P}\{A_1 Z_0 > n\} \sim (A_0^\alpha + A_1^\alpha)\mathbb{P}\{Z_1 > n\}.$$

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Linear Processes[5]

- For the linear process

$$X_k = \sum_{j=0}^{\infty} A_j Z_{k-j},$$

  with $E[Z_k] = 0$ if $\alpha > 1$, it is necessary that the coefficients decay sufficiently fast.

- If

$$\sum |A_j|^{\alpha-\epsilon} < \infty, \quad \text{for some } \epsilon > 0, \quad \alpha \leq 2,$$
$$\sum |A_j|^2 < \infty, \quad \alpha > 2,$$

  then

$$\mathbb{P}\{X_k > n\} \sim \sum_j A_j^\alpha I\{A_j > 0\} \mathbb{P}\{Z_1 > n\}.$$

[5] See [8] for details.

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Moving Average Processes

Random Coefficients

- If A_0, A_1 are random, non-negative, and independent of $\{Z_k\}$ with $E[A_k^{\alpha+\epsilon}] < \infty$, $k = 0, 1$, then **Breiman's Lemma** together with the **one big jump heuristic** implies that

$$X_k = A_0 Z_k + A_1 Z_{k-1}, \qquad k \geq 1$$

satisfies

$$\begin{aligned} \mathbb{P}\{X_1 > n\} &\sim \mathbb{P}\{A_0 Z_1 > n\} + \mathbb{P}\{A_1 Z_0 > n\} \\ &\sim (E[A_0^\alpha] + E[A_1^\alpha])\mathbb{P}\{Z_1 > n\}. \end{aligned}$$

Linear Processes

Random Coefficients[6]

- In the case of random (non-negative) coefficients $\{A_j\}$ that are (essentially) independent of $\{Z_k\}$ the conditions:

$$\sum EA_j^{\alpha-\epsilon} < \infty, \text{ and } \sum EA_j^{\alpha+\epsilon} < \infty, \text{ some } \epsilon > 0,\ \alpha \in (0,1) \cup (1,2),$$

$$E\Big(\sum A_j^{\alpha-\epsilon}\Big)^{\frac{\alpha+\epsilon}{\alpha-\epsilon}} < \infty, \quad \alpha = 1 \text{ or } 2,$$

$$E\Big(\sum A_j^{2}\Big)^{\frac{\alpha+\epsilon}{2}} < \infty, \quad \alpha > 2,$$

imply that

$$\mathbb{P}\{X_k > n\} \sim \sum_j E[A_j^\alpha]\mathbb{P}\{Z_1 > n\}.$$

[6]See [6] for details.

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