Asymptotics and Simulation of Heavy-Tailed Processes

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Outline

Simulation of Heavy-Tailed Processes

- Goal: improve on the computational efficiency of standard Monte Carlo.
- Design: The large deviations analysis lead to a heuristic way to design efficient algorithms.
- Efficiency Analysis: The large deviations analysis can be applied to theoretically quantify the computational performance of algorithms.

Outline

The Performance of Monte Carlo

- Let $X^1, \ldots, X^N$ be independent copies of X.
- The Monte Carlo estimator is

$$\widehat{p} = \frac{1}{N} \sum_{k=1}^{N} I\{X^k > b\}.$$

- Its standard deviation is

$$\text{std} = \frac{1}{\sqrt{N}} \sqrt{p(1-p)}$$

- To obtain a Relative Error (std/p) of 1% you need to take N such that

$$\frac{\frac{1}{\sqrt{N}}\sqrt{p(1-p)}}{p} \leq 0.01 \quad \Leftrightarrow N \geq 10^4 \frac{1-p}{p}$$

Outline

Introduction to Importance Sampling[1]

- The problem with Monte Carlo is that few samples hit the rare event.
- This problem can be fixed by sampling from a distribution which puts more mass on the rare event.
- Must compensate for not sampling from the original distribution.
- Key issue: How to select an appropriate sampling distribution?

[1] See e.g. [1] for an introduction.

Monte Carlo and Importance Sampling

The Basics for Computing $F(A) = \mathbb{P}\{X \in A\}$

Monte Carlo

- Sample $X^1, \dots, X^N$ from F.
- Empirical measure

$$\mathbf{F}_N(\cdot) = \frac{1}{N} \sum_{k=1}^{N} \delta_{X^k}(\cdot).$$

- Plug-in estimator

$$\widehat{p} = \mathbf{F}_N(A).$$

Importance Sampling

- Sample $\widetilde{X}^1, \dots, \widetilde{X}^N$ from $\widetilde{F}$.
- Weighted empirical measure

$$\widetilde{\mathbf{F}}^w_N(\cdot) = \frac{1}{N} \sum_{k=1}^{N} \frac{dF}{d\widetilde{F}}(\widetilde{X}^k)\delta_{\widetilde{X}^k}(\cdot).$$

- Plug-in estimator

$$\widehat{p} = \widetilde{\mathbf{F}}^w_N(A) = \frac{1}{N} \sum_{k=1}^{N} \frac{dF}{d\widetilde{F}}(\widetilde{X}^k) I\{\widetilde{X}^k \in A\}.$$

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Importance Sampling

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- Plug-in estimator

$$\widehat{p} = \widetilde{\mathbf{F}}_N^w(A) = \frac{1}{N}\sum_{k=1}^{N} \frac{dF}{d\widetilde{F}}(\widetilde{X}^k)I\{\widetilde{X}^k \in A\}.$$

Importance Sampling

- The importance sampling estimator is unbiased:

$$\widetilde{E}\left[\frac{dF}{d\widetilde{F}}(\widetilde{X})I\{\widetilde{X} \in A\}\right] = \int_A \frac{dF}{d\widetilde{F}}d\widetilde{F} = \int_A dF = F(A).$$

- Its variance is

$$\begin{aligned}\mathrm{Var}\left(\frac{dF}{d\widetilde{F}}(\widetilde{X})I\{\widetilde{X} \in A\}\right) &= \widetilde{E}\left[\left(\frac{dF}{d\widetilde{F}}\right)^2 I\{\widetilde{X} \in A\}\right] - F(A)^2 \\ &= \int_A \left(\frac{dF}{d\widetilde{F}}\right)^2 d\widetilde{F} - F(A)^2 \\ &= \int_A \frac{dF}{d\widetilde{F}} dF - F(A)^2 \\ &= E\left[\frac{dF}{d\widetilde{F}} I\{X \in A\}\right] - F(A)^2.\end{aligned}$$

Importance Sampling

- The importance sampling estimator is unbiased:

$$\widetilde{E}\left[\frac{dF}{d\widetilde{F}}(\widetilde{X})I\{\widetilde{X}\in A\}\right]=\int_A \frac{dF}{d\widetilde{F}}d\widetilde{F}=\int_A dF=F(A).$$

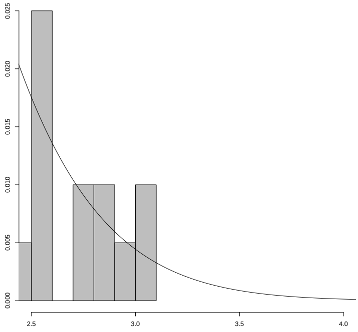
- Its variance is

$$\begin{aligned}\mathrm{Var}\Big(\frac{dF}{d\widetilde{F}}(\widetilde{X})I\{\widetilde{X}\in A\}\Big)&=\widetilde{E}\left[\Big(\frac{dF}{d\widetilde{F}}\Big)^2 I\{\widetilde{X}\in A\}\right]-F(A)^2\\&=\int_A\Big(\frac{dF}{d\widetilde{F}}\Big)^2 d\widetilde{F}-F(A)^2\\&=\int_A\frac{dF}{d\widetilde{F}}dF-F(A)^2\\&=E\left[\frac{dF}{d\widetilde{F}}I\{X\in A\}\right]-F(A)^2.\end{aligned}$$

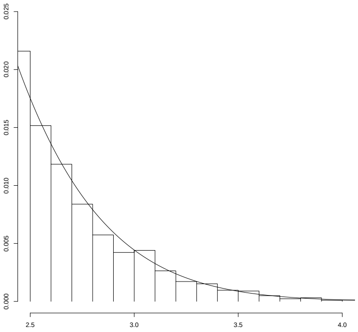
Illustration

Sampling the tail

Monte Carlo



Importance Sampling



Quantifying Efficiency

Variance Based Efficiency Criteria

- Basic idea: variance roughly the size of p^2.
- Embed in a sequence of problems such that $p_n = \mathbb{P}\{X_n \in A\} \to 0$.
- Logarithmic Efficiency: for some $\epsilon > 0$

$$\limsup_{n\to\infty} \frac{\mathrm{Var}(\widehat{p}_n)}{p_n^{2-\epsilon}} < \infty.$$

- Strong Efficiency/Bounded Relative Error:

$$\sup_n \frac{\mathrm{Var}(\widehat{p}_n)}{p_n^2} < \infty.$$

- Vanishing relative error:

$$\limsup_n \frac{\mathrm{Var}(\widehat{p}_n)}{p_n^2} = 0.$$

The Zero-Variance Change of Measure

- There is a best choice of sampling distribution, the zero-variance change of measure, given by

$$F_A(\cdot) = \mathbb{P}\{X_n \in \cdot \mid X_n \in A\}.$$

- For this choice

$$\frac{dF}{dF_A}(x) = \mathbb{P}\{X_n \in A\} I\{X_n \in A\} = p_n I\{X_n \in A\},$$

and hence

$$\mathrm{Var}(\widehat{p}_n) = E\left[p_n I\{X_n \in A\}\right] - F(A)^2 = 0.$$

Example Problem

A Random Walk with Heavy Tails

- Let $\{Z_k\}$ be iid non-negative and regularly varying (index α) with density f.
- Put $S_m = Z_1 + \cdots + Z_m$.
- Compute $\mathbb{P}\{S_m > n\}$.
- The zero-variance sampling distribution is

$$\mathbb{P}\{(Z_1, \ldots, Z_m) \in \cdot \mid S_m > n\}.$$

- By regular variation we know that

$$\mathbb{P}\{(Z_1, \ldots, Z_m) \in \cdot \mid S_m > n\} \to \mu(\cdot),$$

where μ is concentrated on the coordinate axis. But μ and F are singular!

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Outline

The Conditional Mixture Algorithm[2]

- Suppose $S_{i-1} = s$. Sample Z_i as follows
  1. If $s \geq n$, sample Z_i from the original density f.
  2. If $s \leq n$, sample Z_i from the mixture

$$
\begin{aligned}
&p_i f(z) + (1 - p_i)\tilde{f}_i(y \mid s), \quad 1 \leq i \leq m - 1, \\
&\tilde{f}_m(y \mid s), \quad i = m.
\end{aligned}
$$

- Idea: take $\tilde{f}_i$ to produce large values of Z_i.

[2] This algorithm is due to [4]

The Conditional Mixture Algorithm

- We can take $\tilde{f}_i$'s to be the conditional distributions of the form

$$\tilde{f}_i(z \mid s) = \frac{f(z)I\{z > a(n-s)\}}{\mathbb{P}\{Z > a(n-s)\}}, \quad i \leq m-1,$$
$$\tilde{f}_m(z \mid s) = \frac{f(z)I\{z > n-s\}}{\mathbb{P}\{Z > n-s\}},$$

  where $a \in (0,1)$.
- Note: $\tilde{f}_i$ is the conditional distribution of Z given that Z is large.

Outline

The Performance of the Conditional Mixture Algorithm

- The normalized second moment is

$$\frac{E[\widehat{p}_n]}{p_n^2} = \frac{1}{p_n^2}\int \frac{dF}{d\widetilde{F}}(z_1,\dots,z_m)I\{s_m > n\}F(dz_1,\dots,dz_m)$$
$$\sim \frac{1}{m^2\mathbb{P}\{Z_1 > n\}^2}\int \frac{dF}{d\widetilde{F}}(z_1,\dots,z_m)I\{s_m > n\}F(dz_1,\dots,dz_m)$$
$$\sim \frac{1}{m^2}\int \frac{1}{\mathbb{P}\{Z_1 > n\}}\frac{dF}{d\widetilde{F}}(nz_1,\dots,nz_m)I\{s_m > 1\}\frac{F(ndz_1,\dots,ndz_m)}{\mathbb{P}\{Z_1 > n\}}.$$

The Performance of the Conditional Mixture Algorithm

- Weak convergence:

$$\frac{F(n\,\cdot)}{\mathbb{P}\{Z_1 > n\}} \to \sum_{k=1}^{m} \int I\{ze_k \in \cdot\}\mu_\alpha(dz)$$

- The normalized likelihood ratio is bounded:

$$\sup_n \frac{1}{\mathbb{P}\{Z_1 > n\}} \frac{dF}{d\widetilde{F}}(nz_1, \ldots, nz_m) I\{s_m > 1\} < \infty.$$

The Performance of the Conditional Mixture Algorithm

- The above enables us to show that the normalized second moment converges:

$$\lim_n \frac{E[\widehat{p}_n]}{p_n^2} = \frac{1}{m^2}\Big(\sum_{i=1}^{m-1} \frac{a^{-\alpha}}{1-p_i} \prod_{j=1}^{i-1} \frac{1}{p_j} + \prod_{j=1}^{m-1} \frac{1}{p_j}\Big).$$

- It is minimized at

$$p_i = \frac{(m-i-1)a^{-\alpha/2}+1}{(m-i)a^{-\alpha/2}+1},$$

with minimum

$$\frac{1}{m^2}\Big((m-1)a^{-\alpha/2}+1\Big)^2.$$

The Performance of the Conditional Mixture Algorithm

- The conditional mixture algorithm has (almost) vanishing relative error.
- Heavy-tailed heuristics indicate how to design the algorithm.
- Heavy-tailed asymptotics needed to prove efficiency of the algorithm.

MCMC for Rare Events[3]

- Let X be random variable with density f and compute $p = \mathbb{P}\{X \in A\}$, where A is a rare event.
- The zero-variance sampling distribution is

$$F_A(\cdot) = \mathbb{P}\{X \in \cdot \mid X \in A\}, \quad \frac{dF_A}{dx}(x) = \frac{f(x)I\{x \in A\}}{p}.$$

- It is possible to sample from F_A by constructing a Markov chain with stationary distribution F_A (e.g. Gibbs sampler or Metropolis-Hastings).
- Idea: sample from F_A and extract the normalizing constant p.

[3] This part is based on [5]

MCMC for Rare Events

- Let $X_0, \ldots, X_{T-1}$ be a sample of a Markov chain with stationary distribution F_A.
- Consider a non-negative function $v(x)$ with $\int_A v(x)dx = 1$. The sample mean

$$\widehat{q}_T = \frac{1}{T} \sum_{t=0}^{T-1} \frac{v(X_t) I\{X_t \in A\}}{f(X_t)},$$

is an estimator of

$$E_{F_A}\left[\frac{v(X) I\{X \in A\}}{f(X)}\right] = \int_A \frac{v(x)}{f(x)} \frac{f(x)}{p} dx = \frac{1}{p} \int_A v(x)dx = \frac{1}{p}.$$

- Take $\widehat{q}_T$ as the estimator of $1/p$.

MCMC for Rare Events

- The rare-event properties of $\widehat{q}_T$ are determined by the choice of v.
- The large sample properties of $\widehat{q}_T$ are determined by the ergodic properties of the Markov chain.

The Normalized Variance

- The rare-event efficiency (p small) is determined by the normalized variance:

$$
\begin{aligned}
p^2 \mathrm{Var}_{F_A}&\Big(\frac{v(X)}{f(X)} I\{X \in A\}\Big) \\
&= p^2 \Big(E_{F_A}\Big[\Big(\frac{v(X)}{f(X)} I\{X \in A\}\Big)^2\Big] - \frac{1}{p^2}\Big) \\
&= p^2 \Big(\int \frac{v^2(x)}{f^2(x)} \frac{f(x)}{p} dx - \frac{1}{p^2}\Big) \\
&= p \int_A \frac{v^2(x)}{f(x)} dx - 1.
\end{aligned}
$$

The Optimal Choice of v

- It is optimal to take v as $f(x)/p$. Indeed, then

$$p^2 \mathrm{Var}_{F_A}\left(\frac{v(X)}{f(X)} I\{X \in A\}\right) = p \int_A \frac{v^2(x)}{f(x)} dx - 1 = 0.$$

- Insight: take v as an approximation of the conditional density given the event.

Outline

Efficient Sampling for a Heavy-Tailed Random Walk

- Let $\{Z_k\}$ be iid with density f.
- Put $S_n = Z_1 + \cdots + Z_n$.
- Compute $\mathbb{P}\{S_n > a_n\}$.
- Heavy-tailed assumption:

$$\lim_{n\to\infty} \frac{\mathbb{P}\{S_n > a_n\}}{\mathbb{P}\{M_n > a_n\}} = 1.$$

(includes subexponential distributions, see [3]).

Gibbs Sampling of a Heavy-Tailed Random Walk

1. Start from a state with $Z_1 > a_n$.
2. Update the steps in a random order according to $j_1, \ldots, j_n$ (uniformly without replacement).
3. Update Z_{j_k} by sampling from

$$\mathbb{P}\{Z \in \cdot \mid Z + \sum_{i \neq j_k} Z_i > n\}.$$

- The vector $(Z_{t,1}, \ldots, Z_{t,n})'$ forms a uniformly ergodic Markov chain with stationary distribution

$$F_A(\cdot) = \mathbb{P}\{(Z_1, \ldots, Z_n)' \in \cdot \mid Z_1 + \ldots Z_n > a_n\}.$$

Rare-Event Efficiency

The Choice of v

- Take v to be the density of $\mathbb{P}\{(Z_1,\dots,Z_n)' \in \cdot \mid M_n > a_n\}$.
- Then

$$\frac{v(z_1,\dots,z_n)}{f(z_1,\dots,z_n)} = \frac{1}{\mathbb{P}\{M_n > a_n\}} I\{\vee_{i=1}^n z_i > a_n\}.$$

- Rare event efficiency:

$$\begin{aligned} p_n^2 \mathrm{Var}_{F_A}\Big(\frac{v(Z_1,\dots,Z_n)}{f(Z_1,\dots,Z_n)} I\{Z_1+\dots+Z_n > a_n\}\Big) &= \frac{\mathbb{P}\{S_n > a_n\}^2}{\mathbb{P}\{M_n > a_n\}^2}\mathbb{P}\{M_n > a_n \mid S_n > a_n\}\mathbb{P}\{M_n \le a_n \mid S_n > a_n\} \\ &= \frac{\mathbb{P}\{S_n > a_n\}}{\mathbb{P}\{M_n > a_n\}}\Big(1 - \frac{\mathbb{P}\{M_n > a_n\}}{\mathbb{P}\{S_n > a_n\}}\Big) \to 0. \end{aligned}$$

Rare-Event Efficiency

The Choice of v

- Take v to be the density of $\mathbb{P}\{(Z_1, \dots, Z_n)' \in \cdot \mid M_n > a_n\}$.
- Then

$$\frac{v(z_1, \dots, z_n)}{f(z_1, \dots, z_n)} = \frac{1}{\mathbb{P}\{M_n > a_n\}} I\{\vee_{i=1}^n z_i > a_n\}.$$

- Rare event efficiency:

$$\begin{aligned}
p_n^2 \mathrm{Var}_{F_A}&\Big(\frac{v(Z_1, \dots, Z_n)}{f(Z_1, \dots, Z_n)} I\{Z_1 + \dots + Z_n > a_n\}\Big) \\
&= \frac{\mathbb{P}\{S_n > a_n\}^2}{\mathbb{P}\{M_n > a_n\}^2} \mathbb{P}\{M_n > a_n \mid S_n > a_n\} \mathbb{P}\{M_n \le a_n \mid S_n > a_n\} \\
&= \frac{\mathbb{P}\{S_n > a_n\}}{\mathbb{P}\{M_n > a_n\}} \Big(1 - \frac{\mathbb{P}\{M_n > a_n\}}{\mathbb{P}\{S_n > a_n\}}\Big) \to 0.
\end{aligned}$$

Numerical illustration

$n = 5$, $\alpha = 2$, $a_n = an$.

$b = 25$, $T = 10^5$, $\alpha = 2$, $n = 5$, $a = 5$, $p_{\max} = 0.737\text{e-}2$			
	MCMC	IS	MC
Avg. est. Std. dev. Avg. time per batch(s)	1.050e-2 3e-5 12.8	1.048e-2 9e-5 12.7	1.053e-2 27e-5 1.4
$b = 25$, $T = 10^5$, $\alpha = 2$, $n = 5$, $a = 20$, $p_{\max} = 4.901\text{e-}4$			
	MCMC	IS	MC
Avg. est. Std. dev. Avg. time per batch(s)	5.340e-4 6e-7 14.4	5.343e-4 13e-7 13.9	5.380e-4 770e-7 1.5
$b = 20$, $T = 10^5$, $\alpha = 2$, $n = 5$, $a = 10^3$, $p_{\max} = 1.9992\text{e-}7$			
	MCMC	IS	
Avg. est. Std. dev. Avg. time per batch(s)	2.0024e-7 3e-11 15.9	2.0027e-7 20e-11 15.9	
$b = 20$, $T = 10^5$, $\alpha = 2$, $n = 5$, $a = 10^4$, $p_{\max} = 1.99992\text{e-}9$			
	MCMC	IS	
Avg. est. Std. dev. Avg. time per batch(s)	2.00025e-9 7e-14 15.9	2.00091e-9 215e-14 15.9	

Numerical Illustration

$n = 20$, $\alpha = 2$, $a_n = an$.

$b = 25$, $T = 10^5$, $\alpha = 2$, $n = 20$, $a = 20$, $p_{\max} = 1.2437\text{e-}4$			
	MCMC	IS	MC
Avg. est.	1.375e-4	1.374e-4	1.444e-4
Std. dev.	2e-7	3e-7	492e-7
Avg. time per batch(s)	52.8	50.0	2.0
$b = 25$, $T = 10^5$, $\alpha = 2$, $n = 20$, $a = 200$, $p_{\max} = 1.2494\text{e-}6$			
	MCMC	IS	MC
Avg. est.	1.2614e-6	1.2615e-6	1.2000e-6
Std. dev.	4e-10	12e-10	33,166e-10
Avg. time per batch(s)	49.4	48.4	1.9
$b = 20$, $T = 10^5$, $\alpha = 2$, $n = 20$, $a = 10^3$, $p_{\max} = 4.9995\text{e-}8$			
	MCMC	IS	
Avg. est.	5.0091e-8	5.0079e-8	
Std. dev.	7e-12	66e-12	
Avg. time per batch(s)	53.0	50.6	
$b = 20$, $T = 10^5$, $\alpha = 2$, $n = 20$, $a = 10^4$, $p_{\max} = 5.0000\text{e-}10$			
	MCMC	IS	
Avg. est.	5.0010e-10	5.0006e-10	
Std. dev.	2e-14	71e-14	
Avg. time per batch(s)	48.0	47.1	

Outline

Efficient Sampling for a Heavy-Tailed Random Sum

- Let $\{Z_k\}$ be iid with density f.
- Put $S_n = Z_1 + \cdots + Z_n$ and let $\{N_n\}$ be independent of $\{Z_k\}$
- Compute $\mathbb{P}\{S_{N_n} > a_n\}$.
- Heavy-tailed assumption:[4]

$$\lim_{n\to\infty} \frac{\mathbb{P}\{S_{N_n} > a_n\}}{\mathbb{P}\{M_{N_n} > a_n\}} = 1.$$

[4] See e.g. [6] for examples.

Constructing the Gibbs Sampler

- Suppose we are in state $(N_t, Z_{t,1}, \dots, Z_{t,N_t}) = (k_t, z_{t,1}, \dots, z_{t,k_t})$.
- Let $k_t^* = \min\{j : z_{t,1} + \cdots + z_{t,j} > a_n\}$.
- Update the number of steps N_{t+1} from the distribution

$$p(k_{t+1} \mid k_t^*) = \mathbb{P}\{N = k_{t+1} \mid N \geq k^{*t}\}$$

- If $k_{t+1} > k_t$, sample $Z_{t+1,k_t+1}, \dots, Z_{t+1,k_{t+1}}$ independently from F_Z.
- Proceed by updating all steps as before.
- Permute the steps.

Properties of the algorithm

- The Markov chain $\{(N_t, Z_{t,1}, \ldots, Z_{t,N_t})\}$ is uniformly ergodic with stationary distribution

$$F_A(\cdot) = \mathbb{P}\{(Z_1, \ldots, Z_n)' \in \cdot \mid Z_1 + \ldots Z_n > a_n\}.$$

- With v as the density of $\mathbb{P}\{(N, Z_1, \ldots, Z_N) \in \cdot \mid M_N > a_n\}$ we have

$$\begin{aligned} \frac{v(k, z_1, \ldots, z_k)}{f(k, z_1, \ldots, z_k)} &= \frac{1}{\mathbb{P}\{M_N > a_n\}} I\{\max z_1, \ldots, z_k > a_n\} \\ &= \frac{1}{1 - g_N(F_Z(a_n))} I\{\max z_1, \ldots, z_k > a_n\}. \end{aligned}$$

- The algorithm has vanishing normalized variance.

Numerical illustration

Geometric(ρ) number of steps: $\rho = 0.05$, $\alpha = 1$, $a_n = a/\rho$.

$b = 25$, $T = 10^5$, $\alpha = 1$, $\rho = 0.2$, $a = 10^2$, $p_{\max} = 0.990\text{e-}2$			
	MCMC	IS	MC
Avg. est. Std. dev. Avg. time per batch(s)	1.149e-2 4e-5 25.0	1.087e-2 6e-5 11.0	1.089e-2 35e-5 1.2
$b = 25$, $T = 10^5$, $\alpha = 1$, $\rho = 0.2$, $a = 10^3$, $p_{\max} = 0.999\text{e-}3$			
	MCMC	IS	MC
Avg. est. Std. dev. Avg. time per batch(s)	1.019e-3 1e-6 25.8	1.012e-3 3e-6 11.1	1.037e-3 76e-6 1.2
$b = 20$, $T = 10^6$, $\alpha = 1$, $\rho = 0.2$, $a = 5 \cdot 10^7$, $p_{\max} = 2.000000\text{e-}8$			
	MCMC	IS	
Avg. est. Std. dev. Avg. time per batch(s)	2.000003e-8 6e-14 385.3	1.999325e-8 1114e-14 139.9	
$b = 20$, $T = 10^6$, $\alpha = 1$, $\rho = 0.2$, $a = 5 \cdot 10^9$, $p_{\max} = 2.0000\text{e-}10$			
	MCMC	IS	
Avg. est. Std. dev. Avg. time per batch(s)	2.0000e-10 0 358.7	1.9998e-10 13e-14 130.9	

Numerical Illustration

Geometric(ρ) number of steps: $\rho = 0.05$, $\alpha = 1$, $a_n = a/\rho$.

$b = 25$, $T = 10^5$, $\alpha = 1$, $\rho = 0.05$, $a = 10^3$, $p_{\max} = 0.999\text{e-}3$			
	MCMC	IS	MC
Avg. est.	1.027e-3	1.017e-3	1.045e-3
Std. dev.	1e-6	4e-6	105e-6
Avg. time per batch(s)	61.5	44.8	1.3
$b = 25$, $T = 10^5$, $\alpha = 1$, $\rho = 0.05$, $a = 5 \cdot 10^5$, $p_{\max} = 1.9999\text{e-}6$			
	MCMC	IS	MC
Avg. est.	2.0002e-6	2.0005e-6	3.2000e-6
Std. dev.	1e-10	53e-10	55,678e-10
Avg. time per batch(s)	60.7	45.0	1.3

For Further Reading I

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