EXTREME EIGENVALUES OF RANDOM MATRICES

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Collaborators

Almost all results are joint work with

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Examples

Random matrix: Matrix with random entries.

Wigner matrix (W_n): $x_{i,j} = x_{j,i}$.

$$W_n = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1(n-1)} & x_{1n} \\ x_{12} & x_{22} & x_{23} & \cdots & x_{2(n-1)} & x_{2n} \\ & & & \vdots & & \\ x_{1n} & x_{2n} & x_{3n} & \cdots & x_{(n-1)n} & x_{nn} \end{bmatrix}.$$

(Symmetric) Toeplitz matrix (T_n): $x_{i,j} = x_{|i-j|}$.

$$T_n = \begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\ x_1 & x_0 & x_1 & \cdots & x_{n-3} & x_{n-2} \\ x_2 & x_1 & x_0 & \cdots & x_{n-4} & x_{n-3} \\ & & & \vdots & & \\ x_{n-1} & x_{n-2} & x_{n-3} & \cdots & x_1 & x_0 \end{bmatrix}.$$

More examples

Hankel matrix (H_n): $x_{i,j} = x_{i+j}$.

$$H_n = \begin{bmatrix} x_2 & x_3 & x_4 & \cdots & x_n & x_{n+1} \\ x_3 & x_4 & x_5 & \cdots & x_{n+1} & x_{n+2} \\ x_4 & x_5 & x_6 & \cdots & x_{n+2} & x_{n+3} \\ & & & \vdots & & \\ x_{n+1} & x_{n+2} & x_{n+3} & \cdots & x_{2n-1} & x_{2n} \end{bmatrix}.$$

Sample covariance matrix (S): $n^{-1}X_nX_n^t$ where $X_n = ((x_{i,j}))_{1\le i\le p; 1\le j\le n}$ where columns of X are i.i.d.

Other matrices of the form $n^{-1}X_nX_n^t$.

Band matrices, triangular matrices... k-Circulant matrix. Special cases: circulant matrix, reverse circulant matrix, symmetric circulant matrix.

We shall always assume the entries have mean zero variance one unless mentioned otherwise.

k-Circulant matrix

For positive integers k and n, $A_{k,n}$ equals

$$\begin{bmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\ x_{n-k} & x_{n-k+1} & x_{n-k+2} & \cdots & x_{n-k-2} & x_{n-k-1} \\ x_{n-2k} & x_{n-2k+1} & x_{n-2k+2} & \cdots & x_{n-2k-2} & x_{n-2k-1} \\ & & & \vdots & & \\ x_k & x_{k+1} & x_{k+2} & \cdots & x_{k-2} & x_{k-1} \end{bmatrix}$$

$(j+1)$-th row is obtained by giving the j-th row a right circular shift by k positions. Note that all subscripts appearing above are calculated modulo n.

$A_{1,n}$ = Circulant matrix (C_n).

$A_{n-1,n}$ = Reverse circulant matrix (RC_n).

Symmetric circulant matrix (SC_n): Structure same as circulant with first row as a palindrome, $(x_0, x_1, x_2, \ldots, x_2, x_1)$.

Broad aim: Study the behaviour of the eigenvalues, specially when $n \to \infty$.

Bulk behaviour

Convergence of the Empirical spectral distribution (ESD)

$$F_n(x) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}(\lambda_k \leq x)$$

For example,

Wigner: semi-circle law: $[-2,\ 2]$.

S matrix: Marchenko-Pastur law: $[0,\ 4]$.

RC_n: symmetrized square root of χ_2^2 (exponential).

C_n: bivariate standard normal.

For k circulants, limit depends on the value of k (which may also be changing with n).

T_n, H_n: limits exists but not known in closed form, $(-\infty,\ \infty)$.

Edge behaviour

Behaviour of the spectral radius, spectral norm, maximum, minimum, spacings ...

Spectral radius:

$$\text{sp}(A) := \max\Big\{|\lambda| : \lambda \ \text{ is an eigenvalue of } A\Big\},$$

where $|z|$ denotes the modulus of $z \in \mathbb{C}$.

Spectral norm:

$$\|A\| = \sqrt{\lambda_{\text{max}}(A^*A)}$$

where A^* denotes the conjugate transpose of A.

Not many results known for most random matrices.

Almost sure edge behaviour

Wigner: $\lambda_{max} \to 2$ almost surely.

S matrix: $\lambda_{max} \to 4$ almost surely.

For other matrices mentioned so far, the limit is ∞.

T_n: $\frac{\lambda_{max}}{\sqrt{2n \log n}}$ converges in L^{γ} to 0.828.. (the 2–4 operator norm of the sine kernel) when $E|x_1|^{\gamma} < \infty$.

H_n: Not known. Expected to be the same rate as the Toeplitz.

Eigenvalues of circulant

For $0 \leq k < n$,

$$\lambda_k = \sum_{l=0}^{n-1} x_l \Big(\cos \frac{2\pi kl}{n} + i \sin \frac{2\pi kl}{n} \Big) = a_{k,n} + i b_{k,n}.$$

These $\{\lambda_k\}$ are basic building blocks of eigenvalues of general k-circulant matrix.

Suppose $\{x_i\}$ are i.i.d. standard normal. Then for every n, $n^{-1/2} a_{t,n}, n^{-1/2} b_{t,n}$, $1 \leq t \leq (n-1)/2$ are i.i.d. normal with mean zero and variance $1/2$.

Consequently, any subcollection $\{n^{-1}|\lambda_t|^2,\ 1 \leq t < \frac{n-1}{2}\}$, are mutually independent standard exponential random variable.

Eigenvalues of reverse circulant

$$\begin{cases} \eta_0 & = \sum_{t=0}^{n-1} x_t \\ \eta_{n/2} & = \sum_{t=0}^{n-1}(-1)^t x_t, \quad \text{if} \quad n \quad \text{is even} \\ \eta_k = -\eta_{n-k} & = |\lambda_k|, \ 1 \le k \le [\frac{n-1}{2}]. \end{cases}$$

Limiting distribution of spectral radius of RC_n

Bose, Hazra and Saha (2011): Assume $E|x_1|^{2+\delta} < \infty$. Then

$$\frac{\text{sp}(\frac{1}{\sqrt{n}} RC_n) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

Λ: standard Gumbel distribution, $q = q(n) \sim \frac{n}{2}$, $d_q = \sqrt{\ln q}$, $c_q = \frac{1}{2\sqrt{\ln q}}$.

If x_i are Gaussian, then the eigenvalues are essentially i.i.d., each a square root of the χ_2^2. So, the result follows from extreme value theory. For the general case (assume $(2+\delta)$ moment finite), use appropriate normal approximation (Davis and Mikosch (1999)).

What happens for general k-circulant? If $k > 1$ is fixed, different behaviour along subsequences.

Eigenvalues of k-circulant

Let $p_1 < p_2 < \ldots < p_c$ be all the common prime factors of k and n:

$$n = n' \prod_{q=1}^{c} p_q^{\beta_q} \quad \text{and} \quad k = k' \prod_{q=1}^{c} p_q^{\alpha_q}.$$

Here $\alpha_q, \ \beta_q \geq 1$ and $n', \ k', \ p_q$ are pairwise relatively prime.

Let

$$S(x) = \{xk^b \bmod n' : b \geq 0\}, \ 0 \leq x < n'.$$

Eigenvalues of k-circulant matrix (continued)

For $x \neq u$,

$$S(x) = S(u) \text{ or } S(x) \cap S(u) = \phi.$$

Hence, $\{S(x) : 0 \leq x < n'\}$ is a partition of $\mathbb{Z}_{n'} = \{0, 1, \ldots n'\}$.

Label them

$$\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{\ell-1}\} \text{ and } n_i = \#\mathcal{P}_i, \; 0 \leq i < \ell. \tag{0.1}$$

Note that $\sum n_j = n$.

Eigenvalues of k-circulant matrix (continued)

Zhou (1996): The characteristic polynomial of $A_{k,n}$ equals

$$\lambda^{n-n'} \prod_{j=0}^{\ell-1} (\lambda^{n_j} - y_j) \tag{0.2}$$

where

$$y_j := \prod_{t \in \mathcal{P}_j} \lambda_{ty}, \quad j = 0, 1, \ldots, \ell - 1 \quad \text{where} \quad y = n/n'.$$

$n - n'$ zero eigenvalues.

The other eigenvalues are the n_jth roots of y_j.

Each y_j is a product of n_j many $\lambda_k' s$, the indices being different across j. But a λ_k and its conjugate may appear in different y_j.

Bose, Hazra and Saha (2010): Suppose $E|x_i|^{(2+\delta)} < \infty$ for some $\delta > 0$. If $n = k^g + 1$ (g fixed), then

$$\frac{\text{sp}(n^{-1/2}A_{k,n}) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

where $q = q_n = \frac{n}{2g}$ and

$$c_n = \frac{1}{2g^{1/2}(\ln n)^{1/2}},$$

$$d_n = \frac{\ln C_g - \frac{g-1}{2}\ln g}{2g^{1/2}(\ln n)^{1/2}} + \left(\frac{\ln n}{g}\right)^{1/2}\left[1 + \frac{(g-1)\ln\ln n}{4\ln n}\right],$$

$$\text{and} \quad C_g = \frac{1}{\sqrt{g}}(2\pi)^{\frac{g-1}{2}}.$$

Gaussian case: product of i.i.d. exponentials.

Non-Gaussian case: normal approximation.

Tail of product of Exponentials

Let $\{E_i\}$ be i.i.d. standard exponentials. Define

$$H_g(x) = P[E_1 E_2 \cdots E_g > x].$$

Tail behaviour?

Tang (2008) $g = 2$:

$$H_2(x) = e^{-2x^{1/2}} \int_0^\infty \frac{e^{-z}}{\sqrt{z}} \frac{z + 2x^{1/2}}{\sqrt{z^2 + 4zx^{1/2}}} dz \sim \sqrt{\pi} e^{-2x^{1/2}} x^{1/4}.$$

Tail behavior of g fold product

Bose, Hazra and Saha (2010):

$$H_g(x) = C_g x^{\alpha_g} e^{-g x^{\frac{1}{g}}} f_g(x), \quad g \geq 1, \tag{0.3}$$

where for $g \geq 1$,

$$C_g = \frac{1}{\sqrt{g}} (2\pi)^{\frac{g-1}{2}}, \quad \alpha_g = \frac{g-1}{2g} \quad \text{and} \quad f_g(x) \to 1 \quad \text{as} \quad x \to \infty.$$

Other results

Point process convergence known for the spacings of the ordered eigenvalues of SC_n, RC_n, k-circulant with $n = k^2 + 1$.

Dependent observations where x_j is stationary. Eigenvalues scaled by spectral density.

Other combinations of k and n: not known. The sets $\mathcal{P}_j$ are of different sizes and keep changing with n.

Minimum eigenvalue: Need left tail behaviour. Known for Gaussian case.

Heavy tailed entries. Some results are known.

Bose, Hazra and Saha (2012): Extremum of Circulant type matrices: a survey. Journal of Indian Statistical Association, 50, No. 1–2, 21–49.

Other matrices

Wigner: Tracy-Widom distribution.

S matrix: Tracy-Widom distribution.

Smallest eigenvalues: some scattered results.

H_n, T_n: no distributional convergence is known.

More information: see the previous reference.