Some asymptotic results for ruin probabilities in the presence of heavy-tailed claims

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Surplus model

$$U(t) = u + ct - \sum_{k=1}^{N(t)} X_k$$

- t: time
- u: initial surplus
- c: premium rate
- X_k: iid random variables, $EX_1 = \mu$
- $T_1, T_2, \cdots$: times when claims occur
- $\tau_0 = 0, \tau_n = T_n - T_{n-1}$: inter-arrival times, iid random variables $E\tau_1 = \frac{1}{\lambda}$, independent of X_k
- Net profit condition: $c > \lambda\mu$
- $N(t)$ Poisson process $\rightarrow$ Cramer-Lundberg model
- $N(t)$ renewal process $\rightarrow$ Sparre Andersen model

Definitions

- The time of ruin (the first passage time through zero):

$$T_u = \inf_{t \geq 0} \{t : U(t) < 0 \mid U(0) = u\}$$

- The probability of ruin with infinite horizon:

$$\Psi(u) = P(T_u < \infty).$$

- Note: $P(T_u = \infty)$ means that ruin never happens.

Question

- What is the asymptotic behaviour of the ruin probability $\Psi(u)$, as the initial capital (surplus) $u \to \infty$?

Answer: it depends on the distribution of the claims.

Claims distributions

- **Light claims.** Claim sizes $\{X_k\}_{k\geq 0}$ have well-behaved distributions F_X with exponentially bounded tails

$$1 - F_X(x) \leq ce^{-\alpha x}, \ \alpha > 0, c \in \mathbb{R}, \ \forall x \geq 0$$

- **Heavy-tailed claims.** Regularly varying, integrated tail subexponentials.
  Example: claim sizes $\{X_k\}_{k\geq 0}$ have Pareto type distributions

$$1 - F_X(x) = \frac{b^\alpha}{(b+x)^\alpha}, \ \alpha > 0, b > 0, \ \forall x \geq 0$$

Classical compound Poisson/ renewal results

- **Light claims**

$$\Psi(u) \sim k_1 e^{-Ru} \quad \text{as} \quad u \to \infty,$$

where R s.t. $e^{-RU(t)}$ martingale.

(e.g. Cramer, 1930; Sparre Andersen, 1954)

- **Heavy-tailed claims**

$$\Psi(u) \sim k_2 \bar{F}_I(u) \quad \text{as} \quad u \to \infty,$$

where $F_I(x) = \frac{1}{\mu} \int_0^x (1 - F_X(y))dy$.

(e.g. Embrechts & Veraverbeke, 1987; Embrechts et.al, 1997)

Question

- If we invest everything in a risky asset, what is the asymptotic behaviour of the probability of ruin, as the initial capital $u \to \infty$?

Answer: it depends on the parameters of the investments and on the distribution of the claim sizes.

Model with investments in a Geometric Brownian Motion

The surplus model:

$$U(t) = u + ct + a\int_0^t U(s)ds + \sigma\int_0^t U(s)dW_S - \sum_{k=0}^{N(t)} X_k.$$

Geometric Brownian motion (a, σ^2)

- Large volatility : $\rho = \frac{2a}{\sigma^2} - 1 \leq 0$, then

$$\psi(u) = 1, \quad \forall u > 0.$$

- Small volatility: $\rho = \frac{2a}{\sigma^2} - 1 > 0$, then

$$\psi(u) \sim C\, u^{-\rho} + k_n \overline{F}_X(u), \quad u \to \infty.$$

(Norberg & Kalashnikov, 2002; Frolova et.al, 2002; Paulsen, 2002; Albrecher, Constantinescu &Thomann, 2012)

Steps

1. Derivation of an equation for $\Psi(u)$
2. Laplace transform of the equation
3. Exploit regularity at zero of the homogeneous part of the ODE satisfied by the Laplace transform $\hat{\Psi}(s)$
4. Use **Karamata-Tauberian** arguments to establish decay rate of homogeneous part and **Heaviside principle** for the particular solution

Integro-differential equations

$$(-A+\lambda)\Psi(u) = \lambda \int_0^\infty \Psi(u-y)dF_X(y)dy$$

- Classical compound-Poisson model

$$(-c\frac{d}{du}+\lambda)\Psi(u) = \lambda \int_0^\infty \Psi(u-y)dF_X(y)dy$$

- Jump-diffusion model

$$(-(c+au)\frac{d}{du} - \frac{\sigma^2 u^2}{2}\frac{d^2}{du^2}+\lambda)\Psi(u) = \lambda \int_0^\infty \Psi(u-y)dF_X(y)dy$$

Recall

- **Karamata-Tauberian Theorem**

$$\tilde{g}(s) \sim c\ell(1/s)\, s^{\rho}, \quad s \to 0$$

$$g(u) \sim c/\Gamma(1+\rho)\ell(u)\, u^{-\rho}, \quad u \to \infty$$

- **Heaviside Operational Principle**: $-s^*$ rightmost singularity

$$\hat{g}(s) \sim \sum_{k=0}^{\infty} a_k (s+s^*)^k + \hat{f}(s), \quad s \to -s^*$$

$$g(u) \sim f(u), \quad u \to \infty$$

Claims distributions

- **Light claims:** $\hat{\hat{F}}_x(-\mu) = \infty$

- **Heavy-tailed claims:** $\hat{\hat{F}}_x(-\epsilon) = \infty, \forall \epsilon > 0$

Laplace transform equation

$$(-A+\lambda)\Psi(u) = \lambda \int_0^u \Psi(u-y)dF_X(y)dy + \lambda \int_u^\infty dF_X(y)dy$$

- Classical compound Poisson case - linear equation in $\hat{\Psi}(s)$
- Jump-diffusion case - Ordinary differential equation:

$$\underbrace{(-\hat{A}+\lambda)\hat{\Psi}(s) - \lambda \hat{f}_X \hat{\Psi}(s)}_{\text{homogeneous equation - regular at zero}} = \underbrace{\lambda\left(\frac{1}{s} - \frac{\hat{f}_X(s)}{s}\right)}_{\text{particular solution}}$$

Question

- Is the distribution of the inter-arrival times influencing the asymptotic behaviour when investments are made?

Answer: no, as long as from the class of...

Distributions satisfying ODEs with constant coefficients

- Examples:
  - Exponential: $\left(\frac{d}{dt} + \lambda\right) f_\tau(t) = 0$
  - Mixture of 2 exponentials: $\left(\frac{d}{dt} + \lambda_1\right)\left(\frac{d}{dt} + \lambda_2\right) f_\tau(t) = 0$
  - Sum of 2 exponentials: $\left(\frac{d}{dt} + \lambda\right)^2 f_\tau(t) = 0$

- Class larger than phase-type distributions

Extensions

- Fractional Investments
  - The surplus model:

$$U(t) = u + ct + \gamma a \int_0^t U(s)ds + \gamma\sigma \int_0^t U(s)dW_S - \sum_{k=0}^{N(t)} X_k.$$

  - Decay rates depend on γ.
- Other functionals of the risk process (e.g. Laplace transform of the time of ruin, expected discounted penalty function).

Further Questions

- Dependence? Optimal investment?
- What are the surplus models for which one obtains the "in-between" asymptotic behaviours?
- Is the exponential decay of the classical case a limiting case?

Any suggestions are welcome.

THANK YOU FOR YOUR ATTENTION!

H. Albrecher, C. Constantinescu, E. Thomann (2012). Asymptotic Results for Renewal Risk Models with Risky Investments, *Stochastic Processes and their Applications* 122, no. 11, 3767–3789.

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