How strongly do extreme returns cluster?

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Workshop on
"Heavy-tailed Distributions and Extreme Value Theory"
Kolkata, January 17, 2013

Outline

1. Extremal serial dependence
   - Motivation: Modeling financial time series
   - Clustering of extremes
   - Measures of tail dependence
2. Estimating the coefficient of tail dependence
   - Definition of estimators
   - Empirical processes of cluster functionals
   - Asymptotics
   - Bootstrap confidence intervals
3. Application: analysis of DAX returns

Modeling Financial Time Series

Vast variety of models for (univariate) time series of returns
Most important classes for centered returns

$$X_t = \sigma_t \varepsilon_t$$

where ε_t iid with $E(\varepsilon_t) = 0$, $Var(\varepsilon_t) = 1$ and

- GARCH-type: σ_t can be expressed in terms of past innovations ε_s, $s < t$; e.g. GARCH(1,1):
$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$
- SV-type: σ_t driven by another source of randomness; e.g. Taylor's stochastic volatility model
$$\log \sigma_t - \mu = \varphi \cdot (\log \sigma_{t-1} - \mu) + \xi_t.$$
Often $(\xi_t)_{t \in \mathbb{Z}}$ iid Gaussian, independent of heavy-tailed $(\varepsilon_t)_{t \in \mathbb{Z}}$

Not clear whether such models capture extreme value behavior correctly.

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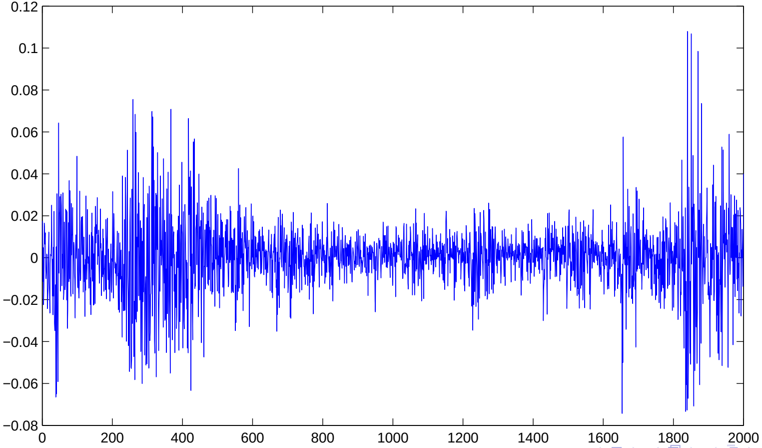
Clusters of extremes

Here we focus on the dependence between extreme observations.

returns on DAX stock index 19.7.2001 – 01.06.2009



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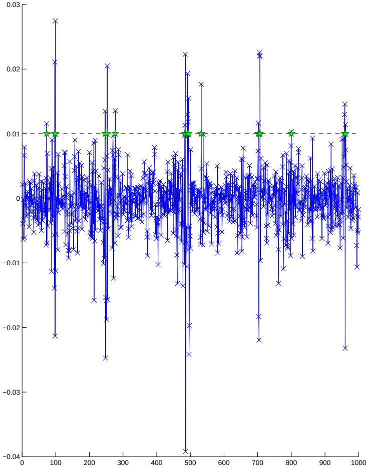

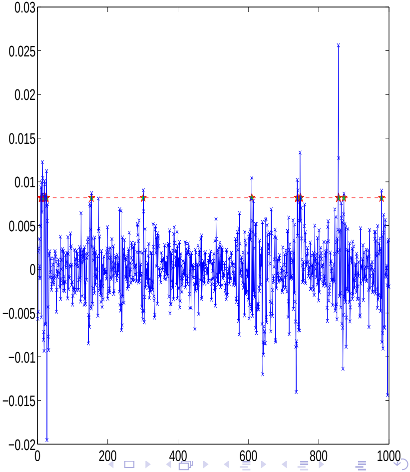

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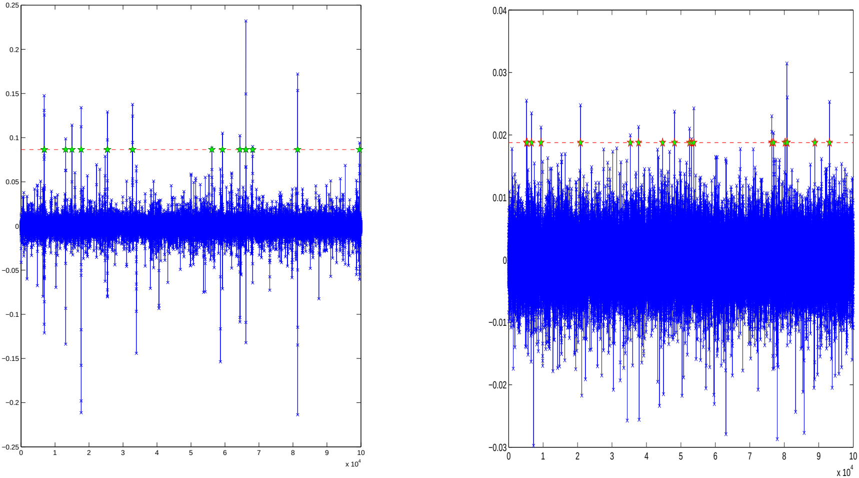

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Qualitatively different asymptotic cluster behavior

E.g.:

- GARCH(p,q): X_t, X_{t+h} asymptotically dependent for all $h > 0$, i.e.

$$\liminf_{u\to\infty} P(X_{t+h} > u \mid X_t > u) > 0$$

- heavy-tailed SV models: under weak conditions, X_t, X_{t+h} asymptotically independent for all $h > 0$, i.e.

$$\lim_{u\to\infty} P(X_{t+h} > u \mid X_t > u) = 0$$

More precise results in the framework of *bivariate regular variation on* $(0,\infty)^2$

Framework

$(X_t)_{t\in\mathbb{Z}}$: stationary time series
F: marginal cdf, assumed eventually continuous

Fix some lag $h > 0$.

(X_t, X_{t+h}) regularly varying on cone $(0,\infty)^2$

$$\iff \frac{P\{(X_t, X_{t+h}) \in uB\}}{P\{(X_t, X_{t+h}) \in (u,\infty)^2\}} \overset{u\to\infty}{\longrightarrow} \nu(B)$$

for some non-degenerate measure ν and all ν-continuous $B \in \mathbb{B}(0,\infty)^2$ bounded away from the axes

$$\iff \frac{P\{X_t > ur, X_{t+h} > us\}}{P\{X_t > u, X_{t+h} > u\}} \overset{u\to\infty}{\longrightarrow} d(r,s) \quad \forall\, r,s > 0$$

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Basic assumption

To allow for more general models, we first standardize the marginal distribution e.g. to standard Pareto (not necessary for GARCH and heavy-tailed SV models):

$$Y_t := \frac{1}{1 - F(X_t)}$$

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Then fct. d necessarily homogeneous of order $-1/\eta$ for some coefficient of tail dependence $\eta = \eta_h \in (0, 1]$ and

$$P\{\min(Y_t, Y_{t+h}) > u\} = P\{Y_t > u, Y_{t+h} > u\} = u^{-1/\eta}\ell(u)$$

for some slowly varying fct. $\ell = \ell_h$.

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Discriminating extremal cluster behavior

Examples:

- GARCH: $\eta_h = 1$, $\lim_{u\to\infty} \ell_h(u) > 0 \quad \forall\, h > 0$
- heavy-tailed SV models: $\eta_h = 1/2$

Objective

Use estimator for so-called *coefficient of tail dependence* η to discriminate between time series models

$$T_i := \min(Y_i, Y_{i+h}) \quad \Longrightarrow \quad 1 - F_T(u) := P\{T_i > u\} = u^{-1/\eta}\ell(u)$$

If T_i, $1 \le i \le n$, were observable, η could be estimated by standard estimators of tail index like ML estimator (in GPD model) or Hill estimator:

$$\tilde\eta_n := \frac{1}{j_n}\sum_{i=1}^{j_n} \log \frac{T_{n-i+1:n}}{T_{n-j_n:n}}$$

However, F and thus $Y_i = 1/(1 - F(X_i))$ are usually unknown

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Estimating the coefficient of tail dependence

Replace F with empirical cdf, i.e. use rank-based version of T_i instead:

$$T_i^{(n)} := \frac{1}{1 - \min(R_i^X, \tilde{R}_{i+h}^X)/(n+1)}, \quad 1 \le i \le n,$$

with R_i^X = rank of X_i among $X_1, \ldots, X_n$
$\tilde{R}_{i+h}^X$ = rank of X_{i+h} among $X_{1+h}, \ldots, X_{n+h}$

Resulting Hill estimator: for suitable sequence $j_n \to \infty$, $j_n/n \to 0$

$$\hat{\eta}_{n,j_n} := \frac{1}{j_n} \sum_{i=1}^{j_n} \log \frac{T_{n-i+1:n}^{(n)}}{T_{n-j_n:n}^{(n)}}$$

Confidence intervals are needed to discriminate between time series models.

Draisma et al. (2004): Asymptotics for analogous estimators for *independent* bivariate vectors (X_i, Y_i); not applicable here because of non-negligible serial dependence.

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Tail empirical processes

Main technical tool to analyze asymptotic behavior of estimators:

limit theorems for suitable empirical processes, in particular

$$Z_n^{(Y)}(x) := k_n^{-1/2} \sum_{i=1}^{n} \Big(1_{\{Y_i > n/(k_n x)\}} - k_n x/n \Big), \quad x > 0,$$

$$Z_n^{(T)}(x) := (n v_n)^{-1/2} \sum_{i=1}^{n} \Big(1_{\{T_i > F_T^{\leftarrow}(1 - v_n x)\}} - v_n x \Big), \quad x > 0,$$

with

$$k_n \to \infty, k_n/n \to 0, \quad v_n := P\{T_i > n/k_n\} = P\{Y_i > n/k_n, Y_{i+h} > n/k_n\}.$$

Special cases of so-called *empirical processes of cluster functionals*, introduced and analyzed by D. and Rootzén (2010). Under suitable conditions, processes jointly converge to centered Gaussian processes.

By inversion, one obtains asymptotics for pertaining tail empirical quantile functions $\big(Y_{n-\lfloor k_n t \rfloor : n}\big)_{t_0 \le t \le 1}$ and $\big(T_{n-\lfloor n v_n t \rfloor : n}\big)_{t_0 \le t \le 1}$ for arbitrary $t_0 > 0$.

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Tail empirical quantile processes

Because with probability tending to 1

$$T_i^{(n)} > x \quad \Longleftrightarrow \quad T_i > \frac{1}{Y_{n+1-\lceil (n+1)/x \rceil:n}}$$

one may conclude limit theorem for $\left(T^{(n)}_{n-\lfloor nv_n t\rfloor:n}\right)_{t_0 \le t \le 1}$ of the type

$$(nv_n)^{1/2}\left(\frac{k_n}{n} T^{(n)}_{n-\lfloor nv_n t\rfloor:n} - t^{-\eta}\right) \longrightarrow Z(t) \quad \text{uniformly for } t \in [t_0, 1].$$

for a Gaussian process Z with cov. fct. c (see next slide)

To derive asymptotic normality of Hill estimator $\hat{\eta}_n$ one needs weighted version:

$$t^{\eta+\nu}(nv_n)^{1/2}\left(\frac{k}{n} T^{(n)}_{n-\lfloor nv_n t\rfloor:n} - t^{-\eta}\right) \longrightarrow t^{\eta+\nu} Z(t) \quad \text{uniformly for } t \in (0, 1],$$

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Conditions

(C1) For all $\iota > 0$ and some decreasing fct. q_1 tending to 0

$$\sup_{x \geq 1} x^{1/\eta - \iota} \left| \frac{P\{Y_1 > ux, Y_{1+h} > ux\}}{P\{Y_1 > u, Y_{1+h} > u\}} - x^{-1/\eta} \right| = O(q_1(u))$$

(C2) $(X_t)_{t\in\mathbb{Z}}$ β-mixing with coefficients β_ℓ satisfying $\beta_{\ell_n} n/r_n \to 0$ for some $\ell_n \to \infty$, $\ell_n = o(r_n)$, $r_n = o\big(\min(n/k_n, k_n^{1/2}\log^{-2} k_n)\big)$, $k_n^{1/2} q_1(n/k_n) \to 0$.

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Asymptotic normality

Here only in the case of asymptotic independence (in particular if $\eta < 1$)

Consider Hill estimator $\hat{\eta}_n$ based on j_n exceedances of $T_i^{(n)}$ over n/k_n.

Recall $v_n := P\{T_i > n/k_n\} = P\{Y_i > n/k_n, Y_{i+h} > n/k_n\}$.

Corollary

Under the conditions given above

$$(nv_n)^{1/2}(\hat{\eta}_n - \eta) \longrightarrow \mathcal{N}_{(0,\sigma^2)} \quad \textit{weakly}$$

with

$$\sigma^2 = \eta^2 \int_0^1 \int_0^1 (st)^{-(\eta+1)} c(s,t) \left(s^\eta ds - \varepsilon_1(ds)\right) \left(t^\eta dt - \varepsilon_1(dt)\right).$$

Under weak conditions, the covariance fct. c is homogeneous of order 1; then $\sigma^2 = \eta^2 c(1,1)$.

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Construction of confidence intervals

In $(nv_n)^{1/2}(\hat\eta_n - \eta) \to \mathcal{N}_{(0,\eta^2 c(1,1))}$ interpret c as cov. fct. of limit of $Z_n^{(T^{(n)})}$ with

$$
\begin{aligned}
Z_n^{(T^{(n)})}(1) &= \Big(nP\{T_1 > \frac{n}{k}\}\Big)^{-1/2} \sum_{i=1}^{n} \Big(1_{\{T_i^{(n)} > \frac{n}{k}\}} - P\{T_1 > \frac{n}{k}\}\Big) \\
&= \Big(nP\{T_1 > \frac{n}{k}\}\Big)^{-1/2} \sum_{i=1}^{m_n} \Big(\sum_{j=1}^{r_n} 1_{\{T^{(n)}_{(i-1)r_n+j} > \frac{n}{k}\}} - r_n P\{T_1 > \frac{n}{k}\}\Big).
\end{aligned}
$$

Use multiplier block bootstrap analog

$$
Z_n^*(1) = \Big(\sum_{l=1}^{n} 1_{\{T_l^{(n)} > \frac{n}{k}\}}\Big)^{-1/2} \sum_{i=1}^{m_n} \zeta_i \Big(\sum_{j=1}^{r_n} 1_{\{T^{(n)}_{(i-1)r_n+j} > \frac{n}{k}\}} - \frac{r_n}{n} \sum_{l=1}^{n} 1_{\{T_l^{(n)} > \frac{n}{k}\}}\Big)
$$

with ζ_i iid independent of $(X_t)_{t\in\mathbb{Z}}$ with $E(\zeta_i) = 0$, $Var(\zeta_i) = 1$.

Theorem

$$
P\{(nv_n)^{1/2} |\hat\eta_n - \eta| \le c_\alpha\} - P_\zeta\{|Z_n^*(1)| \le c_\alpha / \hat\eta_n\} \xrightarrow{P} 0.
$$

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Use multiplier block bootstrap analog

$$Z_n^*(1) = \Big(\sum_{l=1}^{n} 1_{\{T_l^{(n)} > \frac{n}{k}\}}\Big)^{-1/2} \sum_{i=1}^{m_n} \zeta_i \Big(\sum_{j=1}^{r_n} 1_{\{T^{(n)}_{(i-1)r_n+j} > \frac{n}{k}\}} - \frac{r_n}{n} \sum_{l=1}^{n} 1_{\{T_l^{(n)} > \frac{n}{k}\}}\Big)$$

with ζ_i iid independent of $(X_t)_{t\in\mathbb{Z}}$ with $E(\zeta_i) = 0$, $Var(\zeta_i) = 1$.

Theorem

$$P\{(nv_n)^{1/2}|\hat{\eta}_n - \eta| \le c_\alpha\} - P_\zeta\{|Z_n^*(1)| \le c_\alpha/\hat{\eta}_n\} \xrightarrow{P} 0.$$

Construction of confidence intervals

In $(nv_n)^{1/2}(\hat{\eta}_n - \eta) \to \mathcal{N}_{(0,\eta^2 c(1,1))}$ interpret c as cov. fct. of limit of $Z_n^{(T^{(n)})}$ with

$$\begin{aligned} Z_n^{(T^{(n)})}(1) &= \Big(nP\{T_1 > \frac{n}{k}\}\Big)^{-1/2} \sum_{i=1}^{n} \Big(1_{\{T_i^{(n)} > \frac{n}{k}\}} - P\{T_1 > \frac{n}{k}\}\Big) \\ &= \Big(nP\{T_1 > \frac{n}{k}\}\Big)^{-1/2} \sum_{i=1}^{m_n} \Big(\sum_{j=1}^{r_n} 1_{\{T^{(n)}_{(i-1)r_n+j} > \frac{n}{k}\}} - r_n P\{T_1 > \frac{n}{k}\}\Big). \end{aligned}$$

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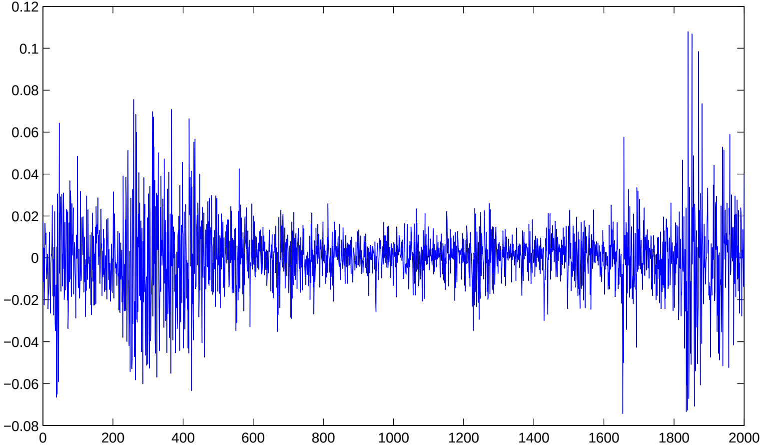
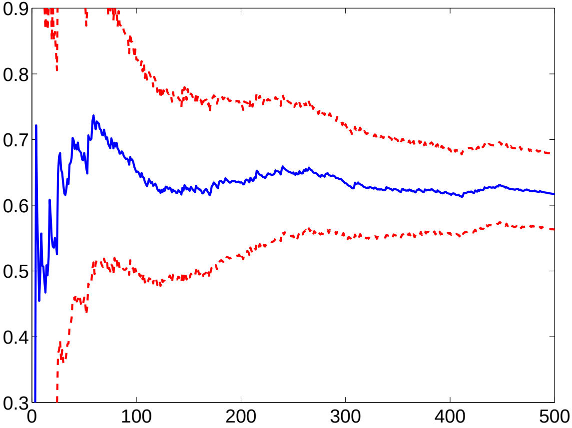
Application: DAX returns

Data: returns on DAX stock index 19.7.2001 – 01.06.2009



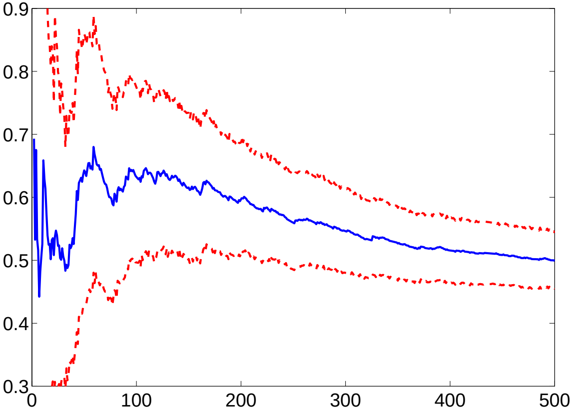
Confidence intervals for η_1 of negative returns



$\hat{\eta}_{n,j}$ (blue) versus j, confidence intervals to level 95% (red)

Estimates lead to rejection of GARCH-models (and possibly standard SV-models)

Confidence intervals for η_1 of positive returns



$\hat{\eta}_{n,j}$ (blue) versus j, confidence intervals to level 95% (red)

Estimates lead to rejection of GARCH-models (and possibly standard SV-models)

Résumé and Outlook

- Empirical processes are very powerful tool to analyze asymptotic behavior of extreme value statistics for time series;
  large class of estimators can be considered in unifying framework;
  in particular, results can be generalized to finite vector of $\hat{\eta}_h$ corresponding to different lags h
- Different quantities describing the serial dependence structure (extremal index, extremogram, ...) may be more appropriate for other purposes; suitable empirical processes of cluster functionals can be used to establish the asymptotic normality of estimators thereof
- Often same techniques yield asymptotics for estimators and for bootstrap versions thereof, that can be used to construct confidence intervals or to calculate critical values in model tests
- If rate of convergence $q_1(u)$ in regular variation condition is slow, there might be a serious bias problem

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Thank you for your attention!