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Model Uncertainty and Risk Aggregation

joint work with
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Mathematical Problem

Assumptions:

$L_1, \ldots, L_d$ one period risks with statistically estimated marginals.

$L_1 + \cdots + L_d$ total loss exposure.

$\mathrm{VaR}_\alpha(L_1 + \cdots + L_d)$ amount of capital to be reserved.

(if $\mathrm{VaR}_\alpha(L_1 + \cdots + L_d) = s$, then $P(L_1 + \cdots + L_d \geq s) \leq 1 - \alpha$)

Task: for a fixed (high) level of probability α, calculate:

$$\overline{\mathrm{VaR}}_\alpha = \sup\left\{\mathrm{VaR}_\alpha(L_1 + \cdots + L_d) : L_j \sim F_j, 1 \leq j \leq d\right\}$$

$$\underline{\mathrm{VaR}}_\alpha = \inf\left\{\mathrm{VaR}_\alpha(L_1 + \cdots + L_d) : L_j \sim F_j, 1 \leq j \leq d\right\}$$

Motivation (QRM)

$$L_1 \sim F_1, \quad L_2 \sim F_2, \quad \ldots, \quad L_d \sim F_d$$

marginal distributions

$d \approx 600$

+

dependence model

=

$$\mathrm{VaR}_\alpha(L_1 + \cdots + L_d)$$

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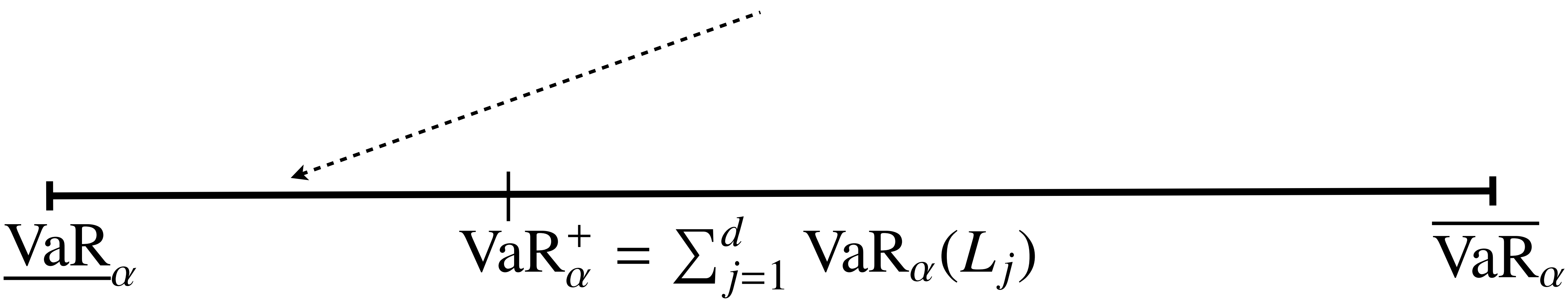

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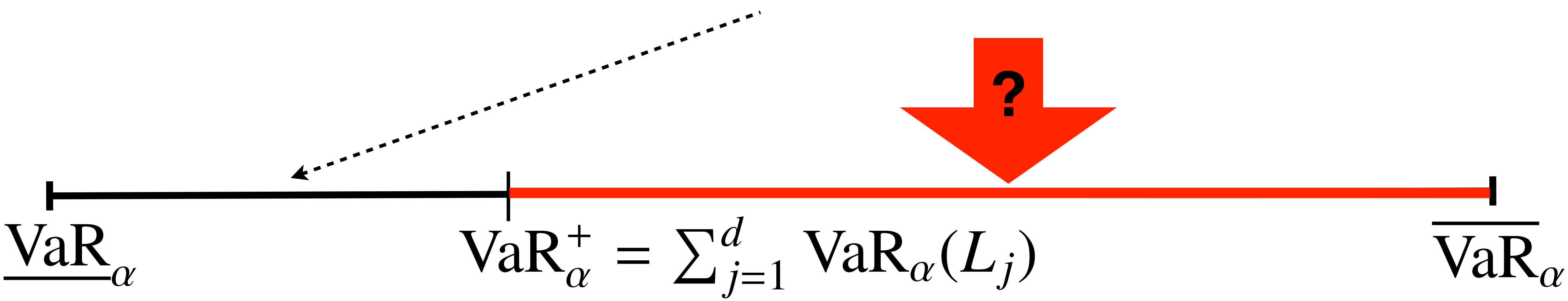

Why useful?

Example: Operational Risk under Basel II.

The ingredients:

- Risk measure: VaR_α
- Holding period: 1 year
- Confidence level: 99.9%, α =0.999
- The data: 8x7 matrix; 8 Business lines, 7 Loss types
- Often: aggregate column-wise ⇒ $\mathrm{VaR}_\alpha(1), \ldots, \mathrm{VaR}_\alpha(8)$

Aggregate: $$\sum_{i=1}^{8} \mathrm{VaR}_\alpha(i) = \mathrm{VaR}_\alpha^+$$

Diversification: $(1-\delta)\mathrm{VaR}_\alpha^+$

- Discussion!

Recall: risk measures

- $\mathrm{VaR}_\alpha(L) = F_L^{-1}(\alpha)$

- $$\mathrm{ES}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\delta(L)\, d\delta$$
  $$= \mathbb{E}[L|L > \mathrm{VaR}_\alpha(L)],\ F_L \text{ continuous}$$

Recall:

- In general, VaR_α is not subadditive, i.e. we may have that:

$$\mathrm{VaR}_\alpha\left(\sum_{i=1}^d L_i\right) > \sum_{i=1}^d \mathrm{VaR}_\alpha(L_i)$$

Typical cases include:

1) Very heavy tailed; 2)Very skew; 3) Special dependence.

Continuation

- VaR_α is subadditive for elliptical distributions
- ES_α is always subadditive
- ES_α is the smallest coherent risk measure larger than VaR_α
- Later: some remarks on backtesting!

  - always possible for VaR_α

  - (almost) impossible for ES_α

On forecasting risk measures

Basic reference: T. Gneiting (2011). Making and evaluating point forecasts. JASA (106), 746-762.

The ingredients:

- A point forecast, functional T
- A scoring function S
- Strict consistency of S
- Elicitability of T relative to a class $\mathfrak{F}$ of distribution functions
- Examples

Some results

Theorem 1 $\mathrm{VaR}_\alpha(F),\ \alpha \in (0,1)$, is elicitable relative to the class $\mathfrak{F}$ of all distribution functions F on the interval $I \subset \mathbb{R}$.

Theorem 2 $\mathrm{ES}_\alpha(F),\ \alpha \in (0,1)$, is <u>not</u> elicitable relative to <u>any</u> class $\mathfrak{F}$ of probability distributions F on the interval $I \subset \mathbb{R}$ that contains the measures with finite support, or the finite mixtures of the absolutely continuous distributions with compact support.

Conclusions

With respect to elicitability, VaR_α is good and ES_α is bad!

Relevant for practice

Remark (Gneiting p.756). "This negative result (Theorem 2) may challenge the use of ES_α as a predictive measure of risk, and may provide a partial explanation for the lack of literature on the evaluation of ES_α forecasts, as opposed to quantile or VaR_α forecasts", and from the BIS Consultative Document (May 2012), Fundamental review of the trading book: Question 8, p.41: " What are the likely operational constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?"

Discussion: ...

Recall Our Mathematical Problem

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Known results

- In the homogeneous case $F_j = F,\ 1 \le j \le d$, the bound $\overline{\mathrm{VaR}}_\alpha$ has been recently given for $d > 2$ in [PR11] and [WW11] under different assumptions.

- In the homogeneous case, $\overline{\mathrm{VaR}}_\alpha$ is very easy to calculate in arbitrary dimensions.

- In the *inhomogeneous* case, the computation of $\overline{\mathrm{VaR}}_\alpha$ poses serious problems. And the computation of $\underline{\mathrm{VaR}}_\alpha$ is not possible.

Timeline to the result

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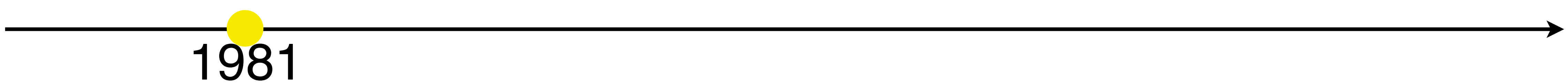

Makarov gives the optimal coupling for the sum of two risks answering a question by Kolmogorov

Timeline to the result

Rüschendorf gives independently the same optimal coupling and the dual solution

1982

1981

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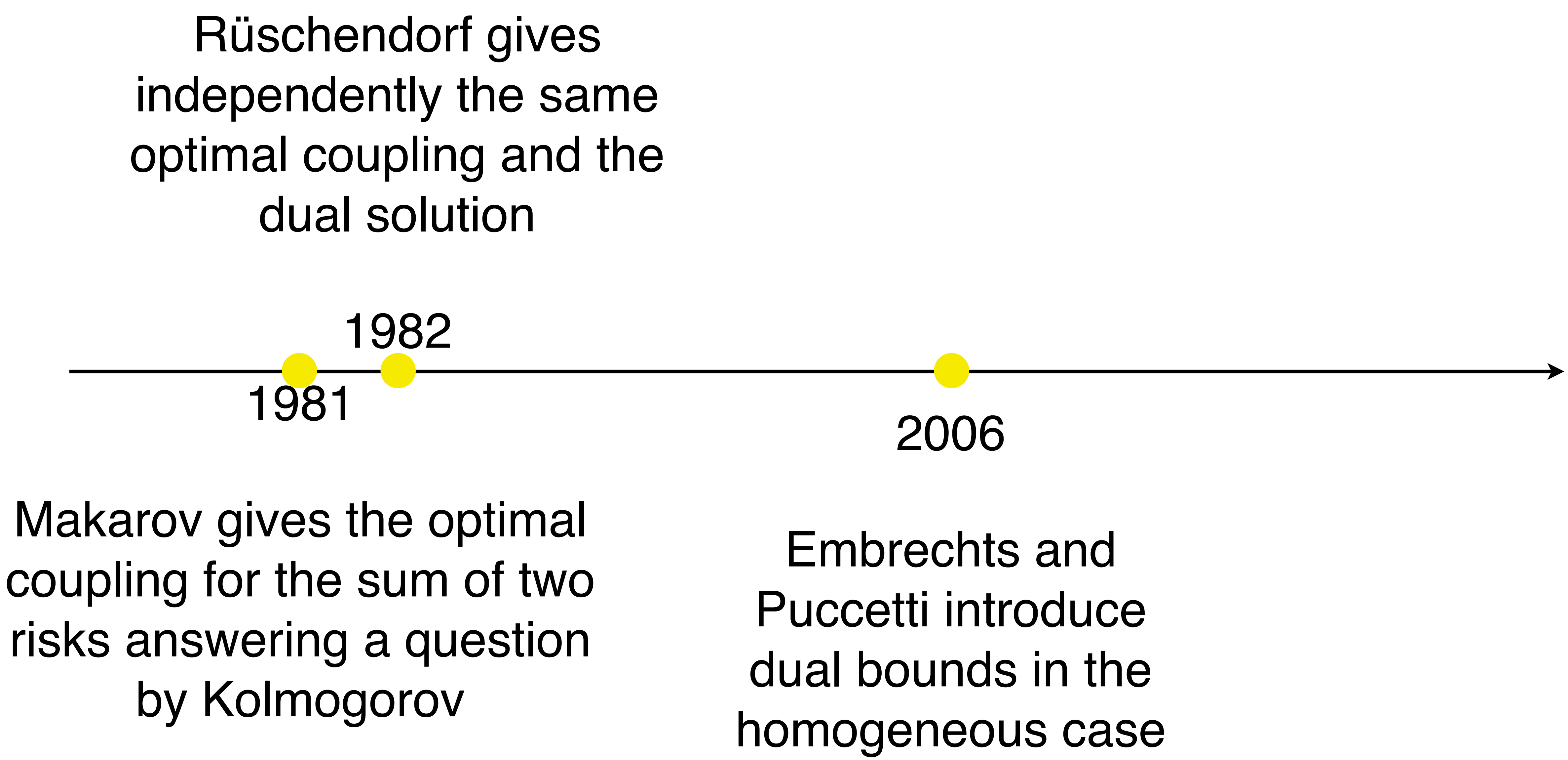

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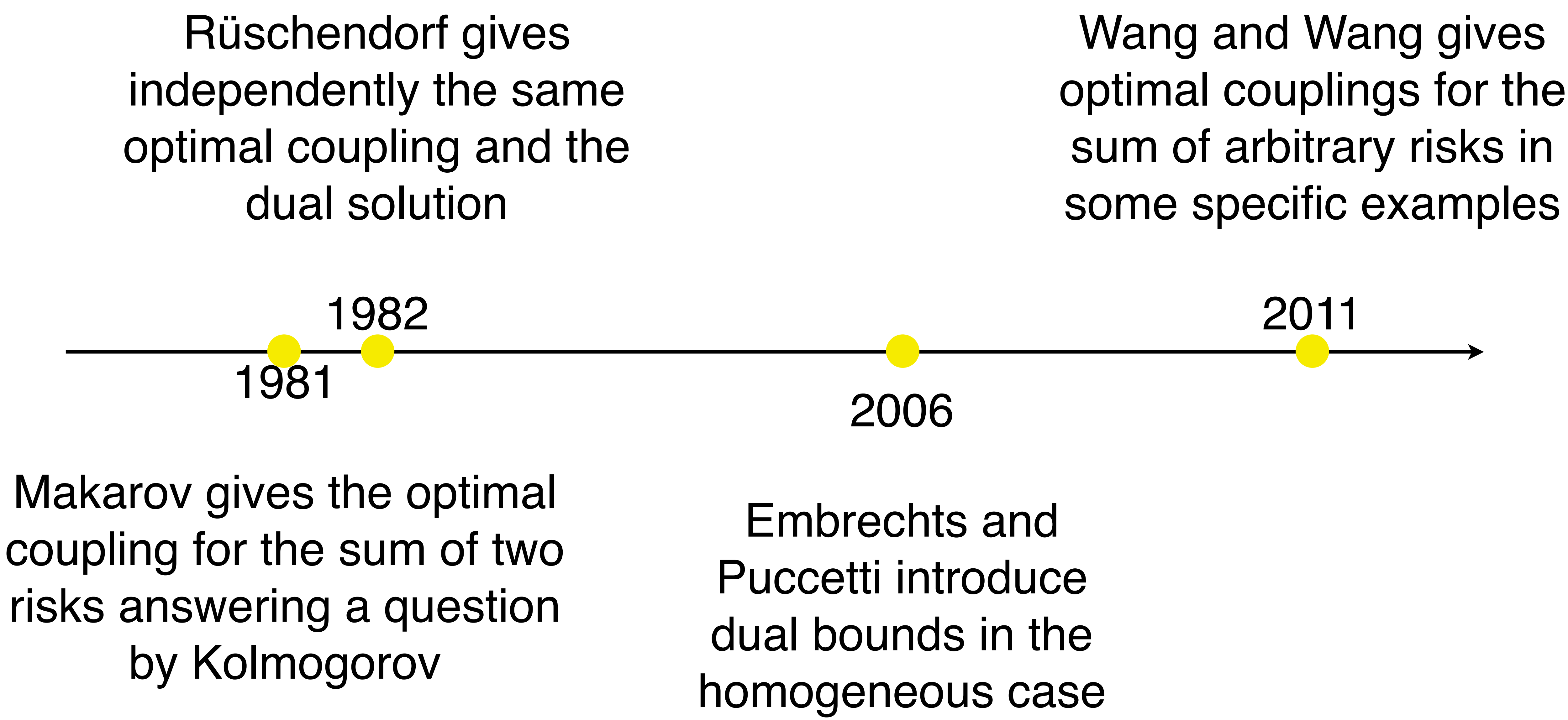
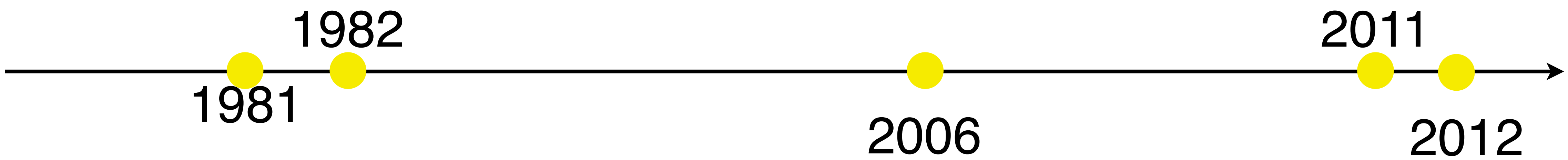


Timeline to the result

Rüschendorf gives independently the same optimal coupling and the dual solution

Wang and Wang gives optimal couplings for the sum of arbitrary risks in some specific examples



Makarov gives the optimal coupling for the sum of two risks answering a question by Kolmogorov

Embrechts and Puccetti introduce dual bounds in the homogeneous case

sharpness of dual bounds is stated for a general class of distributions

$d = 8$	$N = 1.0e05$	*avg time*: 3 secs		
α	$\underline{\mathrm{VaR}}(\alpha)$ (RA range)	$\mathrm{VaR}^+(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (RA range)
0.99	9.00 – 9.00	72.00	141.67	141.66–141.67
0.995	13.13 – 13.14	105.14	203.66	203.65–203.66
0.999	30.47 – 30.62	244.98	465.29	465.28–465.30

$d = 56$	$N = 1.0e05$	*avg time*: 30 secs		
α	$\underline{\mathrm{VaR}}(\alpha)$ (RA range)	$\mathrm{VaR}^+(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (RA range)
0.99	45.82 – 45.82	504	1053.96	1053.80–1054.11
0.995	48.60 – 48.61	735.96	1513.71	1513.49–1513.93
0.999	52.56 – 52.58	1714.88	3453.99	3453.49–3454.48

$d = 648$	$N = 5.0e04$	*avg time*: 10 mins		
α	$\underline{\mathrm{VaR}}(\alpha)$ (RA range)	$\mathrm{VaR}^+(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (exact)	$\overline{\mathrm{VaR}}(\alpha)$ (RA range)
0.99	530.12 – 530.24	5832.00	12302.00	12269.74-12354.00
0.995	562.33 – 562.50	8516.10	17666.06	17620.45-17739.60
0.999	608.08 – 608.47	19843.56	40303.48	40201.48-40467.92

TABLE 1. Estimates for $\overline{\mathrm{VaR}}(\alpha)$ and $\underline{\mathrm{VaR}}(\alpha)$ for random vectors of Pareto(2)-distributed risks.

Application: superadditivity ratio

Define the *superadditivity ratio* as:

$$\delta_\alpha(d) = \frac{\overline{\mathrm{VaR}}_\alpha(L_+)}{\mathrm{VaR}_\alpha^+(L_+)}$$

and investigate its properties as a function of the dimension *d*, the level α and the parameters of the underlying model.

Investigate the limit, given it exists,

$$\delta_\alpha = \lim_{d \to +\infty} \delta_\alpha(d)$$

Examples

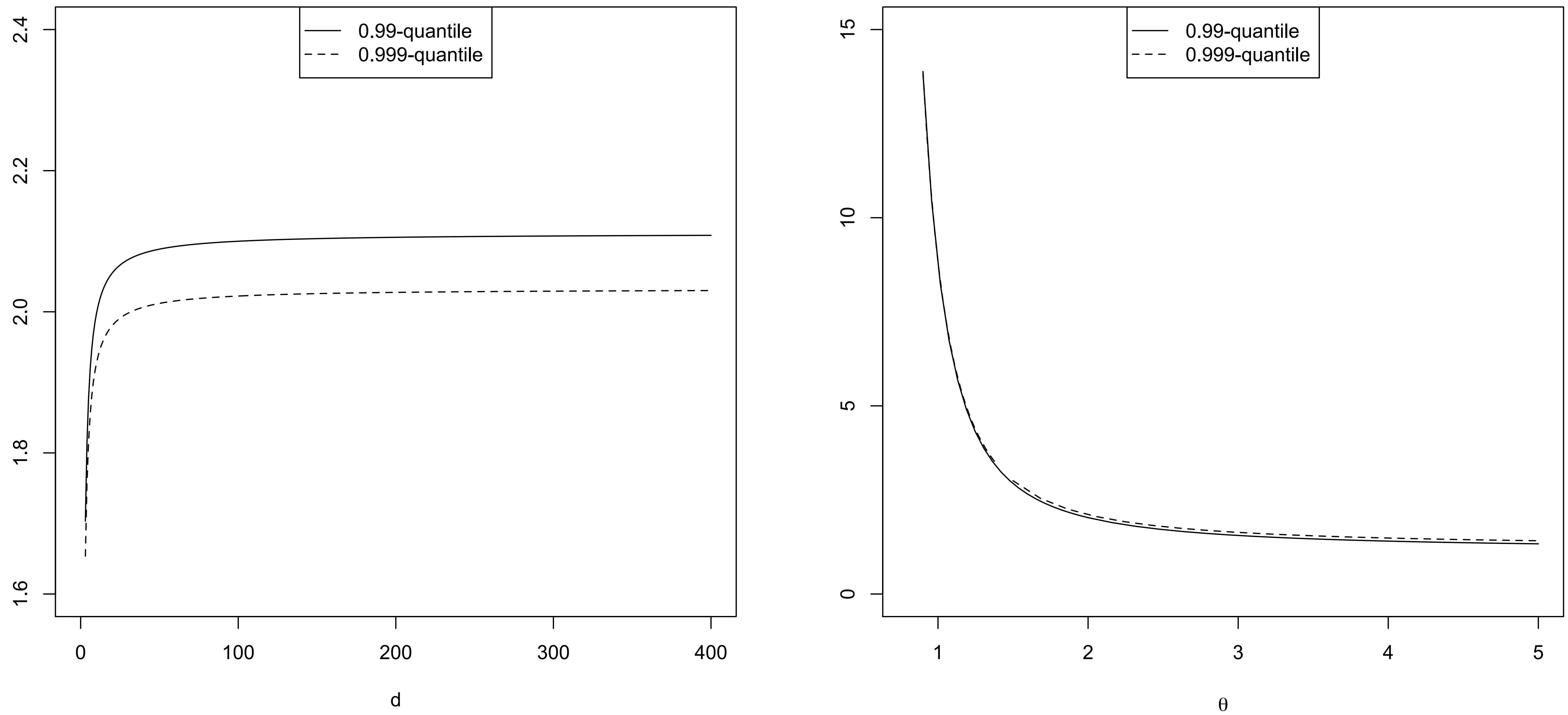


Figure 5: Left: plot of the function $\delta_\alpha(d)$ versus the dimensionality d of the portfolio for a risk vector of Pareto(θ)-distributed risks, for two different quantile levels and $\theta = 2$. Right: Plot of the limit constant δ_α versus the tail parameter θ of the Pareto distribution.

Examples

α	$\theta < 1$	$\theta = 1.1$	$\theta = 1.5$	$\theta = 2$	$\theta = 3$	$\theta = 4$
0.99	$+\infty$	11.15433	3.097393	2.111053	1.637300	1.487223
0.995	$+\infty$	11.08160	3.060219	2.076147	1.603137	1.453967
0.999	$+\infty$	11.01877	3.020209	2.032668	1.555634	1.405445

Values for the constant δ_α for risk portfolios having Pareto(θ) marginals

$$\delta_\alpha = k$$

means that VaR can be k times the comonotonic value if the dimension of the portfolio is large enough.

Conclusions

The rearrangement algorithm calculates numerically sharp bounds for the VaR of a sum of dependent random variables.

- it is accurate, fast and computationally less demanding wrt to the methods in the literature.

- can be used with *inhomogeneous* marginals, in high dimensions.

- computes also the *best-possible* Value-at-Risk.

- can be used with *any* marginal df and *any* quantile level.

- can be used also to compute bounds on the distribution function of different operators such as $\times, \ \min, \ \max$.

Further work

- Find optimal couplings for the best VaR
- Interpret these couplings wrt realistic scenarios
- Add statistical uncertainty
- Compute VaR sharp bounds with some additional dependence information
- Compare and contrast with other approaches: Robust Optimization
- ...

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