High-frequency sampled stable CARMA processes

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joint work with Florian Fuchs

Overview

- Continuous-time ARMA (CARMA) processes

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- Spectral density estimation

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- Spectral density estimation
- Parameter estimation

CARMA Processes

ARMA process

An ARMA(p,q), $p,q \in \mathbb{N}_0$ process $(Y_k)_{k\in\mathbb{Z}}$ is the solution of the difference equation:

$$a(B)Y_k = b(B)\xi_k \quad \text{for } k \in \mathbb{N},$$

where

- B is the backward shift operator,
- $(\xi_k)_{k\in\mathbb{Z}}$ is an iid sequence of random variables,
- $a(z) := z^p + a_1 z^{p-1} + ... + a_p$ with $a_1, ..., a_p \in \mathbb{R}$,
- $b(z) := b_0 z^q + b_1 z^{q-1} + ... + b_q$ with $b_0, ..., b_q \in \mathbb{R}$.

CARMA Process

Motivation:

A continuous-time ARMA(p,q) (CARMA(p,q)) process $(Y(t))_{t\in\mathbb{R}}$, $p,q\in\mathbb{N}_0$, $p>q$, is the solution of the stochastic differential equation:

$$a(D)Y(t)=b(D)DL(t)\quad \text{for } t\in\mathbb{R},$$

where

- D is the differential-operator (after t),
- $(L(t))_{t\in\mathbb{R}}$ is a Lévy process,
- $a(z):=z^p+a_1z^{p-1}+...+a_p$ with $a_1,...,a_p\in\mathbb{R}$,
- $b(z):=b_0z^q+b_1z^{q-1}+...+b_q$ with $b_0,...,b_q\in\mathbb{R}$.

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- $b = (b_q \cdots b_{q-p+1})' \in \mathbb{R}^p \quad (b_j = 0 \text{ for } j < 0)$.

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- $b = (b_q \cdots b_{q-p+1})' \in \mathbb{R}^p$ ($b_j = 0$ for $j < 0$).
- $\{z \in \mathbb{C} : a(z) = 0\} \subseteq (-\infty, 0) + i\mathbb{R}$.
- $\int_{\|\mathbf{x}\|>1} \log \|x\| \, \nu_L(dx) < \infty$.

CARMA Process

Let

$$\mathbf{X}(t) = \int_{-\infty}^{t} \mathrm{e}^{A(t-s)} e_p \, \mathrm{d}L(s) \qquad \text{for } t \in \mathbb{R}$$

in $\mathbb{R}^p$ with $e_p = (0, \dots, 0, 1)' \in \mathbb{R}^p$ be the solution of the stochastic differential equation

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then a (causal, stationary) CARMA(p,q) process is defined as

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Example: Ornstein-Uhlenbeck process

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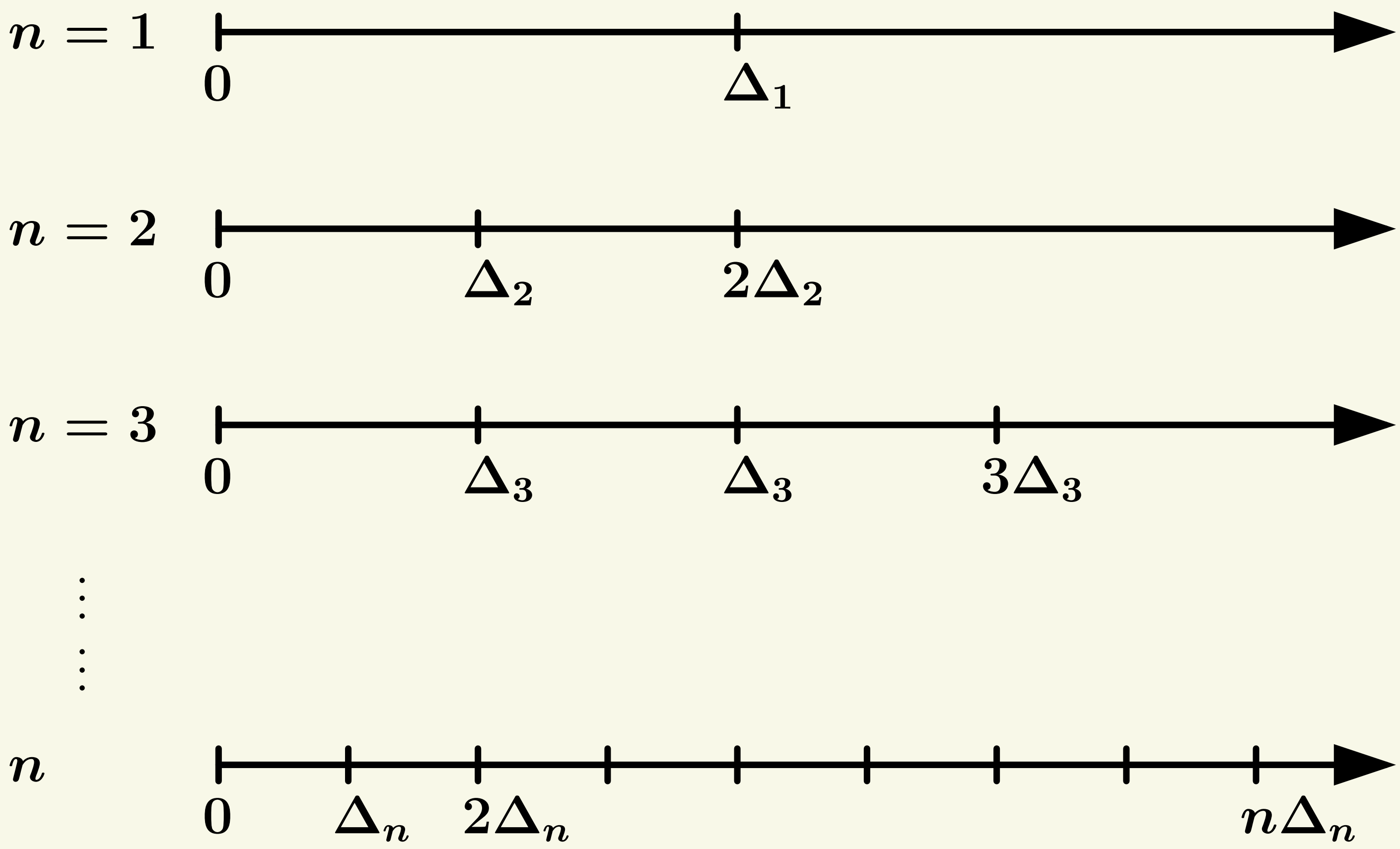
Spectral Density Estimation

Observation Scheme

High-frequency observations



$$\Delta_n \downarrow 0 \text{ as } n \to \infty \qquad \text{and} \qquad \lim_{n\to\infty} n\Delta_n = \infty.$$

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Then

$$\lim_{n\to\infty}\Delta_n g_{\Delta_n}(\omega\Delta_n)\,\mathbf{1}_{[-\frac{\pi}{\Delta_n},\frac{\pi}{\Delta_n}]}(\omega) = g_Y(\omega), \quad \omega\in\mathbb{R}.$$

Normalized Periodogram

Theorem (Fasen and Fuchs (2013))

The (normalized) periodogram of the sampled sequence $Y^{\Delta_n} := (Y(k\Delta_n))_{k\in\mathbb{N}}$ is

$$I_{n,Y^{\Delta_n}}(\omega) = \left| n^{-1/\alpha} \sum_{k=1}^{n} Y(k\Delta_n)\, \mathrm{e}^{-i\omega k} \right|^2, \quad \omega \in [-\pi, \pi],$$

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Then as $n \to \infty$,

$$\Delta_n^{2-\frac{2}{\alpha}} I_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathscr{D}} \frac{|b(i\omega)|^2}{|a(i\omega)|^2} \cdot \left| \int_{[0,1)} \mathrm{e}^{2\pi i s}\, \mathrm{d}L_s \right|^2, \quad \omega \in \mathbb{R}\setminus\{0\}.$$

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$\Rightarrow$ If $\alpha = 2$ this means that as $n \to \infty$,

$$\Delta_n I_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathscr{D}} \frac{\pi}{\sigma_L^2} g_Y(\omega) \cdot \left| \int_{[0,1)} e^{2\pi i s} \, dL_s \right|^2.$$

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where $\left(S_j^{\Re}(\widetilde{\omega}), S_j^{\Im}(\widetilde{\omega})\right)_{j\in\{1,\ldots,m\}}$ is a $(2m)$-dimensional stable random vector with characteristic function

$$\mathbb{E}\left[\exp\left\{i\left(\sum_{j=1}^{m} \theta_j S_j^{\Re}(\widetilde{\omega}) + v_j S_j^{\Im}(\widetilde{\omega})\right)\right\}\right] = \exp\{-\sigma_L^\alpha \cdot K_{\widetilde{\omega}}(\widetilde{\theta}, \widetilde{v})\},$$

$\widetilde{\theta} = (\theta_1, \ldots, \theta_m)$, $\widetilde{v} = (v_1, \ldots, v_m)$,

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1. If $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Z}$ (i.e. there is no $h \in \mathbb{Z}^m$, $h \neq 0$, such that $\langle h, \widetilde{\omega} \rangle = 0$),

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2. If $\omega_1, \dots, \omega_m$ are linearly dependent over $\mathbb{Z}$, then there is an $s \in \{1, \dots, m-1\}$ such that

$$K_{\widetilde{\omega}}(\widetilde{\theta}, \widetilde{v}) = \frac{1}{\lambda^{(m-s)}(\mathscr{M})} \int_{\mathscr{M}} \left| \sum_{j=1}^{m} \theta_j \cos(2\pi x_j) + v_j \sin(2\pi x_j) \right|^{\alpha} \mathrm{d}\lambda^{(m-s)}(x_1, \dots, x_m),$$

where $\mathscr{M} = \mathscr{M}(\omega_1, \dots, \omega_m)$ is a $(m-s)$-dimensional linear manifold in $[0,1)^m$.

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$\alpha = 2$: Then as $n \to \infty$,

$$\Delta_n I_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathscr{D}} 2\pi g_Y(\omega)\left(\frac{N_1^2}{2} + \frac{N_2^2}{2}\right) \stackrel{d}{=} 2\pi g_Y(\omega) E,$$

where N_1 and N_2 are i.i.d. standard normal random variables and E is a standard exponential random variable.

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Then as $n \to \infty$,

$$\Delta_n \widehat{I}_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathscr{D}} \left(\int_0^\infty f^2(s)\,\mathrm{d}s\right)^{-1} \frac{|b(i\omega)|^2}{|a(i\omega)|^2} \cdot \frac{[S^{\mathscr{R}}]^2 + [S^{\mathscr{I}}]^2}{S^2},$$

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$$\widehat{T}_{n,Y^{\Delta_n}}(\omega) = \sum_{|k|\leq m_n} W_n(k)\,\widehat{I}_{n,Y^{\Delta_n}}\left(\omega + \frac{k}{n}\right), \quad \omega \in [-\pi,\pi],$$

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where

- m_n is a sequence in $\mathbb{N}$ such that

$$m_n \to \infty \qquad \text{and} \qquad \frac{m_n}{n\Delta_n} \to 0 \qquad \text{as } n \to \infty.$$

Notation

The sequence of weight functions $W_n : \mathbb{Z} \to \mathbb{R}$ satisfies the following conditions:

- $W_n(k) = W_n(-k), \qquad W_n(k) \geq 0, \qquad \forall k \in \mathbb{N},$
- $\sum_{|k| \leq m_n} W_n(k) = 1, \qquad \forall n \in \mathbb{N},$
- $\max_{|k| \leq m_n} W_n^2(k) = o\left(\frac{1}{m_n}\right)$ as $n \to \infty$.

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Then as $n \to \infty$,

$$\Delta_n \widehat{T}_{n,Y^{\Delta_n}}(\omega \Delta_n) \xrightarrow{\mathbb{P}} \left(\int_0^\infty f^2(s)\,\mathrm{d}s\right)^{-1} \frac{|b(i\omega)|^2}{|a(i\omega)|^2}, \quad \omega \in \mathbb{R}\backslash\{0\}.$$

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$$\Delta_n T_{n,Y^{\Delta_n}}(\omega\Delta_n) \xrightarrow{\mathbb{P}} \frac{\pi}{\sigma_L^2} g_Y(\omega) =: \Psi(\omega), \quad \omega \in \mathbb{R}\setminus\{0\}.$$

Parameter estimation

Motivation

Let

$$a(z) := z^p + a_1 z^{p-1} + ... + a_p,$$
$$b(z) := z^q + b_1 z^{q-1} + ... + b_q,$$

and the normalized power transfer function (spectral density)

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$$\Psi(\omega) := \frac{|b(i\omega)|^2}{|a(i\omega)|^2} = \frac{\prod_{k=1}^{q} (\omega + i\,\mu_k)(\omega - i\,\overline{\mu_k})}{\prod_{j=1}^{p} (\omega + i\,\lambda_j)(\omega - i\,\overline{\lambda_j})}$$

where $\mu_1, \ldots, \mu_q$ are the zeros of $b(\cdot)$ and $\lambda_1, \ldots, \lambda_p$ are the zeros of $a(\cdot)$.

Motivation

Assumption

The zeros $\mu_1, ..., \mu_q$ and $\lambda_1, ..., \lambda_p$ are all distinct and possess strictly negative real parts.

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Define

$$\Psi(\omega;\theta) := \frac{|(i\omega)^q + b_1(i\omega)^{q-1} + \dots + b_q|^2}{|(i\omega)^p + a_1(i\omega)^{p-1} + \dots + a_p|^2}$$

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Then the parameters of the CARMA process are identifiable from $\Psi(\cdot;\theta)$.

Estimation Heuristic

Suppose we have observed the CARMA(p,q) process on the time grid $\{\Delta_n, ..., n\Delta_n\}$. Then we choose $m \in \mathbb{N}$ different frequencies $\omega_j \in (0, \pi/\Delta_n)$, $j = 1, ..., m$, Suppose we have observed the CARMA(p,q) process on the time grid $\{\Delta_n, \ldots, n\Delta_n\}$. Then we choose $m \in \mathbb{N}$ different frequencies $\omega_j \in (0, \pi/\Delta_n)$, $j = 1, \ldots, m$, and solve the constrained nonlinear least squares problem

$$\widehat{\theta} := \underset{\theta \in \Theta}{\operatorname{argmin}} \sum_{j=1}^{m} \left| \log\left(C_\theta \cdot \Psi_\theta(\omega_j)\right) - \log\left(\underbrace{\Delta_n \widehat{T}_{n, Y^{\Delta_n}}(\omega_j \Delta_n)}_{\xrightarrow{\mathbb{P}} C_{\theta^*} \cdot \Psi_{\theta^*}(\omega_j)} \right) \right|^2 .$$

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$\rightsquigarrow$ variance stabilizing technique

Example

Consider a CARMA$(2,1)$ process which is the strictly stationary solution to

$$\left(\mathsf{D}^2 + a_1\mathsf{D} + a_2\right) Y_t = \left(\mathsf{D} + \mu\right) \mathsf{D} L_t, \quad t \in \mathbb{R},$$

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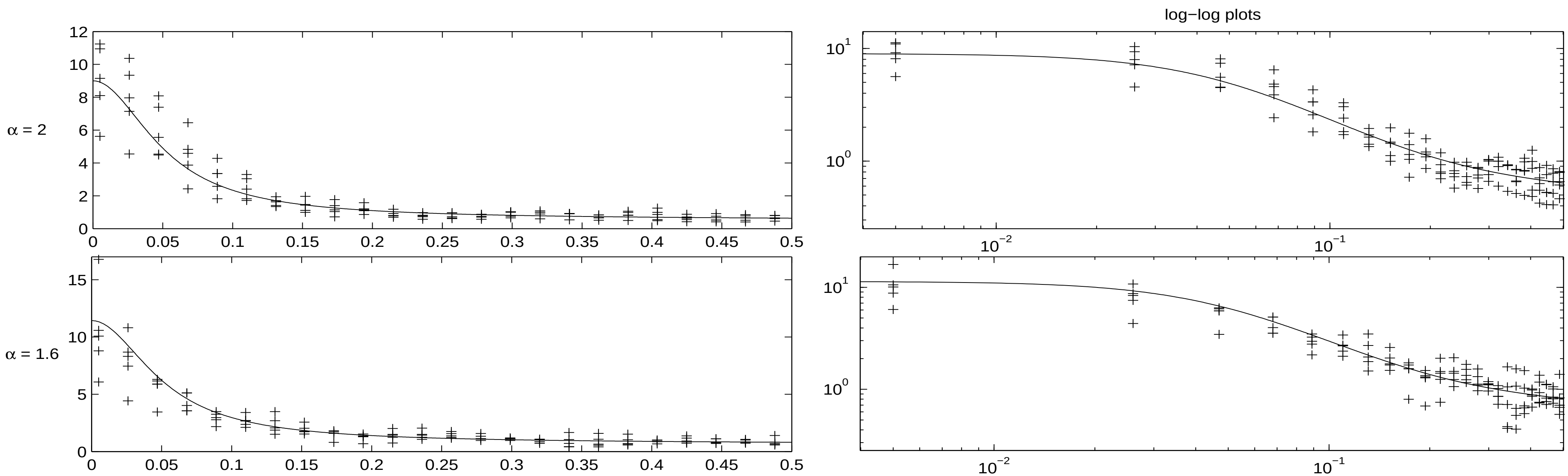
i.e. $a(z) = z^2 + a_1 z + a_2 = (z - \lambda_1)(z - \lambda_2)$ and $b(z) = z + \mu$.

In this case the normalized power transfer function can be written as

$$\left(\int_0^\infty f^2(s)\,\mathrm{d}s\right)^{-1} \Psi(\omega) = C(a_1, a_2, \mu) \cdot \frac{\omega^2 + \mu^2}{\omega^4 + (a_1^2 - 2a_2)\omega^2 + a_2^2}$$

with $C(a_1, a_2, \mu) = \left(\int_0^\infty f^2(s)\,\mathrm{d}s\right)^{-1} = 2\frac{a_1 a_2}{\mu^2 + a_2}$.

Simulation



Smoothed periodogram values plotted against frequencies for five selected time series (pluses) in the Gaussian case (on top) and the 1.6-stable case (below). The true spectral density and normalized power transfer function is plotted as a solid line, respectively.

$n = 15000$; $\Delta_n = 0.1$; $m = 300$; $m_n = \lfloor \sqrt{n\Delta_n} \rfloor = 38$; $W_n(k) = \frac{1}{2m_n+1}$ for $|k| \le m_n$.

Simulation Study

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		σ_L	a_1	a_2	μ
	True	1.5	2.0	0.1	0.2
	Mean	1.5127	2.0859	0.1182	0.2159
$\alpha = 2$	Bias	0.0127	0.0859	0.0182	0.0159
	Std. dev.	0.0392	0.1204	0.0358	0.0366
	Mean	-	2.0580	0.1108	0.2185
$\alpha = 1.8$	Bias	-	0.0580	0.0108	0.0185
	Std. dev.	-	0.1240	0.0372	0.0378
	Mean	-	2.0626	0.1079	0.2127
$\alpha = 1.6$	Bias	-	0.0626	0.0079	0.0127
	Std. dev.	-	0.1130	0.0315	0.0361
	Mean	-	2.0659	0.1101	0.2129
$\alpha = 1.4$	Bias	-	0.0659	0.0101	0.0129
	Std. dev.	-	0.1151	0.0311	0.0329

Simulation study for different values of α, based on 250 sample paths each. $n = 15000$; $\Delta_n = 0.1$; $m = 300$; $m_n = \lfloor \sqrt{n\Delta_n} \rfloor = 38$; $W_n(k) = \frac{1}{2m_n+1}$ for $|k| \leq m_n$.

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References

[1] FASEN, V. (2012). Limit theory for high frequency sampled MCARMA models. Submitted for publication.

[2] FASEN, V. AND FUCHS, F. (2012). Spectral estimates for high-frequency sampled CARMA processes. Submitted for publication.

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End

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Thank you!