Extreme eigenvalues of random matrices with dependent entries

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Wigner matrix with dependent entries

On going works with
Arijit Chakraborty, Parthanil Roy and Deepayan Sarkar

Wigner Matrix W_n: symmetric with IID random variables.

$$W_n = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1(n-1)} & x_{1n} \\ x_{12} & x_{22} & x_{23} & \cdots & x_{2(n-1)} & x_{2n} \\ & & & \vdots & & \\ x_{1n} & x_{2n} & x_{3n} & \cdots & x_{(n-1)n} & x_{nn} \end{bmatrix}.$$

Problems of interest

- The limiting spectral distribution of empirical spectral distribution:

$$L_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}$$

- Spectral radius: $Sp(A) = \max\{|\lambda_i| : \lambda_i \text{ eigenvalues of } A\}$.
- Spectral Norm: $\|A\| = \sqrt{\lambda_{\max}(A^* A)}$.
- If A is self-adjoint matrix ($A = A^*$) then $sp(A) = \|A\|$.
- Universality of the results in light tailed case.
- Heavy Tailed case: behavior depends on the tail of the input sequence.

The model

- Suppose that $\{X_{i,j} : i, j \geq 1\}$ is a family of i.i.d. random variables such that

$$P(|X_{11}| > \cdot) \in RV_{-\alpha} \text{ for some } \alpha > 0\,.$$

- $\{c_{i,j} : 0 \leq i, j \leq N\}$ are real numbers.
- Define

$$Y_{k,l} := \sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij} X_{i+k,j+l},\ 1 \leq k \leq l\,.$$

- For $k > l$, set

$$Y_{k,l} := Y_{l,k}\,.$$

- For $n \geq 1$, let A_n denote the $n \times n$ matrix whose (i, j)-th entry is $Y_{i,j}$.

The problem

- Problem: To find the asymptotics of $\|A_n\|$ as $n \to \infty$.
- Similar models with dependence considered for Covariance matrix by Davis, Pffafel, Stelzer (2011)
- i.i.d. entries + heavy tailed ($0 \leq \alpha \leq 2$) - Soshnikov
- i.i.d. entries + heavy tailed ($2 \leq \alpha \leq 4$)-Ben Arous and Péche.

The result

Define

$$
\begin{aligned}
b(t) &:= \inf\left\{x : P(|X_{11}| > x) \le t^{-1}\right\},\ t > 0\,,\\
C &:= \begin{bmatrix} 0 & \dots & 0 & c_{NN} & \dots & c_{N0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & c_{0N} & \dots & c_{00} \\ c_{NN} & \dots & c_{0N} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{N0} & \dots & c_{00} & 0 & \dots & 0 \end{bmatrix}_{(2N+1)\times(2N+1)} .
\end{aligned}
$$

The result (contd.)

Theorem

If $0 < \alpha < 1$, *then*

$$\frac{\|A_n\|}{b(n^2/2)} \Longrightarrow \|C\| Z \,,$$

as $n \to \infty$, *where* Z *is a Fréchet* (α) *random variable, with c.d.f.*

$$P(Z \le x) = \exp\left(-x^{-\alpha}\right), \; x > 0 \,.$$

Some remarks about the proof

- We look at $A_n^{2r} = U_n + V_n$.
- U_n contains all the $2r$-th power and V_n other crossed terms.
- $\|V_n\|_\infty = o_P(b(n)^{2r})$. (Our proof fails here for $\alpha > 1$).
- U_n gives the contributing terms as $r \to \infty$.
- $1 \leq \alpha \leq 4$ remains open!!
- $\alpha > 4$?? Light tailed case- Gaussian??

Light tailed: LSD

Theorem (A. Chakraborty, R, D. Sarkar)

The limiting spectral distribution for light tailed entries converge to compactly supported measure μ.
The Stieltjes transform $\mathcal{G}$ of μ, defined by

$$\mathcal{G}(z) := \int_{\mathbb{R}} \frac{\mu(dx)}{z-x},\ z \in \mathbb{C} \setminus \mathbb{R}\,,$$

is given by

$$\mathcal{G}(z) = \int_0^1 \mathcal{H}(z,x)dx$$

where $\mathcal{H}$ is a function from $\mathbb{C} \times [0,1]$ to $\mathbb{C}$, satisfying

$$z\mathcal{H}(z,x) = 1 + \mathcal{H}(z,x)\int_0^1 f(x,y)\mathcal{H}(z,y)dy,\ 0 \le x \le 1\,.$$

Largest Eigenvalue of Sample Autocovariance matrix

On going works with
Bikramjit Das and Souvik Ghosh

- Let $(X_i)_{i \in \mathbb{Z}}$ be a stationary process with mean $\mu = \mathrm{E}\, X_i$,
- $\gamma_k = \mathrm{E}[(X_0 - \mu)(X_k - \mu)]$, $k \in \mathbb{Z}$, its autocovariances.
- $$\Sigma_n = (\gamma_{i-j})_{1 \le i,j \le n}$$
  is the autocovariance matrix of $(X_1, \dots, X_n)$.
- Given observations $X_1, X_2, \cdots, X_n$ with $E[X_i] = 0$ consider
  $$\hat{\Sigma}_n = (\hat{\gamma}_{i-j})_{1 \le i,j \le n}, \text{ where } \hat{\gamma}_k = \frac{1}{n} \sum_{i=|k|+1}^{n} X_{i-|k|} X_i. \quad (2.1)$$
- Aim: Asymptotics of the spectral norm of $\widehat{\Sigma_n}$.

Existing Works

- Wu and Pourhamadi (2009) : $\|\widehat{\Sigma_n} - \Sigma\| \nrightarrow 0$ in probability.
- Xiao and Wu (2012):

$$\lim_{n\to\infty} \mathrm{P}\left[C^{-1}\log n \le \|\widehat{\Sigma_n} - \Sigma\| \le C\log n\right] \to 1.$$

- Related works on Autocovariance matrix estimation: Basak and Bose (2012) and Cai, Zhang and Zhou (2010).
- Banded consistent estimator can also be formed-Xiao and Wu (2012) and McMurry and Politis(2010)

$X_t = \sum_{i=0}^{\infty} a_i \epsilon_{t-i}$.
Conjecture 1:

$$\lim_{n\to\infty} \frac{\|\hat{\Sigma}_n\|}{2\pi \max_\theta f(\theta) \log n} = C(?) \quad \text{almost surely.} \tag{2.2}$$

Conjecture 2:

$$\lim_{n\to\infty} \frac{\|\hat{\Sigma}_n\|}{2\pi \max_\theta f(\theta) \log n} = C \quad \text{in probability.} \tag{2.3}$$

Conjecture 3:

$$\frac{\|\hat{\Sigma}_n\| - a_n}{b_n} \implies \Lambda \text{ where } \Lambda(x) = \exp(-e^{-x}).$$

$$\hat{\Sigma}_n = \begin{bmatrix} \hat{\gamma}_0 & \hat{\gamma}_1 & \cdots & \hat{\gamma}_{n-2} & \hat{\gamma}_{n-1} \\ \hat{\gamma}_1 & \hat{\gamma}_0 & \hat{\gamma}_1 & \cdots & \hat{\gamma}_{n-2} \\ \vdots & \hat{\gamma}_1 & \hat{\gamma}_0 & \ddots & \vdots \\ \hat{\gamma}_{n-2} & \vdots & \ddots & \ddots & \hat{\gamma}_1 \\ \hat{\gamma}_{n-1} & \hat{\gamma}_{n-2} & \cdots & \hat{\gamma}_1 & \hat{\gamma}_0 \end{bmatrix} = ((\hat{\gamma}_{|i-j|}))_{0 \le i,j \le n}.$$

Motivation coming from works of Sen and Virag (2012) on Toeplitz matrices.

Brockwell and Davis: $\hat{\Sigma}_n = \frac{1}{n} TT'$ where T is the following $n \times 2n$ matrix:

$$T_{n\times 2n} = \begin{bmatrix} 0 & 0 & \cdots & 0 & X_0 & X_1 & \cdots & X_{n-1} \\ 0 & 0 & \cdots & X_0 & X_1 & \cdots & X_{n-1} & 0 \\ \vdots & & & & & & \ddots & \vdots \\ 0 & 0 & X_0 & X_1 & \cdots & X_{n-1} & 0 & \cdots \\ 0 & X_0 & X_1 & \cdots & X_{n-1} & 0 & 0 & \cdots \end{bmatrix}$$

T is $n \times 2n$ principal submatrix of a Reverse Circulant matrix. A reverse circulant matrix of order $2n \times 2n$ is of the following form,

$$\mathbf{R}_{2n} = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{2n-2} & a_{2n-1} \\ a_1 & a_2 & a_3 & \dots & a_{2n-1} & a_0 \\ a_2 & a_3 & a_4 & \dots & a_0 & a_1 \\ & & & \vdots & & \\ a_{2n-1} & a_0 & a_1 & \dots & a_{2n-3} & a_{2n-2} \end{bmatrix}$$

Now if you take $a_j = 0$ for $0 \leq j \leq n-1$ and $a_{j+n} = X_j$ for $0 \leq j \leq n$

Now note that any Reverse circulant of order $2n \times 2n$ can be written in the block form as

$$\begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_2 & \mathbf{H}_1 . \end{bmatrix}$$

With the above choice of a_j we can write $T = [\mathbf{H}_1 \quad \mathbf{H}_2]$. So we can write,

$$\begin{bmatrix} \hat{\Sigma}_n & O_n \\ O_n & O_n \end{bmatrix} = \frac{1}{n} \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ O_n & O_n \end{bmatrix} \begin{bmatrix} \mathbf{H}_1 & O_n \\ \mathbf{H}_2 & O_n \end{bmatrix} = \frac{1}{n} \mathbf{Q}_{2n} \mathbf{R}_{2n} \mathbf{R}_{2n} \mathbf{Q}_{2n}$$

where $\mathbf{Q}_{2n}$ denote the following projection matrix

$$\mathbf{Q}_{2n} = \left[\begin{array}{cc} \mathbf{I}_n & 0_n \\ 0_n & 0_n \end{array} \right].$$

- $\hat{\Sigma}_n$ has the same non zero eigenvalues as $2\,\mathbf{P}_{2n}\,\mathbf{D}^{\dagger}_{2n}\,\mathbf{P}_{2n}$.
- $\mathbf{P}_{2n} = \mathbf{U}^{*}_{2n}\,\mathbf{Q}_{2n}\,\mathbf{U}_{2n}$.
- $\mathbf{D}^{\dagger}_{2n} = \mathrm{diag}(d_0, d_1, \ldots, d_{2n-1})$.

$$d_k = \frac{1}{2n}\Big| \left[\sum_{j=0}^{n-1} X_j \exp(\frac{2\pi ijk}{2n})\right] \Big|^2.$$

- $d_k = d_{2n-k}$.

- $\| \mathbf{P}_{2n} D_{2n}^{\dagger} \mathbf{P}_{2n} \| \leq \max_{1 \leq k \leq n} d_k$.
- $$d_k = \frac{1}{2n} \Big| \left[\sum_{j=0}^{n-1} X_j \exp(\frac{2\pi ijk}{2n}) \right] \Big|^2 = C_k^2 + S_k^2.$$
- For $\{X_j\}$ i.i.d. Gaussian, C_j, S_j approximately χ^2 with 1-degree of freedom.
- d_k is approximately exponential, but having dependence.
- For k-l odd,

$$Cov(C_k, S_l) = \frac{1}{n} \left[\cot(\frac{\pi(k+l)}{2n}) + \cot(\frac{\pi(l-k)}{2n}) \right]$$

Suppose $\{X_j\}$ are i.i.d. Gaussian. Using the methods of Berman (1964), and using, $\sum_{1\leq \text{k-l: odd} \leq n} \text{Cov}(C_k, S_l) \sim \log n$

Theorem

$$\mathrm{P}\left[\max_{1\leq k\leq n-1} d_k < 2x + 2\log n\right] \to \exp(-e^{-x}).$$

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Corollary

$$\frac{\|\widehat{\Sigma_n}\|}{2\pi \max_\theta f(\theta)} = O_P(2 \log n).$$

For $X_t = \sum_{j=1}^{\infty} \psi_j Z_{t-j}$ one has to assume $\sum_{j=1}^{\infty} j^{\frac{1}{2}} |\psi_j| < \infty$ and $f(\lambda) > 0$ for all $\lambda \in [0, \pi]$.
We use some results on Periodogram from Walker(1965) to show the above theorem.

Concluding remarks

- Using $\| \mathsf{P}_{2n} \| \leq 1$ loses most of information about the constant.
- $\mathsf{P}_{2n} \sim \Pi : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$.
- $\| \mathsf{P}_{2n} \, \mathsf{D}_{2n}^{\dagger} \, \mathsf{P}_{2n} \| \approx \|\Pi_{2n} \, \mathsf{D}_{2n}^{\dagger} \, \Pi_{2n}\| \approx \log n \|\Pi_{2n}^{\dagger}\Pi_{2n}^{\dagger}\|$.
- $\|\Pi_{2n}^{\dagger}\Pi_{2n}^{\dagger}\| \approx 0.8$.

Thank you for your attention ☺