Asymptotic Independence of Stochastic Volatility Models

Anja Janßen

University of Hamburg

ISI Kolkata
January 2013

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Overview

1. Stochastic Volatility Models
   - General definition
   - Extremal dependence structure
2. Second order behavior
   - Hidden regular variation and coefficient of tail dependence
   - Breiman's lemma for hidden regular variation
3. SV models with heavy-tailed volatility sequence
   - General behavior
   - Weibull-type log-volatilities

General definition of SV models

- Many common models for financial time series are of the form

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$

where $\epsilon_t, t \in \mathbb{Z}$, are i.i.d. standardized innovations and $(\sigma_t)_{t \in \mathbb{Z}}$, is referred to as a "volatility" sequence.

- Sometimes

$$\sigma_t \in \sigma(X_t, X_{t-1}, \dots, \sigma_{t-1}, \sigma_{t-2}, \dots), \quad t \in \mathbb{Z},$$

e.g. for GARCH models.

- Alternative: Volatility sequence $(\sigma_t)_{t \in \mathbb{Z}}$ depends on an additional source of randomness $\Rightarrow$ SV models!

General definition of SV models

- Many common models for financial time series are of the form

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$

where $\epsilon_t, t \in \mathbb{Z}$, are i.i.d. standardized innovations and $(\sigma_t)_{t\in\mathbb{Z}}$, is referred to as a "volatility" sequence.

- Sometimes

$$\sigma_t \in \sigma(X_t, X_{t-1}, \dots, \sigma_{t-1}, \sigma_{t-2}, \dots), \quad t \in \mathbb{Z},$$

e.g. for GARCH models.

- Alternative: Volatility sequence $(\sigma_t)_{t\in\mathbb{Z}}$ depends on an additional source of randomness $\Rightarrow$ SV models!

General definition of SV models

- Many common models for financial time series are of the form

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$

where $\epsilon_t, t \in \mathbb{Z}$, are i.i.d. standardized innovations and $(\sigma_t)_{t\in\mathbb{Z}}$, is referred to as a "volatility" sequence.

- Sometimes

$$\sigma_t \in \sigma(X_t, X_{t-1}, \ldots, \sigma_{t-1}, \sigma_{t-2}, \ldots), \quad t \in \mathbb{Z},$$

e.g. for GARCH models.

- Alternative: Volatility sequence $(\sigma_t)_{t\in\mathbb{Z}}$ depends on an additional source of randomness $\Rightarrow$ SV models!

Taylor's SV model

A very common specification is

Taylor's lognormal SV model (1982)

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$

$$\log(\sigma_t^2) - \mu = \phi(\log(\sigma_{t-1}^2) - \mu) + \xi_t, \quad t \in \mathbb{Z},$$

where $\xi_t, t \in \mathbb{Z}$, are i.i.d. standard normal, independent of $(\epsilon_t)_{t\in\mathbb{Z}}$ and $|\phi| < 1$.

$\Rightarrow$ Volatility sequence has a log-normal distribution.

With regard to real data examples, heavy-tailed (power law) marginals are a preferable feature of models for financial time series.

Taylor's SV model

A very common specification is

Taylor's lognormal SV model (1982)

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$

$$\log(\sigma_t^2) - \mu = \phi(\log(\sigma_{t-1}^2) - \mu) + \xi_t, \quad t \in \mathbb{Z},$$

where $\xi_t, t \in \mathbb{Z}$, are i.i.d. standard normal, independent of $(\epsilon_t)_{t\in\mathbb{Z}}$ and $|\phi| < 1$.

$\Rightarrow$ Volatility sequence has a log-normal distribution.

With regard to real data examples, heavy-tailed (power law) marginals are a preferable feature of models for financial time series.

Taylor's SV model

A very common specification is

Taylor's lognormal SV model (1982)

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$
$$\log(\sigma_t^2) - \mu = \phi(\log(\sigma_{t-1}^2) - \mu) + \xi_t, \quad t \in \mathbb{Z},$$

where $\xi_t, t \in \mathbb{Z}$, are i.i.d. standard normal, independent of $(\epsilon_t)_{t \in \mathbb{Z}}$ and $|\phi| < 1$.
$\Rightarrow$ Volatility sequence has a log-normal distribution.
With regard to real data examples, heavy-tailed (power law) marginals are a preferable feature of models for financial time series.

SV models with heavy-tailed innovation sequence

Breiman's lemma - "the heaviest tail wins"

- If $|\epsilon_t|$ is regularly varying with index $-\alpha$, i.e.

$$c(u)P(|\epsilon_t| > u) \to 1, \quad u \to \infty,$$

  for a regularly varying function $c(\cdot)$ with index α
- and $\sigma_t \geq 0$ independent of ϵ_t with $E(\sigma_t^{\alpha+\delta}) < \infty$ for some $\delta > 0$, it holds that

$$c(u)P(\sigma_t|\epsilon_t| > u) \to E(\sigma_t^{\alpha}), \quad u \to \infty,$$

i.e. $|X_t| = \sigma_t|\epsilon_t|$ is tail-equivalent to $|\epsilon_t|$.

$\Rightarrow$ Common model specification: Taylor's log-normal SV model with heavy-tailed innovations.

SV models with heavy-tailed innovation sequence

Breiman's lemma - "the heaviest tail wins"

- If $|\epsilon_t|$ is regularly varying with index $-\alpha$, i.e.

$$c(u)P(|\epsilon_t| > u) \to 1, \quad u \to \infty,$$

  for a regularly varying function $c(\cdot)$ with index α

- and $\sigma_t \geq 0$ independent of ϵ_t with $E(\sigma_t^{\alpha+\delta}) < \infty$ for some $\delta > 0$, it holds that

$$c(u)P(\sigma_t|\epsilon_t| > u) \to E(\sigma_t^{\alpha}), \quad u \to \infty,$$

i.e. $|X_t| = \sigma_t|\epsilon_t|$ is tail-equivalent to $|\epsilon_t|$.

$\Rightarrow$ Common model specification: Taylor's log-normal SV model with heavy-tailed innovations.

Extremal dependence structure

What do we know about joint extremal behavior of

$$\begin{pmatrix} X_0 \\ X_h \end{pmatrix} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_h \end{pmatrix}, \quad h > 0?$$

Multivariate Breiman (Basrak, Davis, Mikosch (2002))

- Random vector $\mathbf{X} \in \mathbb{R}^d$ multivariate regularly varying with index $-\alpha$, i.e.

$$c(u)P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu(\cdot)$$

for a regularly varying function $c(\cdot)$ with index α and a measure μ on $\mathbb{R}^d \setminus \{\mathbf{0}\}$,

- random $q \times d$ matrix $\mathbf{A}$, independent of $\mathbf{X}$, with $0 < E(\|\mathbf{A}\|^{\alpha+\delta}) < \infty$ for some $\delta > 0$. Then

$$c(u)P(u^{-1}\mathbf{A}\mathbf{X} \in \cdot) \xrightarrow{v} \tilde{\mu}(\cdot) := E\left[\mu \circ \mathbf{A}^{-1}(\cdot)\right].$$

Extremal dependence structure

What do we know about joint extremal behavior of

$$\begin{pmatrix} X_0 \\ X_h \end{pmatrix} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_h \end{pmatrix}, \quad h > 0?$$

Multivariate Breiman (Basrak, Davis, Mikosch (2002))

- Random vector $\mathbf{X} \in \mathbb{R}^d$ multivariate regularly varying with index $-\alpha$, i.e.

$$c(u)P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu(\cdot)$$

  for a regularly varying function $c(\cdot)$ with index α and a measure μ on $\mathbb{R}^d \setminus \{\mathbf{0}\}$,
- random $q \times d$ matrix $\mathbf{A}$, independent of $\mathbf{X}$, with $0 < E(\|\mathbf{A}\|^{\alpha+\delta}) < \infty$ for some $\delta > 0$. Then

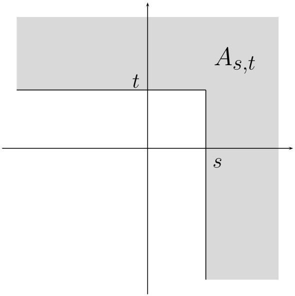
$$c(u)P(u^{-1}\mathbf{A}\mathbf{X} \in \cdot) \xrightarrow{v} \tilde{\mu}(\cdot) := E\left[\mu \circ \mathbf{A}^{-1}(\cdot)\right].$$

Extremal dependence structure

$$c(u)P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu(\cdot) \Rightarrow c(u)P(u^{-1}\mathbf{AX} \in \cdot) \xrightarrow{v} \tilde{\mu}(\cdot) := E\left[\mu \circ \mathbf{A}^{-1}(\cdot)\right]$$

Application to Taylor's log-normal SV model:

- $\begin{pmatrix} X_0 \\ X_h \end{pmatrix} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_h \end{pmatrix}$
- (ϵ_0, ϵ_t) bivariate regularly varying with μ on $[-\infty, \infty] \times [-\infty, \infty] \setminus \{(0,0)\}$ concentrated on the axes $\Rightarrow \mu(A_{s,t}) = c(s^{-\alpha} + t^{-\alpha})$.



$\Rightarrow (X_0, X_h)$ is regularly varying with

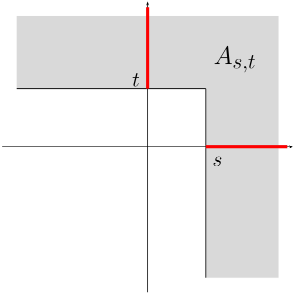
$$\tilde{\mu}(A_{s,t}) = E\left[\mu \circ \begin{pmatrix} \sigma_0^{-1} & 0 \\ 0 & \sigma_h^{-1} \end{pmatrix} (A_{s,t})\right] = cE(\sigma_h^{\alpha})(s^{-\alpha} + t^{-\alpha})$$

Extremal dependence structure

$$c(u)P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu(\cdot) \;\Rightarrow\; c(u)P(u^{-1}\mathbf{A}\mathbf{X} \in \cdot) \xrightarrow{v} \tilde{\mu}(\cdot) := E\left[\mu \circ \mathbf{A}^{-1}(\cdot)\right]$$

Application to Taylor's log-normal SV model:

- $\begin{pmatrix} X_0 \\ X_h \end{pmatrix} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_h \end{pmatrix}$
- (ϵ_0, ϵ_t) bivariate regularly varying with μ on $[-\infty,\infty] \times [-\infty,\infty] \setminus \{(0,0)\}$ concentrated on the axes $\Rightarrow \mu(A_{s,t}) = c(s^{-\alpha} + t^{-\alpha})$.



$\Rightarrow (X_0, X_h)$ is regularly varying with

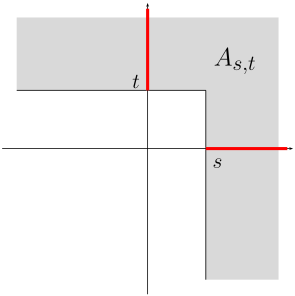
$$\tilde{\mu}(A_{s,t}) = E\left[\mu \circ \begin{pmatrix} \sigma_0^{-1} & 0 \\ 0 & \sigma_h^{-1} \end{pmatrix}(A_{s,t})\right] = cE(\sigma_h^{\alpha})(s^{-\alpha} + t^{-\alpha})$$

Extremal dependence structure

$$c(u)P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu(\cdot) \Rightarrow c(u)P(u^{-1}\mathbf{AX} \in \cdot) \xrightarrow{v} \tilde{\mu}(\cdot) := E\left[\mu \circ \mathbf{A}^{-1}(\cdot)\right]$$

Application to Taylor's log-normal SV model:

- $\begin{pmatrix} X_0 \\ X_h \end{pmatrix} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_h \end{pmatrix}$
- (ϵ_0, ϵ_t) bivariate regularly varying with μ on $[-\infty, \infty] \times [-\infty, \infty] \setminus \{(0,0)\}$ concentrated on the axes
  $\Rightarrow \mu(A_{s,t}) = c(s^{-\alpha} + t^{-\alpha})$.



$\Rightarrow (X_0, X_h)$ is regularly varying with

$$\tilde{\mu}(A_{s,t}) = E\left[\mu \circ \begin{pmatrix} \sigma_0^{-1} & 0 \\ 0 & \sigma_h^{-1} \end{pmatrix} (A_{s,t})\right] = cE(\sigma_h^{\alpha})(s^{-\alpha} + t^{-\alpha})$$

Hidden regular variation and coefficient of tail dependence

Hidden regular variation (Resnick (2002))

A multivariate regularly varying vector $\mathbf{X} \in \mathbb{R}_+^d$ with limit measure μ concentrated on the axes shows hidden regular variation (HRV) on $(0,\infty]^d$ if a non-zero measure μ^0 on $(0,\infty]^d$ exists, such that

$$c^0(u)P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu^0(\cdot), \quad u \to \infty,$$

for a regularly varying function $c^0(\cdot)$ with index α^0.

Coefficient of tail dependence (Ledford & Tawn (1998))

If $\mathbf{X}$ is standardized to index -1 of regular variation, we call $\eta = 1/\alpha^0 \in (0,1]$ the coefficient of tail dependence.

$\Rightarrow$ Stochastic independence of X_1, X_2 implies $\eta = 1/2$ for (X_1, X_2) since $c^0(u) = (P(X_1 > u)P(X_2 > u))^{-1}$ is regularly varying with index 2.

Hidden regular variation and coefficient of tail dependence

Hidden regular variation (Resnick (2002))

A multivariate regularly varying vector $\mathbf{X} \in \mathbb{R}_+^d$ with limit measure μ concentrated on the axes shows hidden regular variation (HRV) on $(0, \infty]^d$ if a non-zero measure μ^0 on $(0, \infty]^d$ exists, such that

$$c^0(u)P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu^0(\cdot), \quad u \to \infty,$$

for a regularly varying function $c^0(\cdot)$ with index α^0.

Coefficient of tail dependence (Ledford & Tawn (1998))

If $\mathbf{X}$ is standardized to index -1 of regular variation, we call $\eta = 1/\alpha^0 \in (0, 1]$ the coefficient of tail dependence.

$\Rightarrow$ Stochastic independence of X_1, X_2 implies $\eta = 1/2$ for (X_1, X_2) since $c^0(u) = (P(X_1 > u)P(X_2 > u))^{-1}$ is regularly varying with index 2.

Breiman's lemma for hidden regular variation

- Remember the multivariate version of Breiman's lemma for a multivariate regularly varying vector and a random matrix. Does there exist an analogue for HRV?
- In the MRV setting, sets must be bounded away from $\mathbf{0}$, for HRV they must be bounded away from the axes. Set $\mathbb{F}^d = \{\mathbf{x} \in \mathbb{R}^d_{0,+} : \min(x_1, \ldots, x_d) = 0\}$.
- Define $d(\mathbf{x}, B) := \min_{\mathbf{y} \in B} \|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{x} \in \mathbb{R}^d, B \subset \mathbb{R}^d$, and $\mathcal{N}^d := \{\mathbf{x} \in \mathbb{R}^d_{0,+} : d(\mathbf{x}, \mathbb{F}^d) = 1\}$.

  $\Rightarrow$ For a $d \times d$ matrix $\mathbf{A}$ define

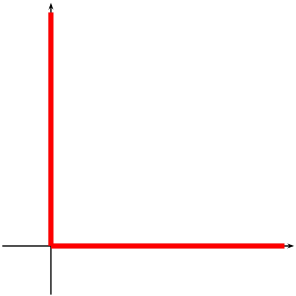
$$\tau(\mathbf{A}) := \sup_{\mathbf{x} \in \mathcal{N}^d} d(\mathbf{A}\mathbf{x}, \mathbb{F}^d) \in [0, \infty].$$

Breiman's lemma for hidden regular variation

- Remember the multivariate version of Breiman's lemma for a multivariate regularly varying vector and a random matrix. Does there exist an analogue for HRV?
- In the MRV setting, sets must be bounded away from **0**, for HRV they must be bounded away from the axes. Set $\mathbb{F}^d = \{\mathbf{x} \in \mathbb{R}^d_{0,+} : \min(x_1, \ldots, x_d) = 0\}$.
- Define $d(\mathbf{x}, B) := \min_{\mathbf{y} \in B} \|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{x} \in \mathbb{R}^d$, $B \subset \mathbb{R}^d$, and $\mathcal{N}^d := \{\mathbf{x} \in \mathbb{R}^d_{0,+} : d(\mathbf{x}, \mathbb{F}^d) = 1\}$.

  $\Rightarrow$ For a $d \times d$ matrix $\mathbf{A}$ define

$$\tau(\mathbf{A}) := \sup_{\mathbf{x} \in \mathcal{N}^d} d(\mathbf{A}\mathbf{x}, \mathbb{F}^d) \in [0, \infty].$$

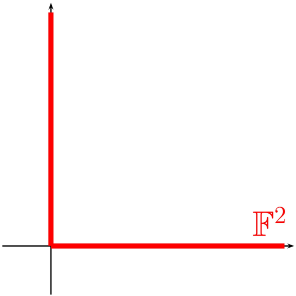


Breiman's lemma for hidden regular variation

- Remember the multivariate version of Breiman's lemma for a multivariate regularly varying vector and a random matrix. Does there exist an analogue for HRV?
- In the MRV setting, sets must be bounded away from $\mathbf{0}$, for HRV they must be bounded away from the axes. Set $\mathbb{F}^d = \{\mathbf{x} \in \mathbb{R}^d_{0,+} : \min(x_1, \ldots, x_d) = 0\}$.
- Define $d(\mathbf{x}, B) := \min_{\mathbf{y} \in B} \|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{x} \in \mathbb{R}^d$, $B \subset \mathbb{R}^d$, and $\mathcal{N}^d := \{\mathbf{x} \in \mathbb{R}^d_{0,+} : d(\mathbf{x}, \mathbb{F}^d) = 1\}$.

  $\Rightarrow$ For a $d \times d$ matrix $\mathbf{A}$ define

$$\tau(\mathbf{A}) := \sup_{\mathbf{x} \in \mathcal{N}^d} d(\mathbf{A}\mathbf{x}, \mathbb{F}^d) \in [0, \infty].$$

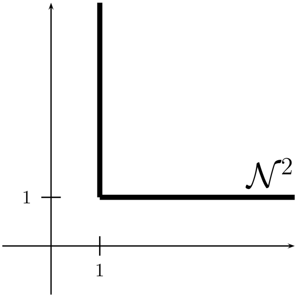


Breiman's lemma for hidden regular variation

- Remember the multivariate version of Breiman's lemma for a multivariate regularly varying vector and a random matrix. Does there exist an analogue for HRV?
- In the MRV setting, sets must be bounded away from $\mathbf{0}$, for HRV they must be bounded away from the axes. Set $\mathbb{F}^d = \{\mathbf{x} \in \mathbb{R}^d_{0,+} : \min(x_1, \ldots, x_d) = 0\}$.
- Define $d(\mathbf{x}, B) := \min_{\mathbf{y} \in B} \|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{x} \in \mathbb{R}^d$, $B \subset \mathbb{R}^d$, and $\mathcal{N}^d := \{\mathbf{x} \in \mathbb{R}^d_{0,+} : d(\mathbf{x}, \mathbb{F}^d) = 1\}$.

$\Rightarrow$ For a $d \times d$ matrix $\mathbf{A}$ define

$$\tau(\mathbf{A}) := \sup_{\mathbf{x} \in \mathcal{N}^d} d(\mathbf{A}\mathbf{x}, \mathbb{F}^d) \in [0, \infty].$$

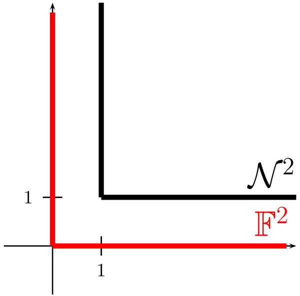


Breiman's lemma for hidden regular variation

- Remember the multivariate version of Breiman's lemma for a multivariate regularly varying vector and a random matrix. Does there exist an analogue for HRV?
- In the MRV setting, sets must be bounded away from $\mathbf{0}$, for HRV they must be bounded away from the axes. Set $\mathbb{F}^d = \{\mathbf{x} \in \mathbb{R}^d_{0,+} : \min(x_1, \ldots, x_d) = 0\}$.
- Define $d(\mathbf{x}, B) := \min_{\mathbf{y} \in B} \|\mathbf{x} - \mathbf{y}\|$ for $\mathbf{x} \in \mathbb{R}^d$, $B \subset \mathbb{R}^d$, and $\mathcal{N}^d := \{\mathbf{x} \in \mathbb{R}^d_{0,+} : d(\mathbf{x}, \mathbb{F}^d) = 1\}$.

  $\Rightarrow$ For a $d \times d$ matrix $\mathbf{A}$ define

$$\tau(\mathbf{A}) := \sup_{\mathbf{x} \in \mathcal{N}^d} d(\mathbf{A}\mathbf{x}, \mathbb{F}^d) \in [0, \infty].$$



Breiman's lemma for hidden regular variation

Multivariate Breiman for hidden regular variation (J. (2011))

- Random vector $\mathbf{X} \in \mathbb{R}_+^d$ showing hidden regular variation on $(0, \infty]^d$ with index $-\alpha^0$ such that

$$c^0(u)P(u^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \mu^0(\cdot)$$

- random invertible $d \times d$ matrix $\mathbf{A}$, independent of $\mathbf{X}$ with $\tau(\mathbf{A}) > 0$ almost surely and $E(\tau(\mathbf{A})^{\alpha^0+\delta}) < \infty$ for some $\delta > 0$. Then

$$c^0(u)P(u^{-1}\mathbf{A}\mathbf{X} \in \cdot) \xrightarrow{v} \tilde{\mu}^0(\cdot) := E\left[\mu^0 \circ \mathbf{A}^{-1}(\cdot)\right].$$

Implications for "classical" SV models

- Let $(\sigma_t)_{t\in\mathbb{Z}}$ a light-tailed volatility sequence and $(\epsilon_t)_{t\in\mathbb{Z}}$ i.i.d. standardized regularly varying innovations independent of the volatilities.
- For $h > 0$, vector (ϵ_0, ϵ_h) shows HRV with coefficient of tail dependence $\eta = 1/2$ $(\alpha^0 = 2)$.
- For invertible 2×2-matrix $\mathbf{\Sigma}_h = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix}$ one can show that $\tau(\mathbf{\Sigma}_h) = \max(\sigma_0, \sigma_h)$, thus $E(\tau(\mathbf{\Sigma}_h)^{2+\delta})$ exists for light-tailed volatilities.

$\Rightarrow$ Aforementioned result implies that $\begin{pmatrix} X_0 \\ X_h \end{pmatrix} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_h \end{pmatrix}$ has the same coefficient of tail dependence $\eta = 1/2$.

Implications for "classical" SV models

- Let $(\sigma_t)_{t\in\mathbb{Z}}$ a light-tailed volatility sequence and $(\epsilon_t)_{t\in\mathbb{Z}}$ i.i.d. standardized regularly varying innovations independent of the volatilities.
- For $h > 0$, vector (ϵ_0, ϵ_h) shows HRV with coefficient of tail dependence $\eta = 1/2$ $(\alpha^0 = 2)$.
- For invertible 2×2-matrix $\mathbf{\Sigma}_h = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix}$ one can show that $\tau(\mathbf{\Sigma}_h) = \max(\sigma_0, \sigma_h)$, thus $E(\tau(\mathbf{\Sigma}_h)^{2+\delta})$ exists for light-tailed volatilities.

$\Rightarrow$ Aforementioned result implies that $\begin{pmatrix} X_0 \\ X_h \end{pmatrix} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_h \end{pmatrix} \begin{pmatrix} \epsilon_0 \\ \epsilon_h \end{pmatrix}$ has the same coefficient of tail dependence $\eta = 1/2$.

General second order behavior of SV models

- Previous section has shown: Product of light-tailed volatility and heavy-tailed innovations "inherits" second order extremal behavior of the innovations.
- It is natural to assume the innovations to be independent, in contrast to the volatility terms.

⇒ A heavy-tailed volatility sequence and light-tailed innovations would offer us more flexibility with respect to the finer modeling of the extremal dependence structure.
cf. also Mikosch and Rezapur (2013)

General second order behavior of SV models

- Previous section has shown: Product of light-tailed volatility and heavy-tailed innovations "inherits" second order extremal behavior of the innovations.
- It is natural to assume the innovations to be independent, in contrast to the volatility terms.

$\Rightarrow$ A heavy-tailed volatility sequence and light-tailed innovations would offer us more flexibility with respect to the finer modeling of the extremal dependence structure.
cf. also Mikosch and Rezapur (2013)

Weibull-type log-volatilities

- Assume that

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$
$$\log(\sigma_t) - \mu = \sum_{j=0}^{\infty} \alpha_j \xi_{t-j}, \quad t \in \mathbb{Z}.$$

- $\xi_t, t \in \mathbb{Z}$, i.i.d. with distribution such that

$$P(\xi_t > z) \sim K z^{\alpha} e^{-z}, \quad z \to \infty,$$

for a real constant $\alpha \neq -1$ and a positive constant K and $P(\xi_t < z) = o\left(e^z\right), z \to -\infty$ (i.e. Exponential distribution).
- with $\alpha_i \in [0,1], \max_{i \in \mathbb{N}}\{\alpha_i\} = 1$, $\alpha_i = o(i^{-\theta}), i \to \infty$ for some $\theta > 1$.
- Innovations $\epsilon_t, t \in \mathbb{Z}$, i.i.d. such that $E(|\epsilon_t|^{1+\delta}) < \infty$.

Special case: Weibull-type AR(1) log-volatilities

- Assume that

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$
$$\log(\sigma_t) - \mu = \phi(\log(\sigma_{t-1}) - \mu) + \xi_t, \quad t \in \mathbb{Z}.$$

  with the same assumptions on the distributions of $\epsilon_t, \xi_t, t \in \mathbb{Z}$ as before and $\phi \in (0,1)$.

- Then obviously $\alpha_i = \phi^i \in [0,1], i \in \mathbb{N}$, $\max_{i \in \mathbb{N}}\{\alpha_i\} = 1$ and $\alpha_i = o(i^{-\theta}), i \to \infty$ or some $\theta > 1$.
- This may be regarded as an extension of Taylor's "standard" SV model.

Special case: Weibull-type AR(1) log-volatilities

- Assume that

$$X_t = \sigma_t \epsilon_t, \quad t \in \mathbb{Z},$$
$$\log(\sigma_t) - \mu = \phi(\log(\sigma_{t-1}) - \mu) + \xi_t, \quad t \in \mathbb{Z}.$$

  with the same assumptions on the distributions of $\epsilon_t, \xi_t, t \in \mathbb{Z}$ as before and $\phi \in (0,1)$.
- Then obviously $\alpha_i = \phi^i \in [0,1], i \in \mathbb{N}$, $\max_{i\in\mathbb{N}}\{\alpha_i\} = 1$ and $\alpha_i = o(i^{-\theta}), i \to \infty$ or some $\theta > 1$.
- This may be regarded as an extension of Taylor's "standard" SV model.

Stationary distribution of this model

- It follows from Rootzén (1986) that the corresponding MA(∞) process is well defined and that

$$P(\ln(\sigma_t) - \mu > z) \sim \hat{K} z^{\hat{\alpha}} e^{-z}, \quad z \to \infty,$$

for certain constants $\hat{K} > 0, \hat{\alpha} \in \mathbb{R}$.

- Thus, σ_t is regularly varying with index -1 (model can be generalized to index $-\alpha$ by writing $\frac{1}{\alpha}\xi_t$ instead ξ_t).
- Extremal behavior? Follows again from Rootzén that $(\ln(\sigma_0), \ln(\sigma_h))$ is asymptotically independent for all $h > 0$, same holds true for (σ_0, σ_h) and then by multivariate Breiman also for (X_0, X_h)

Stationary distribution of this model

- It follows from Rootzén (1986) that the corresponding MA(∞) process is well defined and that

$$P(\ln(\sigma_t) - \mu > z) \sim \hat{K} z^{\hat{\alpha}} e^{-z}, \quad z \to \infty,$$

for certain constants $\hat{K} > 0, \hat{\alpha} \in \mathbb{R}$.

- Thus, σ_t is regularly varying with index -1 (model can be generalized to index $-\alpha$ by writing $\frac{1}{\alpha}\xi_t$ instead ξ_t).
- Extremal behavior? Follows again from Rootzén that $(\ln(\sigma_0), \ln(\sigma_h))$ is asymptotically independent for all $h > 0$, same holds true for (σ_0, σ_h) and then by multivariate Breiman also for (X_0, X_h)

Stationary distribution of this model

- It follows from Rootzén (1986) that the corresponding MA(∞) process is well defined and that

$$P(\ln(\sigma_t) - \mu > z) \sim \hat{K} z^{\hat{\alpha}} e^{-z}, \quad z \to \infty,$$

for certain constants $\hat{K} > 0, \hat{\alpha} \in \mathbb{R}$.

- Thus, σ_t is regularly varying with index -1 (model can be generalized to index $-\alpha$ by writing $\frac{1}{\alpha}\xi_t$ instead ξ_t).
- Extremal behavior? Follows again from Rootzén that $(\ln(\sigma_0), \ln(\sigma_h))$ is asymptotically independent for all $h > 0$, same holds true for (σ_0, σ_h) and then by multivariate Breiman also for (X_0, X_h)

Second order behavior of this model

- We are interested in the asymptotic behavior of

$$
\begin{aligned}
& P(\ln(\sigma_t) > \ln(x), \ln(\sigma_{t+h}) > \ln(x)) \\
= \quad & P(\sigma_t > x, \sigma_{t+h} > x) \\
\overset{\mu=0}{=} \quad & P\left(e^{\sum_{j=0}^{\infty} \xi_{t-j}\alpha_j} > x, e^{\sum_{j=0}^{\infty} \xi_{t+h-j}\alpha_j} > x\right) \\
= \quad & P\left(\prod_{j=0}^{\infty} \left(e^{\xi_{t-j}}\right)^{\alpha_j} > x, \prod_{j=0}^{\infty} \left(e^{\xi_{t+h-j}}\right)^{\alpha_j} > x\right),
\end{aligned}
$$

where we know that $e^{\xi_t}, t \in \mathbb{Z}$, are i.i.d. regularly varying with index -1.

A general result for weighted power products

Let $Y_1, Y_2, \dots$ be i.i.d. regularly varying random variables with index -1. Let $\alpha_i, \beta_i, i \in \mathbb{N}$, be two non-negative sequences. Then

$$P(\prod_{i=1}^{\infty} Y_i^{\alpha_i} > x, \prod_{j=1}^{\infty} Y_j^{\beta_j} > x) \sim P(Y_s > x^{\kappa_s}) P(Y_t > x^{\kappa_t})$$

where $s, t \in \mathbb{N}, \kappa_s, \kappa_t \geq 0$ are such that

$$\alpha_s \kappa_s + \alpha_t \kappa_t \geq 1, \;\; \beta_s \kappa_s + \beta_t \kappa_t \geq 1$$

and

$$\kappa_s + \kappa_t \to \min!$$

if a unique solution to this optimization problem exists.

"The *most efficient* tail combination wins".

$\Rightarrow$ In our AR(1) model, this gives us that the coefficient of tail dependence for vectors of lag h is equal to $\frac{1}{2-\phi^h}$.

A general result for weighted power products

Let $Y_1, Y_2, \ldots$ be i.i.d. regularly varying random variables with index -1. Let $\alpha_i, \beta_i, i \in \mathbb{N}$, be two non-negative sequences. Then

$$P\Big(\prod_{i=1}^{\infty} Y_i^{\alpha_i} > x, \prod_{j=1}^{\infty} Y_j^{\beta_j} > x\Big) \sim P(Y_s > x^{\kappa_s})P(Y_t > x^{\kappa_t})$$

where $s, t \in \mathbb{N}, \kappa_s, \kappa_t \geq 0$ are such that

$$\alpha_s\kappa_s + \alpha_t\kappa_t \geq 1, \;\; \beta_s\kappa_s + \beta_t\kappa_t \geq 1$$

and
$$\kappa_s + \kappa_t \to \min!$$

if a unique solution to this optimization problem exists.
"The *most efficient* tail combination wins".

$\Rightarrow$ In our AR(1) model, this gives us that the coefficient of tail dependence for vectors of lag h is equal to $\frac{1}{2-\phi^h}$.

A general result for weighted power products

Let $Y_1, Y_2, \ldots$ be i.i.d. regularly varying random variables with index -1. Let $\alpha_i, \beta_i, i \in \mathbb{N}$, be two non-negative sequences. Then

$$P(\prod_{i=1}^{\infty} Y_i^{\alpha_i} > x, \prod_{j=1}^{\infty} Y_j^{\beta_j} > x) \sim P(Y_s > x^{\kappa_s})P(Y_t > x^{\kappa_t})$$

where $s, t \in \mathbb{N}, \kappa_s, \kappa_t \geq 0$ are such that

$$\alpha_s\kappa_s + \alpha_t\kappa_t \geq 1, \;\; \beta_s\kappa_s + \beta_t\kappa_t \geq 1$$

and

$$\kappa_s + \kappa_t \to \min!$$

if a unique solution to this optimization problem exists.
"The *most efficient* tail combination wins".

$\Rightarrow$ In our AR(1) model, this gives us that the coefficient of tail dependence for vectors of lag h is equal to $\frac{1}{2-\phi^h}$.

Résumé and Outlook

- "Classic" SV models with heavy-tailed innovations are (just like GARCH(p, q) models) limited to a very specific range of extremal behavior.
- SV models with heavy-tailed volatility sequence share nice probabilistic properties of well-known models while allowing for a finer modelling of the extremal dependence structure.
- Also needed for applications: Efficient estimation techniques.

Résumé and Outlook

- "Classic" SV models with heavy-tailed innovations are (just like GARCH(p, q) models) limited to a very specific range of extremal behavior.
- SV models with heavy-tailed volatility sequence share nice probabilistic properties of well-known models while allowing for a finer modelling of the extremal dependence structure.
- Also needed for applications: Efficient estimation techniques.

Literature

Basrak, B., Davis, R. A. and Mikosch, T.
Regular variation of GARCH processes
Stoch. Proc. Appl. **99**, 95–115 (2002)

Ledford, A.W. and Tawn, J.A.
Statistics for near independence in multivariate extreme values
Biometrika **83**, 169–187 (1996)

Resnick, S.I.:
Hidden regular variation, second order regular variation and asymptotic independence.
Extremes, **5**, 303–336 (2002)

Rootzén, H.:
Extreme value theory for moving average processes.
Ann. Probab. **14**, 612–652 (1986)