Regular variation for measures

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Regular variation

For a probability distribution P on a space $\mathbf{S}$ such as $\mathbf{R}^d$, a Banach space, a sequence space, a function space such as $C[0,1]$ and $D[0,1]$, etc., regular variation can be defined as follows:

there exists a nonzero measure μ that assigns finite mass to sets bounded away from 0 and a set E bounded away from 0 such that

$$\lim_{t\to\infty} \frac{P(tA)}{P(tE)} = \mu(A)$$

for all Borel sets A bounded away from 0 such that $\mu(\partial A) = 0$.

It follows that $\mu(\lambda A) = \lambda^{-\alpha}\mu(A)$ for all Borel sets $A \subset \mathbf{R}^d \setminus \{0\}$ and $\lambda > 0$.

Some thoughts

Looks like weak convergence.

It appears that only multiplication by a scalar is the relevant structure of the space that is needed.

In order to develop useful results with analogs in the weak-convergence theory (mapping theorems, characterizations of relative compactness, Portmanteau theorems, etc.) it is desired to have some structure such as a separable and complete metric space.

Perhaps one is interested in sets bounded away from some other subset rather than the origin (if such a point exists). Possibility to consider hidden regular variation.

By having a flexible definition of multiplication by a scalar, possibility to allow for different tail decay in different "directions".

Suggested setting

Let $(\mathbb{S}, d)$ be a complete separable metric space. Require that multiplication by a scalar is a mapping $(\lambda, x) \mapsto \lambda x$ from $(0, \infty) \times \mathbf{S}$ into $\mathbf{S}$ such that

(A1) the mapping $(\lambda, x) \mapsto \lambda x$ is continuous,

(A2) $1x = x$ and $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x$.

Fix a closed cone $\mathbf{C} \subset \mathbf{S}$ and set $\mathbf{O} = \mathbf{S} \setminus \mathbf{C}$. Require that

(A3) $d(x, \mathbf{C}) < d(\lambda x, \mathbf{C})$ if $\lambda > 1$ and $x \in \mathbf{O}$.

Some weak convergence theory

Let $\mathbf{M_O}$ be the class of Borel measures on $\mathbf{O}$ whose restriction to $\mathbf{C}^r$ is finite for $r > 0$, where $\mathbf{C}^r = \{x \in \mathbf{S} : d(x, \mathbf{C}) < r\}$.

$\mu_n \to \mu$ in $\mathbf{M_O}$ is equivalent to $\int f \mathrm{d}\mu_n \to \int f \mathrm{d}\mu$ for all real-valued bounded and continuous functions f on $\mathbf{O}$ for which there exists $r > 0$ such that f vanishes on $\mathbf{C}^r$.

$\mu_n \to \mu$ in $\mathbf{M_O}$ is equivalent to $\mu_n(A) \to \mu(A)$ for all A bounded away from $\mathbf{C}$ with $\mu(\partial A) = 0$.

$\mu_n \to \mu$ in $\mathbf{M_O}$ is equivalent to the existence of a sequence $r_i \downarrow 0$ such that the restrictions $\mu_n^{(r_i)}, \mu^{(r_i)}$ of μ_n, μ to $\mathbf{O} \setminus \mathbf{C}^{r_i}$ satisfy $\mu_n^{(r_i)} \to \mu^{(r_i)}$ weakly on $\mathbf{O} \setminus \mathbf{C}^{r_i}$.

$\mathbf{M_O}$ is metrizable as a complete and separable metric space.

Regular variation

A sequence $\{\nu_n\}_{n\geq 1}$ in $\mathbf{M_O}$ is regularly varying if there exists a sequence $\{c_n\}_{n\geq 1}$ of positive numbers which is regularly varying and a nonzero $\mu \in \mathbf{M_O}$ such that $c_n\nu_n \to \mu$ in $\mathbf{M_O}$ as $n \to \infty$.

A measure $\nu \in \mathbf{M_O}$ is regularly varying if the sequence $\{\nu(n\cdot)\}_{n\geq 1}$ in $\mathbf{M_O}$ is regularly varying: $c_n\nu(n\cdot) \to \mu(\cdot)$ in $\mathbf{M_O}$ as $n \to \infty$.

Equivalently,

There exist nonzero $\mu \in \mathbf{M_O}$ and regularly varying function c such that $c(t)\nu(t\cdot) \to \mu(\cdot)$ in $\mathbf{M_O}$ as $t \to \infty$.

There exist nonzero $\mu \in \mathbf{M_O}$ and set $E \subset \mathbf{O}$ bounded away from $\mathbf{C}$ such that $\nu(tE)^{-1}\nu(t\cdot) \to \mu(\cdot)$ in $\mathbf{M_O}$ as $t \to \infty$.

Scaling property

If $\nu \in \mathbf{M}_{\mathbf{O}}$ is regularly varying, according to any of the equivalent statements, then the limit measure μ satisfies

$$\mu(\lambda A) = \lambda^{-\alpha}\mu(A)$$

for some $\alpha \geq 0$ and all Borel sets $A \in \mathbf{O}$.

The regular variation index α is determined if we specify what is meant by multiplication by a scalar (not enough to specify ν).

Example 1

Let $\mathbf{S} = \mathbf{R}^2$ and let $\mathbf{C} = \{0\}$. Let X_1 be Pa(γ_1) and X_2 be Pa(γ_2) and independent. Define $(\lambda, (x_1, x_2)) \mapsto (\lambda^{1/\gamma_1} x_1, \lambda^{1/\gamma_2} x_2)$.

For $a, b > 0$

$$\begin{aligned} t^2 \mathbf{P}((X_1, X_2) \in t[(a, \infty) \times (b, \infty)]) &= t\mathbf{P}(X_1 > t^{1/\gamma_1} a) t\mathbf{P}(X_2 > t^{1/\gamma_2} b) \\ &= a^{-\gamma_1} b^{-\gamma_2}. \end{aligned}$$

The limit measure therefore has the scaling property. Here,

$$\begin{aligned} \mu(\lambda[(a, \infty) \times (b, \infty)]) &= \mu((\lambda^{1/\gamma_1} a, \infty) \times (\lambda^{1/\gamma_1} b, \infty)) \\ &= \lambda^{-2} a^{-\gamma_1} b^{-\gamma_2} \\ &= \lambda^{-2} \mu((a, \infty) \times (b, \infty)). \end{aligned}$$

Example 2

Let $\mathbf{S} = \mathbf{R}^2$ and let $\mathbf{C} = \mathbf{R} \times \{0\}$. Let X_1 be N$(0, 1)$ and X_2 be Pa(γ) and independent. Define $(\lambda, (x_1, x_2)) \mapsto (x_1, \lambda x_2)$.

For $a, b > 0$

$$\begin{aligned} t^\gamma \mathbf{P}((X_1, X_2) \in t[(a, \infty) \times (b, \infty)]) &= t^\gamma \mathbf{P}(X_1 > a)\mathbf{P}(X_2 > tb) \\ &= (1 - \Phi(a))b^{-\gamma}. \end{aligned}$$

The limit measure therefore has the scaling property. Here,

$$\begin{aligned} \mu(\lambda[(a, \infty) \times (b, \infty)]) &= \mu((a, \infty) \times (\lambda b, \infty)) \\ &= \lambda^{-\gamma}(1 - \Phi(a))b^{-\gamma} \\ &= \lambda^{-\gamma}\mu((a, \infty) \times (b, \infty)). \end{aligned}$$