On Adam Jakubowski's approach to proving asymptotic results for regularly varying sequences [1]

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Regularly varying stationary sequence

- **An $\mathbb{R}^d$-valued stationary sequence (X_t) is regularly varying with index $\alpha > 0$ if its finite-dimensional distributions are regularly varying with index α:**
- **For every $k \geq 1$, there exists a non-null Radon measure μ_k on $\overline{\mathbb{R}}_0^d$ which does not charge infinite points such that**

$$\frac{P(x^{-1}(X_1,\ldots,X_k) \in \cdot)}{P(|X_0| > x)} \xrightarrow{v} \mu_k(\cdot)\,.$$

The measures μ_k have the property $\mu_k(t\cdot) = t^{-\alpha}\mu_k(\cdot)$, $t > 0$, for some $\alpha > 0$.

- **If f is a continuous mapping such that $f^{-1}(\{0\}) = \{0\}$, then**

$$\frac{P(f(x^{-1}(X_1,\ldots,X_k)) \in A)}{P(|X_0| > x)} \to \mu_k(f^{-1}(A))\,.$$

- **For example,**

$$\frac{P(x^{-1}S_k \in A)}{P(|X_0| > x)} \to \mu_k(\{\mathbf{x} \in \mathbb{R}^{dk} : x_1 + \cdots + x_k \in A\}) .$$

Examples of regularly varying stationary sequences

Linear processes.

- **A linear process**

$$X_t = \sum_j \psi_j Z_{t-j}, \quad t \in \mathbb{Z},$$

  is regularly varying with index $\alpha > 0$ if the iid sequence (Z_t) is regularly varying with index α, under conditions on (ψ_j) which are close to those in the 3-series theorem M. and Samorodnitsky (2000)

- **Regular variation of X_0 is in general not sufficient for regular variation of Z_0.** Jacobsen, M., Samorodnitsky, Rosiński (2009, 2012)

Solutions to stochastic recurrence equation.

- For an iid sequence $((A_t, B_t))_{t\in\mathbb{Z}}$, $A_t > 0$, the stochastic recurrence equation

$$X_t = A_t X_{t-1} + B_t\,, \quad t \in \mathbb{Z}\,,$$

has a unique stationary solution

$$X_t = B_t + \sum_{i=-\infty}^{t-1} A_t \cdots A_{i+1} B_i\,, \quad t \in \mathbb{Z},$$

provided $E \log A_0 < 0$, $E \log^+ |B_0| < \infty$.

- The sequence (X_t) is regularly varying with index α which is the unique solution to $EA_0^\kappa = 1$, $\kappa > 0$, (given this solution exists) Kesten (1973), Goldie (1991) and for some $c_\pm \geq 0$, $c_+ + c_- > 0$,

$$P(X_0 > x) \sim c_+\, x^{-\alpha}\,, \quad P(X_0 \leq -x) \sim c_-\, x^{-\alpha}\,, \quad x \to \infty\,.$$

- **The GARCH(1, 1) process** Bollerslev (1986) satisfies a stochastic recurrence equation: for an iid standard normal sequence (Z_t), positive parameters $\alpha_0, \alpha_1, \beta_1$,

$$\sigma_t^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta_1)\sigma_{t-1}^2 \,.$$

The process $X_t = \sigma_t Z_t$ is regularly varying with index α satisfying $E(\alpha_1 Z_0^2 + \beta_1)^{\alpha/2} = 1$.

- **α-Stable stationary processes**, $\alpha \in (0, 2)$.

- **Max-stable stationary processes with α-Fréchet marginals**, $\alpha > 0$.

Looking backwards

- **The approaches by**
  - **Davis and Hsing (1995)**
  - **Jakubowski (1993, 1997)**
- **Asymptotic theory for general regularly varying sequences**
  - **Central limit theory with α-stable limits.**
  - **Point process convergence.**
  - **Convergence of maxima to α-Fréchet law.**
  - **Large deviations.**

The Davis and Hsing (1995) approach for point processes

- Assume $P(|X_0| > a_n) \sim n^{-1}$. Consider the point processes

$$N_{nj} = \sum_{t=1}^{j} \varepsilon_{a_n^{-1} X_t}\,, \quad j = 1,\ldots,n\,, \quad N_{nn} = N_n\,.$$

with state space $\overline{\mathbb{R}}_0^d$.

- *Mixing condition* $\mathcal{A}(a_n)$: There exists a sequence $m = m_n \to \infty$ such that $k_n = [n/m] \to \infty$ and

$$E\mathrm{e}^{-\int f dN_n} - \left(E\mathrm{e}^{-\int f dN_{nm}}\right)^{k_n} \to 0\,, \quad f \in \mathbb{C}_K^+\,,$$

- *Anti-clustering condition* (AC): For (m_n) from $\mathcal{A}(a_n)$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\Big(\max_{k\le|t|\le m_n} |X_t| > \delta a_n \;\Big|\; |X_0| > \delta a_n\Big) = 0\,, \quad \delta > 0\,.$$

- **If (X_t) is regularly varying, (AC) and $\mathcal{A}(a_n)$ hold then**

$$N_n \xrightarrow{d} N = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{\Gamma_i^{-1/\alpha} Q_{ij}} \,,$$

(Γ_i) is an increasing enumeration of the points of a homogeneous Poisson process on $(0, \infty)$ with intensity γ, $(Q_{ij})_{j\ge 1}$, $i = 1, 2, \ldots$, are iid sequences of points Q_{ij} such that $\sup_{j\ge 1} |Q_{ij}| = 1$ a.s. and γ is the extremal index of the sequence $(|X_t|)$.

- **α-stable limit theory and large deviations are consequences.**
- **Alternative proof by Basrak and Segers (2009), using the tail chain approach. Also show $\gamma > 0$.**

The Jakubowski (1993,1995) approach to α-stable limits

- Based on Bartkiewicz, Jakubowski, M., Wintenberger (2011).
- Partial sum process for $d = 1$, $EX_0 = 0$ for $\alpha \in (1,2)$,

$$S_0 = 0 \quad \text{and} \quad S_n = X_1 + \cdots + X_n \,, \quad n \geq 1 \,.$$

- *Mixing condition* (MX): There exists a sequence

$m = m_n \to \infty$ such that $k_n = [n/m] \to \infty$ and

$$E\mathrm{e}^{ita_n^{-1}S_n} - \Big(E\mathrm{e}^{ita_n^{-1}S_m}\Big)^{k_n} \to 0 \,, \quad t \in \mathbb{R} \,.$$

- *Anti-clustering condition* (AC'): For (m_n) as in (MX)

$$\lim_{l\to\infty} \limsup_{n\to\infty} k_n \sum_{l \leq j \leq m} E\big|h(xa_n^{-1}(S_j - S_l))h(xa_n^{-1}X_1)\big| = 0 \,, \quad x \in \mathbb{R} \,,$$

where $h(y) = (y \wedge 2) \vee (-2)$.

- The characteristic function of an α-stable random variable ξ_α for $\alpha \in (0,1)\cup(1,2)$, $b_+, b_- \ge 0$, $b_+ + b_- > 0$,

$$\psi_\alpha(t) = \exp(-|t|^\alpha \chi_\alpha(t, b_+, b_-))\,, \quad t \in \mathbb{R}\,,$$

where $\chi_\alpha(t, b_+, b_-)$ is given by

$$\frac{\Gamma(2-\alpha)}{1-\alpha}\Big((b_+ + b_-)\,\cos(\pi\alpha/2) - i\,\mathrm{sign}(t)(b_+ - b_-)\,\sin(\pi\,\alpha/2)\Big)\,, \quad t \in \mathbb{R}\,,$$

- Step 1. Under (MX), $a_n^{-1} S_n \stackrel{d}{\to} \xi_\alpha$ if and only if $(a_n^{-1}\sum_{i=1}^{k_n} S_{mi}) \stackrel{d}{\to} \xi_\alpha$ for iid copies S_{mi} of S_m.

- Equivalently,

$$k_n \log E\mathrm{e}^{ita_n^{-1}S_m} \sim k_n\big(E\mathrm{e}^{ita_n^{-1}S_m} - 1\big) \to \log E\mathrm{e}^{it\xi_\alpha}\,, \quad t \in \mathbb{R}\,.$$

- Step 2. Under (AC'),

$$\lim_{k\to\infty}\limsup_{n\to\infty}\Big|k_n\big(E\mathrm{e}^{ita_n^{-1}S_m} - 1\big) - n\,\big((E\mathrm{e}^{ita_n^{-1}S_k} - 1) - (E\mathrm{e}^{ita_n^{-1}S_{k-1}} - 1)\big)\Big| = 0\,.$$

Since S_k is regularly varying for every fixed $k \geq 1$, $t \in \mathbb{R}$,

$$n\left((E e^{ita_n^{-1}S_k} - 1) - (E e^{ita_n^{-1}S_{k-1}} - 1)\right) \to \log \widetilde{\psi}_{\alpha,k}(t) - \log \widetilde{\psi}_{\alpha,k-1}(t)\,,$$

where

$$\widetilde{\psi}_{\alpha,k}(t)/\widetilde{\psi}_{\alpha,k-1}(t)$$

$$= \exp\left(-|t|^\alpha \chi_\alpha(t, b_+(k) - b_+(k-1), b_-(k) - b_-(k-1))\right),$$

and

$$\lim_{n\to\infty} n\,P(S_k > a_n) = b_+(k) \quad \text{and} \quad \lim_{n\to\infty} n\,P(S_k \leq -a_n) = b_-(k)\,.$$

- **Step 3.** In view of Steps 1 and 2, $a_n^{-1} S_n \xrightarrow{d} \xi_\alpha$ is proved if we can show that the limiting cluster index

$$\lim_{k\to\infty} (b_\pm(k) - b_\pm(k-1)) = b_\pm$$

exists and is finite.

Large deviations for a regularly varying sequence, $\alpha > 0$

- If (X_t) is regularly varying, the following limit exists:

$$\lim_{x\to\infty} \frac{P(S_{k+1} > x) - P(S_k > x)}{P(|X_0| > x)} = b_+(k+1) - b_+(k)\,.$$

- Assume $\lim_{k\to\infty}(b_+(k+1) - b_+(k)) = b_+$ exists.

- Using Jakubowski's idea for fixed $k \geq 2$,

$$\Big|\frac{P(S_n > x)}{n\,P(|X_0| > x)} - b_+\Big|$$

$$\leq \Big|\frac{P(S_n > x) - n\,(P(S_{k+1} > x) - P(S_k > x))}{n\,P(|X_0| > x)}\Big|$$

$$+\Big|\frac{P(S_{k+1} > x) - P(S_k > x)}{P(|X_0| > x)} - b_+\Big|\,.$$

- Show that the green part is negligible.

- **Truncate the $|X_t|$'s from below at the level $\delta\, x$:** $\underline{S}_k$
- **Use lemma of Jakubowski (1993):**

$$\frac{|P(\underline{S}_n > x) - n\,(P(\underline{S}_{k+1} > x) - P(\underline{S}_k > x))|}{n\,P(|X_0| > x)}$$

$$\leq 3\,\frac{k\,P(|X_0| > \delta\,x)}{n\,P(|X_0| > x)} + 2\sum_{j=k}^{n}\frac{P(|X_j| > \delta\,x, |X_0| > \delta\,x)}{P(|X_0| > x)}$$

- **Theorem. 1. AC_α.** There exist $\delta_k \downarrow 0$ as $k \to \infty$ and a sequence of sets $\Lambda_n \subset (0,\infty)$, $n = 1, 2, \ldots$, with $b_n = \inf \Lambda_n$ such that $n\, P(|X| > b_n) \to 0$ as $n \to \infty$ and

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{x\in\Lambda_n} \delta_k^{-\alpha} \sum_{j=k}^{n} P(|X_j| > x\delta_k \mid |X_0| > x\delta_k) = 0\,.$$

2. For the sequences (Λ_n), (δ_k) as above, and a sequence (ε_k) satisfying $\varepsilon_k = o(k^{-1})$ and $(k+1)\delta_k \le \varepsilon_k$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{x\in\Lambda_n} \frac{P\Big(\sum_{i=1}^n X_i I_{\{|X_i|\le\delta_k\, x\}} > \varepsilon_k x\Big)}{n\, P(|X_0| > x)} = 0.$$

Then the large deviation principle holds:

$$\lim_{n\to\infty} \sup_{x\in\Lambda_n} \Big|\frac{P(S_n > x)}{n\, P(|X_0| > x)} - b_+\Big| = 0\,,$$

Regularly varying Markov chains

- Let $X_t = f(\Phi_t)$, $t \in \mathbb{Z}$, be regularly varying, (Φ_t) an (irr., aper.) stationary Markov chain satisfying $\mathbf{AC}_\alpha$ and $\mathbf{DC}_p$ for every $p < \alpha$:

  There exist constants $\beta \in (0,1)$, $b > 0$ and a small set A such that for any y,

$$E(|X_1|^p \mid \Phi_0 = y) \le \beta\,|f(y)|^p + b\,I_A(y).$$

- This condition implies geometric β-mixing of (X_t).

- Then
  - $b_+ = \lim_{k\to\infty}(b_+(k+1) - b_+(k))$ exists
  - 2. of Theorem is satisfied Due to an exponential inequality of Bertail and Clémencon (2009)
  - and the large deviation principle holds

$$\lim_{n\to\infty} \sup_{x\in\Lambda_n} \left| \frac{P(S_n > x)}{n\,P(|X| > x)} - b_+ \right| = 0\,,$$

for $\Lambda_n = (b_n, \mathrm{e}^{s_n})$, where $s_n = o(n)$ and $b_n/n^{0.5+\delta} \to \infty$ for $\alpha \geq 2$, $b_n/n^{1/\alpha+\delta} \to \infty$ for $\alpha \in (0, 2)$, any $\delta > 0$.

- Due to regeneration

$$S_n = \sum_{i=1}^{\tau_A} X_i + \sum_{k=1}^{N_A(n)} S(k) + \cdots ,$$

where τ_A is the first time the Markov chain hits the set A and $(S(k))$ are block sums over independent cycles.

- The iid random variables $S(k)$ are regularly varying with index α and by Nagaev's results

$$\lim_{n\to\infty} \sup_{x\in\Lambda_n} \left| \frac{P(S_n > x)}{n\, P_A(S_A > x)} - \frac{1}{E\tau_A} \right| = 0 \,.$$

- Also

$$\frac{P(S_n > x)}{n\, P(|X| > x)} \sim b_+ + \frac{P(S_n > x\,, \tau_A > n)}{n\, P(|X| > x)}$$

and $P(\tau_A > n) \le \mathrm{e}^{-\kappa n} E \mathrm{e}^{\kappa \tau_A}$.

Moving towards the multivariate case

- Assume that $(\mathbf{X}_t)$ is an $\mathbb{R}^d$-valued function of a (aper., irr.) Markov chain.
- Then DC$_p$, $p < \alpha$, holds for every sequence $(\theta'\mathbf{X}_t)$, $\theta \in \mathbb{S}^{d-1}$, in modified form

$$E(|\theta'\mathbf{X}_1|^p \mid \Phi_0 = y) \le E(|\mathbf{X}_1|^p \mid \Phi_0 = y) \le \beta\,|f(y)|^p + b\,I_A(y).$$

- By regular variation of $(\mathbf{X}_t)$ the limits

$$b_k(\theta) = \lim_{x\to\infty} \frac{P(\theta' S_k > x)}{P(|\mathbf{X}_0| > x)}, \quad \theta \in \mathbb{S}^{d-1},$$

exist and DC$_p$, $p < \alpha$, implies that

$$b(\theta) = \lim_{k\to\infty} (b_{k+1}(\theta) - b_k(\theta)), \quad \theta \in \mathbb{S}^{d-1}.$$

- We call b the cluster index.

- The following large deviation principle holds

$$\lim_{n\to\infty} \sup_{x\in\Lambda_n} \left| \frac{P(\theta' S_n > cx)}{n\, P(|X_0| > x)} - c^{-\alpha} b(\theta) \right| = 0\,, \quad c > 0\,, \theta \in \mathbb{S}^{d-1}\,,$$

for $\Lambda_n = (b_n, e^{s_n})$, where $s_n = o(n)$.

- Consider the measures on the Borel σ-field of $\overline{\mathbb{R}}_0^d$

$$\mu_n(\cdot) = \frac{P(b_n^{-1} S_n \in \cdot)}{n P(|X_0| > b_n)}\,, \quad n \geq 1\,.$$

- The latter limit relation implies that for sets A bounded away from zero

$$\sup_{n\geq 1} \mu_n(A) < \infty\,,$$

hence the measures (μ_n) are vaguely tight and

$$\mu_n(\{x : \theta' x > c\}) \to c^{-\alpha} b(\theta)\,, \quad c > 0\,, \theta \in \mathbb{S}^{d-1}\,.$$

- Subsequential limits of (μ_n) coincide on the sets $\{x : \theta' x > c\}$.

- If α is not an integer, or α is an odd integer and X is symmetric, or $b(\theta) = 0$ for $\theta \in U$, $U \subset \mathbb{S}^{d-1}$ open, one can show

  Basrak, Davis, M. (2002), Boman, Lindskog (2008), Klüppelberg, Pergamenchikov (2007)

  that

$$\mu_n(\cdot) = \frac{P(b_n^{-1} S_n \in \cdot)}{nP(|\mathrm{X}| > b_n)} \xrightarrow{v} \mu(\cdot)$$

for a limit measure μ which is uniquely determined by the relations

$$\mu(\{\mathrm{x} : \theta' \mathrm{x} > c\}) = c^{-\alpha} b(\theta)\,, \quad c > 0\,, \quad \theta \in \mathbb{S}^{d-1}\,,$$

and which has the property $\mu(tA) = t^{-\alpha}\mu(A)$, $t > 0$.

- **For an iid $\mathbb{R}^d$-valued regularly varying sequence (X_n) with index α and limit measure ν, i.e.** $\dfrac{P(x^{-1}\mathrm{X}_0 \in \cdot)}{P(|\mathrm{X}_0| > x)} \xrightarrow{v} \nu(\cdot)$ **, one has** $\mu = \nu$. **Hult, Lindskog, M. Samorodnitsky (2005)**

An interpretation of the cluster index

- **Basrak, Segers (2009) prove that an $\mathbb{R}^d$-valued regularly varying stationary sequence satisfies the relation**

$$P(x^{-1}(X_{-h},\ldots,X_h) \in \cdot \mid |X_0| > x) \xrightarrow{w} P((Y_{-h},\ldots,Y_h) \in \cdot)\,, \quad h \geq 0\,,$$

and the limiting vector has representation

$$(Y_{-h},\ldots,Y_h) = |Y_0|\,(\Theta_{-h},\ldots,\Theta_h)\,,$$

where $|Y_0|$ is independent of $(\Theta_{-h},\ldots,\Theta_h)$ and $P(|Y_0| > x) = x^{-\alpha}$, $x > 1$.

- Then for $\theta \in \mathbb{S}^{d-1}$, $k \geq 1$,

$$
\begin{aligned}
b_{k+1}(\theta) - b_k(\theta) &= \lim_{x \to \infty} \frac{P(\theta' S_{k+1} > x) - P(\theta' S_k > x)}{P(|X_0| > x)} \\
&= E\Big[\Big(\theta' \sum_{t=0}^{k} \Theta_t\Big)_+^{\alpha} - \Big(\theta' \sum_{t=1}^{k} \Theta_t\Big)_+^{\alpha}\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
b(\theta) &= \lim_{k \to \infty} (b_{k+1}(\theta) - b_k(\theta)) \\
&= E\Big[\Big(\theta' \sum_{t=0}^{\infty} \Theta_t\Big)_+^{\alpha} - \Big(\theta' \sum_{t=1}^{\infty} \Theta_t\Big)_+^{\alpha}\Big]
\end{aligned}
$$

Concluding remarks

- Balan and Louhichi (2009) use the Jakubowski approach to point process convergence.
- M., Wintenberger (2011) use the Jakubowski approach to prove large deviations for general regularly varying sequences for $d = 1$.
- M., Wintenberger (2011) use the Jakubowski approach to prove large deviations and stable limit theory for regularly varying functions of a Markov chain for $d \geq 1$.
- **Hypothesis:** The Jakubowski approach can be used for functionals acting on a regularly varying sequence (X_t), e.g. ruin, supremum of random walk, length of longest strange segment, maxima,... And

- **Different functionals lead to different cluster indices.**