Heavy tailed branching process with immigration

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Model

$$\theta_t \circ x := \theta_t(x) = \sum_{i=1}^{x} A_i^{(t)} , \qquad x \in \mathbb{N}_0$$

θ_t maps an integer x into a random integer with an interpretation that each of x individuals in the $(t-1)$th generation leaves behind a random number of children, and all these numbers are independent and have the same distribution as some generic random variable, say A.
To introduce immigration in the model, we assume that another i.i.d. sequence

$$(B, B_t\,, t \in \mathbb{Z})$$

$$X_t = \theta_t \circ X_{t-1} + B_t \qquad \text{for each } t \geq 1$$

or in an alternative notation

$$X_t = \sum_{i=1}^{X_{t-1}} A_i^{(t)} + B_t \qquad \text{for each } t \geq 1$$

Motivation

Queueing theory - Altman (2002, 2004, 2005), Fiems and Altman (2002)

- polling systems - Resing (1993)
- infinite server queues
- processor sharing queues
- packet forwarding in delay-tolerant mobile ad-hoc networks

Time series theory (INteger AutoRegressive (INAR) processes) - Alosh and Al-zaid (1987), Dion, Gauthier and Latour (1995)

Markov chain theory - Segers and Janßen (2012)

Stationary distribution

Denote

$$f(z) = E(z^A)\ , \qquad g(z) = E(z^B)$$

Following Foster and Williamson (1971), the Markov chain (X_t) is ergodic with unique stationary distribution if and only if

$$\int_0^1 \frac{1-g(s)}{f(s)-s} ds < \infty$$

In terms of moments, sufficient conditions are given in Seneta (1970). If

$$0 < \mu := E(A) < 1 \qquad \text{and} \qquad E(\ln(1+B)) < \infty$$

then the chain is ergodic with unique stationary distribution:

$$X_t \overset{d}{=} B_t + \sum_{k=1}^{\infty} \theta_t^{(t-k)} \circ \cdots \circ \theta_{t-k+1}^{(t-k)}(B_{t-k}) =: B_t + \sum_{k=1}^{\infty} \bigotimes_{i=0}^{k-1} \theta_{t-i}^{(t-k)}(B_{t-k}) =: \sum_{k=0}^{\infty} C_{t,k}$$

where $(C_{t,k})_{k \in \mathbb{N}_0}$ is a sequence of independent random variables.

Another Markov chain

We can also consider a Markov chain $(X'_t)_{t\in\mathbb{N}_0}$ which evolves as

$$X'_t = \max\{\theta_t \circ X'_{t-1}, B_t\} \qquad \text{for each } t \geq 1$$

The unique stationary distribution of (X'_t) exists since $X'_t \leq X_t$ and it is given by:

$$X'_t \stackrel{d}{=} \sup\{B_t, \theta_t^{(t-k)} \circ \cdots \circ \theta_{t-k+1}^{(t-k)}(B_{t-k}) : k = 1, 2, \ldots\} = \sup\{C_{t,k} : k \geq 0\}$$

Goal:

- Identify the *tail behaviour of the distribution of the stationary solution* X_t under assumption that the generic size of immigration B or generic size of offsprings A has regularly varying distribution
- CLT for the heavy tailed partial sums
- prove that partial maxima have Fréchet limiting distribution

Regularly varying immigration

$$0 < \mu = E(A) < 1$$

$$P(B > x) = x^{-\alpha} L(x)$$

for some $\alpha \in (0, 2)$ and a slowly varying function L.

We consider here the case $\alpha \in (0, 2)$ only. For $\alpha > 2$ proofs become much more involved, however, a technique is clearly suggested by the case $\alpha \in [1, 2)$ For $\alpha \in [1, 2)$ we also assume that

$$E(A^2) < \infty$$

In particular, it means that:

$$P(A > x) = o(P(B > x))$$

Then:

$$P\left(\sum_{i=1}^{B} A_i > x\right) \sim P(B > x/\mu)$$

Main result

Theorem 1

$$\lim_{x\to\infty} \frac{P(X_t > x)}{P(B > x)} = \lim_{x\to\infty} \frac{P(X'_t > x)}{P(B > x)} = \sum_{k=0}^{\infty} \mu^{k\alpha}$$

Notations:

$$\tilde{A}^{(k)} = \theta_k \circ \cdots \circ \theta_1 \circ 1$$

$$\tilde{A}_i^{(k)} \stackrel{d}{=} \sum_{j=1}^{A} \tilde{A}_j^{(k-1)}$$

$$C_{t,k} \stackrel{d}{=} \sum_{j=1}^{B_{t-k}} \tilde{A}_j^{(k)}$$

Recall that:

$$X_t \stackrel{d}{=} \sum_{k=0}^{\infty} C_{t,k}$$

Idea of the proof

Step 1: Large deviation results give:

$$P(C_{t,k} > x) = P\left(\bigotimes_{i=0}^{k-1} \theta_{t-i}^{(t-k)}(B_{t-k}) > x\right) \sim P(B > x/\mu^k)$$

Let

$$X_{t,m} = \sum_{k=0}^{m} C_{t,k}$$

and

$$X'_{t,m} = \max\{C_{t,k} : k = 0, \ldots, m\}$$

Then

$$\lim_{x\to\infty} \frac{P(X_{t,m} > x)}{P(B > x)} = \lim_{x\to\infty} \frac{P(X'_{t,m} > x)}{P(B > x)} = \sum_{k=0}^{m} \mu^{k\alpha}$$

Idea of the proof

Step 2:

$$\liminf_{x\to\infty} \frac{P(\sum_{k=0}^{\infty} C_{t,k} > x)}{P(B > x)} \geq \lim_{x\to\infty} \frac{P(\sum_{k=0}^{m} C_{t,k} > x)}{P(B > x)} \geq \sum_{k=0}^{\infty} \mu^{k\alpha}$$

Step 3: To establish an upper bound for the tail of X_t it is enough to show that

$$\lim_{k_0\to\infty} \limsup_{x\to\infty} \frac{P(\sum_{k=k_0}^{\infty} C_{t,k} > x)}{P(B > x)} = 0$$

Similarly, to obtain an upper bound for the tail of X_t', it is enough to show that

$$\lim_{k_0\to\infty} \limsup_{x\to\infty} \frac{P(\sup_{k\geq k_0} C_{t,k} > x)}{P(B > x)} = 0$$

Observe that:

$$\frac{P(\sum_{k=k_0}^{\infty} C_{t,k} > x)}{P(B > x)}$$

$$\leq \frac{P(\sup_{k\geq k_0} B_{t-k} > x(1-\varepsilon)/\mu^k)}{P(B > x)} + \frac{P(\sum_{k=k_0}^{\infty} C_{t,k}\mathbf{1}_{\{B_{t-k} < x(1-\varepsilon)/\mu^k\}} > x)}{P(B > x)}$$

Idea of the proof

The first term on the right hand side is bounded above by:

$$\sum_{k=k_0}^{\infty} \frac{P(B > x(1-\varepsilon)/\mu^k)}{P(B > x)}$$

One can use the Potter's bounds to see that its limit is zero if we let first x and then k_0 converge to ∞.

Upper bound of the second term

Case $0 < \alpha < 1$: We apply the Markov inequality, Karamata's theorem in combination with the Potter's bounds.

Case $1 \leq \alpha < 2$: Note that it sufficient to prove that:

$$\lim_{k_0\to\infty} \limsup_{x\to\infty} \frac{P(\sum_{k=k_0}^{\infty} C_{t,k} > \sqrt{x})}{P(B > \sqrt{x})} = \lim_{k_0\to\infty} \limsup_{x\to\infty} \frac{P((\sum_{k=k_0}^{\infty} C_{t,k})^2 > x)}{P(B^2 > x)} = 0$$

Idea of the proof

Repeating a similar argument as for $\alpha \in (0,1)$, we obtain

$$\frac{P\left(\left(\sum_{k=k_0}^{\infty} C_{t,k}\right)^2 > x\right)}{P(B^2 > x)}$$

$$\leq \frac{P(\sup_{k\geq k_0} B_{t-k}^2 > x(1-\varepsilon)/\mu^{2k})}{P(B^2 > x)} + \frac{P\left(\left(\sum_{k=k_0}^{\infty} C_{t,k}\mathbf{1}_{\{B_{t-k}^2 < x(1-\varepsilon)/\mu^{2k}\}}\right)^2 > x\right)}{P(B^2 > x)}$$

The first term could be treated using the Potter's bound since B^2 is regularly varying. Using Markov inequality, the second one can be bounded above by:

$$\frac{E\left(\sum_{k=k_0}^{\infty} C_{t,k}\mathbf{1}_{\{B_{t-k}^2 < x(1-\varepsilon)/\mu^{2k}\}}\right)^2}{xP(B^2 > x)} \leq \frac{E\left(\sum_{k=k_0}^{\infty} C_{t,k}^2\mathbf{1}_{\{B_{t-k}^2 < x(1-\varepsilon)/\mu^{2k}\}}\right)}{xP(B^2 > x)} +$$

$$+ \frac{E\left(\sum_{\substack{k,l=k_0\\ k\neq l}}^{\infty} C_{t,k}C_{t,l}\mathbf{1}_{\{B_{t-k}^2 < x(1-\varepsilon)/\mu^{2k}\}}\mathbf{1}_{\{B_{t-l}^2 < x(1-\varepsilon)/\mu^{2k}\}}\right)}{xP(B^2 > x)} =: J_1(k_0) + J_2(k_0)$$

Idea of the proof

$$
\begin{aligned}
J_1(k_0) &= \frac{E\left(\sum_{k=k_0}^{\infty} C_{t,k}^2 \mathbf{1}_{\{B_{t-k}^2 < x(1-\varepsilon)/\mu^{2k}\}}\right)}{xP(B^2 > x)} \\
&\leq \sum_{k=k_0} \frac{\sum_{n \leq (x(1-\varepsilon))^{1/2}/\mu^k} E\left(\sum_{j=1}^{n} \tilde{A}_j^{(k)}\right)^2 P(B_{t-k} = n)}{xP(B^2 > x)} \\
&\leq \sum_{k=k_0}^{\infty} C(k+1)\mu^k \frac{E\left(B\mathbf{1}_{\{B^2 < x(1-\varepsilon)/\mu^{2k}\}}\right)}{xP(B^2 > x)} + \sum_{k=k_0}^{\infty} \mu^{2k} \frac{E\left(B^2\mathbf{1}_{\{B^2 < x(1-\varepsilon)/\mu^{2k}\}}\right)}{xP(B^2 > x)} \\
&= J_{11}(k_0) + J_{12}(k_0)
\end{aligned}
$$

Since B^2 is regularly varying with index $\alpha/2 \in (0,1)$, Karamata's theorem applies again and we finally have

$$
\lim_{k_0 \to \infty} \limsup_{x \to \infty} J_{12}(k_0) = 0
$$

Idea of the proof

If $E(B) < \infty$, then $J_{11}(k_0)$ is bounded by

$$J_{11}(k_0) \leq C \sum_{k=k_0}^{\infty} (k+1)\mu^k \frac{E(B)}{xP(B^2 > x)}$$

and hence goes to 0 as $x \to \infty$. Slightly more complex is the case $E(B) = \infty$ (when $\alpha = 1$) and we skip it here.
Likewise,

$$\begin{aligned} J_2(k_0) &= \frac{\sum_{\substack{k,l=k_0 \\ k \neq l}}^{\infty} E\left(C_{t,k}\mathbf{1}_{\{B_{t-k}^2 < x(1-\varepsilon)/\mu^{2k}\}}\right) E\left(C_{t,l}\mathbf{1}_{\{B_{t-l}^2 < x(1-\varepsilon)/\mu^{2k}\}}\right)}{xP(B^2 > x)} \\ &\leq \sum_{\substack{k,l=k_0 \\ k \neq l}}^{\infty} E\left((k+1)\mu^k B_{t-k}\mathbf{1}_{\{B_{t-k}^2 < x(1-\varepsilon)/\mu^{2k}\}}\right) \\ &\qquad E\left((l+1)\mu^l B_{t-l}\mathbf{1}_{\{B_{t-l}^2 < x(1-\varepsilon)/\mu^{2k}\}}\right) \frac{1}{xP(B^2 > x)} \end{aligned}$$

and

$$\lim_{k_0 \to \infty} \limsup_{x \to \infty} J_2(k_0) = 0$$

Regularly varying offspring

$$0 < \mu := E(A) < 1$$

$$P(A > x) = x^{-\alpha} L(x)$$

for some $\alpha \in (1, 2)$ and a slowly varying function L.
We consider here the case $\alpha \in (1, 2)$ only. For $\alpha > 2$ the proof of the main result of this subsection could be adopted.

$$\lim_{x \to \infty} \frac{P(B > x)}{P(A > x)} = c$$

where c is finite (possible equal 0) constant. If $c > 0$ we need also to assume that B is consistently varying.

Theorem 2

$$\lim_{x \to \infty} \frac{P(X_t > x)}{P(A > x)} = \lim_{x \to \infty} \frac{P(X_t' > x)}{P(A > x)} = \sum_{k=0}^{\infty} \psi_k$$

Asymptotics for maxima

Denote in the sequel by (a_n) a sequence of constants such that for any $u > 0$ as $n \to \infty$:

$$nP(X_0 > a_n u) \to u^{-\alpha}$$

Theorem 3 Under assumptions of Model I or Model II with $\alpha \neq 1$, as $n \to \infty$ it holds that

$$P\left(\frac{M_n}{a_n} \leq x\right) \to \exp\left(-(1 - \mu^{\alpha})x^{-\alpha}\right)$$

for every $x \geq 0$ where $M_n = \max(X_1, \ldots, X_n)$.

Idea of the proof

We describe the asymptotic behavior of the following point processes

$$N_n = \sum_{i=1}^{n} \delta_{(i/n,\, X_i/a_n)} \qquad \text{for all } n \in \mathbb{N}.$$

It turns out by Theorem 2.3 in Basrak, Krizmanić and Segers (2012) that there exist a point processes $N^{(u)}$, $u > 0$ on the space $[0, 1] \times (u, \infty)$ with compound Poisson structure such that as $n \to \infty$

$$N_n\Big|_{[0,1]\times(u,\infty)} \xrightarrow{d} N^{(u)}$$

Behavior of large values

Theorem 4 Under assumptions of Model I or Model II with $\alpha \neq 1$ there is a compound Poisson process N° on $[0,1]$ such that

$$N_n^\circ = \sum_{i=1}^{n} \delta_{\frac{i}{n}} \mathbf{1}_{\{X_i > a_n\}} \xrightarrow{d} N^\circ, \qquad n \to \infty.$$

Moreover, the limiting process N° has the following representation

$$N^\circ \stackrel{d}{=} \sum_{i=1}^{\infty} \kappa_i \delta_{T_i}$$

where $\sum_i \delta_{T_i}$ is a homogeneous Poisson point process on the interval $[0,1]$ with intensity θ and $(\kappa_i)_{i\geq 1}$ is a sequence of i.i.d. random variables with values in $\mathbb{N}$ independent of it. Finally, random variables κ_i have geometric distribution:

$$P(\kappa_1 = k) = \mu^{-\alpha(k-1)}(1-\mu^{-\alpha})$$

for all $k \in \mathbb{N}$.

Asymptotics for sums

Let

$$S_n = X_1 + \cdots + X_n$$

Theorem 5 For $\alpha > 2$ as $n \to \infty$

$$\frac{1}{\sqrt{n}} (S_n - nEB/(1-\mu)) \xrightarrow{d} N(0, \sigma^2)$$

where $\sigma^2 = E(X_0)^2 + \sum_{i=1}^{\infty} E(X_0 X_i) < \infty$. Under the assumptions of Model I, for $\alpha \in (0,1)$

$$\frac{S_n}{a_n} \xrightarrow{d} \mathcal{S}_\alpha$$

Similarly, when $\alpha \in (1,2)$ under assumptions of either Model I or II and under additional condition (3.2) of Davis and Hsing (1995):

$$\frac{S_n - E\frac{X_i}{a_n}\mathbf{1}_{\{\frac{|X_i|}{a_n} \le 1\}}}{a_n} \xrightarrow{d} \mathcal{S}_\alpha$$

THANK YOU
for Your Attention !!!