MOMENTS OF MAXIMA OF STABLE PROCESSES AND FIELDS

Snigdha Panigrahi

ISI Kolkata

Based on an ongoing joint work with Parthanil Roy (ISI Kolkata) and Yimin Xiao (Michigan State University)

17th of January, 2013

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$$0 < \alpha < 2 \implies P(X > x) \sim C_\alpha x^{-\alpha} \implies \text{ no } \alpha th \text{ moment}$$

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$\alpha = 2 \implies$ Gaussian RF. But we assume: $0 < \alpha < 2$.

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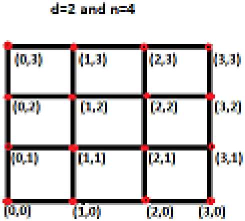

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Open Problem (Xiao (2010)): Can we give "sharp lower and upper bound" on $E|M_n|^\beta$ for $\beta \in (0, \alpha)$?

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Smoothness of $\{Y_{(u,v)}\}_{(u,v)\in\mathbb{R}^2}$ depends on how small the quantity

$$E\left(\max_{1\le i\le 2^n}\max_{1\le j\le 2^n}\left|Y_{\left(\frac{i+1}{2^n},\frac{j}{2^n}\right)} - Y_{\left(\frac{i}{2^n},\frac{j}{2^n}\right)}\right|^{\beta}\right)$$

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Assume $\{Y_{(u,v)}\}_{(u,v)\in\mathbb{R}^2}$ has stationary increments and is self-similar.

Motivation

$$
\begin{aligned}
&E\left(\max_{1\le i\le 2^n}\max_{1\le j\le 2^n}\left|Y_{\left(\frac{i+1}{2^n},\frac{j}{2^n}\right)}-Y_{\left(\frac{i}{2^n},\frac{j}{2^n}\right)}\right|^{\beta}\right)\\
&=2^{-nH}E\left(\max_{0\le i\le 2^n-1}\max_{0\le j\le 2^n-1}\left|Y_{(i+1,j)}-Y_{(i,j)}\right|^{\beta}\right)\\
&=2^{-nH}E\left(\max_{0\le i\le (2^n-1)}\max_{0\le j\le (2^n-1)}\left|X_{(i,j)}\right|^{\beta}\right), \qquad (1)
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$$=2^{-nH}E\left(\max_{0\le i\le 2^n-1}\max_{0\le j\le 2^n-1}\left|Y_{(i+1,j)}-Y_{(i,j)}\right|^{\beta}\right)$$
$$=2^{-nH}E\left(\max_{0\le i\le (2^n-1)}\max_{0\le j\le (2^n-1)}\left|X_{(i,j)}\right|^{\beta}\right), \qquad (1)$$

NOTE: This quantity is nothing but $2^{-nH}E|M_{2^n}|^{\beta}$ and therefore, a sharp upper bound on $E|M_{2^n}|^{\beta}$ can help determine the smoothness of the paths.

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An independently scattered and σ-additive

$$M : \mathcal{S}_0 \to L^0(\Omega)$$

is called an $S\alpha S$ Random Measure on $(S, \mathcal{S})$ with control measure μ if

$$M(A) \sim S\alpha S((\mu(A)^{1/\alpha}).$$

Non-singular $\mathbb{Z}^d$-action on S

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4. $\mu \circ \phi_t^{-1} \sim \mu \ \ \forall \ t \in \mathbb{Z}^d$.

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$$f_t(s) = \pm \left(\frac{d\mu \circ \phi_t}{d\mu}(s)\right)^{1/\alpha} f \circ \phi_t(s)$$

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NOTE: We say that the random field $\{X_t\}_{t \in \mathbb{Z}^d}$ is generated by $\{\phi_t\}_{t \in \mathbb{Z}^d}$ and f.

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$\{\phi_t\}_{t \in \mathbb{Z}^d}$ is called a conservative $\mathbb{Z}^d$-action if it has no wandering set of positive measure.

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This was formalized by **Samorodnitsky (2004)** for $d = 1$. **Roy and Samorodnitsky (2008)** established a generalization of this for $d > 1$. Continuous parameter extension is handled in another paper of **Samorodnitsky (2004)** ($d = 1$ case) and a paper of **Roy (2010)** ($d > 1$ case).

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Theorem

Let $0 < \beta < \alpha$.
If $\{\phi_t\}_{t \in \mathbb{Z}^d}$ *is conservative then,*

$$E\left(M_n^\beta\right) = o(n^{d\beta/\alpha}).$$

Otherwise, for some $a > 0$,

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An Example

$S = \mathbb{R},\ \mu = Leb,\ d = 2,\ \phi_{(i,j)}(s) = s + i - j$ (measure preserving),
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Define $X_{(i,j)} = \int_R f \circ \phi_{(i,j)}(s) M(ds), \quad i, j \in \mathbb{Z}$.

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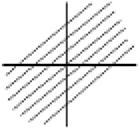

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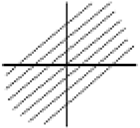

NOTE: $j = i + k,\ \phi_{(i,j)} = \phi_{(0,k)} \implies X_{(i,j)} = X_{(0,k)}$.

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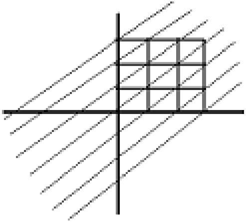
$$M_{n+1} = \max_{0 \le i,j \le n} |X_{(i,j)}| = \max_{-n \le k \le n} |X_{(0,k)}| \stackrel{d}{=} \max_{0 \le k \le 2n} |X_{(0,k)}|.$$



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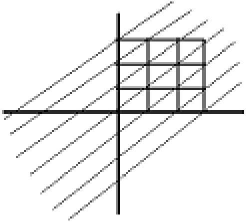

Since $\{X_{(0,k)}\}_{k \in \mathbb{Z}}$ is generated by $\{\phi_{(0,k)}\}_{k \in \mathbb{Z}}$, which is not conservative, we have,

$$E\,(M_n)^{\beta} \sim cn^{\beta/\alpha} \;\; for \;\; some \;\; c > 0.$$

Understanding the example

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In general, if $K = \left\{t \in \mathbb{Z}^d : \phi_t = id_S\right\}$, is it true that $\mathbb{Z}^d/K \cong \mathbb{Z}^p$ for some $p \leq d$?

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An Algebraic Fact: $\bar{F}$ has an isomorphic copy F sitting inside $\mathbb{Z}^d$.

Result on Moments of Extreme Values for Conservative Flows

Clearly $p = rank(F) \leq d$. It is the effective dimension of the random field. This notion was formalized by **Roy and Samorodnitsky (2008)** (discrete parameter case) and extended by **Chakrabarty and Roy (2011)** (continuous parameter case).

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References

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