Weighted Sums of Regularly Varying Random Variables With Dependent Weights

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Generic set-up

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- In random coefficient linear processes.

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In particular, when $\{\Theta_t\}$ and $\{X_t\}$ are NOT independent?

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- $\{X_t\}$ i.i.d. $RV_{-\alpha}$.
- $$\sum_{t=1}^{\infty} c_t^{\delta} < \infty \quad \text{for some } 0 < \delta < \alpha \wedge 1.$$
- Then $\sum_{t=1}^{\infty} c_t X_t$ regularly varying with index $-\alpha$.

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- First must study behaviour of product
$$\Theta_t X_t.$$

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$$P(\Theta X > x) \sim E(\Theta^{\alpha}) P(X > x) \text{ as } x \to \infty,$$

hence $P(\Theta X > x) \in RV_{-\alpha}$.

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When $\alpha \geq 1$, *there exists* $\varepsilon > 0$ *such that*

$$\sum_t [E(\Theta_t^{\alpha+\varepsilon})]^{\frac{1}{\alpha+\varepsilon}} < \infty \quad \textit{and} \quad \sum_t [E(\Theta_t^{\alpha-\varepsilon})]^{\frac{1}{\alpha+\varepsilon}} < \infty.$$

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$$\textit{Then} \quad \lim_{x\to\infty} \frac{P[\sum_{t=1}^{\infty} \Theta_t X_t > x]}{P[X_1 > x]} = \sum_{t=1}^{\infty} E(\Theta_t^{\alpha}).$$

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Then $\sum_{t=1}^{\infty} \Theta_t X_t^+$ *and* $\sup_{1 \le n < \infty} \sum_{t=1}^{n} \Theta_t X_t$ *are regularly varying with index* $-\alpha$.

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- $E(\Theta^\alpha) < \infty$ *and* $P\{\Theta > x\} = o(P\{X > x\}).$

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$$P[\sup_{1\leq n<\infty} \sum_{t=1}^{n} \Theta_t X_t > x] \sim P[X_\infty > x] \sim P[X_1 > x] \sum_{t=1}^{\infty} E[\Theta_t^{\alpha}]$$

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and $X_\infty = \sum_{t=1}^{\infty} \Theta_t X_t^+$ *is almost surely finite.*

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Next obvious question

- What if $\{X_t\}$ and $\{\Theta_t\}$ are not independent?

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- $$1 + \theta\phi_1(x)\phi_2(y) \geq 0.$$

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$$P(XY > x) = \bar{H}(x) \sim \{E(Y^\alpha) + \theta d_1 E(\phi_2(Y)Y^\alpha)\}\bar{F}(x).$$

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Y^*_θ, *independent of* X^*, *with*

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- $E(Y^{\alpha}) < \infty$.

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$$P[\sup_{1\leq m\leq n} \sum_{i=1}^{m} X_i \prod_{j=1}^{i} Y_j > x] \sim \frac{1-\{E(Y^\alpha)\}^n}{1-E(Y^\alpha)}[E(Y^\alpha) + \theta d_1 E(\phi_2(Y)Y^\alpha)]\bar{F}(x).$$

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- Bivariate Sarmanov family includes many generalizations of Farlie-Gumbel-Morgenstern (FGM) distributions.

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$$P[\sup_{n \geq 1} \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j > x] \sim \frac{1}{1 - E(Y^\alpha)} [E(Y^\alpha) + \theta d_1 E\{\phi_2(Y) Y^\alpha\}] \bar{F}(x).$$

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  $\Psi(x) = P(\sup_{n \geq 1} S_n > x)$.

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- **Lower bound:**
  Define $S_n = \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j$,
  $\Psi(x, m) = P(\sup_{1 \leq n \leq m} S_n > x)$, and
  $\Psi(x) = P(\sup_{n \geq 1} S_n > x)$.

$$\Psi(x) \geq \Psi(x, m) = P\left[\sup_{1 \leq n \leq m} \sum_{i=1}^{n} X_i Y_i \prod_{j=1}^{i-1} Y_j > x\right].$$

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  Let $m \to \infty$ to get lower bound.

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$$\lim_{x\to\infty} \frac{\Psi((1-\delta)x, m)}{\overline{F}(x)} =$$

$$(1-\delta)^{-\alpha} \frac{1-\{E(Y^\alpha)\}^m}{1-E(Y^\alpha)} E(Y^\alpha) + \theta d_1 E(\phi_2(Y) Y^\alpha).$$

- Motivated by the principle of one large jump,

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B should be small compared to A.
Get upper bound for B separately for $\alpha < 1$ and $\alpha \geq 1$.

- $\alpha < 1$,

$$\frac{B}{\overline{F}(x)} \leq C \sum_{t=m+1}^{\infty} \frac{P[\Theta_t Z_t > x]}{\overline{F}(x)}.$$

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$\alpha \geq 1$,

$$\frac{B}{\overline{F}(x)} \leq \sum_{t=m+1}^{\infty} \frac{P[\Theta_t Z_t > x]}{\overline{F}(x)}$$

$$+ C[\sum_{t=m+1}^{\infty} (\frac{P[\Theta_t Z_t > x]}{\overline{F}(x)})^{\frac{1}{\alpha+\varepsilon}}]^{\alpha+\varepsilon}.$$

- $$\frac{P[\Theta_t Z_t > x]}{\overline{F}(x)} \leq B_t$$

  for all large values of x, where

  $$B_t = BE[\Theta_t^\alpha] = B\{E[Y_1^\alpha]\}^{t-1}$$

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- $$\frac{P[\Theta_t Z_t > x]}{\overline{F}(x)} \leq B_t$$
  for all large values of x, where
  $$B_t = BE[\Theta_t^{\alpha}] = B\{E[Y_1^{\alpha}]\}^{t-1}$$
  with
  $$\sum_{t=m+1}^{\infty} B_t < \infty \quad \text{for } \alpha < 1$$
  and
  $$\sum_{t=m+1}^{\infty} B_t^{\frac{1}{\alpha+\varepsilon}} < \infty \quad \text{for } \alpha \geq 1.$$

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Thank you ☺