The Art of Seeking Hidden Risks

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1. Background.

Suppose

$$\boldsymbol{X} = (X_1, \ldots, X_d)$$

is a *risk vector*. Imagine X_i is

- loss from ith asset in portfolio;
- concentration of ith pollutant;
- car maker's warranty exposure over a month for ith car model in lineup.

Goal: Estimate the probability of a *risk region* $\mathcal{R}$

$$P[\boldsymbol{X} \in \mathcal{R}]$$

where $\mathcal{R}$ is beyond the range of observed data.

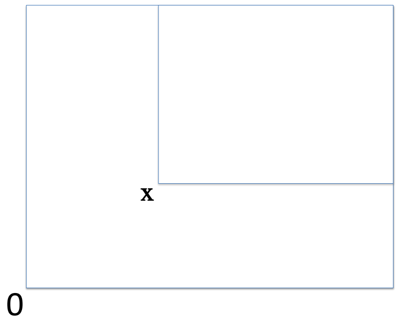
Example:

$d = 2$ and

$\mathcal{R} = (\mathbf{x}, \boldsymbol{\infty}] = (x_1, \infty] \times (x_2, \infty]$

and

$P[\boldsymbol{X} \in \mathcal{R}] = P[X_1 > x_1, X_2 > x_2].$



Risk contagion: Can two or more components of the risk vector $\boldsymbol{X}$ be simultaneously large? Typically,

large=beyond the range of the data.

1.1. Asymptotic method for estimation

- Estimating probabilities of risk regions beyond the range of the data requires an assumption that enables extrapolation.
- Usual assumption: $\boldsymbol{X}$ is in the domain of attraction (DOA) of an extreme value distribution; ie, $\exists\, a_i(n) > 0, b_i(n) \in \mathbb{R}, i = 1, \ldots, d; n \geq 1$ such that if $\{\boldsymbol{X}(m), m \geq 1\}$ are iid copies of $\boldsymbol{X}$, then

$$P\Big[\bigvee_{m=1}^{n} \frac{\boldsymbol{X}(m) - \boldsymbol{b}(n)}{\boldsymbol{a}(n)} \leq \mathbf{x}\Big] = \Big(P\Big[\frac{X_i - b_i(n)}{a_i(n)} \leq x_i,\ i = 1, \ldots, d\Big]\Big)^n \to G(\mathbf{x})$$

where G is a multivariate EV distribution *with non-degenerate marginals*. Equivalently,

$$nP\Big[\frac{\boldsymbol{X} - \boldsymbol{b}(n)}{\boldsymbol{a}(n)} \in \cdot\,\Big] \to \nu(\cdot) \qquad \text{(DOA)}$$

where

$$\nu\big([-\boldsymbol{\infty}, \mathbf{x}]^c\big) = -\log G(\mathbf{x}).$$

- If one is determined to use asymptotic methods and $\mathcal{R}$ is the risk region, (DOA) yields a method to estimate the risk probability:

$$P[\boldsymbol{X} \in \mathcal{R}] = P\Big[\frac{\boldsymbol{X} - \boldsymbol{b}(n)}{\boldsymbol{a}(n)} \in \frac{\mathcal{R} - \boldsymbol{b}(n)}{\boldsymbol{a}(n)}\Big] \approx \frac{1}{n}\hat{\nu}\Big(\frac{\mathcal{R} - \hat{\boldsymbol{b}}}{\hat{\boldsymbol{a}}}\Big).$$

- Standardized version of (DOA) which expresses the condition as multivariate regular variation on $\mathbb{E} := [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$: Set

$$U_i(x) = \frac{1}{P[X_i > x]}$$

and

$$\boldsymbol{X}^* = \big(U_i(X_i), i = 1, \dots, d\big).$$

Then marginal convergence in (DOA) to non-degenerate EV distributions plus (DOA) is equivalent to

$$nP\Big[\frac{\boldsymbol{X}^*}{n} \in \cdot\,\Big] \to \nu^*(\cdot) \qquad \text{(StandRegVarE)}$$

on $\mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$ where for $t > 0$,

$$\nu^*(t\,\cdot) = t^{-1}\nu^*(\cdot).$$

This is just transformation to Pareto scale.

1.2. Curse of asymptotic independence

- If in (DOA), the limit G is a product

$$G(\mathbf{x}) = \prod_{i=1}^{d} G_i(x_i), \qquad \text{(AsyIndep)}$$

we say $\boldsymbol{X}$ possesses *asymptotic independence*.

- Unintended consequence: (AsyIndep) $\Rightarrow$

$$\nu(\{\mathbf{x} : x_i > y_i(0), x_j > y_j(0)\}) = 0,$$

for all $1 \leq i < j \leq d$ and thus such an *asymptotic* model has no risk contagion since we estimate

$$P[\text{ two or more components of } \boldsymbol{X} \text{ are large simultaneously }] \approx 0.$$

- In standardized form: (StandRegVarE)+(AsyIndep) mean when $d = 2$,

$$\nu^*\big(\mathbb{E}_0\big) = \nu^*\big((\mathbf{0}, \boldsymbol{\infty}]\big) = 0,$$

and ν^* concentrates on the axes through 0.

- Can we improve on this asymptotic method?

1.3. How common is (AsyIndep)?

- For $d = 2$: If $\boldsymbol{X}$ satisfies (DOA) and $X_1 \perp\!\!\!\perp X_2$ then $\boldsymbol{X}$ possesses (AsyIndep).
- If $\boldsymbol{X} = (X_1, \dots, X_d)$ is Gaussian with

$$\mathrm{corr}(X_i, X_j) = \rho(i, j) < 1,$$

then $\boldsymbol{X}$ possesses (AsyIndep) (Sibuya, 1960). Here the marginals of $\boldsymbol{X}$ are Gaussian and

$$G(\mathbf{x}) = \prod_{i=1}^{d} \exp\{-e^{-x_i}\}.$$

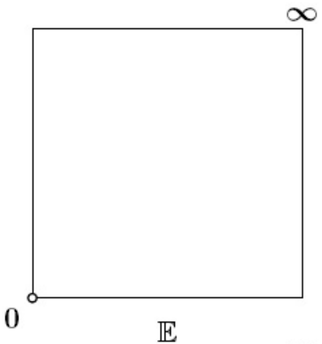
- So using the Gaussian dependence copula means you are exposed to (AsyIndep) and lack of risk contagion.
- Let $U \sim U(0, 1)$ and define

$$\boldsymbol{X} = \Big(\frac{1}{U}, \frac{1}{1-U}\Big).$$

Since $1/U$ and $1/(1-U)$ cannot be simultaneously large, $\boldsymbol{X}$ possesses (AsyIndep). The marginals of $\boldsymbol{X}$ are Pareto and

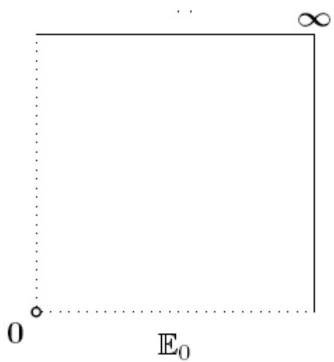
$$G(\mathbf{x}) = \exp\{-(x_1^{-1} + x_2^{-1})\}, \qquad \mathbf{x} > \mathbf{0}.$$

2. Strategy



(AsyIndep) + (StandRegVarE) implies the limit measure $\nu^*(\cdot)$ in Pareto scale concentrates on the axes through $\mathbf{0}$.

<u>Hint:</u> Consider the complement of the support of ν^* and seek a lower order regular variation on this new set.



Since ν^* concentrates on axes and puts zero mass on interior of quadrant, seek (hidden) regular variation on the interior $\mathbb{E}_0 = (\mathbf{0}, \boldsymbol{\infty}]$ with index < 1. This would allow non-zero estimate of

$$P[\boldsymbol{X} > \mathbf{x}].$$

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Example. For $d = 2$: If $\boldsymbol{X} = (X_1, X_2)$ and $X_1 \perp\!\!\!\perp X_2$, X_1, X_2 iid with

$$P[X_i > y] = y^{-1}, \quad y > 1.$$

Then for $x_1 > 0$, $x_2 > 0$, as $n \to \infty$

$$nP[X_i > nx_i] \to x_i^{-1}, \qquad i = 1, 2,$$
$$nP[X_1 > nx_1, X_2 > nx_2] \to 0,$$

so $\boldsymbol{X}$ is regularly varying on $\mathbb{E}$ with index 1 and limit measure concentrating on the axes, and

$$nP[X_1 > \sqrt{n}x_1, X_2 > \sqrt{n}x_2] = \sqrt{n}P[X_1 > \sqrt{n}x_1] \cdot \sqrt{n}P[X_2 > \sqrt{n}x_2]$$
$$\to \frac{1}{x_1 x_2}, \qquad x_1 > 0, x_2 > 0,$$

so $\boldsymbol{X}$ is regularly varying on $\mathbb{E}_0$ with index 2 and limit measure giving positive mass to $(\mathbf{x}, \boldsymbol{\infty}]$.

Conclude for this example:

- $\boldsymbol{X}$ is regularly varying on $\mathbb{E} = [\mathbf{0}, \infty] \setminus \{\mathbf{0}\}$ with index 1 (scale by n) and limit measure concentrating on lines through $\{\mathbf{0}\}$, and giving zero mass to $(\mathbf{0}, \infty]$.
- $\boldsymbol{X}$ is regularly varying on $\mathbb{E}_0 = (\mathbf{0}, \infty]$ with index 2 (scale by $\sqrt{n}$) and the limit measure gives positive mass to $(\mathbf{0}, \infty]$.

Summary:

Lesson: If the support (eg, axes) of the limit measure is less than the full space (eg, $\mathbb{E}$):

- peel away the support (axes);
- look for extreme value behavior on what's left (eg, $\mathbb{E} \setminus \{\text{axes}\} = \mathbb{E}_0$).

3. Directions to pursue

Antecedents: Das et al. (2011), Draisma et al. (2004), Heffernan and Resnick (2005), Ledford and Tawn (1996, 1997), Maulik and Resnick (2005), Mitra and Resnick (2011a,b), Resnick (2002)

1. Hidden regular variation (HRV)
   (a) HRV for $d = 2$.
   (b) HRV for $d > 2$. Possibly seek regular variation on a progression of decreasing of cones. Must decide how to specify sequence of cones.
2. Hidden domain of attraction (HDA):
   - $\boldsymbol{X}$ satisfies (DOA) so that $\boldsymbol{X}^*$ satisfies (StandRegVarE).
   - (AsyIndep) holds so limit measure $\nu^*(\cdot)$ for $\boldsymbol{X}^*$ concentrates on the axes through $\mathbf{0}$.
   - However extreme value behavior other than regular variation holds in the interior of the state space. Eg, $\vee_{i=1}^d X_i^*$ has a regularly varying distribution but $\wedge_{i=1}^d X_i^*$ has a distribution in a one dimensional domain of attraction other than Fréchet.

3. More general unifying theory: Seek lower order regular variation on complement of support of the limit measure.
    - Asymptotic full dependence: limit measure concentrates on the diagonal. Remove diagonal and seek regular variation on what is left. Do we need a new theory?
    - Sequence of regular variation properties on successively smaller cones.
4. What is the unit sphere? What takes the place of the transformation to polar coordinates?
5. Mass on lines through $\boldsymbol{\infty}$?
    - For standardized regular variation on $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$, limit measures have a scaling property which precludes mass on lines through $\boldsymbol{\infty}$.
    - On smaller cones such as $(0, \infty]^2$, this is no longer true.
    - Mass on lines through $\boldsymbol{\infty}$ invalidates convergence to types:
        - Under one normalization get a limit measure with mass on lines through $\boldsymbol{\infty}$ but
        - under another normalization all mass on $(0, \infty)^2$.
    - Exclude mass on lines through $\boldsymbol{\infty}$? Give up on the one-point uncompactification.

6. Estimation?

- Non-parametric approach: Does the rank transform uncover all the hidden structure?
- What sub-cones do we examine?
- How to automate in high dimensions?
- How should we infer the support of the limit measure?

4. Hidden Regular Variation

4.1. $d = 2$

Suppose $\boldsymbol{X} = (X_1, X_2)$ satisfies (DOA) and

$$U_i(x) = \frac{1}{P[X_i > x]}, \quad \boldsymbol{X}^* = \big(U_1(X_1), U_2(X_2)\big).$$

So $\boldsymbol{X}^*$ satisfies (StandRegVarE) on $\mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$; ie,

$$nP\Big[\frac{\boldsymbol{X}^*}{n} \in \cdot\Big] \to \nu^*(\cdot).$$

$\boldsymbol{X}^*$ has *hidden regular variation* on $\mathbb{E}_0 = (0, \infty]^2$ if in addition to (StandRegVarE):

- There is a measure $\nu_0^*(\cdot)$ on $\mathbb{E}_0$; and a
- There is a sequence $b_0(n) \to \infty$ such that $b_0(n)/n \to 0$; and
- On $\mathbb{E}_0$

$$nP[\frac{\boldsymbol{X}^*}{b_0(n)} \in \cdot\,] \to \nu_0^*(\cdot). \qquad \text{(HRV E0)}$$

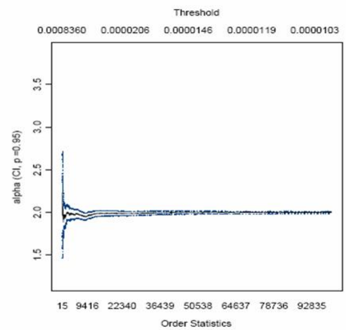
Consequences

- Because $b_0(n) = o(n)$, $\boldsymbol{X}^*$ and hence $\boldsymbol{X}$ must have (AsyIndep).
- For some $\alpha_0 \geq 1$,
$$b_0(n) \in RV_{1/\alpha_0}.$$
- Hence to identify α_0 or detect HRV:
$$P[X_1^* \vee X_2^* > x] \in RV_{-1}, \qquad P[X_1^* \wedge X_2^* > x] \in RV_{-\alpha_0}.$$

Example 1: $\boldsymbol{X}^* = (X_1, X_2)$, $X_1 \perp\!\!\!\perp X_2$ and
$$P[X_i > x] = x^{-1}, \quad x > 1,\ i = 1, 2.$$
Then $\alpha_0 = 2$. Consider $\boldsymbol{X}_1, \dots, \boldsymbol{X}_{5000}$ iid.

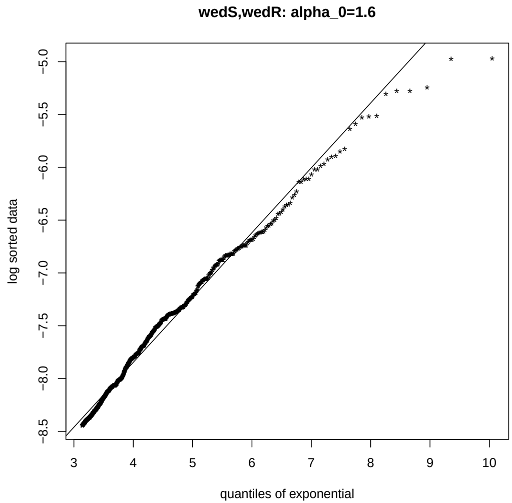
5000 pairs of iid Pareto;
$$\alpha = 1;\ \alpha_0 = 2.$$
Hill plot for minima of components.

<u>Conclude:</u> Maybe it is possible to detect HRV.



Example 2: UNC Wed (S,R): Response data where S is size of response and R is average transmission rate= size/(download time).

- Need non-standard model.
- Standardize using rank method. (Now marginal $\alpha = 1$.)
- QQ plot of minimum component of rank transformed data using 1000 upper order statistics for UNC Wed (S,R).
- Method yields $\alpha = 1$ and estimated $\hat{\alpha}_0 = 1.6$.
- Conclude: For $d = 2$, detection is not hopeless.



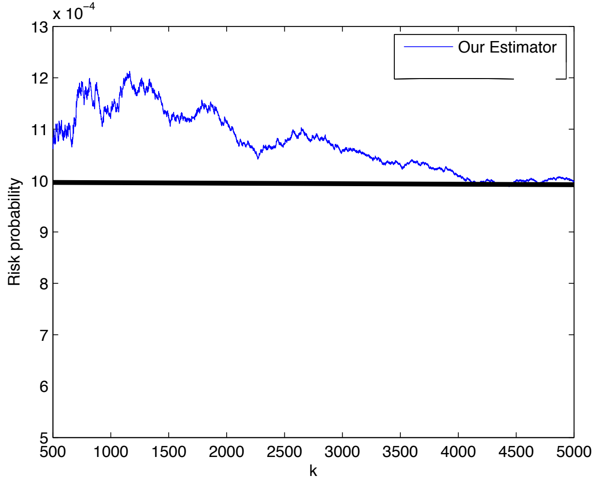
Example 3: Risk calculations.

Simulate data: $\{((X_1(n), X_2(n)); 1 \le n \le 5000\}$ iid where

- $X_1(n) \perp\!\!\!\perp X_2(n)$ for each n;
- $X_1(n) \sim \mathrm{Par}(1)$, $X_2(n) \sim \mathrm{Par}(2)$.
- Estimate the risk probability (exact value=0.001)

  $$P[X_1 > 100, X_2 > \sqrt{10}]$$

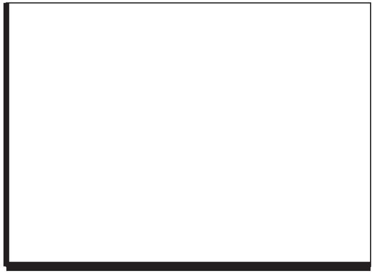
  with *spectral distribution estimator*.
- Conclude: At least in nice cases, this can work.



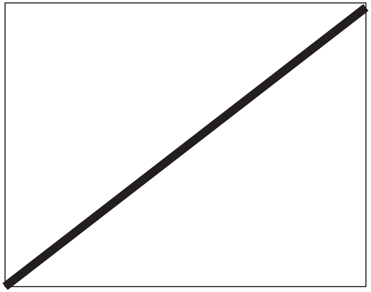
5. General approach.

(Das et al., 2011)
Compare and contrast the two situations thought to be at opposite ends of the spectrum for regularly varying distributions when $d = 2$.

1. **Asymptotic independence:** limit measure $\nu(\cdot)$ concentrates on axes through $\mathbf{0}$.



2. **Asymptotic full dependence:** limit measure $\nu(\cdot)$ concentrates on the diagonal.



- In both cases, the limit measure has a *support* far smaller than $\mathbb{E} = [\mathbf{0}, \infty] \setminus \{\mathbf{0}\}$.
- For HRV, remove support and seek a regular variation property on the complement of the support $(\mathbf{0}, \infty]$ (when $d = 2$).
- Standard case regular variation implies limit measure $\nu^*(\cdot)$ has scaling property:

$$\nu^*(c\cdot) = c^{-1}\nu^*(\cdot), \quad c > 0,$$

which implies

$$\text{support } \nu^* = \text{closed cone.}$$

- This suggests unifying both asymptotic independence and asymptotic full dependence and ... under one theory:
  - Identify support of the limit measure $\nu^*(\cdot)$.
  - Seek lower order regular variation on the complement of the support.

5.1. Regular variation on cones.

Abandon the one point uncompactification of the positive quadrant; exclude lines through ∞. Let $\mathbb{S}$ be CSMS and suppose $F_1 \subset \mathbb{S}$ closed (cone) containing $\mathbf{0}$ and define

$$\mathbb{S}_{F_1} = \mathbb{S} \setminus F_1.$$

$\to$ The random element $\boldsymbol{X} \in \mathbb{S}$ has a distribution with a regularly varying tail on $\mathbb{S}_{F_1}$ if $\exists\, b(t) \uparrow \infty$ and measure $\nu \not\equiv 0$ on $\mathbb{S}_{F_1}$ such that

$$tP[\frac{\boldsymbol{X}}{b(t)} \in \cdot\,] \to \nu(\cdot), \qquad \text{in } M^*(\mathbb{S}_{F_1}).$$

Let F_2 be another closed (cone) containing $\mathbf{0}$ and set

$$\mathbb{S}_{F_1 \cup F_2} = \mathbb{S} \setminus (F_1 \cup F_1).$$

$\to$ The random $\boldsymbol{X}$ has a distribution with hidden regular variation on $\mathbb{S}_{F_1 \cup F_2}$ if there is regular variation on $\mathbb{S}_{F_1}$ AND if $\exists\, b_1(t) \uparrow \infty$ and a measure $\nu_1(\cdot) \not\equiv 0$ on $\mathbb{S}_{F_1 \cup F_2}$ such that

$$tP[\frac{\boldsymbol{X}}{b_1(t)} \in \cdot\,] \to \nu_1(\cdot), \qquad \text{in } M^*(\mathbb{S}_{F_1 \cup F_2}),$$

AND

$$b(t)/b_1(t) \to \infty$$

(which makes the behavior on $\mathbb{S}_{F_1 \cup F_2}$ hidden).

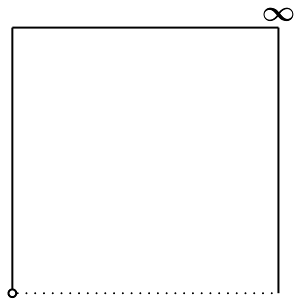
Examples for $d = 2$:

1. Regular variation on the positive quadrant with conditional extreme value (CEV) model:

$$\begin{aligned} \mathbb{S} =& [\mathbf{0}, \infty), \;\; F_1 = \{\mathbf{0}\}, \\ \mathbb{S}_{F_1} =& [\mathbf{0}, \boldsymbol{\infty}) \setminus \{\mathbf{0}\}. \end{aligned}$$

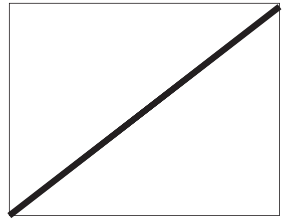
CEV on $\mathbb{D}_{\sqcap}$:

$$\begin{aligned} \mathbb{F}_2 =& \{(x, 0) : x > 0\}, \\ \mathbb{S}_{F_1 \cup F_2} =& [\mathbf{0}, \boldsymbol{\infty}) \setminus \big(\{\mathbf{0}\} \cup \{(x, 0) : x > 0\}\big) \\ =& [0, \infty) \times (0, \infty) \\ =:& \mathbb{D}_{\sqcap}. \end{aligned}$$



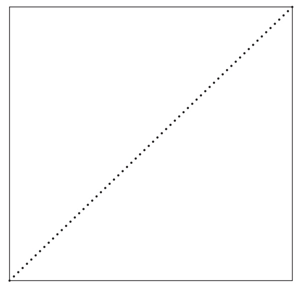
2. Asymptotic full dependence:

Regular variation on $[\mathbf{0},\boldsymbol{\infty})\backslash\{\mathbf{0}\}$ with limit measure concentrating on diagonal.



$$\begin{aligned} \mathbb{S} =& [\mathbf{0},\infty),\ \mathbb{F}_1 = \{\mathbf{0}\}, \\ \mathbb{S}_{F_1} =& [\mathbf{0},\boldsymbol{\infty}) \setminus \{\mathbf{0}\}. \end{aligned}$$

Remove diagonal:



$$\begin{aligned} \mathbb{F}_2 =& \{(x,x) : x > 0\}, \\ \mathbb{S}_{F_1 \cup F_2} =& \mathbb{C} \setminus (\mathbb{F} \cup \mathbb{F}_1) \\ =& [\mathbf{0},\boldsymbol{\infty}) \setminus \{(x,x) : x \geq 0\} \end{aligned}$$

Example 3 (continued): Asymptotic full dependence.

Suppose $\boldsymbol{X} = (X_1, X_2)$ is regularly varying on $[\mathbf{0}, \boldsymbol{\infty}) \setminus \{\mathbf{0}\}$ with asymptotic full dependence so the limit measure $\nu(\cdot)$ concentrates on $\{(x,x) : x > 0\}$. Suppose

$$X_i = \text{ one period loss of financial instrument } I_i.$$

Construct the portfolio:

- Buy one unit of I_1. (Go long.)
- Sell one unit of I_2. (Go short.)

One period loss for the portfolio is

$$L = X_1 - X_2$$

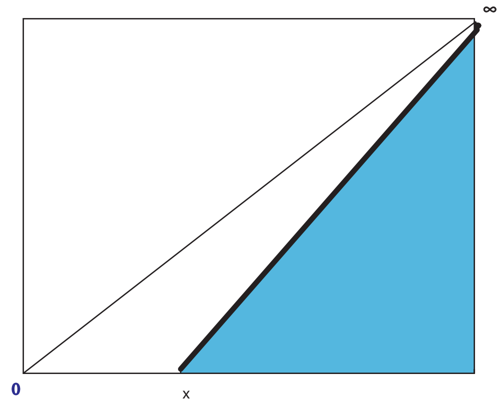
and for large x, seek

$$P[X_1 - X_2 > x].$$

Under asymptotic full dependence, limit measure concentrates on the line $\{(x,x) : x > 0\}$ so we estimate probability as 0:

$$\widehat{P}[X_1 - X_2 > x] = 0.$$

Conclude: A more general theory has applicability.



5.2. Consequences:

Useful to have a more general umbrella of HRV that includes:

- Asymptotic independence.
- Asymptotic full dependence.
- Other cases where the support of limit measure is strictly smaller than the state space.
- Stochastic processes.
- In high dimensional spaces want the possibility of a nested sequence cones each of which has a regular variation property.

5.3. Topology: General approach.

What topology is appropriate? What are the *bounded* sets? Modify Hult and Lindskog (2006):

- $\mathbb{S} =$ CSMS.
- $F \in \mathcal{F}(\mathbb{S})$ (closed subset; often a closed cone).
- State space=$\mathbb{S}_F := \mathbb{S} \setminus F$.
- Tail regions: any subset of $\mathbb{S}_F$ which is *bounded away from* F; ie, $R \subset \mathbb{S}_F$ is a tail region if

$$d_{\mathbb{S}}(R, F) > 0.$$

- $M^*(\mathbb{S}_F) =$ measures on $\mathbb{S}_F$ which are finite on sets bounded away from F.
- $\mathcal{C}(\mathbb{S}_F) =$ bounded, positive, continuous functions whose supports are bounded away from F.
- Topology on $M^*(\mathcal{S}_F)$ is smallest topology which makes

$$\mu \mapsto \mu(f)$$

from

$$M^*(\mathbb{S}_F) \mapsto \mathbb{R}_+$$

continuous.

- Example: $\mathbb{S} = \mathbb{R}_+^\infty$, $F = F^{(j)}$, $j \geq 0$, where

$$F^{(j)} = \{\mathbf{x} := (x_1, x_2, \dots) \in \mathbb{R}_+^\infty : \sum_{j=1}^{\infty} \epsilon_{x_j}(0,\infty) \leq j\}$$
$$= \{\mathbf{x} : \text{ at most } j \text{ components} > 0\}.$$

So

$$F^{(0)} = \{\mathbf{0}\}$$
$$F^{(1)} = \text{lines through } \mathbf{0}, \text{ including } \mathbf{0}$$
$$= \bigcup_{j=1}^{\infty} \{0\}^{j-1} \times (0,\infty) \times \{0\}^\infty \cup \{\mathbf{0}\},$$
$$\vdots$$

- Mapping theorem: Let $T : \mathbb{S}_1 \setminus F_1 \mapsto \mathbb{S}_2 \setminus F_2$ be continuous and satisfy: If $D_2 \in \mathcal{F}_2(\mathbb{S}_2 \setminus F_2)$ is bounded away from F_2 then $T^{-1}(D_2)$ is bounded away from F_1. Then $\hat{T} : M^*(\mathbb{S}_1 \setminus F_1) \mapsto M^*(\mathbb{S}_2 \setminus F_2)$ defined by
$$\hat{T}(\mu_1) = \mu_1 \circ T^{-1}$$
is continuous.

- Mapping theorem variant: Suppose $T : \mathbb{S}_1 \mapsto \mathbb{S}_2$ is *uniformly* continuous and suppose for $F_1 \in \mathcal{F}(S_1)$ we have $TF_1 \in \mathcal{F}(S_2)$. Then $\hat{T} : M^*(\mathbb{S}_1 \setminus F_1) \mapsto M^*(\mathbb{S}_2 \setminus F_2)$ is continuous.

- Example: $\mathbb{S} = \mathbb{R}_+^\infty$ and $\boldsymbol{X} = (X_1, X_2 \dots)$ has iid components with each X_i having a regularly varying tail with scaling function $b(t)$. Then as $t \to \infty$, for $j \geq 1$

$$tP[\boldsymbol{X}/b(t^{1/j}) \in \cdot] \to \nu^{(j)} \text{ in } M^*(\mathbb{R}_+^\infty \setminus F^{(j-1)})$$

and $\nu^{(j)}$ concentrates on $F^{(j)} \setminus F^{(j-1)}$, the sequences with j positive components. Define

$$\text{CUMSUM} : \mathbb{R}_+^\infty \mapsto \mathbb{R}_+^\infty$$

by

$$\text{CUMSUM}(\mathbf{x}) = (x_1, x_1 + x_2, \dots)$$

and then

$$tP[\text{CUMSUM}(\boldsymbol{X})/b(t^{1/j}) \in \cdot] \to \nu^{(j)} \circ \text{CUMSUM}^{-1}$$

in $M^*(\mathbb{R}_+^\infty \setminus \text{CUMSUM}(F^{(j-1)}))$ and $\nu^{(j)} \circ \text{CUMSUM}^{-1}$ concentrates on $\text{CUMSUM}(F^{(j)} \setminus F^{(j-1)})$, the set of sequences with j jumps.

6. Remarks.

- Practical?
  - Limitations of asymptotic methods: rates of convergence?
  - Instead of estimating a risk probability as 0, estimate is a very small number.
- Need for more formal inference for estimation including confidence statements.
- General HRV technique requires knowing the support of the limit measure. Estimate support?
- High dimension problems? How to sift through different possible subcones?
- How to go from standard to more realistic non-standard case; still some inference problems.

7. Hidden Domain of attraction (HDA)

(Mitra and Resnick, 2011a)
Recall: **Example 1:** Let $U \sim U(0,1)$ and define

$$\boldsymbol{X} = (X_1, X_2) = \left(\frac{1}{U}, \frac{1}{1-U}\right).$$

Properties:

- $\boldsymbol{X}$ satisfies (StandRegVarE) on $\mathbb{E} = [0,\infty]^2 \setminus \{\boldsymbol{0}\}$.
- $\boldsymbol{X}$ possesses (AsyIndep).
- $X_1 \wedge X_2 \leq 2$ so $\boldsymbol{X}$ cannot have HRV. So is there an asymptotic regime that might help compute risk probabilities?

BUT

- $X_1 \wedge X_2 \leq 2$ belongs to the doa of the (reversed) Weibull EV distribution;
- A property akin to HRV holds on a cone but ...
- the cone is not a subcone of $\mathbb{E}$).

Blood and guts:

For $\{(x_1,x_2)\in(-\infty,\infty]^2 : x_1+x_2\le 0\}$, and large n,

$$
\begin{aligned}
nP\Big[n(X_1-2)>x_1,\, n(X_2-2)>x_2\Big]&\\
&=nP\Big[1-\frac{1}{2+x_2/n}<U<\frac{1}{2+x_1/n}\Big]\\
&=n\Big(\frac{1}{2+x_1/n}-(1-\frac{1}{2+x_2/n})\Big)\\
&=\frac{n}{2}\Big(-\frac{x_1+x_2}{2n}+O(\frac{1}{n^2})\Big)\\
&\to-(x_1+x_2)/4 \quad (n\to\infty),
\end{aligned}
$$

and if $x_1+x_2\ge 0$,

$$nP\Big[n(X_1-2)>x_1,\; n(X_2-2)>x_2\Big]\to 0.$$

Hmmmm! Suggests concept of *hidden domain of attraction (HDA)*:

7.1. HDA: Standard case; $d = 2$.

Simple case: Suppose $d = 2$ and $\boldsymbol{X} = (X_1, X_2)$ satisfies

- $X_1 \stackrel{d}{=} X_2$.
- (DOA) holds with $\boldsymbol{b}_n = (b_n, b_n) = b_n \mathbf{1}$ and $\boldsymbol{a}_n = (a_n, a_n) = a_n \mathbf{1}$ with $a_n > 0$ and

$$nP\Big[\frac{\boldsymbol{X} - b_n \mathbf{1}}{a_n} \in \cdot\Big] \to \nu(\cdot),$$

on $[-\infty, \infty]^2 \setminus \{-\boldsymbol{\infty}\}$ or $[0, \infty]^2 \setminus \{\mathbf{0}\}$.

- (AsyIndep) holds:

$$e^{-\nu([\boldsymbol{\infty}, \mathbf{x}]^c)} = G_1(x_1) G_2(x_2),$$

and additionally,

- there exist positive scaling and real centering constants $\{c_n\}$ and $\{d_n\}$ and a non-zero measure ν_0 on a cone $\mathbb{E}_0 = (\mathbf{0}, \boldsymbol{\infty}]$ or $(-\boldsymbol{\infty}, \boldsymbol{\infty}]$ such that

$$nP\left[(\boldsymbol{X} - d_n \mathbf{1})/c_n \in \cdot\,\right] \to \nu_0(\cdot) \qquad (n \to \infty). \qquad \text{(ConvE0)}$$

Then $\boldsymbol{X}$ possesses standard case HDA.

Notes:

1. It is not necessarily the case that the cone $\mathbb{E}_0 \subset \mathbb{E}$. For

$$\boldsymbol{X} = \Big(\frac{1}{U}, \frac{1}{1-U}\Big),$$

we have

$$\mathbb{E} = [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}, \quad \mathbb{E}_0 = \{\mathbf{x} \in (-\boldsymbol{\infty}, \boldsymbol{\infty}] : x_1 + x_2 \le 0.\}.$$

2. Since $X_1 \wedge X_2$ is in a doa, set

$$U^{\wedge}(x) = \frac{1}{P[X_1 \wedge X_2 > x]}.$$

Then (ConvE0) can be expressed as *standard regular variation* on $(\mathbf{0}, \boldsymbol{\infty}]$:

$$nP\left[\left(\frac{U^{\wedge}(X_1)}{n}, \frac{U^{\wedge}(X_2)}{n}\right) \in \cdot\right] \to \tilde{\nu}_0(\cdot),$$

where $\tilde{\nu}_0(\cdot)$ is obtained from $\nu_0(\cdot)$ and satisfies

$$\tilde{\nu}^0(c\cdot) = c^{-1}\tilde{\nu}^0(\cdot), \qquad c > 0. \tag{homog}$$

(homog) allows disintegration of $\tilde{\nu}^0(\cdot)$ as a product measure in the correct coordinate system and permits definition of a spectral measure.

3. Suppose $X_1 \overset{d}{\neq} X_2$ but $\boldsymbol{X}$ satisfies (DOA). Standardize:

$$U_i(x) = \frac{1}{P[X_i > x]}, \quad i = 1, 2$$

and set

$$\boldsymbol{X}^* = (X_1^*, X_2^*) = (U_1(X_1), U_2(X_2)).$$

Since $\boldsymbol{X}$ satisfies (DOA), $\boldsymbol{X}^*$ satisfies (StandRegVarE). Assume also (AsyIndep). Can now apply the HDA definition to $\boldsymbol{X}^*$:

- Ask if there exist positive scaling and real centering constants $\{c_n\}$ and $\{d_n\}$ and a non-zero measure ν_0^* on a cone E_0 such that (ConvE0) holds with $\boldsymbol{X}^*$ replacing $\boldsymbol{X}$.
- If so, set

$$U^{*\wedge}(x) = \frac{1}{P[X_1^* \wedge X_2^* > x]},$$

and then set

$$\boldsymbol{X}^{**} = (U^{*\wedge}(X_1^*), U^{*\wedge}(X_2^*) = \big(U^{*\wedge} \circ U_1(X_1), U^{*\wedge} \circ U_2(X_2)\big)$$

- Conclude:
  - The distribution of $\boldsymbol{X}^*$ is standard regularly varying on $[\boldsymbol{0}, \boldsymbol{\infty}] \setminus \{\boldsymbol{0}\}$ and (AsyIndep) holds.
  - $\boldsymbol{X}^{**}$ has a distribution standard regularly varying on $(\boldsymbol{0}, \boldsymbol{\infty}]$.
  - Ingenuity may be required to do estimation.

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