Maxima of bivariate triangular arrays: Between asymptotic complete dependence and asymptotic independence lies the conditional extreme value model

Joyjit Roy
In collaboration with Sidney I. Resnick

School of Operations Research and Information Engineering
Cornell University

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Outline

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Some basic facts

- If $\{(X_{in}, Y_{in}, i \leq n, n \geq 1\}$ is a triangular array of bivariate r.v.'s where each row is iid with distribution identical to $(X^{(n)}, Y^{(n)})$, then

$$\left(\frac{\bigvee_{i=1}^{n} X_{in} - b(n)}{a(n)}, \frac{\bigvee_{i=1}^{n} Y_{in} - b(n)}{a(n)} \right) \xrightarrow{d} F$$

$$\iff \quad \mathbf{P}^n \left[\left(\frac{X^{(n)} - b(n)}{a(n)}, \frac{Y^{(n)} - b(n)}{a(n)} \right) \leq \cdot \right] \xrightarrow{w} F(\cdot).$$

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- [Sibuya, 1960] If $(X, Y) \sim N\left(\left(\begin{array}{c} 0 \\ 0 \end{array} \right), \left(\begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right) \right)$ and $b(n)$ is defined by $b(n) = n\phi(b(n))$, then

$$\text{for } 0 \leq \rho < 1, \lim_{n\to\infty} \mathbf{P}^n[b(n)(X - b(n)) \leq x, b(n)(Y - b(n)) \leq y] = \exp(-e^{-x} - e^{-y})$$

$$\text{for } \rho = 1, \lim_{n\to\infty} \mathbf{P}^n[b(n)(X - b(n)) \leq x, b(n)(Y - b(n)) \leq y] = \exp(-e^{-\min\{x,y\}}).$$

Hüsler-Reiss' celebrated result

- [Hüsler-Reiss, 1989] If $\{(X^{(n)}, Y^{(n)}), n \geq 1\}$ is a sequence of bivariate normal random variables with

$$(X^{(n)}, Y^{(n)}) \sim N\left(\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left(\begin{array}{cc} 1 & \rho(n) \\ \rho(n) & 1 \end{array}\right)\right)$$

and $0 < \rho(n) < 1, (1-\rho(n)) \log n \to \lambda^2 \in (0, \infty)$ as $n \to \infty$, then

$$\lim_{n\to\infty} \mathbf{P}^n[X^{(n)} \leq b(n) + \frac{x}{b(n)}, Y^{(n)} \leq b(n) + \frac{y}{b(n)}] = H_\lambda(x, y)$$

where

$$H_\lambda(x, y) = \exp\left[-\Phi(\lambda + \frac{x-y}{2\lambda})e^{-y} - \Phi(\lambda + \frac{y-x}{2\lambda})e^{-x}\right].$$

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where
$$H_\lambda(x, y) = \exp\left[-\Phi(\lambda + \frac{x-y}{2\lambda})e^{-y} - \Phi(\lambda + \frac{y-x}{2\lambda})e^{-x}\right].$$
- H_λ is max-stable and
$$\lim_{\lambda\to 0} H_\lambda(x, y) = \exp(-e^{-\min\{x,y\}}) \quad \text{and} \quad \lim_{\lambda\to\infty} H_\lambda(x, y) = \exp(-e^{-x} - e^{-y}).$$

A further motivating example

- If $\{(X^{(n)}, Y^{(n)}), n \geq 1\}$ is a sequence of random variables such that

$$(X^{(n)}, Y^{(n)}) \stackrel{d}{=} (\min\left(\frac{X}{1-p(n)}, \frac{Y}{p(n)}\right), Y) \text{ where } (X, Y) \text{ are iid } Exp(1)$$

and $0 < p(n) < 1$ with $(1 - p(n)) \log n \to c \in \mathbb{R}$ as $n \to \infty$, then

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{P}^n[X^{(n)} \leq x + \log n, Y^{(n)} \leq y + \log n] \\
&= \begin{cases} \exp(-e^{-y}) & \text{if} \quad x \geq y + c \\ \exp(-e^{-x} - e^{-y}(1 - e^{-c})) & \text{if} \quad x < y + c \end{cases} \\
&=: G_c(x, y)
\end{aligned}$$

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- G_c is max-stable, continuous and

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Some points to note about Hüsler-Reiss' result

- If (X, Y) are iid $N(0, 1)$, then

$$(X^{(n)}, Y^{(n)}) = (\rho(n)Y + \sqrt{1 - \rho(n)^2}X, Y) := (f_n(X, Y), Y).$$

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- The crux of the result lies in proving the following
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$$\lim_{n\to\infty} n\mathbf{P}[X^{(n)} > b(n) + \frac{x}{b(n)}] = e^{-x}$$
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$$\lim_{n\to\infty} n\mathbf{P}[Y^{(n)} > b(n) + \frac{y}{b(n)}] = e^{-y}$$
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$$\lim_{n\to\infty} n\mathbf{P}[X^{(n)} > b(n) + \frac{x}{b(n)}, Y^{(n)} > b(n) + \frac{y}{b(n)}] = \int_y^\infty \bar{\Phi}(\lambda + \frac{x-z}{2\lambda})e^{-z}\,dz$$

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- Interestingly

$$\int_{-\infty}^{\infty} \bar{\Phi}(\lambda + \frac{x-z}{2\lambda})e^{-z}\,dz = e^{-x}.$$

Delving further into the proof of the Hüsler-Reiss example

- Note that

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&n\mathbf{P}[X^{(n)} > b(n) + \frac{x}{b(n)}, Y^{(n)} > b(n) + \frac{y}{b(n)}] \\
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&\quad = n\mathbf{P}\left[(X, b(n)(Y - b(n))) \in \left\{(w,z)\middle| w > \frac{b(n)(1-\rho(n))}{\sqrt{(1-\rho(n)^2)}} + \frac{x - \rho(n)z}{b(n)\sqrt{(1-\rho(n)^2)}}, z > y\right\}\right]
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- **(CEV)** Using the fact that (X, Y) are independent standard normal random variables,we have that

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\lim_{n\to\infty} n\mathbf{P}\left[X \leq x, b(n)(Y - b_n) > y\right] = \Phi(x)\exp(-y).
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- **(FC)** Also, the assumptions on $\rho(n)$ and the fact that $b(n) \sim \sqrt{2\log n}$, implies that as $n \to \infty$,

$$\frac{b(n)(1-\rho(n))}{\sqrt{(1-\rho(n)^2)}} + \frac{x-\rho(n)z}{b(n)\sqrt{(1-\rho(n)^2)}} \to \lambda + \frac{x-z}{2\lambda}.$$

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- Note that

$$nP[X^{(n)} > b(n) + \frac{x}{b(n)}, Y^{(n)} > b(n) + \frac{y}{b(n)}]$$

$$= nP[b(n)(\rho(n)X + \sqrt{1-\rho(n)^2}Y - b(n)) > x, b(n)(Y - b(n)) > y]$$

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- This leads to

$$\lim_{n\to\infty} nP[X^{(n)} > b(n) + \frac{x}{b(n)}, Y^{(n)} > b(n) + \frac{y}{b(n)}] = \int_y^\infty \bar{\Phi}(\lambda + \frac{x-z}{2\lambda})e^{-z}\,dz.$$

Our setup and the questions I will try to answer

- If $(X^{(n)}, Y^{(n)}) = (f_n(X,Y), Y)$, where $Y \in \mathcal{D}(G_\gamma)$ with scaling and centering functions $a(\cdot) > 0, b(\cdot)$, then how to formulate conditions similar to **(CEV)** and **(FC)** such that

$$n\mathbf{P}\left[\frac{f_n(X,Y) - b(n)}{a(n)} > x \frac{Y - b(n)}{a(n)} > y\right]$$

converges to a limit as $n \to \infty$ and to get a grasp on the behavior of the limit.

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- Formulate a condition similar to $f_n(X,Y) \to Y$ such that we can get convergence of

$$n\mathbf{P}\left[\frac{f_n(X,Y)-b(n)}{a(n)} > x\right]$$

- We will see that this is enough to generate classes of examples for Hüsler-Reiss type results.

Theorem 1

- **(CEV)** For the bivariate random vector (X, Y), assume that there exists scaling and centering functions $\alpha(\cdot) > 0, a(\cdot) > 0, \beta(\cdot)$ and $b(\cdot)$ and a Radon measure ξ on $[-\infty, \infty] \times (l^{\gamma}, u^{\gamma}]$, such that in $\mathbb{M}_{+}([-\infty, \infty] \times (l^{\gamma}, u^{\gamma}])$

$$n\mathbf{P}\left[\left(\frac{X-\beta(n)}{\alpha(n)}, \frac{Y-b(n)}{a(n)}\right) \in \cdot\right] \xrightarrow{v} \xi(\cdot)$$

where $\xi([-\infty, \infty] \times (y, u^{\gamma}]) = -\log(G_{\gamma}(y))$, for $y \in (l^{\gamma}, u^{\gamma})$ and ξ satisfies the non-degeneracy conditions for the conditional extreme value model.

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  where $\xi([-\infty, \infty] \times (y, u^\gamma]) = -\log(G_\gamma(y))$, for $y \in (l^\gamma, u^\gamma)$ and ξ satisfies the non-degeneracy conditions for the conditional extreme value model.
- **(FC)** Consider the sequence of functions $\{f_n : \mathbb{R}^2 \mapsto \mathbb{R}, n = 1, 2, \ldots\}$ where each f_n is nondecreasing in both variables. Suppose there exists for every $y \in (l^\gamma, u^\gamma)$, a nondecreasing function $f(\cdot, y)$ so that as $n \to \infty$,

$$\frac{f_n(\beta(n) + \alpha(n)u, b(n) + a(n)y) - b(n)}{a(n)} \to f(u, y)$$

  for every continuity point $u \in \mathbb{R}$ of the function $f(\cdot, y)$.

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- **(CC)** There exists a non-null Radon measure ν on $[-\infty, \infty] \times (l^\gamma, u^\gamma]$ such that in $\mathbb{M}_+([-\infty, \infty] \times (l^\gamma, u^\gamma])$

$$nP\left[\left(\frac{f_n(X, Y) - b(n)}{a(n)}, \frac{Y-b(n)}{a(n)}\right) \in \cdot\right] \xrightarrow{v} \nu(\cdot)$$

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- **(CC)** There exists a non-null Radon measure ν on $[-\infty, \infty] \times (l^\gamma, u^\gamma]$ such that in $\mathbb{M}_+([-\infty, \infty] \times (l^\gamma, u^\gamma])$

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  where $\nu([-\infty, \infty] \times (y, u^\gamma]) = -\log(G_\gamma(y))$, for $y \in (l^\gamma, u^\gamma)$.
- If **(CEV)** and **(FC)** hold, then **(CC)** also holds. In fact we have

$$\nu((x, \infty] \times (y, u^\gamma]) = \xi(\{(w, z) \mid w > f^{\leftarrow}_{(\cdot, z)}(x), u^\gamma \geq z > y\}).$$

Theorem 1 contd.

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- **(AN & I)** For some sequence $y_n \to l^\gamma$ and for all $x > l$,

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and

$$\nu((l^\gamma,\infty] \times (l,u^\gamma]) = \xi(\{(w,z) \mid w > f^{\leftarrow}_{(\cdot,z)}(x), u^\gamma \geq z > l^\gamma\}) < \infty.$$

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- **(MC)** There exists a Radon measure μ on $(l^\gamma, \infty]$ such that in $\mathbb{M}_+((l,\infty]])$

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- If **(CEV)**, **(FC)** and **(AN & I)** all hold then **(MC)** holds. In fact we have

$$\mu((x,\infty]) = \xi(\{(w,z) \mid w > f^{\leftarrow}_{(\cdot,z)}(x), u^\gamma \geq z > l^\gamma\}).$$

Applications to the case of spherically symmetric r.v.'s (Gumbel)

Suppose $R > 0$ has distribution function F such that $1 - F$ is Γ-varying with auxiliary function f and (X, Y) are defined as

$$(X, Y) \stackrel{d}{=} (R \cos \theta, R \sin \theta)$$

where θ is uniformly distributed in $(-\pi, \pi)$.

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- [Berman,1992] X, Y have identical distribution function G which is also Γ-varying with auxiliary function f and

$$n\mathbf{P}\left[\left(\frac{X}{\sqrt{b(n)a(n)}}, \frac{Y - b(n)}{a(n)}\right) \in \cdot\right] \stackrel{v}{\to} \xi(\cdot) \text{ in } \mathbb{M}_+\Big([-\infty,\infty] \times (-\infty,\infty]\Big),$$

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  where $\xi\Big([-\infty, x] \times (y, \infty]\Big) = \Phi(x)e^{-y}$.

- [Hashorva, 2005] If $b(n) = G^{\leftarrow}(1 - \frac{1}{n})$, $a(n) = f(b(n))$ and

$$(1 - \rho(n))\frac{b(n)}{a(n)} \to 2\lambda^2 \text{ as } n \to \infty,$$

  then the sequence of independent random variables $\{(X^{(n)}, Y^{(n)}), n \geq 1\}$ with

$$(X^{(n)}, Y^{(n)}) = (\rho(n)Y + \sqrt{1 - \rho(n)^2}X, Y)$$

$$\lim_{n\to\infty} \mathbf{P}^n[X^{(n)} \leq b(n) + a(n)x, Y^{(n)} \leq b(n) + a(n)y] = H_\lambda(x, y).$$

Applications to the case of spherically symmetric r.v.'s (Reversed Weibull)

Suppose $R > 0$ has distribution function F with right endpoint 1 such that $F \in \mathcal{D}(\Psi_{\alpha^*})$ for some $\alpha^* > 0$ and (X, Y) are defined as

$$(X, Y) \stackrel{d}{=} (R\cos\theta, R\sin\theta)$$

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- [Berman,1992] X, Y have identical distribution function G are identically distributed and the identical distribution function $G \in \mathcal{D}(\Psi_{\alpha*+\frac{1}{2}})$ and G has right endpoint 1 and

$$n\mathbf{P}\left[\left(\frac{X}{\sqrt{a(n)}}, \frac{Y-1}{a(n)}\right) \in \cdot\right] \stackrel{v}{\to} \xi^*(\cdot) \text{ in } \mathbb{M}_+\Big([-\infty,\infty] \times (-\infty, 0]\Big),$$

where $\xi^*\left([-\infty, x] \times (y, 0]\right) = \psi_{\alpha^*}\left(\frac{x}{\sqrt{2(-y)}}\right)(-y)^{\alpha*+\frac{1}{2}}$ and

$$\psi_{\alpha^*}(z) = \begin{cases} 0, & \text{if} \quad z < -1, \\ \frac{\Gamma(\alpha^*+3/2)}{\Gamma(\alpha^*+1)\sqrt{\pi}} \int\limits_{-1}^{z} (1-s^2)^{\alpha^*}\, ds, & \text{if} \quad z \in [-1, 1], \\ 1, & \text{if} \quad z > 1. \end{cases}$$

Applications to the case of spherically symmetric r.v.'s (Reversed Weibull)

- If $a(n) = 1 - G^{\leftarrow}(1 - \frac{1}{n})$ and

$$\frac{(1-\rho(n))}{a(n)} \to 2\lambda^2 \text{ as } n \to \infty,$$

then the sequence of independent random variables $\{(X^{(n)}, Y^{(n)}), n = 1, 2, \dots\}$ with .

$$(X^{(n)}, Y^{(n)}) \stackrel{d}{=} (\rho(n)Y + \sqrt{1-\rho(n)^2}X, Y).$$

has the property that for $(x, y) \in (-\infty, 0)^2$,

$$\lim_{n\to\infty} \mathbf{P}^n[X^{(n)} \le 1 + a(n)x, Y^{(n)} \le 1 + a(n)y] = H_{\alpha^*+1/2,\lambda}(x, y).$$

where

$$H_{\alpha,\lambda}(x,y) = \exp\left[-(-x)^{\alpha}\psi_{\alpha-\frac{1}{2}}\left(\frac{1}{\sqrt{2(-x)}}\left(\lambda + \frac{y-x}{2\lambda}\right)\right)\right.$$
$$\left. -(-y)^{\alpha}\psi_{\alpha-\frac{1}{2}}\left(\frac{1}{\sqrt{2(-y)}}\left(\lambda + \frac{x-y}{2\lambda}\right)\right)\right]$$

A more general example

Assume that

- In $\mathbb{M}_+([-\infty,\infty]\times(-\infty,\infty])$

$$nP\left[\left(\frac{X}{\alpha(n)},\frac{Y-b(n)}{a(n)}\right)\in\cdot\right]\xrightarrow{v}\xi(\cdot)$$

  where $\xi([-\infty,\infty]\times(y,\infty])=e^{-y}$, for $y\in\mathbb{R}$ and ξ satisfies the non-degeneracy conditions for the conditional extreme value model.

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Define $f_n(X,Y)=c(n)X+d(n)Y$ where $0<c(n),d(n)<1$ and $c(n)\frac{\alpha(n)}{a(n)}\to\tau>0$ and $\frac{b(n)}{a(n)}(1-d(n))\to\tau^2$.

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$$\mathbf{P}^n\left[\frac{f_n(X,Y)-b(n)}{a(n)}\leq x,\frac{Y-b(n)}{a(n)}\leq y\right]$$
$$\to\exp(-e^{-y}-\xi(\{(w,z)\mid w>\tau+\frac{x-z}{\tau},y\geq z\})$$

A simple problem

- If for each n, $\{(X_i^{(n)}, Y_i^{(n)}),\ i = 1, 2, \ldots, n\}$ are iiid $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho(n) \\ \rho(n) & 1 \end{pmatrix}\right)$ where $(1 - \rho(n)) \log n \to \lambda^2$ as $n \to \infty$, then

$$\left(b(n)(\bigvee_{i=1}^{n} X_i^{(n)} - b(n)), b(n)(\bigvee_{i=1}^{n} Y_i^{(n)} - b(n))\right) \xrightarrow{d} H_\lambda.$$

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- Consider an iid. sample $\{(X_i, Y_i),\ i = 1, 2, \dots, m\}$ where each (X_i, Y_i) has distribution $N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$ with $\rho > 0$.

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- Using classical extreme value theory there are two possible limit distributions for the scaled and centered maxima of $\{(X_i, Y_i),\ i = 1, 2, \dots, m\}$. Depending on whether $\rho = 1$ or $0 < \rho < 1$, we must have the limit distribution as $F_0(x, y) = \exp(-e^{-\min\{x,y\}})$ (complete asymptotic dependence) or $F_\infty(x, y) = \exp(-e^{-x} - e^{-y})$ (complete asymptotic independence) respectively. Either model would not give satisfactory estimates for values of ρ significantly greater than 0 but not close to 1.

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- We propose using H_{λ_m} as an approximation in these cases where $\lambda_m = \sqrt{(1-\rho)\log m}$.

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Table : Actual value and estimates of $m\mathbf{P}[X > x, Y > y]$ where $x = R\cos\theta, y = R\sin\theta$ and $(X, Y) \sim BVN(0, 0, 1, 1, \rho)$, $m = 1000$. A value of 0 in the table indicates that the actual value is less than 1×10^{-30}.

		$\theta = \pi/10$		
		Actual	HR estimate	CD estimate
$\rho = 0.7$	$R = 7$	1.39e-008	1.61e-005	1.61e-005
	$R = 8$	1.39e-011	8.32e-007	8.32e-007
	$R = 9$	0.00e+000	4.30e-008	4.30e-008
$\rho = 0.8$	$R = 7$	1.39e-008	1.61e-005	1.61e-005
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	$R = 9$	0.00e+000	4.30e-008	4.30e-008
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		$\theta = \pi/4$		
		Actual	HR estimate	CD estimate
$\rho = 0.7$	$R = 7$	1.17e-005	4.95e-004	3.30e-003
	$R = 8$	1.13e-007	5.47e-005	3.64e-004
	$R = 9$	6.19e-010	6.04e-006	4.02e-005
$\rho = 0.8$	$R = 7$	3.25e-005	7.92e-004	3.30e-003
	$R = 8$	4.05e-007	8.74e-005	3.64e-004
	$R = 9$	2.96e-009	9.65e-006	4.02e-005
$\rho = 0.9$	$R = 7$	8.88e-005	1.34e-003	3.30e-003
	$R = 8$	1.40e-006	1.48e-004	3.64e-004
	$R = 9$	1.33e-008	1.63e-005	4.02e-005