Functional Central Limit Theorem for Heavy Tailed Stationary Infinitely Divisible Processes Generated by Conservative Flows

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Let $\mathbf{X} = (X_1, X_2, \ldots)$ be a stationary stochastic process.

A (functional) central limit theorem for such a process is a statement of the type

$$\left(\frac{1}{c_n} \sum_{k=1}^{\lceil nt \rceil} X_k - h_n t,\ 0 \le t \le 1 \right) \Rightarrow \Big(Y(t),\ 0 \le t \le 1 \Big).$$

- $\mathbf{Y} = \Big(Y(t),\ 0 \le t \le 1 \Big)$ is a non-degenerate process.
- By the Lamperti theorem, $\mathbf{Y}$ is self-similar with stationary increments.

We consider a class of stationary symmetric infinitely divisible processes withn regularly varying tails.

The main ergodic-theoretical property will be that of pointwise dual ergodicity.

The length of memory will be determined by the rate of growth of wandering rate sequence.

It will have one parameter, $0 < 1 - \beta < 1$, that will determine the limiting process $\mathbf{Y}$.

The limiting process

Let $0 < \beta < 1$. We start with inverse process

$$M_\beta(t) = S_\beta^{\leftarrow}(t) = \inf\{u \geq 0: \ S_\beta(u) \geq t\}, \ t \geq 0 .$$

- $\big(S_\beta(t), t \geq 0\big)$ is a (strictly) β-stable subordinator.
- $\big(M_\beta(t), \ t \geq 0\big)$ is called *the Mittag-Leffler process*.

The Mittag-Leffler process has a continuous and non-decreasing version.

- It is self-similar with exponent β.
- Its increments are neither stationary nor independent.
- All of its moments are finite.

$$E \exp\{\theta M_\beta(t)\} = \sum_{n=0}^{\infty} \frac{(\theta t^\beta)^n}{\Gamma(1+n\beta)}, \quad \theta \in \mathbb{R}.$$

Define

$$Y_{\alpha,\beta}(t) = \int_{\Omega' \times [0,\infty)} M_\beta\big((t-x)_+, \omega'\big)\, dZ_{\alpha,\beta}(\omega', x), \quad t \geq 0.$$

- $Z_{\alpha,\beta}$ is a SαS random measure on $\Omega' \times [0,\infty)$ with control measure $P' \times \nu$.
- ν a measure on $[0,\infty)$ given by $\nu(dx) = (1-\beta)x^{-\beta}\, dx$.
- M_β is a Mittag-Leffler process defined on $(\Omega', \mathcal{F}', P')$.

The process $\big(Y_{\alpha,\beta}(t),\ t \geq 0\big)$ is a well defined SαS process with stationary increments.

It is self-similar with exponent of self-similarity

$$H = \beta + (1-\beta)/\alpha\,.$$

We call it the β-Mittag-Leffler (or β-ML) fractional SαS motion.

A connection: $\hat{\beta}$-stable local time fractional SαS motion.

Let $\hat{\beta} = (1-\beta)^{-1} \in (1,\infty)$.

If $\hat{\beta} \in (1,2)$, a $\hat{\beta}$-stable local time fractional SαS motion was introduced in Dombry and Guillotin-Plantard (2009).

$$\hat{Y}_{\alpha,\beta}(t) = \int_{\Omega' \times \mathbb{R}} L_t(x,\omega')\, d\hat{Z}_\alpha(\omega',x), \quad t \geq 0;$$

- $\hat{Z}_\alpha$ is a SαS random measure on $\Omega' \times \mathbb{R}$ with control measure $P' \times \text{Leb}$
- $(L_t(x),\ t \geq 0,\ x \in \mathbb{R})$ is a jointly continuous local time process of a symmetric $\hat{\beta}$-stable Lévy process.

In this range, the ML fractional SαS motion coincides, distributionaly, with the $\hat{\beta}$-stable local time fractional SαS motion.

One can view the ML fractional SαS motion as

an extension of the $\hat{\beta}$-stable local time fractional SαS motion from the range $1 < \hat{\beta} \leq 2$ to the range $1 < \hat{\beta} < \infty$.

A bit of ergodic theory

Let $(E, \mathcal{E}, \mu)$ be a σ-finite, **infinite** measure space.

Let $T : E \to E$ be a measurable map that preserves the measure μ.

When the entire sequence $T, T^2, T^3, \ldots$ of iterates of T is involved, we will sometimes refer to it as a flow.

The dual operator $\widehat{T}$ is an operator $L^1(\mu) \to L^1(\mu)$ defined by

$$\widehat{T} f = \frac{d(\nu_f \circ T^{-1})}{d\mu},$$

with ν_f a signed measure on $\big(E, \mathcal{E}\big)$ given by $\nu_f(A) = \int_A f \, d\mu$, $A \in \mathcal{E}$.

The dual operator satisfies the relation

$$\int_E \widehat{T} f \cdot g \, d\mu = \int_E f \cdot g \circ T \, d\mu$$

for $f \in L^1(\mu)$, $g \in L^\infty(\mu)$.

An ergodic conservative measure preserving map T is called pointwise dual ergodic if there is a sequence of positive constants $a_n \to \infty$ such that

$$\frac{1}{a_n}\sum_{k=1}^{n} \widehat{T}^k f \to \int_E f \, d\mu \quad \text{a.e.}$$

for every $f \in L^1(\mu)$.

Pointwise dual ergodicity rules out invertibility of the map T.

The stationary process X

We consider infinitely divisible processes of the form

$$X_n = \int_E f_n(x) dM(x), \quad n = 1, 2, \ldots .$$

- M is a homogeneous symmetric infinitely divisible random measure on a $(E, \mathcal{E})$.
- μ has an infinite, σ-finite, control measure μ and local Lévy measure ρ: for every $A \in \mathcal{E}$ with $\mu(A) < \infty$, $u \in \mathbb{R}$,

$$Ee^{iuM(A)} = \exp\left\{-\mu(A) \int_{\mathbb{R}} (1 - \cos(ux))\, \rho(dx)\right\} .$$

The functions f_n, $n = 1, 2, \ldots$ are deterministic functions of the form

$$f_n(x) = f \circ T^n(x) = f\big(T^n x\big),\ x \in E,\ n = 1, 2, \ldots :$$

- $f : E \to \mathbb{R}$ is a measurable function, satisfying certain integrability assumptions;
- $T : E \to E$ a pointwise dual ergodic map.

We assume that the local Lévy measure ρ has a regularly varying tail with index $-\alpha$, $0 < \alpha < 2$:

$$\rho(\cdot, \infty) \in RV_{-\alpha} \text{ at infinity.}$$

With a proper integrability assumption on the function f:

the process X has regularly varying finite-dimensional distributions, with the same tail exponent $-\alpha$.

Theorem

Assume that the normalizing sequence (a_n) in the pointwise dual ergodicity is regularly varying with exponent $0 < \beta < 1$ and that $\mu(f) = \int f \, d\mu \neq 0$. Then for some sequence (c_n) that is regularly varying with exponent $\beta + (1-\beta)/\alpha$,

$$\frac{1}{c_n} \sum_{k=1}^{\lfloor n \cdot \rfloor} X_k \Rightarrow |\mu(f)| Y_{\alpha,\beta} \quad \text{in } D[0,\infty) \, .$$

Example Consider an irreducible null recurrent Markov chain with state space $\mathbb{Z}$ and transition matrix $P = (p_{ij})$.

Let $\{\pi_j, j \in \mathbb{Z}\}$ be the unique invariant measure of the Markov chain that satisfies $\pi_0 = 1$.

Define a σ-finite measure on $(E, \mathcal{E}) = \big(\mathbb{Z}^{\mathbb{N}}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}})\big)$ by

$$\mu(\cdot) = \sum_{i \in \mathbb{Z}} \pi_i P_i(\cdot)\,,$$

Let $T: \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^{\mathbb{N}}$ be the left shift map

$$T(x_0, x_1, \dots) = (x_1, x_2, \dots) \ \text{ for } \{x_k, k = 0, 1, \dots\} \in \mathbb{Z}^{\mathbb{N}}.$$

Let $A = \left\{x \in \mathbb{Z}^{\mathbb{N}} : x_0 = 0\right\}$ and the corresponding first entrance time $\varphi(x) = \min\{n \geq 1 : \ x_n = 0\}$, $x \in \mathbb{Z}^{\mathbb{N}}$.

Assume that

$$P_0(\varphi \geq k) \in RV_{-\beta} \,.$$

The assumptions of the theorem hold, for example, if $f = \mathbf{1}_A$.

The length of memory in the process $\mathbf{X}$ is quantified by the tail of the first return time φ.

In the general case, the length of memory is still quantified by a single parameter β. It is also related to return times.