# Minimal spectral representations of infinitely divisible and max–infinitely divisible processes

Stilian Stoev (sstoev@umich.edu)
University of Michigan, Ann Arbor

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Joint work with Zakhar Kabluchko, Ulm University, Germany

# Spectral Representations

# Max-id processes

**Definition**

$\{X_t, t \in T\}$ is max-id if for all $n$,

$$\{X_t, t \in T\} \stackrel{d}{=} \Big\{ \max_{i=1,\cdots,n} X_t^{(i,n)},\ t \in T \Big\},$$

for some iid $\{X_t^{(i,n)}, t \in T\},\ i = 1, \cdots, n$.

- Ch. 5, Resnick (1987): Max-id laws in $\mathbb{R}^n$ have the form:

  $$F(x) = \exp\{-\mu(-\infty, x]^c\},\ x \in \mathbb{R}^n,$$

  for some $\sigma-$finite measure $\mu$ on $\mathbb{R}^n$ – Balkema and Resnick [1977], Gerritse [1986] and Vatan [1985].
- Balkema et al. [1993]: spectral representations for max-id processes.

**Note:** WLOG we will suppose

$$\mathrm{essinf}(X_t) = 0, \quad \text{for all } t \in T.$$

# The Poisson calculus for max-id processes

- Let $(E, \mathcal{E}, \mu)$ be a $\sigma-$finite measure space.
- Let $\Pi_\mu = \{U_i,\ i \in \mathbb{N}\}$ be a Poisson random measure on $(E, \mathcal{E})$ with intensity $\mu$.

**Definition**

$\mathcal{L}^\vee(E, \mathcal{E}, \mu)$ is the set of non-negative measurable functions $f : E \to \mathbb{R}_+$, such that

$$\mu\{f > a\} < \infty, \quad \text{for all } a > 0.$$

Define

$$I^\vee(f) \equiv \int_E^\vee f d\Pi_\mu := \sup_{U \in \Pi_\mu} f(U).$$

**Note:** For $f \in \mathcal{L}^\vee(E, \mu)$, we have $\mathrm{essinf}(I^\vee(f)) = 0$ and

$$P(I^\vee(f) \le x) = P(\Pi_\mu \cap \{f > x\}) = \emptyset) = e^{-\mu\{f>x\}}, \quad (x > 0).$$

# Spectral representations

For $f_t \in \mathcal{L}^\vee(E, \mu)$, $t \in T$ it is easy to see that $X_t := \int_E^\vee f_t d\Pi_\mu,\ t \in T$ is a max-id process with fidi

$$P\{X_{t_i} \le x_i,\ i = 1, \cdots, k\} = \exp\Big\{ -\mu\Big( \cup_{i=1}^k \{f_{t_i} > x_i\}\Big)\Big\},\ (x_i \ge 0)$$

Conversely:

**Definition**

$\{X_t,\ t \in T\}$ satisfies Condition S, if extists a countable $T_0 \subset T$, s.t.

$$\forall t,\ X_t = \mathrm{plim} X_{t_n},\ \text{for some } t_n \in T_0.$$

**Theorem (Balkema et al. [1993] & Kabluchko and S. [2012])**

*If $\{X_t,\ t \in T\}$ is max-id satisfies Condition S and $\mathrm{essinf}(X_t) = 0$, then there exist $\{f_t,\ t \in T\} \subset \mathcal{L}^\vee(\mathbb{R}, \mathrm{Leb})$*

$$\{X_t,\ t \in T\} \stackrel{d}{=} \{I^\vee(f_t),\ t \in T\}.$$

**Note:** $\{f_t,\ t \in T\}$ above is called a spectral representation of $X$.

# Examples

- (mixed moving maxima)

$$X_t = \bigvee_i F_i(t - U_i), \ t \in \mathbb{R}^2.$$

where the PPP $\Pi_\mu = \{(U_i, F_i(\cdot))\}$ has intensity $\mu(du, dF) = duP(dF)$, where $P$ is the law of a random field $F = \{F(t)\}_{t \in \mathbb{R}^2}$.
Think of $U_i$ as storm locations and $F_i(\cdot)$ as storm profiles.

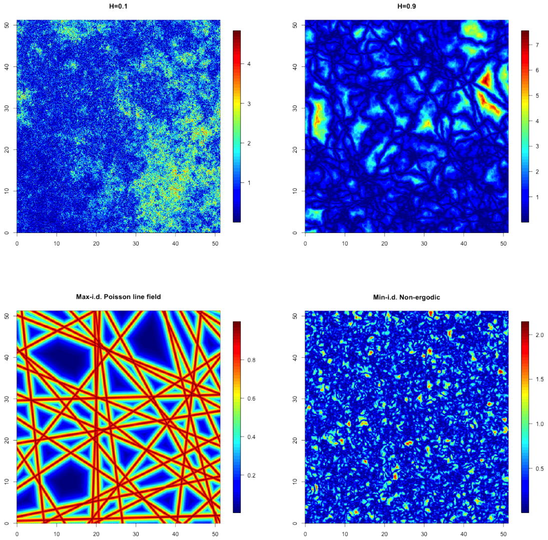
- (Penrose type processes) $\xi = \{\xi(t)\}_{t \in \mathbb{R}}$ process with stationary increments. Then,

$$X_t := \min_i |U_i + \xi_i(t)|, \quad t \in \mathbb{R},$$

is stationary min-id, where $(U_i, \xi_i(\cdot))$ is a PPP with intensity $dudP_\xi$.

- See Kabluchko and S. [2012] for more examples: max-stable processes, Poisson lines, $\cup$-id random sets.

# Pictures of max-id random fields



# Minimality

# A "new" minimality concept

**Definition**

The spec rep $\{f_t\}_{t\in T} \subset \mathcal{L}^{\vee}(E, \mathcal{E}, \mu)$ of $X$ is minimal if:
*(i)* $\sigma\{f_t, t \in T\} = \mathcal{E}$ (mod $\mu$)
*(ii)* $\mathrm{supp}\{f_t,\ t \in T\} = E$ (mod $\mu$) i.e. there is no $A \in \mathcal{E}$ with $\mu(A) > 0$ s.t. $f_t = 0$ on $A$.

**Theorem (Kabluchko and S. [2012])**

*Under Condition S the max-id process $X$ has a minimal spec rep on the space $\mathcal{L}^{\vee}(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$, for some $\sigma$-finite Borel measure $\mu$.*

**Notes:**

1. The proof is not innovative – book-keeping + use of prior results of Balkema et al. [1993], Vatan [1985], and Kuratowski's Thm.
2. The definition is the "right one" because of the following uniqueness result.

# Uniqueness

**Theorem (Kabluchko and S. [2012])**

*If $\{f_t\}_{t\in T} \subset \mathcal{L}^\vee(E,\mathcal{E},\mu)$ and $\{g_t\}_{t\in T} \subset \mathcal{L}^\vee(F,\mathcal{F},\nu)$ are two minimal reps of $X$, then exists a measure space isomorphism $\Phi : (E,\mu) \to (F,\nu)$, s.t.*

$$\forall t,\ \ f_t = g_t \circ \Phi, \quad mod\ \mu.$$

*Moreover, $\Phi$ is mod $\mu$ unique.*

**Notes:**

1. Similar results are well-known in the stable and max–stable cases under a somewhat different minimality concept. They imply important structural results through connections with non-singular flows and ergodic theory: Hardin Jr. [1982], Rosiński [1995], Samorodnitsky [2005], Roy and Samorodnitsky [2008], Roy [2010], de Haan and Pickands III [1986], Kabluchko [2009], Wang and Stoev [2010].
2. How to use this uniqueness result?

# Stationry max-id processes and measure-preserving flows

If $X$ is stationary,

$$\{f_t\}_{t\in\mathbb{R}} \quad \text{and} \quad \{g_t\}_{t\in\mathbb{R}} := \{f_{t-\tau}\}_{t\in\mathbb{R}},$$

are both minimal spec reps and then, for any $\tau \in \mathbb{R}$:

$$g_t \circ \Phi_\tau \equiv f_{t-\tau} \circ \Phi_\tau = f_t \pmod{\mu}$$

Uniqueness yields the flow property:

$$\Phi_{t+s} = \Phi_t \circ \Phi_s \pmod{\mu}$$

and using "standard" techniques (Mackey [1962]) one can get a measurable version of the flow $\{\phi_t\}_{t\in\mathbb{R}}$, defined everywhere.

**Notes:**

1. Now $\phi_t : E \to E$ is measure-preserving! Not just non-singular...
2. Hence the spec rep has the flow representation:

$$f_t = f_0 \circ \phi_t \quad (\bmod\ \mu)$$

# The precise statement

**Theorem (Kabluchko and S. [2012])**

*Let $X = \{X_t,\ t \in \mathbb{R}^d\}$ be continuous in probability, stationary max-id random field. Then*

$$X \stackrel{d}{=} \left\{ \int_E^{\vee} f_0 \circ \phi_t d\Pi_\mu \right\}_{t \in \mathbb{R}^d},$$

*where $f_0 \in \mathcal{L}^{\vee}(E, \mathcal{E}, \mu)$ and $\{\phi_t\}_{t \in \mathbb{R}^d}$ is a measurable, measure preserving action on a $\sigma$-finite Borel space $(E, \mathcal{E}, \mu)$.*

**Notes:**

1. Recall that for convenience, we are assuming throughout:

$$\operatorname{essinf}(X_t) = 0, \quad t \in \mathbb{R}^d.$$

   The result trivially extends to other max-id processes.
2. The spec rep $\{f_0 \circ \phi_t\}_{t \in \mathbb{R}^d}$ may me chosen to be minimal.

# Max-stable case

# Max-stable case

Let $E = (0, \infty) \times F$ and $\Pi_\mu = \{(\epsilon_i, V_i)\}_{i \in \mathbb{N}}$ be a PPP with intensity

$$\mu(dx, dv) = dx\nu(dv). \quad \text{Define} \quad f_t(x, v) := x^{-1} g_t(v),$$

where $g_t \in L^1_+(F, \nu)$. Then

$$X_t := \int_E f_t d\Pi_\mu \equiv \bigvee_{i \in \mathbb{N}} \frac{g_t(V_i)}{\epsilon_i}, \quad t \in T$$

is a max-stable process.

**Notes:**

1. $X_t$ is well-defined because $g_t \in L^1_+(\nu)$ implies $f_t \in \mathcal{L}^\vee(E, \mu)$. Indeed, for all $a > 0$

$$\mu\{f_t > a\} = \int_F \int_0^\infty \mathbb{I}(g_t(v) > ax) dx \nu(dv) = a^{-1} \int_F g_t(v) \nu(dv) < \infty.$$

2. Max-stability follows from thinning.

# Max-stable case (cont'd)

More precisely, the fidi of $X$ are:

$$P(X_{t_i} \le x_i,\ i = 1, \cdots, k) = \exp\{-\mu(\cup_i \{f_{t_i} > x_i\}\}$$
$$= \exp\left\{ - \int_F \left( \int_0^\infty \max_{i=1,\cdots,k} \mathbb{I}(g_{t_i}(v) > x_i x\} dx \right) \nu(dv) \right\}$$
$$= \exp\left\{ - \int_F ( \max_{i=1,\cdots,k} g_{t_i}/x_i ) d\nu \right\}.$$

# Non-singular flows in the max-stable case

Recall

**Fact**

*For a stationary max-stable 1-Fréchet process, we have*

$$X \stackrel{d}{=} \left\{ \int_F^{\vee} \left( g_0 \circ \varphi_t(v) \frac{d\nu \circ \varphi_t}{d\nu}(v) \right) M_1(dv) \right\}_{t \in \mathbb{R}}$$

*where $M_1$ is 1-Fréchet sup-measure on $(F, \mathcal{F})$ with control measure $\nu$, $g_0 \in L^1_+(\nu)$ and $\{\varphi_t\}$ is a non-singular flow on $F$.*

**Notes:**

1. The flow $\varphi_t : F \to F$ is non-singular if $\nu \circ \varphi_t \sim \nu$ and hence the above Radon-Nikodym derivative $d\nu \circ \varphi_t / d\nu$ makes sense.
2. Question: $X$ is max-id, so what is its measure-preserving flow representation in terms of a PPP?!

# From max-stable to max-id

- Let $g_t \in L^1_+(F, \nu)$ and $M_1$ be a 1-Fréchet sup-measure.
- Define the max-stable process

$$X_t = \int_F^{\vee} g_t(v) M_1(dv), \ \ t \in T.$$

- What is the PPP spec rep of $X = \{X_t\}_{t \in T}$ as a max-id process?

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Let $E := (0, \infty) \times F$ and $\Pi_\mu = \{(\epsilon_i, V_i)\}_{i \in \mathbb{N}}$ be a PPP with intensity $\mu(dx, dv) = dx\nu(dv)$

- Note that $M_1(B) := \bigvee_i \epsilon_i^{-1} \mathbb{I}_B(V_i)$ is an independently scattered 1−Fréchet sup-measure.
- Thus,

$$\int_{(0,\infty)\times F}^\vee x^{-1} g_t(v) \Pi_\mu(dx, dv) = \int_F^\vee g_t(v) M_1(dv).$$

- This gives the natural max-id spec rep of a max-stable process.

# Dorothy Maharam's construction

If $\varphi_t : F \to F$ is a non-singular flow on $(F, \nu)$, then

$$\phi_t(x, v) := \left(\frac{d\nu \circ \varphi_t}{d\nu}(v)^{-1}x, \varphi_t(v)\right)$$

is a measure-preserving flow on

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Now, as above, define $f_t(x, v) := x^{-1} g_t(v)$ and note that

$$f_t(x, v) = f_0 \circ \phi_t(x, v) = x^{-1} \frac{d\nu \circ \varphi_t}{d\nu}(v) g_t \circ \varphi_t(v)$$

Since $\bigvee_i \epsilon_i^{-1} \mathbb{I}_B(V_i) = M_1(B)$, recall that

$$\int_{(0,\infty)\times F}^{\vee} x^{-1} g_t(v) \Pi_\mu(dx, dv) = \int_F^{\vee} g_t(v) M_1(dv),$$

The max-id spec rep is generated by a measure-preserving flow!

## Old and new minimality

Let $X = \{X_t\}_{t\in T}$ be max-stable with spec rep

$$\{X_t\}_{t\in T} \stackrel{d}{=} \{\int_F^{\vee} g_t(v)M_1(dv)\}_{t\in T}, \quad (g_t \in L_+^1(F, \mathcal{F}\nu)).$$

Recall that $\{g_t\}_{t\in T}$ is minimal if:

1. (ratio $\sigma$-alg) $\rho\{g_t,\ t \in T\} := \sigma\{g_t/g_s,\ t, s \in T\} \sim \mathcal{F}$ (mod $\nu$).
2. (full support) $\mathrm{supp}\{g_t,\ t \in T\} = F$ (mod $\nu$).

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**Lemma (" Old" $\Rightarrow$ " New")**

*If $\{g_t\}_{t \in T} \subset L_+^1(F, \nu)$ is minimal then $\{f_t\}_{t \in T} \subset \mathcal{L}^{\vee}(E, \mu)$ is minimal.*

**Notes:**

1. "Old" $\Rightarrow$ "New" is great news! Because all "old" resuts on max-stable proc can be reproduced with the "new" tools.
2. Proof is a nice exercise.
3. **Open problem:** I don't know if "New" $\Rightarrow$ "Old".

## You may wonder...

What about the sum infinitely divisible case?

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- Define the class $\mathcal{L}^+(E,\mu) \ni f : \int_E 1 \wedge |f|^2 d\mu < \infty$.
- Let $\Pi_\mu = \{U_i,\ i \in \mathbb{N}\}$ be a PPP on $(E,\mu)$ and define

$$I^+(f) := \mathrm{plim}_{\epsilon\downarrow 0}\Big(\sum_{U\in\Pi} f(U)\mathbb{I}(|f(U)|>\epsilon) - \int_E f\mathbb{I}(\epsilon<|f|\le 1)d\mu\Big).$$

**Theorem (Kabluchko and S. [2012])**

*Under Condition S, any sum-id process X has a minimal spec rep*

$$\{X_t\}_{t\in T} \stackrel{d}{=} \{I^+(f_t) + c_t\}_{t\in T},$$

*for some constants $c_t$, over a Borel $\sigma$-finite $(E,\mu)$.*

**Notes:** In close paralel with the max-id case:

- minimal spec reps are unique.
- stationary processes corresond to measure-preserving flows.

# Main messages and contributions

- Provide PPP-based stochastic integal representations for both sum and max-id processes.
- Unifing and simple notion of a minimal spec rep was developed.
- Stationary sum and max-id processes can be associated with measure-preserving flows.
- Tools for classification and ergodic theory decompositions!
- Clarified/cleaned-up a bit the theory on spec rep's of continuous-time sum-id processes.

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# Shift-invariant $\cup$-id random sets

A random set $A$ is $\cup$-id if for all $n \in \mathbb{N}$

$$A \stackrel{d}{=} A_{1,n} \cup \cdots \cup A_{n,n},$$

for some iid $A_{i,n}$, $i = 1, \cdots, n$.
**Note:** $X_t := \mathbb{I}_A(t)$ is a max-id process.

### Theorem

*If $A$ is a shift-invariant $\cup$-id random set in $\mathbb{R}^d$ that is continuous in probability. Then exists a $\sigma$-finite Borel space $(E, \mu)$ with a PPP $\Pi_\mu$ and a measure-preserving action $\varphi_t : E \to E$, such that*

$$A \stackrel{d}{=} \{t \in R^d \ : \ \Pi_\mu \cap \varphi_t(A_0) \neq \emptyset\},$$

*for some non-random $A_0$ with $\mu(A_0) < \infty$.*