Martingale Problems

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Outline

1. Introduction
   - Definition
   - Well-posedness
2. Examples-Finite Dimensions
   - Brownian Motion
   - Poisson Process
   - Diffusions
   - Markov Jump Processes
3. Examples - Infinite Dimensions
   - Hilbert Space Valued Diffusion
   - Measure valued processes

Solution of Martingale Problem

Definition 1.1

An E- valued measurable process $(X_t)_{\{t\geq 0\}}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a solution of the martingale problem for (A, μ) with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$ if

1. $\mathcal{L}(X_0) = \mu$
2. *for every $f \in D(A)$*

$$M_t^f = f(X_t) - \int_0^t Af(X_s)ds$$

is a $(\mathcal{G}_t)_{t\geq 0}$ - martingale.

General Setup

- State space - E - a complete, separable metric space
  - $M(E)$ - real valued, measurable functions on E
  - $B(E)$ - real valued, bounded, measurable functions on E
  - $C(E)$ - real valued, continuous functions on E
  - $C_b(E)$ - real valued, bounded, continuous functions on E
- operator A on $M(E)$ with domain $D(A)$
  - $\mathcal{B}(E)$ - Borel σ-field on E
  - $\mathcal{P}(E)$ - space of probability measures on $(E, \mathcal{B}(E))$
- Initial measure $\mu \in \mathcal{P}(E)$
- For any process $(X_t)_{\{t\geq 0\}}$, $(\mathcal{F}_t^X)_{t\geq 0}$ will denote its natural filtration. i.e.

$$\mathcal{F}_t^X = \sigma(X_s : 0 \leq s \leq t)$$

In Definition 1.1 if $(\mathcal{G}_t)_{t\geq 0} = (\mathcal{F}_t^X)_{t\geq 0}$, the σ-fields are dropped from the statement

Well-Posedness

- Solution of a martingale problem is defined only in a weak sense

Definition 1.2

Uniqueness holds for the martingale problem for (A, μ) if any two solutions of the martingale problem have the same distributions

Definition 1.3

The martingale problem for (A, μ) is well-posed if

1. *there exists a solution X of the martingale problem for (A, μ)*
2. *Uniqueness of solution holds for the martingale problem*

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Example 1 - Brownian Motion

Let B be a Standard Brownian Motion.
Then
$M_t^1 = B_t$ and $M_t^2 = B_t^2 - t$ are martingales.
Let

$$E = \mathbb{R}, D(A) = \{f_1, f_2\}$$
$$f_1(x) = x, f_2(x) = x^2$$
$$Af_1(x) \equiv 0, Af_2(x) \equiv 1$$

Then for $i = 1, 2$

$$M_t^i = f_i(B_t) - \int_0^t Af_i(B_s)ds.$$

$\therefore (B_t)_{\{t \geq 0\}}$ is a solution of the martingale problem for (A, δ_0)

Example 1 - Brownian Motion (Contd.)

Conversely,
Let $(X_t)_{\{t\geq 0\}}$ be a continuous solution of the martingale problem for (A, δ_0)
We have X_t and $X_t^2 - t$ are martingales
i.e. X_t is a continuous martingale with $\langle X\rangle_t = t$.
Define $g_s(x) = e^{isx}$ where $i = \sqrt{-1}$. Note $|g_s(x)| \leq 1$.
Then $g_s'(x) = isg_s(x), g_s''(x) = -s^2 g_s(x)$
By Ito's formula

$$dg_s(X_t) = isg_s(X_t)dX_t - \frac{1}{2}s^2 g_s(X_t)dt.$$

The stochastic integral is a martingale M_t. Then for $0 \leq r < t$

$$e^{isX_t} = e^{isX_r} + M_t - M_r - \frac{1}{2}s^2 \int_r^t e^{isX_u}du$$

Example 1 - Brownian Motion (Contd.)

Let $A \in \mathcal{F}_r^X$. Multiplying by $e^{-isX_r}\mathbb{I}_A$

$$\mathbb{I}_A e^{is(X_t - X_r)} = \mathbb{I}_A + e^{-isX_r}\mathbb{I}_A(M_t - M_r) - \frac{1}{2}s^2 \int_r^t \mathbb{I}_A e^{is(X_u - X_r)} du$$

Taking expectations (of conditional expectations)

$$\mathbb{E}\left[\mathbb{I}_A e^{is(X_t - X_r)}\right] = \mathbb{P}(A) + 0 - \frac{1}{2}s^2 \int_r^t \mathbb{E}\left[\mathbb{I}_A e^{is(X_u - X_r)}\right] du$$

Let $h(t) = \mathbb{E}\left[\mathbb{I}_A e^{is(X_t - X_r)}\right]$. Then

$$h(t) = \mathbb{P}(A) - \frac{1}{2}s^2 \int_r^t h(u)du$$

$$h'(t) = -\frac{1}{2}s^2 h(t) \text{ with } h(r) = \mathbb{P}(A)$$

$$\mathbb{E}\left[\mathbb{I}_A e^{is(X_t - X_r)}\right] = h(t) = \mathbb{P}(A)e^{-\frac{1}{2}s^2(t-r)}.$$

Example 1 - Brownian Motion (Contd.)

Since this holds for all $A \in \mathcal{F}_r^X$, we get

$$\mathbb{E}\left[e^{is(X_t - X_r)} | \mathcal{F}_r^X\right] = e^{-\frac{1}{2}s^2(t-r)} \text{a.s.}$$

This implies

$$(X_t - X_r) \coprod \mathcal{F}_r^X \text{ independent increments}$$

$$(X_t - X_r) \sim N(0, t-r) \text{ Stationary, Gaussian}$$

Thus X_t is a Brownian motion.

This is Levy's Charachterization of Brownian Motion

- The martingale problem for (A, δ_0) is well-posed in the class of continuous processes

Levy's Charachterization Theorem

Levy's Charachterization Theorem

Let $(X_t)_{\{t\geq 0\}}$ *be a continuous* $\mathbb{R}^d$ *valued process, with* $(X_t = (X_t^{(1)}, \ldots, X_t^{(d)}))$*, such that for every* $1 \leq k, j \leq d$

1. $M_t^{(k)} = X_t^{(k)} - X_0^{(k)}$ *is a continuous local martingale*
2. $\langle M^{(k)}, M^{(j)} \rangle_t = \delta_{kj} t$ *i.e.* $M_t^{(k)} M_t^{(j)} - \delta_{kj} t$ *is a continuous local martingale*

Then $(X_t)_{\{t\geq 0\}}$ *is a d- dimensional Brownian Motion.*

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Then $(X_t)_{\{t\geq 0\}}$ is a d- dimensional Brownian Motion.

(local martingale (M_t):) $\exists$ a sequence of stop-times $\tau_n \uparrow \infty$ such that for every $n \geq 1$, the stopped process (M_t^n) defined by

$$M_t^n = M_{t\wedge\tau_n}$$

is a martingale.

Example 2 - Compensated Poisson Process

- Let $(N_t)_{\{t \geq 0\}}$ be a Poisson Process with intensity 1
- Define $\tilde{N}_t = N_t - t$, (compensated Poisson process)
- Using independent increment propoerty of N, it follows that $\tilde{N}_t$ and $\tilde{N}_t^2 - t$ are martingales
- $(\tilde{N}_t)_{\{t \geq 0\}}$ is also a solution of the martingale problem for (A, δ_0) of Example 1
- The martingale problem for (A, δ_0) is not well-posed though uniqueness holds in the class of continuous solutions.

Example 3 - Diffusions

Let $b(x) = (b_i(x))_{1 \le i \le d}, \sigma(x) = ((\sigma_{ij}(x)))_{1 \le i,j \le d}$ be measurable functions, and
$(W_t)_{\{t \ge 0\}} = (W_t^{(1)}, \ldots, W_t^{(d)})_{\{t \ge 0\}}$ be a d -dimensional Standard Brownian Motion.
Suppose that (the d - dimensional process) X is a solution of the Stochastic Differential Equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

$$dX_t^{(i)} = b_i(X_t)dt + \sum_{j=1}^{d} \sigma_{ij}(X_t)dW_t^{(j)} \quad 1 \le i \le d$$

$$X_t^{(i)} = X_0^{(i)} + \int_0^t b_i(X_s)ds + \sum_{j=1}^{d} \int_0^t \sigma_{ij}(X_s)dW_s^{(j)} \quad 1 \le i \le d$$

Example 3 - Diffusions (Contd.)

Then, by Ito's formula, for $f \in C_b^2(\mathbb{R}^d)$

$$df(X_t) = \sum_{i=1}^{d} \partial_i f(X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(X_t) d\langle X^{(i)}, X^{(j)} \rangle_t$$

Example 3 - Diffusions (Contd.)

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Example 3 - Diffusions (Contd.)

Then, by Ito's formula, for $f \in C_b^2(\mathbb{R}^d)$

$$
\begin{aligned}
df(X_t) &= \sum_{i=1}^{d} \partial_i f(X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(X_t) d\langle X^{(i)}, X^{(j)} \rangle_t \\
&= \sum_{i=1}^{d} \partial_i f(X_t) b_i(X_t) dt + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(X_t) (\sigma\sigma^T)_{ij}(X_t) dt \\
&\quad + \sum_{i=1}^{d} \partial_i f(X_t) \sigma_{ij}(X_t) dW_t^{(j)}.
\end{aligned}
$$

Example 3 - Diffusions (Contd.)

Then, by Ito's formula, for $f \in C_b^2(\mathbb{R}^d)$

$$\begin{aligned} df(X_t) &= \sum_{i=1}^{d} \partial_i f(X_t) dX_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(X_t) d\langle X^{(i)}, X^{(j)} \rangle_t \\ &= \sum_{i=1}^{d} \partial_i f(X_t) b_i(X_t) dt + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(X_t)(\sigma\sigma^T)_{ij}(X_t) dt \\ &\quad + \sum_{i=1}^{d} \partial_i f(X_t) \sigma_{ij}(X_t) dW_t^{(j)}. \end{aligned}$$

Let $Af(x) = \sum_{i=1}^{d} \partial_i f(x) b_i(x) + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{ij} f(x)(\sigma\sigma^T)_{ij}(x)$

Example 3 - Diffusions (Contd.)

Thus for all $f \in D(A) = C_b^2(\mathbb{R}^d)$

$$f(X_t) - \int_0^t Af(X_s)ds$$

is a martingale.

Or, X_t is a solution of the (A, μ) martingale problem where $\mu = \mathcal{L}(X_0)$.

Example 3 - Diffusions (Contd.)

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Or, X_t is a solution of the (A, μ) martingale problem where $\mu = \mathcal{L}(X_0)$.

Converse!!!

Stroock-Varadhan Theory of Martingale Problems

Example 3: Diffusions - Martingale Charachterization

- $E = \mathbb{R}^d$; $D(A) = C_b^2(\mathbb{R}^d)$
- $b(x) = (b_i(x))_{1\le i\le d}, \sigma(x) = ((\sigma_{ij}(x)))_{1\le i\le d, 1\le j\le d}$ be measurable functions
- $Af(x) = \sum_{i=1}^d \partial_i f(x) b_i(x) + \frac{1}{2}\sum_{i,j=1}^d \partial_{ij} f(x)(\sigma\sigma^T)_{ij}(x)$

Theorem 1

Let $(X_t)_{\{t\ge 0\}}$ (defined on some $(\Omega, \mathcal{F}, \mathbb{P})$) be a continuous $\mathbb{R}^d$ valued solution of the martingale problem for (A, μ). Then $\exists$ a d-dimensional Brownian motion $(W_t)_{\{t\ge 0\}}$, defined possibly on an extended probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that $(X_t)_{\{t\ge 0\}}$ solves the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad \mathcal{L}(X_0) = \mu. \tag{1}$$

Example 3: Diffusions - Martingale Charachterization

Proof. Let $a(x) = (\sigma\sigma^T)(x)$.
If $f_k(x) = x^k, g_{kl}(x) = x^k x^l \in D(A)$, then

$$M_t^k = X_t^k - \int_0^t b_k(X_s)ds \tag{2}$$

is a martingale, since $\partial_i f_k \equiv \delta_{ik}, \partial_{ij} f_k \equiv 0$.

Example 3: Diffusions - Martingale Charachterization

Proof. Let $a(x) = (\sigma\sigma^T)(x)$.
If $f_k(x) = x^k, g_{kl}(x) = x^k x^l \in D(A)$, then

$$M_t^k = X_t^k - \int_0^t b_k(X_s)ds \quad (2)$$

is a martingale, since $\partial_i f_k \equiv \delta_{ik}, \partial_{ij} f_k \equiv 0$.
Using the functions g_{kl} and their partial derivatives we can write

$$M_t^k M_t^l - \int_0^t a_{kl}(X_s)ds$$

as sum of martingales

$$g_{kl}(X_t) - \int_0^t Ag_{kl}(X_s)ds, \qquad \int_0^t Z_s^k dM_s^l, \qquad \int_0^t Z_s^l dM_s^k$$

where $Z_s^j = \int_0^s b_j(X_r)dr, j = k, l$.

Example 3: Diffusions - Martingale Charachterization

Thus

$$\langle M^k, M^l \rangle_t = \int_0^t a_{kl}(X_s)ds \tag{3}$$

However, $f_k, g_{kl} \notin D(A)$.
Define $f_{k,n}, g_{kl,n} \in D(A)$:
Let $B(0, n) =$ the ball of radius n with center 0

$$f_{k,n}(x) = x^k, g_{kl,n}(x) = x^k x^l \text{ on } B(0,n)$$
$$f_{k,n}(x) = g_{kl,n}(x) = 0 \text{ on } B(0,n+1)^c$$

Then using stop-times

$$\tau_n = \inf\{t \geq 0 : X_t \notin B(0,n)\}$$

we see that (2) is a local martingale and such that (3) holds.

Example 3: Diffusions - Martingale Charachterization

Now, if σ is invertible, define

$$W_t = \int_0^t \sigma^{-1}(X_s)dM_s, \text{ or}$$

$$W_t^i = \sum_{k=1}^d \int_0^t \sigma_{ik}^{-1}(X_s)dM_s^k, \qquad 1 \leq i \leq d$$

Then, (2) $\implies$ $(W_t^k)_{\{t\geq 0\}}$ is a local martingale; (3) $\implies$

$$\langle W^i, W^j\rangle_t = \sum_{k,l=1}^d \left\langle \int_0^t \sigma_{ik}^{-1}(X_s)dM_s^k, \int_0^t \sigma_{jl}^{-1}(X_s)dM_s^l \right\rangle$$

$$= \sum_{k,l=1}^d \int_0^t \left(\sigma_{ik}^{-1} a_{kl} (\sigma^T)_{lj}^{-1}\right)(X_s)ds$$

$$= \delta_{ij} t$$

Example 3: Diffusions - Martingale Charachterization

Thus Levy's Charachterization theorem implies that W is a d-dimensional Brownian motion.
Finally,

$$\int_0^t \sigma(X_s)dW_s = \int_0^t dM_s = X_t - \int_0^t b(X_s)ds$$

and hence X is a solution of the SDE (1).

Example 3: Diffusions - Martingale Charachterization

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Finally,

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and hence X is a solution of the SDE (1).

- When σ is singular - it is possible that the space Ω may not be rich enough to hold a Brownian motion.
- Intutively, we plug in another Brownian motion *wherever* σ is degenerate

Example 3: Diffusions - Martingale Charachterization

- Consider $(\Gamma, \mathcal{G}, \mathbb{Q})$ and a Brownian motion $(B_t)_{\{t\geq 0\}}$ defined on it
- Get $d \times d$ (measurable) matrices $\rho(x), \eta(x)$ satisfying
  - $\rho a \rho^T + \eta\eta^T = I_d$
  - $\rho\eta = 0$
  - $(I_d - \sigma\rho)(I_d - \sigma\rho)^T = 0$
- Define on $(\Omega, \mathcal{F}, \mathbb{P}) \otimes (\Gamma, \mathcal{G}, \mathbb{Q})$

$$W_t = \int_0^t \rho(X_s)dM_s + \int_0^t \eta(X_s)dB_s$$

- Then W is a Brownian motion on the extended space and (1) holds. □

Example 4: Markov Jump Process

- Let $\mu(x,\Gamma)$ be a transition function on $E \times \mathcal{E}$ and $\lambda > 0$.
- Let $\{Y_0, Y_1, Y_2, \ldots\}$ be a Markov chain
  - $\mathbb{P}(Y_0 \in \Gamma) = \nu(\Gamma)$
  - $\mathbb{P}\left(Y_{k+1} \in \Gamma \middle| Y_0, \ldots, Y_k\right) = \mu(Y_k, \Gamma)$
- Let N be a Poisson process with intensity λ, independent of Y.
- Define X by

$$X_t = Y_{N_t} \quad t \geq 0$$

  Then X is a process which jumps at exponential times and the jump is dictated by the transition function $\mu(\cdot,\cdot)$.
- Define

$$Pf(x) = \int_E f(y)\mu(x,dy)$$

Example 4: Markov Jump Process (Contd.)

- Note for $F_1 \in \mathcal{F}_t^N$ and $F_2 \in \mathcal{F}_l^Y$

$$\begin{aligned}\mathbb{E}\left[f(Y_{k+N_t})\mathbb{I}_{F_1\cap F_2\cap\{N_t=l\}}\right] &= \mathbb{E}\left[f(Y_{k+l})\mathbb{I}_{F_1\cap F_2\cap\{N_t=l\}}\right]\\ &= \mathbb{E}\left[P^k f(Y_l)\mathbb{I}_{F_2}\right]\mathbb{P}(F_1\cap\{N_t=l\})\\ &= \mathbb{E}\left[f(X_t)\mathbb{I}_{F_1\cap F_2\cap\{N_t=l\}}\right]\end{aligned}$$

- $F_1 \cap F_2 \cap \{N_t = l\}$ generate $\mathcal{F}_t = \mathcal{F}_t^N \vee \mathcal{F}_t^X$
- Thus

$$\mathbb{E}\left[f(Y_{k+N_t})|\mathcal{F}_t\right] = P^k f(X_t) \text{ a.s.}$$

- Using, independent increments of N

$$\begin{aligned}\mathbb{E}\left[f(X_{t+s})|\mathcal{F}_t\right] &= \mathbb{E}\left[f\left(Y_{N_{t+s}-N_t+N_t}\right)|\mathcal{F}_t\right]\\ &= \sum_{k=0}^{\infty} e^{-\lambda s}\frac{(\lambda s)^k}{k!}P^k f(X_t)\end{aligned}$$

Example 4: Markov Jump Process (Contd.)

Finally

$$T_t = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} P^k$$

defines a one parameter operator semigroup
Generator A : $T_t = e^{tA}$

$$A = \lambda(P - I)$$

$$Af(x) = \lambda \int_E (f(y) - f(x))\, \mu(x, dy)$$

X is a solution of the martingale problem for (A, ν)

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Example 5: Hilbert Space Valued Diffusion

- Let $E = H$, a real, separable Hilbert space, with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.
- $\mathcal{L}_2(H,H)$ - the space of Hilbert Schmidt operators on H *i.e.* $\Sigma \in \mathcal{L}_2(H,H)$ iff $\|\Sigma\|_{HS} = \sum_i (\Sigma\phi_i, \Sigma\phi_i) < \infty$, Hilbert Schmidt norm
- Let $\sigma : H \to \mathcal{L}_2(H,H)$, $b : H \to H$ be measurable

$$
\begin{aligned}
\|\sigma(h)\|_{HS} &\leq K \\
\|b(h)\| &\leq K \\
\|\sigma(h_1) - \sigma(h_2)\|_{HS} &\leq K\|h_1 - h_2\| \\
\|b(h_1) - b(h_2)\| &\leq K\|h_1 - h_2\|
\end{aligned}
$$

for all $h, h_1, h_2 \in H$.

Example 5: Hilbert Space Valued Diffusion (Contd.)

- Fix a Complete OrthoNormal System $\{\phi_i : i \geq 1\}$ in H
- Let $P_n : H \to \mathbb{R}^n$ be defined by

$$P_n(h) = ((h, \phi_1), \ldots, (h, \phi_n)).$$

- $D(A) = \{f \circ P_n : f \in C_c^2(\mathbb{R}^n), n \geq 1\}$,

$$[A(f \circ P_n)](h) = \frac{1}{2} \sum_{i,j=1}^{n} (\sigma^*(h)\phi_i, \sigma^*(h)\phi_j)\partial_{ij} f \circ P_n(h) + \sum_{i=1}^{n} (b(h), \phi_i)\partial_i f \circ P_n(h)$$

Example 5: Hilbert Space Valued Diffusion (Contd.)

- The martingale problem for (A, μ) is well-posed
- The unique solution X is continuous a.s.
- $\exists$ Cylindrical Brownian motion B on some $(\Omega, \mathcal{F}, \mathbb{P})$
  - (B_t, h) is a 1-dimensional Brownian Motion for all $h \in H$
  - $\mathbb{E}\left[(B_t, h_1)(B_t, h_2)\right] = (h_1, h_2)$ for all $h_1, h_2 \in H$
- It is a Hilbert space valued diffusion. *i.e.*

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

for some Cylindrical Brownian motion B

Example 6: Branching Brownian Motion

Initial Configuration Individuals in the population are scattered in $\mathbb{R}^d$

Spatial Motion Each individual, during its lifetime, moves in $\mathbb{R}^d$ according to a Brownian motion, independently of all other particles

Branching rate, α Each individual has an exponentially distributed lifetime α

Branching mechanism, Φ When the individual dies, it leaves behind at the same location a random number of offsprings with probability generating function

$$\Phi(s) = \sum_{l=0}^{\infty} p_l s^l$$

Example 6: Branching Brownian Motion (Contd.)

Let X denote such a process

- state space $E' = \{(k, x_1, \dots, x_k) : k = 0, 1, 2, \dots, x_i \in \mathbb{R}^d\}$.
- Consider functions $f(k, x_1, \dots, x_k) = \prod_{i=1}^{k} g(x_i)$ on E
- Generator of Brownian motion - $L_1 = \frac{1}{2}\Delta$
- Generator for the Branching process $L_2 h(k) = \sum_{l=0}^{\infty} \alpha k p_l \left(h(k - 1 + l) - h(k)\right)$
- In the absence of branching

$$A_1 \left(\prod_{i=1}^{k} g(x_i) \right) = \sum_{j=1}^{k} L_1 g(x_j) \prod_{i \neq j} g(x_i)$$

so that $f(X_t) - \int_0^t A_1 f(X_s) ds$ is a martingale

Example 6: Branching Brownian Motion (Contd.)

- In presence of branching but no motion

$$A_2\left(\prod_{i=1}^{k} g(x_i)\right) = \sum_{j=1}^{k} \alpha\left(\Phi(g(x_j)) - g(x_j)\right)\prod_{i\neq j} g(x_i)$$

- Independence of branching and motion suggest that the "martingale problem operator" for X should be

$$A = A_1 + A_2$$

Example 6: Branching Brownian Motion (Contd.)

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- Independence of branching and motion suggest that the "martingale problem operator" for X should be

$$A = A_1 + A_2$$

- E' is cumbersome to work with
- order of particles is not important

Example 6: Branching Brownian Motion (Contd.)

- $E = \left\{ \sum_{i=1}^{k} \delta_{x_i} : k = 0, 1, 2, \ldots ; x_i \in \mathbb{R}^d \right\}$
- $E \subset \mathcal{M}(\mathbb{R}^d)$, space of positive finite measures on $\mathbb{R}^d$
- For $\mu = \sum_{i=1}^{k} \delta_{x_i}$

$$\prod_{i=1}^{k} g(x_i) = e^{\langle \log g, \mu \rangle}$$

- $$\mathcal{A} e^{\langle \log g, \mu \rangle} = e^{\langle \log g, \mu \rangle} \left\langle \frac{L_1 g + \alpha(\Phi(g) - g)}{g}, \mu \right\rangle$$
- $\xi_t = \sum \delta_{X_t^i}$ is a solution of the martingale problem for $\mathcal{A}$