Martingale Problems

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Outline

1. Markov Processes
   - Introduction
   - Associated Semigroups and Generators
2. Martingale Problem
   - Preliminary results & Definitions
   - Markovian Solutions
   - Path Properties
3. Independence
   - Time-Inhomogeneous Martingale Problem
   - Jump Perturbations

Markov Processes

Let X be a E-valued process.

- X satisfies the Markov property if

$$\mathbb{P}\left(X_{t+r} \in \Gamma | \sigma(X_s : 0 \le s \le t)\right) = \mathbb{P}\left(X_{t+r} \in \Gamma | X_t\right)$$

- A function $P(t, x, \Gamma)$ is called a transition function
    - if $P(t, x, \cdot)$ is a probability measure for all $(t, x) \in [0, \infty) \times E$
    - $P(0, x, \cdot) = \delta_x$ for all $x \in E$
    - $P(\cdot, \cdot, \Gamma)$ is measurable for all $\Gamma \in \mathcal{E}$

  (Interpretation) $\mathbb{P}\left(X_t \in \Gamma | X_0 = x\right) = P(t, x, \Gamma)$

- X is a Markov process if it *admits a transition function* so that

$$\mathbb{P}\left(X_{t+s} \in \Gamma | \mathcal{F}_t^X\right) = P(s, X_t, \Gamma) \quad \forall t, s \ge 0, \Gamma \in \mathcal{E}$$

  Equivalently

$$\mathbb{E}\left[f(X_{t+s}) | \mathcal{F}_t^X\right] = \int f(y) P(s, X_t, dy) \quad \forall f \in C_b(E)$$

Associated Semigroups

Define

$$T_t f(x) = \int f(y)P(t,x,dy) \quad = \mathbb{E}\left[f(X_t)|X_0 = x\right]$$

$$T_t T_s = T_{t+s} \quad \text{semigroup property}$$

$$T_t f \geq 0 \text{ whenever } f \geq 0 \quad \text{positivity}$$

$$\|T_t f\| \leq \|f\| \quad \text{contraction}$$

$$\mathbb{E}\left[f(X_{t+s})|\mathcal{F}_t^X\right] = T_s f(X_t)$$

Generator

- A contraction semigroup $\{T_t\}$ satisfying

$$\lim_{t \downarrow 0} T_t f = T_0 f = f$$

is called a Strongly continuous contraction semigroup

- The Generator L of a strongly continuous contraction semigroup is defined as follows.

$$D(L) = \left\{ f : \lim_{t \downarrow 0} \frac{T_t f - f}{t} \text{ exists} \right\}.$$

$$Lf = \lim_{t \downarrow 0} \frac{T_t f - f}{t}.$$

$$T_t f - f = \int_0^t T_s L f ds \quad \forall f \in D(L) \tag{1}$$

Martingale Problem

Proposition 1

Let X, P, (T_t) and L be as above. Then X is a solution of the martingale problem for L.

Proof. Fix $f \in D(L)$ and let $M_t = f(X_t) - \int_0^t Lf(X_s)ds$.

$$\begin{aligned}
\mathbb{E}\left[M_{t+s}|\mathcal{F}_t^X\right] &= \mathbb{E}\left[f(X_{t+s})|\mathcal{F}_t^X\right] - \int_0^{t+s} \mathbb{E}\left[Lf(X_u)|\mathcal{F}_t^X\right] du \\
&= T_s f(X_t) - \int_t^{t+s} T_{u-t} Lf(X_t)du - \int_0^t Lf(X_u)du \\
&= T_s f(X_t) - \int_0^s T_u Lf(X_t)du - \int_0^t Lf(X_u)du \\
&= f(X_t) - \int_0^t Lf(X_u)du = M_t \qquad \text{(using (1))}
\end{aligned}$$

Outline

Finite Dimensional Distributions

Lemma 1

A process X is a solution to the martingale problem for A if and only if

$$\mathbb{E}\left[\left(f(X_{t_{n+1}}) - f(Xt_n) - \int_{t_n}^{t_{n+1}} Af(X_s)ds\right)\prod_{k=1}^{n} h_k(X_{t_k})\right] = 0 \quad (2)$$

for all $f \in D(A), 0 \leq t_1 < t_2 < \ldots < t_{n+1}, h_1, h_2, \ldots, h_n \in B(E)$, and $n \geq 1$.

Thus, (being a) solution of the martingale problem is a finite dimensional property.

Thus if X is a solution and Y is a modification of X, then Y is also a solution.

The space $D([0,\infty),E)$

- $D([0,\infty),E)$ - the space of all E valued functions on $[0,\infty)$ which are right continuous and have left limits
- Skorokhod topology on $D([0,\infty),E)$
- $D([0,\infty),E)$ - complete, separable, metric space
- $\mathcal{S}_E$, the Borel σ-field on $D([0,\infty),E)$.
- $\theta_t(\omega) = \omega_t$ - co-ordinate process
- r.c.l.l. process - process taking values in $D([0,\infty),E)$

r.c.l.l. Solutions

Definition 2.1

A probability measure $P \in \mathcal{P}(D([0,\infty),E))$ is solution of the martingale problem for (A,μ) if there exists a $D([0,\infty),E)$- valued process X with $\mathcal{L}(X) = P$ and such that X is a solution to the martingale problem for (A,μ)
Equivalently,
$P \in \mathcal{P}(D([0,\infty),E))$ is a solution if θ defined on $(D([0,\infty),E),\mathcal{S}_E,P)$ is a solution

- For a r.c.l.l. process X defined on some $(\Omega,\mathcal{F},\mathbb{P})$, we will use the dual terminology
  - X is a solution
  - $\mathbb{P} \circ X^{-1}$ is a solution

Well-posedness - Definitions

Definition 2.2

The martingale problem for (A, μ) is well - posed in a class $\mathcal{C}$ of processes if there exists a solution $X \in \mathcal{C}$ of the martingale problem for (A, μ) and if $Y \in \mathcal{C}$ is also a solution to the martingale problem for (A, μ), then X and Y have the same finite dimensional distributions. i.e. uniqueness holds

- When $\mathcal{C}$ is the class of all measurable processes then we just say that the martingale problem is well - posed.

Definition 2.3

The martingale problem for A is well - posed in $\mathcal{C}$ if the martingale problem for (A, μ) is well-posed for all $\mu \in \mathcal{P}(E)$.

Well-posedness in $D([0,\infty),E)$

- finite dimensional distributions characterize the probability measures on $D([0,\infty),E)$

Definition 2.4

The $D([0,\infty),E)$ - martingale problem for (A,μ) is well - posed if there exists a solution $P \in \mathcal{P}(D([0,\infty),E))$ of the $D([0,\infty),E)$ - martingale problem for (A,μ) and if Q is any solution to the $D([0,\infty),E)$ - martingale problem for (A,μ) then $P = Q$.

Bounded-pointwise convergence

Definition 2.5

- *Let* $f_k, f \in B(E)$. f_k *converge bounded-ly and pointwise to* f $f_k \xrightarrow{bp} f$ *if* $\|f_k\| \leq M$ *and* $f_k(x) \to f(x)$ *for all* $x \in E$.
- *A class of functions* $\mathcal{U} \subset B(E)$ *bp-closed if* $f_k \in \mathcal{U}, f_k \xrightarrow{bp} f$ *implies* $f \in \mathcal{U}$.
- *bp-closure*($\mathcal{U}$) *- the smallest class of functions in* $B(E)$ *which contains* $\mathcal{U}$ *and is bp-closed.*

i.e. $\mathcal{E}_0$ - a field; $\sigma(\mathcal{E}_0) = \mathcal{E}$
$\mathcal{H}$ - class of all $\mathcal{E}_0$-simple functions
Then bp-closure($\mathcal{H}$) $=$ class of all bounded, $\mathcal{E}$ - measurable functions.

Separability condition

Definition 2.6

The operator A satisfies the separability condition if

- *There exists a countable subset $\{f_n\} \subset D(A)$ such that*

$$bp - closure(\{(f_n, Af_n) : n \geq 1\}) \supset \{(f, Af) : f \in D(A)\}.$$

- Let $A_0 = A|_{\{f_n\}}$, the restriction of A to $\{f_n\}$
- Solution of martingale problem for A
  $\implies$ solution of martingale problem for A_0

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- Let $A_0 = A|_{\{f_n\}}$, the restriction of A to $\{f_n\}$
- Solution of martingale problem for A
  $\Longleftrightarrow$ solution of martingale problem for A_0 from Lemma 1
  Use Dominated convergence Theorem to show that the set of all $\{(g, Ag)\}$ satisfying (2) is bp-closed

Markov Family of Solutions

Theorem 1

Let A be an operator on $C_b(E)$ satisfying the separability condition. Suppose the $D([0,\infty),E)$ - martingale problem for (A,δ_x) is well-posed for each $x \in E$. Then

1. $x \mapsto P_x(C)$ *is measurable for all* $C \in \mathcal{S}_E$.
2. *For all* $\mu \in \mathcal{P}(E)$, *the* $D([0,\infty),E)$ *- martingale problem for* (A,μ) *is well - posed, with the solution* P_μ *given by*

$$P_\mu(C) = \int_E P_x(C)\mu(dx).$$

3. *Under* P_μ, θ_t *is a Markov process with transition function*

$$P(s,x,F) = P_x(\theta_s \in F). \tag{3}$$

Proof of (1)

- Choose $M \subset C_b(E)$ - countable such that $B(E) \subset$ bp-closure(M).
- Let $H = \Big\{\eta :$

$$\eta(\theta) = (f_n(\theta_{t_{m+1}}) - f_n(\theta_{t_m}) - \int_{t_m}^{t_{m+1}} Af_n(\theta_s)ds) \prod_{k=1}^{m} h_k(\theta_{t_k})$$

where $h_1, h_2, \ldots, h_m \in M, 0 \leq t_1 < t_2 \ldots < t_{m+1} \subset \mathbb{Q}\Big\}$

- H is countable
- Lemma 1 $\implies \mathcal{M}_1 = \cap_{\eta \in H}\{P : \int \eta dP = 0\}$ is the set of solutions of the martingale problem for A.
- $P \mapsto \int \eta dP$ is continuous. Hence $\mathcal{M}_1$ is Borel set

Proof of (1) (Contd.)

- $G : \mathcal{P}(D([0,\infty), E)) \to \mathcal{P}(E)$

  $$G(P) = P \circ \theta(0)^{-1}.$$

  G is continuous
- $\mathcal{M} = \mathcal{M}_1 \cap G^{-1}(\{\delta_x : x \in E\}) = \{P_x : x \in E\}$ is Borel
- Well-posedness $\implies$ G restricted to $\mathcal{M}$ is one-to-one mapping onto $\{\delta_x : x \in E\}$.
- $G^{-1} : \{\delta_x : x \in E\} \mapsto \mathcal{M}$ is Borel
- $G(P_x) = \delta_x$. Hence $\delta_x \mapsto P_x$ is measurable
- $x \mapsto P_x = x \mapsto \delta_x \mapsto P_x$ is measurable $\square$

Proof of (2)

- For $F \in \mathcal{E}$

$$P_\mu \circ \theta_0^{-1}(F) = \int_E P_x \circ \theta_0^{-1}(F)\mu(dx) = \int_E \delta_x(F)\mu(dx) = \mu(F).$$

- For $\eta \in H$,

$$\int_{D([0,\infty),E)} \eta dP_\mu = \int_E \int_{D([0,\infty),E)} \eta dP_x \mu(dx) = 0.$$

Hence P_μ is a solution to the martingale problem for (A, μ).

- Let Q be another solution of the $D([0,\infty), E)$– martingale problem for (A, μ)
- Let Q_ω be the regular conditional probability of Q given θ_0.

Proof of (2) (Contd.)

- Fix $\eta \in H, h \in C_b(E)$. Define $\eta'(\theta) = \eta(\theta)h(\theta_0)$.
- $\eta' \in H$. Thus

$$\mathbb{E}^Q[\eta(\theta)h(\theta_0)] = \mathbb{E}^Q[\eta'] = 0.$$

- Since this holds for all $h \in C_b(E)$,

$$\mathbb{E}^{Q_\omega}[\eta] = \mathbb{E}^Q[\eta|\theta_0] = 0 \text{ a.s. - } Q.$$

- Since H is countable, $\exists$ ONE Q-null set N_0 satisfying

$$\mathbb{E}^{Q_\omega}[\eta] = 0 \qquad \forall \omega \notin N_0$$

- Q_ω is a solution of the martingale problem for A initial distribution $\delta_{\theta_0(\omega)}$.
- Well - posedness implies

$$Q_\omega = P_{\theta_0(\omega)} \text{ a.s.}[Q]$$

Hence $Q = P_\mu$.

Proof of (3)

- Fix s. Let $\theta'_t = \theta_{t+s}$.
- Let Q'_ω be the regular conditional probability distribution of θ' (under P_x) given $\mathcal{F}_s$.
- Q'_ω is a solution to the martingale problem for $(A, \delta_{\theta_s(\omega)})$.
- Well-posedness $\implies$ $Q'_\omega(\theta'_t \in F) = P(t, \theta_s(\omega), F)$ (See (3))
- Hence for $f \in B(E)$,

$$\mathbb{E}^{P_x} f(\theta_{t+s}) = \mathbb{E}^{P_x}\left[\mathbb{E}^{P_x}\left[f(\theta_{t+s})|\mathcal{F}_s\right]\right]$$
$$= \mathbb{E}^{P_x}\left[\int_E f(y)P(t, \theta_s(\cdot), dy)\right]$$
$$= \int_E \int_E f(y_2)P(t, y_1, dy_2)P(s, x, dy_1).$$

- $P(s+t, x, F) = P_x(\theta_{t+s} \in F) = \int_E P(t, y, F)P(s, x, dy)$ □

One-dimensional equality

Theorem 2

Suppose that for each $\mu \in \mathcal{P}(E)$, any two solutions X and Y (defined respectively on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$) of the martingale problem for (A, μ) have the same one-dimensional distributions. Then X and Y have the same finite dimensional distributions, i.e. the martingale problem is well - posed.

Proof. To show

$$\mathbb{E}^{\mathbb{P}_1}\left[\prod_{k=1}^{m} f_k(X_{t_k})\right] = \mathbb{E}^{\mathbb{P}_2}\left[\prod_{k=1}^{m} f_k(Y_{t_k})\right] \tag{4}$$

for all $0 \leq t_1 < t_2 < \ldots t_m, f_1, f_2, \ldots, f_m \in B(E)$ and $m \geq 1$.

Induction argument

- Case: $m = 1$ - true by hypothesis
- Assume that the Induction hypothesis (4) is true for $m = n$
- Fix $0 \leq t_1 < t_2 < \ldots t_n, f_1, f_2, \ldots, f_n \in B(E), f_k > 0$.
- Define

$$\mathbb{Q}_1(F_1) = \frac{\mathbb{E}^{\mathbb{P}_1}[\mathbb{I}_{F_1} \prod_{k=1}^n f_k(X_{t_k})]}{\mathbb{E}^{\mathbb{P}_1}[\prod_{k=1}^n f_k(X_{t_k})]} \quad \forall\ F_1 \in \mathcal{F}_1$$

$$\mathbb{Q}_2(F_2) = \frac{\mathbb{E}^{\mathbb{P}_2}[\mathbb{I}_{F_2} \prod_{k=1}^n f_k(Y_{t_k})]}{\mathbb{E}^{\mathbb{P}_2}[\prod_{k=1}^n f_k(Y_{t_k})]} \quad \forall\ F_2 \in \mathcal{F}_2$$

- Let $\tilde{X}_t = X_{t_n+t}$, $\tilde{Y}_t = Y_{t_n+t}$.
- Fix $0 \leq s_1 < s_2 < \ldots, s_{m+1} = t, h_1, h_2, \ldots, h_m \in B(E)$ and $f \in D(A)$.

Induction argument (Contd.)

$$\eta(\theta) = \left(f(\theta_{s_{m+1}}) - f(\theta_{s_m}) - \int_{s_m}^{s_{m+1}} Af(\theta_s)ds \right) \prod_{k=1}^{m} h_k(\theta_{t_k})$$

$$\mathbb{E}^{\mathbb{P}_1}\left[\eta(X_{t_n+\cdot}) \prod_{k=1}^{n} f_k(X_{t_k})\right] = \mathbb{E}^{\mathbb{P}_1}\left[\left(f(X_{s_{m+1}+t_n}) - f(X_{s_m+t_n}) - \int_{t_n+s_m}^{t_n+s_{m+1}} Af(X_u)du \right) \prod_{j=1}^{m} h_j(X_{t_n+s_j}) \prod_{k=1}^{n} f_k(X_{t_k})\right] = 0.$$

Induction argument (Contd.)

Hence

$$\mathbb{E}^{\mathbb{Q}_1}[\eta(\tilde{X})] = \frac{\mathbb{E}^{\mathbb{P}_1}[\eta(X_{t_n+\cdot})\prod_{k=1}^{n} f_k(X_{t_k})]}{\mathbb{E}^{\mathbb{P}_1}[\prod_{k=1}^{n} f_k(X_{t_k})]} = 0.$$

Similarly $\mathbb{E}^{\mathbb{Q}_2}[\eta(\tilde{Y})] = 0.$

- $\tilde{X}$ and $\tilde{Y}$ are solutions of the martingale problems for $(A, \mathcal{L}(\tilde{X}_0))$ and $(A, \mathcal{L}(\tilde{Y}_0))$ respectively.

$$\mathbb{E}^{\mathbb{Q}_1}[f(\tilde{X}_0)] = \frac{\mathbb{E}^{\mathbb{P}_1}[f(X_{t_n})\prod_{k=1}^{n} f_k(X_{t_k})]}{\mathbb{E}^{\mathbb{P}_1}[\prod_{k=1}^{n} f_k(X_{t_k})]}$$

$$= \frac{\mathbb{E}^{\mathbb{P}_2}[f(Y_{t_n})\prod_{k=1}^{n} f_k(Y_{t_k})]}{\mathbb{E}^{\mathbb{P}_2}[\prod_{k=1}^{n} f_k(Y_{t_k})]} = \mathbb{E}^{\mathbb{Q}_2}[f(\tilde{Y}_0)] \quad \forall f \in B(E).$$

This equality follows from induction hypothesis for $m = n$

Induction argument (Contd.)

- Hence $\tilde{X}$ and $\tilde{Y}$ have the same initial distribution.
- One-dimensional uniqueness implies

$$\mathbb{E}^{\mathbb{Q}_1}[f(\tilde{X}_t)] = \mathbb{E}^{\mathbb{Q}_2}[f(\tilde{Y}_t)] \quad \forall\ t \geq 0, f \in B(E).$$

-

$$\mathbb{E}^{\mathbb{P}_1}[f(X_{t_n+t}) \prod_{k=1}^{n} f_k(X_{t_k})] = \mathbb{E}^{\mathbb{P}_2}[f(X_{t_n+t}) \prod_{k=1}^{n} f_k(X_{t_k})]$$

- Induction Hypothesis (4) is true for $m = n+1$
  set $t_{n+1} = t_n + t$

$\square$

Semigroup associated with the Martingale Problem

- Suppose A satisfies the conditions of Theorem 1.
- Associate the Markov semigroup $(T_t)_{t\geq 0}$ with A -

$$T_t f(x) = \int_E f(y) P(t, x, dy)$$

The following theorem can be proved exactly as the previous one.

Strong Markov Property

Theorem 3

Suppose that the $D([0,\infty), E)$- martingale problem for A is well - posed with associated semigroup T_t
Let X, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, be a solution of the martingale problem for A (with respect to $(\mathcal{G}_t)_{t\geq 0}$). Let τ be a finite stop time. Then for $f \in B(E), t \geq 0$,

$$\mathbb{E}[f(X_{\tau+t})|\mathcal{G}_\tau] = T_t f(X_\tau)$$

In particular

$$P((X_{\tau+t} \in \Gamma)|\mathcal{G}_\tau) = P(t, X\tau, \Gamma)\forall\ \Gamma \in \mathcal{E}$$

r.c.l.l. modification

Definition 2.7

Let D be a class of functions on E

1. *D is measure determining if $\int f d\mathbb{P} = \int f d\mathbb{Q}$ forall $f \in D$ implies $\mathbb{P} = \mathbb{Q}$.*
2. *D separates points in E if $\forall x \neq y$ $\exists g \in D$ such that $g(x) \neq g(y)$.*

Theorem 4

Let E be a compact metric space. Let A be an operator on $C(E)$ such that $D(A)$ is measure determining and contains a countable subset that separates points in E. Let X, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, be a solution to the martingale problem for A. Then X has a modification with sample paths in $D([0, \infty), E)$.

Proof

Proof.

- Let $\{g_k : k \geq 1\} \subset D(A)$ separate points in E.
- Define
$$M_k(t) = g_k(X_t) - \int_0^t Ag_k(X_s)ds$$
M_k is a martingale for all k.
- Then for all t
$$\lim_{\substack{s\uparrow t\\ s\in\mathbb{Q}}} M_k(s)\ ,\ \lim_{\substack{s\downarrow t\\ s\in\mathbb{Q}}} M_k(s) \text{ exist a.s.}$$
- Hence $\exists\, \Omega' \subset \Omega$ with $P(\Omega') = 1$ and
$$\lim_{\substack{s\uparrow t\\ s\in\mathbb{Q}}} g_k(X_s(\omega))\ ,\ \lim_{\substack{s\downarrow t\\ s\in\mathbb{Q}}} g_k(X_s(\omega)) \text{ exist } \forall \omega \in \Omega', t \geq 0, k \geq 1$$

Proof (contd.)

- Fix $t \geq 0, \{s_n\} \subseteq \mathbb{Q}$ with $s_n > t, \lim_{n\to\infty} s_n = t$ and $\omega \in \Omega'$.
- Since E is compact, $\exists$ a subsequence $\{s_{n_i}\}$ such that $\lim_{i\to\infty} X_{s_{n_i}}(\omega)$ exists.
- Clearly

$$g_k\left(\lim_{i\to\infty} X_{s_{n_i}}(\omega)\right) = \lim_{\substack{s\downarrow t \\ s\in\mathbb{Q}}} g_k(X_s(\omega)) \qquad \forall\ k.$$

- Since $\{g_k : k \geq 1\}$ separate points in E, $\lim_{\substack{s\downarrow t \\ s\in\mathbb{Q}}} X_s(\omega)$ exists.
- Similarly $\lim_{\substack{s\uparrow t \\ s\in\mathbb{Q}}} X_s(\omega)$ exists
- Define

$$Y_t(\omega) = \lim_{\substack{s\downarrow t \\ s\in\mathbb{Q}}} X_s(\omega).$$

Proof (contd.)

- For $\omega \in \Omega'$, $Y_t(\omega)$ is r.c.l.l. &

$$Y_t^-(\omega) = \lim_{\substack{s \uparrow t \\ s \in \mathbb{Q}}} X_s(\omega)$$

- Define Y suitably for $\omega \notin \Omega'$
- Then Y has sample paths in $D([0, \infty), E)$.
- Since X is a solution to the martingale problem for A, for $f \in D(A)$, a measure determining set

$$\mathbb{E}[f(Y_t)|\mathcal{F}_t^X] = \lim_{\substack{s \downarrow t \\ s \in \mathbb{Q}}} \mathbb{E}[f(X_s)|\mathcal{F}_t^X] = f(X_t).$$

- $\implies X = Y$ a.s.

Outline

Definitions

For $t \geq 0$, let $(A_t)_{t\geq 0}$ be linear operators on $M(E)$ with a common domain $D \subset M(E)$.

Definition 3.1

A measurable process X defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a solution to the martingale problem for $(A_t)_{t\geq 0}$ with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$ if for any $f \in D$

$$f(X_t) - \int_0^t A_s f(X_s) ds$$

is a $(\mathcal{G}_t)$ - martingale.
Let $\mu \in \mathcal{P}(E)$. The martingale problem for $((A_t)_{t\geq 0}, \mu)$ is well-posed if there exists an unique solution for the martingale problem

Space-Time Process

- Let $E^0 = [0, \infty) \times E$.
- Let $X_t^0 = (t, X_t)$
- Define

$$D(A^0) = \left\{ g(t,x) = \sum_{i=1}^{k} h_i(t) f_i(x) \;\; h_i \in C_c^1([0,\infty)), f_i \in D \right\}$$

$$A^0 g(t,x) = \sum_{i=1}^{k} [f_i(x)\partial_t h_i(t) + h_i(t) A_t f_i(x)]$$

Theorem 5

X is a solution to the martingale problem for $(A_t)_{t\geq 0}$ if and only if X^0 is a solution to the martingale problem for A^0 .

Proof.

- Let X be a solution (with respect to a filtration $(\mathcal{G}_t)_{t\geq 0}$) to the martingale problem for $(A_t)_{t\geq 0}$.
- Let $fh \in D(A^0)$
- For $0 < s < t$, let $g(t) = \mathbb{E}[f(X_t)|\mathcal{G}_s]$.

$$g(t) - g(s) = \int_s^t \mathbb{E}[A_u f(X_u)|\mathcal{G}_s] du$$

- Then

$$\begin{aligned} g(t)h(t) - g(s)h(s) &= \int_s^t \partial_u \left[g(u)h(u)\right] du \\ &= \int_s^t \left\{h(u)\mathbb{E}[A_u f(X_u)|\mathcal{G}_s] + g(u)\partial_u h(u)\right\} du \\ &= \int_s^t \mathbb{E}\left[A^0(fh)(X_u^0)|\mathcal{G}_s\right] du. \end{aligned}$$

Proof (Contd.)

- $fh(X^0(t)) - \int_0^t A^0 fh(X^0(s))ds$ is a martingale
- X^0 is a solution to the martingale problem for A^0.
- The converse follows by taking $h = 1$ on $[0, T]$, $T > 0$.

□

A more General Result

- State spaces E_1 and E_2
- Operators A_1 on $M(E_1)$ and A_2 on $M(E_2)$
- Solutions X_1 and X_2
- Define

$$D(A) = \{f_1 f_2 : f_1 \in D(A_1), f_2 \in D(A_2)\}$$

$$A(f_1 f_2) = (A_1 f_1) f_2 + f_1 (A_2 f_2)$$

- (X_1, X_2) is a solution of the martingale problem for A

Theorem 6

Suppose uniqueness holds for the martingale problem for A_1, A_2. Then uniqueness holds for the martingale problem for A.

A perturbed operator

- Let A be an operator with $D(A) \subset C_b(E)$.
- Let $\lambda > 0$ and let $\eta(x, \Gamma)$ be a transition function on $E \times \mathcal{E}$.
- Let

$$Bf(x) = \lambda \int_E (f(y) - f(x))\eta(x, dy) \; f \in B(E).$$

Theorem 7

Suppose that for every $\mu \in \mathcal{P}(E)$, there exists a solution to the $D([0, \infty), E)$ martingale problem for (A, μ). Then for every $\mu \in \mathcal{P}(E)$ there exists a solution to the martingale problem for $(A + B, \mu)$.

Proof

Proof.

- For $k \geq 1$, let $\Omega_k = D([0,\infty), E), \Omega_k^0 = [0,\infty)$
- Let $\Omega = \prod_{k=1}^{\infty} \Omega_k \times \Omega_k^0$
- Let θ_k and ξ_k denote the co-ordinate random variables
- Borel σ-fields - $\mathcal{F}_k, \mathcal{F}_k^0$
- Let $\mathcal{F}$ be the product σ-field on Ω.
- Let $\mathcal{G}_k$ the σ-algebra generated by cylinder sets $C_1 \times \prod_{i=k+1}^{\infty}(\Omega_i \times \Omega_i^0)$, where $C_1 \in \mathcal{F}_1 \otimes \mathcal{F}_1^0 \otimes \ldots \otimes \mathcal{F}_k \otimes \mathcal{F}_k^0$.
- Let $\mathcal{G}^k$ be the σ-algebra generated by $\prod_{i=1}^{k}(\Omega_i \times \Omega_i^0) \times C_2$, where $C_2 \in \mathcal{F}_k \otimes \mathcal{F}_k^0 \otimes \ldots$.

Perturbed Solution X

- X evolves in E as a solution to the martingale problem for A till an exponentially distributed time with parameter λ which is independent of the past.
- At this time if the process is at x, it jumps to y with probability $\eta(x, dy)$ and then continues evolving as a solution to the martingale problem for (A, δ_y).
- To put this in a mathematical framework, we consider that between the k^{th} and the $(k+1)^{th}$ jump (dictated by B), the process lies in Ω_k.
- The k^{th} copy of the exponential time is a random variable in Ω_k^0.

Proof (Contd.)

- Let P_x, P_μ be solutions of the martingale problems for (A, δ_x), (A, μ) respectively.
- Let γ be the exponential distribution with parameter λ.
- Fix $\mu \in \mathcal{P}(E)$. Define, for $\Gamma_1 \in \mathcal{F}_1, \ldots, \Gamma_k \in \mathcal{F}_k$, $F_1 \in \mathcal{F}_1^0, \ldots, F_k \in \mathcal{F}_k^0$,

$$P_1(\Gamma_1) = P_\mu(\Gamma_1) \quad ; \quad P_1^0(\theta_1, F_1) = \gamma(F_1)$$

$$\vdots \qquad \vdots$$

$$P_k(\theta_1, \xi_1, \ldots, \theta_{k-1}, \xi_{k-1}, \Gamma_k) = \int_E P_x(\Gamma_k)\eta(\theta_{k-1}(\xi_{k-1}), dx)$$

$$P_k^0(\theta_1, \ldots, \xi_{k-1}, \theta_k, F_k) = \gamma(F_k)$$

Proof(Contd.)

- $P_1 \in \mathcal{P}(\Omega_1)$ and $P_1^0, P_2, P_2^0, \ldots$ are transition probability functions.
- $\exists$ an unique P on $(\Omega, \mathcal{F})$ satisfying
  For $C \in \mathcal{G}_k$ and $C' \in \mathcal{G}^{k+1}$

$$P(C \cap C') = \mathbb{E}\left[\int_C P(C'|\theta_{k+1}(0) = x)\eta(\theta_k(\xi_k), dx)\right].$$

- Define $\tau_0 = 0, \tau_k = \sum_{i=1}^k \xi_i$
  $N_t = k$ for $\tau_k \leq t < \tau_{k+1}$.
  Note that N is a Poisson process with parameter λ.

Proof(Contd.)

- Define
$$X_t = \theta_{k+1}(t - \tau_k), \; \tau_k \le t < \tau_{k+1}$$
- $\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^N$.
- For $f \in D(A)$
$$f(\theta_{k+1}((t \vee \tau_k) \wedge \tau_{k+1} - \tau_k)) - f(\theta_{k+1}(0)) - \int_{\tau_k}^{(t \vee \tau_k) \wedge \tau_{k+1}} Af(\theta_{k+1}(s - \tau_k))ds$$
is an $(\mathcal{F}_t)_{t \ge 0}$ martingale.
- Summing over k we get
$$f(X_t) - f(X_0) - \int_0^t Af(X(s))ds - \sum_{k=1}^{N(t)} (f(\theta_{k+1}(0)) - f(\theta_k(\xi_k)))$$
is an $(\mathcal{F}_t)_{t \ge 0}$ martingale.

Proof(Contd.)

- Also, the following are $(\mathcal{F}_t)_{t\geq 0}$ martingales.

$$\sum_{k=1}^{N_t}\left(f(\theta_{k+1}(0)) - \int_E f(y)(\eta(\theta_k(\xi_k), dy))\right)$$

$$\int_0^t \int_E (f(y) - f(X_{s-}))\,\eta(X_{s-}, dy)d(N_s - \lambda s)$$

- Hence

$$f(X_t) - f(X_0) - \int_0^t (Af(X_s) + Bf(X_s))\,ds$$

is an $(\mathcal{F}_t)_{t\geq 0}$ martingale.

$\square$