Coalition Formation Rules

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Abstract

We examine a scenario where a set of agents has to be sub-divided into coalitions. We assume that agents have preferences over coalitions that contain them. A coalition formation rule (CFR) takes a profile of such preferences and assigns agents to coalitions. We explore the properties of such rules. We find sufficient conditions for a CFR to satisfy the top-coalition property, which is a form of a unanimity condition of top preferences. We also show some impossibility results.

1 Preliminaries

Let N be a finite set of agents $\{1, 2, ..., n\}$. Let $\mathcal{N}$ be the set of all subsets of N. We call each subset a *coalition*. For each individual $i \in N$, let $\mathcal{N}_i$ be the set of all subsets of N that contain i.

For every agent i, we assume that there corresponds a weak binary relation R_i over $\mathcal{N}_i$, with strict component P_i. We interpret R_i as agent i's preferences over coalitions that contain him or her. For any $A \subseteq \mathcal{N}_i$, denote $\text{top}(A, R_i)$ as agent i's top preference in A.

Let $\mathcal{R}$ be the set of all such binary relations. A preference profile $\{R_1, ..., R_n\}$ is an element of $\mathcal{R}^n$, one relation for each agent.

Let $\{S_k\} \subset \mathcal{N}$ such that:

1. $S_j \neq \phi, \forall j$.
2. For every $i \in N$, $i \in S_j$ for some j.
3. For every $j \neq k$, $S_j \cap S_k = \phi$.

Then $\{S_k\}$ is a *partition* of N. Let Σ be the set of all partitions of N. For every agent i, we denote σ_i as the element of $\{S_k\}$ that contains i.

A *coalition formation rule* is a function $f : R^n \to \Sigma$ that assigns, for every preference profile, a partition of N.

2 Top-Coalition Property

Definition 1 *A coalition formation rule f satisfies the* top-coalition property (TC) *if, for any profile of preferences $R \in \mathcal{R}^n$ and any group of agents $T \subseteq N$,*

$$\forall i \in T,\ top(\mathcal{N}, R_i) = T \implies f_i(R) = T, \forall i \in T.$$

In other words, if a coalition of agents mutually consider that coalition to be their top preference, then a coalition formation rule satisfying TC must award them that coalition. Note that we put no restrictions on the size of T. In particular, T may be a singleton set.

Rodriguez-Alvarez (2004) shows that a coalition formation rule that satisfies strategy-proofness, individual rationality, non-bossiness, and a domain condition he calls voters' sovereignty, also satisfies TC. But these conditions are not necessary. Indeed, we may define coalition formation rules satisfying TC that violate each of the above conditions. See the examples below.

- *If a group of agents agree on a top preference, award it to them. Otherwise, for all agents not assigned their top preference, award them the largest remaining coalition.* It is easy to see that this rule satisfies TC but does not satisfy IR, VS or SP.
- *If a group of agents agree on a top preference, award it to them. Otherwise, for all agents not assigned their top preference, award them their top remaining preference in the order determined by agent 1's preference over them.* It is easy to see that this rule satisfies TC but does not satisfy NB.

3 Necessary conditions for the top-coalition property

Definition 2 *(Individual Rationality (IR)) A coalition formation rule f is called* individually rational *if, for every $R \in \mathcal{R}^n$ and every $i \in N$, $f_i(R) R_i \{i\}$.*

Definition 3 *(Voluntariness (V)) A coalition formation rule f is called* voluntary *if, for every $R \in \mathcal{R}^n$ and every $i \in N$, $top(\mathcal{N}, R_i) = \{i\} \implies f_i(R) = \{i\}$.*

That is, an individually rational coalition formation rule always assigns agents a coalition that they weakly prefer to being alone, whereas a voluntary coalition formation rule assigns the singleton coalition to every agent whose top preference is to remain alone. It is easy to see that voluntariness is a weakening of individual rationality. That is, IR implies V but V does not necessarily imply IR.

In our definition of the top-coalition property, we made no assumption on the size of the coalition that coordinates its top preference. In the event that we permit coalitions of cardinality 1, we immediately get voluntariness from the definition. Hence V is a necessary condition.

Definition **4** *(Freedom of Association (FA)) A coalition formation rule f satisfies FA if, for any coalition $T \subseteq N$, there exists a partition $\sigma \in \Sigma$ and a preference profile $R \in \mathcal{R}^n$ such that $T \in \sigma$ and $f(R) = \sigma$.*

Under FA, agents are free to associate themselves in any coalition. It is easy to see that FA is a weaker condition than VS as in Rodriguez-Alvarez (2004). Under VS, any partition may be generated by f under an appropriate preference profile. Here we stipulate that any coalition has at least one partition that contains it and a preference profile that generates it. That is, VS implies FA but FA does not necessarily imply VS.

We would like to respect freedom of association, since there is no reason a priori to distinguish some coalitions from others. In various contexts, there may be reason to limit the domain of partitions or, correspondingly, the range of the coalition formation rule. But in general, and to our purposes here, FA is necessary.

To see this formally, suppose that FA does not hold. So there is a coalition $T \subseteq N$ that may never form. Consider the following preference profile $R \in \mathcal{R}$: $\text{top}(\mathcal{N}, R_i) = T$ for all $i \in T$ and $\text{top}(\mathcal{N}, R_j) = \{j\}$ for all $j \notin T$. Here, $\text{top}(\mathcal{N}, R_i) = T$ for all $i \in T$ but $f_i(R) \neq T$ for all $i \in T$. Then TC does not hold.

In the way that TC is formulated, there are no other necessary conditions. TC applies only to those situations where agents coordinate on their top preference. In other words, agents who do not coordinate on a top preference may be assigned any coalition whatever, and the rule may still satisfy TC.

4 Sufficient conditions for the top-coalition property

Definition **5** *(Mutuality (M)) A coalition formation rule is said to be* mutual *if, for any $R \in \mathcal{R}$ and any $i, j \in N$, $i \in top(\mathcal{N}, R_j)$ and $j \in top(\mathcal{N}, R_i) \implies \{i, j\} \subseteq f_i(R)$ and $\{i, j\} \subseteq f_j(R)$.*

That is, a coalition formation rule is mutual if any pair of agents that include each other in their top preference is assigned the same coalition.

Definition **6** *(Connectedness (Co)) A coalition formation rule f is said to be* connected *if, for any $R \in \mathcal{R}$, $T \in f(R)$, and any $S \subset T$, $i \in top(\mathcal{N}, R_j)$ for some $i \in S$, $j \in T \setminus S$.*

A coalition formation rule is connected if, no matter how you split any coalition that it assigns, there is at least one agent in the two sides of that split that lists another agent from the other side in his top preference.

Lemma 1 *A coalition formation rule f that satisfies Co and M satisfies TC.*

Proof: Let f be a coalition formation rule that satisfies Co and M. Let $R \in \mathcal{R}$ be a preference profile, and let $T \subseteq N$ be a coalition such that $\text{top}(\mathcal{N}, R_i) = T$ for all $i \in T$. We have to show that $f_i(R) = T$ for all $i \in T$.
Consider any $i, j \in T$. Clearly, $i \in top(\mathcal{N}, R_j)$ and $j \in top(\mathcal{N}, R_i)$. Since f satisfies M, this implies that $\{i,j\} \subseteq f_i(R)$ and $\{i,j\} \subseteq f_j(R)$. Since i and j are arbitrary, $\{i,j\} \subseteq f_i(R)$ and $\{i,j\} \subseteq f_j(R)$ for all $i, j \in T$. Thus $T \subseteq f_i(R)$ for all $i \in T$.
Now, suppose that $f_i(R) = S$ for all $i \in T$ such that $T \subset S$. In particular, $f_j(R) = S$ for all $j \in S \setminus T$. So consider the split $(T, S \setminus T)$. For any agent $j \in S \setminus T$, there is no agent $i \in T$ with $j \in top(\mathcal{N}, R_i)$ (since $\text{top}(\mathcal{N}, R_i) = T$ for all $i \in T$). That is, we have found a subcoalition that violates Co. This is a contradiction. Since $S \supset T$ was arbitrary, this is true for all such S.
Hence $f_i(R) = T$ for all $i \in T$, and TC is satisfied.

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Note: Rodriguez-Alvarez (2004) shows that the combined properties of SP, NB, IR and VS are sufficient for a coalition formation rule to satisfy TC.

5 Some properties of coalition formation rules

In this section we list some desirable properties for a coalition formation rule to satisfy.

Definition 7 *(Inclusivity (I)) A coalition formation rule f is called* inclusive *if, for every $R \in \mathcal{R}^n$ and every $i \in N$, $top(\mathcal{N}, R_i) \neq \{i\} \implies f_i(R) \neq \{i\}$.*

Inclusivity imposes the restriction that every agent whose top preference is not to be alone is assigned to a non-singleton coalition. In other words, V and I together imply that only those agents whose top preference is to be alone are assigned to singleton coalitions.

Definition 8 *(Strategy-proofness (SP)) A coalition formation rule f is called* strategy-proof *if, for every $R \in \mathcal{R}^n$, every $i \in N$, and every R'_i, $f_i(R) R_i f_i(R'_i, R_{-i})$.*

That is, no agent can make himself or herself better off by falsely reporting some other preference ordering. This is a standard definition.

Definition 9 *(Cohesiveness (C)) A coalition formation rule f is called* cohesive *if, for every $R \in \mathcal{R}^n$, every $i \in N$, and every R'_i, $f_i(R'_i, R_{-i}) P_i f_i(R) \implies f_j(R'_i, R_{-i}) R_j f_j(R) \forall j \in N$.*

Cohesiveness imposes the condition on coalition formation rules that an agent may consider it profitable to mis-report his or her own preferences only when doing so leaves all other agents at least as well off as before. It is clear that a strategy-proof rule satisfies cohesiveness trivially, since no agent ever finds it profitable to mis-report preferences. However, cohesiveness is a weaker condition in that it allows the possibility of mis-reporting, but only when doing so leads to a socially "better" outcome.

Definition **10** *(Non-bossiness (NB)) A coalition formation rule f is called* non-bossy *if, for every $R \in \mathcal{R}^n$, every $i \in N$, and every R'_i, $f_i(R'_i, R_{-i}) = f_i(R) \implies f(R'_i, R_{-i}) = f(R)$.*

That is, if by changing his or her preference an agent does not change his or her own outcome, then this change in preference should not result in a change in outcome for any other agent either.

Definition **11** *(Weak Efficiency (WE)) A partition σ'* is dominated by *a partition σ if $\sigma_i R_i \sigma'_i$ for all $i \in N$ and $\sigma_j P_j \sigma'_j$ for some $j \in N$. A coalition formation rule f is* weakly efficient *if, for every $R \in \mathcal{R}^n$, $f(R)$ is not dominated by any other partition.*

Definition **12** *(Weak Coalitional Efficiency (WCE)) A partition σ'* is dominated via coalition *$T \subseteq N$ by a partition σ if:*

- $T \in \sigma$
- $T P_i \sigma'_i$ *for all* $i \in T$

A coalition formation rule f is weakly coalitionally efficient *if, for every $R \in \mathcal{R}^n$, $f(R)$ is not dominated via any coalition T by any other partition.*

In other words, a partition is dominated by another partition if there is a coalition that forms under the second partition such that each of its members strictly prefer it to what they were getting under the first partition. Note that we say nothing about the preferences of agents outside this coalition, hence this is a weak form of this statement.

Lemma **2** *There is no coalition formation rule f satisfying TC, V, C, and I.*

Proof: We prove this lemma with an example. Suppose there are four agents. Let f satisfy TC, V, C and I. Consider the (incomplete) preference profile as given in Table 1:

Table 1: A preference profile R

1	2	3	4
$\{1,3\}$	$\{2,3\}$	$\{1,3\}$	$\{2,4\}$
	$\{2\}$		$\{4\}$
	$\{2,4\}$		

TC implies that agents 1 and 3 get the coalition $\{1,3\}$. Condition I implies that $f_2(R) \neq \{2\}$ and $f_4(R) \neq \{4\}$. Therefore $f_2(R) = f_4(R) = \{2,4\}$.

Table 2: A preference profile R' with agent 2's preference modified

1	2	3	4
$\{1,3\}$	$\{2\}$	$\{1,3\}$	$\{2,4\}$
	$\{2,3\}$		
	$\{2,4\}$		

But consider the profile R' (Table 2), in which we change only agent 2's preference.

As before, TC implies that agents 1 and 3 get the coalition $\{1,3\}$. But now V implies that $f_2(R) = \{2\}$. So $f_4(R) = \{4\}$.

Now, $\{2\} = f_2(R')P_2f_2(R) = \{2,4\}$. So agent 2 considers it profitable to mis-report using R'_2. But $\{2,4\} = f_4(R)P_4f_4(R') = \{4\}$. This violates C, which requires agent 4 to be at least as well off when 2 mis-reports.

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6 Side-Track

The top-coalition property applies only to those situations where there are agents who coordinate on a top preference of coalition. Even when satisfied, it leaves open the question of what to do with agents who are unable to do so. If we assume voluntariness, these are precisely those agents whose top preference contains at least one other agent besides themselves, i.e., with $|top(\mathcal{N}_i, R_i)| \geq 2$. Also, we can say that at least one of the other agents in each agent's top preference must have a different ranking (because otherwise they would be coordinated and TC would apply).