CHOICE UNDER SOCIAL CONSTRAINTS

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Abstract

Agents choose a best element from the available set. But the available set may itself depend on the choices of other agents. We call such a scenario *constrained choice* and design a model to explore its consequences.

1 PRELIMINARIES

Let X be a finite set of alternatives $\{a, b, c...\}$. Let $\mathcal{X}$ be the set of all subsets of X. Let N be a finite set of individuals, with typical elements $\{1, 2, ...\}$. For each individual $i \in N$, we assume that there corresponds a preference ordering R_i over X, with strict component P_i. We interpret R_i as agent i's preferences over X. Denote the set of strict preference profiles as $\mathcal{P}$. It is a well-known result that, if P_i is a strict ordering, for every $A \in \mathcal{X}$, the set $\{x : xP_iy, \forall y \in A\}$ is well-defined and singleton for all agents i. So, given a subset A of $\mathcal{X}$, and a subset M of N, we may define a vector $\mathbf{b} \in X^M$, such that $\mathbf{b_i}$ is agent i's best element in A.

2 SOCIAL CONSTRAINTS

In this section we wish to explore the possibility that the set of actions (alternatives) available to an agent depends on the actions of everyone else. In other words, when the vector $\mathbf{x}_{-i}$ corresponds to the actions of all agents other than i, then the subset of actions available to player i is some function of $\mathbf{x}_{-i}$, i.e., a subset of X.

Formally, a social constraint $c : X^{N-1} \to \mathcal{X}$ is a correspondence that maps the vector of other agents' chosen alternatives to a subset of X. This subset represents the choice set available to the agent. Also note that the best element $\mathbf{b_i}$ may not belong to $c_i(\mathbf{x_{i-1}})$. A profile of social constraints C is an N-dimensional vector of social constraints. The set of all profiles of social constraints is $\mathcal{C}$.

Some examples of social constraints:

1. (Strict Exclusivity SE)) An agent may only pick an alternative that no other agent has picked. This is the case when, for example, assigning office space or picking cookies from a tray, such that the item goes only to one agent who picks it. Under this interpretation we define c^{SE} as follows:
   For any $\mathbf{x}_{-i}$, $c_i^{SE}(\mathbf{x}_{-i}) = X - \bigcup x_{-i}$, for all $i \in N$.

2. (Strict Inclusivity (SI)) An agent is free to pick from only those alternatives that some other agent has also picked. This is the case in situations where, for example, a new member of a club must pick his or her affiliations upon joining. It is also the case when individuals are assigned a religion at birth. Under this interpretation we may define c^{SI} as follows:
   For any $\mathbf{x}_{-i}$, $c_i^{SI}(\mathbf{x}_{-i}) = \bigcup x_{-i}$, for all $i \in N$.

3. (Dictatorship (D)) There is an agent $J \in N$ such that the feasible set of all other agents is the choice that J makes. So we may define c^D as follows:
   There exists a $J \in N$ such that, for any $\mathbf{x}_{-i}$, $c_i^D(\mathbf{x}_{-i}) = \mathbf{x}_J$, for all $i \neq J$.

3 ALLOCATIONS AND MECHANISMS

An **allocation** is an injective map from N to X. A **feasible allocation** $y \in X^N$ is such that $y_i \in c_i(y_{-i})$ for all $i \in N$. For any profile of social constraints C, the set of all feasible allocations is Y^C.

An **allocation mechanism** $f : \mathcal{P}X\mathcal{C} \to X^N$ assigns an allocation for all admissible profiles of preferences P and social constraints C.

DEFINITION **1** *(Feasibility) A mechanism f is* **feasible** *if, for any profile of preferences P and social constraints C, $f_i(P, C) \in c_i(f_{-i}(P, C))$ for all $i \in N$. A feasible mechanism that respects Strict Exclusivity is called* **exclusive***. while one that respects Strict Inclusivity is called* **inclusive***.*

DEFINITION **2** *(Strategyproofness) A mechanism f is* **manipulable** *if there is an agent i, a preference ordering P_i', and a profile of preferences P such that $f_i((P_i', P_{-i}), C) P_i f_i(P, C)$. That is, agent i can make himself strictly better off by misreporting his preferences. A mechanism that is not manipulable is* **strategyproof***.*

DEFINITION **3** *(Group-strategyproofness) A mechanism f is* **group-manipulable** *if there is a subset of agents M, preference orderings P_M', and a profile of preferences P such that $f_i((P_i', P_{-i}), C) P_i f_i(P, C)$ for all $i \in M$. A mechanism that is not group-manipulable is* **group-strategyproof***.*

It is clear that every mechanism that is group-strategyproof is also strategyproof (in the case that M consists of one agent). But every group-strategyproof mechanism need not be strategyproof. In fact, Papai (1998) proves that a mechanism is group-strategyproof if and only if it is strategyproof and non-bossy.

DEFINITION **4** *(Efficiency) A mechanism f is* **weakly efficient** *if, for any profile of preferences P and social constraints C, there is no allocation $y \in X^N$ such that $y_i P_i f_i(P, C)$ for all $i \in N$.*

DEFINITION **5** *(Neutrality) A mechanism f is* **neutral** *if, for all preference profiles P, social constraints C, and permutations π of X, $f(\pi P, C) = \pi f(P, C)$.*

This means that the "real" outcome of a neutral mechanism is independent of the names of the indivisible goods.

DEFINITION **6** *(Nonbossiness) A mechanism f is* **nonbossy** *if, for all individuals $i \in N$, for all preferences R'_i, and all preference profiles P, and social constraints C, $f((R'_i, R_{-i}), C) = f(R, C)$ when $f_i((R'_i, R_{-i}), C) = f_i(R, C)$.*

This means that an individual cannot change the outcome of the mechanism without changing the outcome for himself at the same time.

4 EXCLUSIVITY

This problem is similar to the House Allocation Problem without initial private endowments, in that it is a question of allocating a finite number of indivisible goods among a finite number of agents that have a strict preference ordering over the goods, but do not pre-own any of them.

In the case of initial private endowments or property rights, the **Top Trading Cycle Algorithm (TTCA)** is Pareto efficient and strategyproof. It is also stable (the allocation lies in the core of the relevant preference profile, i.e., it is coalitionproof). Moreover, there is a unique core matching for every preference profile (Debasis Mishra online notes). But it is not applicable to the non-ownership case, except by random endowments. Instead, we propose an alternate algorithm that has certain desirable advantages over the TTCA. But it is not without its own problems, as we will see below.

DEFINITION **7** **Exclusive Algorithm (EA):** *Agents are ordered according to a fixed index. In any round, all agents point to the alternative they most prefer. For any alternative, if there is only one agent that points to it, it is assigned to that agent. If more than one agent points to a given alternative, it is assigned to the agent with the lowest index. The unassigned alternatives and remaining agents survive to the next round, which proceeds the same way until all agents have been allocated an alternative.*

It is easy to see that, as with the TTCA, EA is strategyproof. Moreover, it may be shown that EA satisfies neutrality and the non-bossy condition. So it must be a serial dictatorship (Svensson (1999)). Indeed:

LEMMA **1** *EA is a serial dictatorship in the special case when all individual preference orderings coincide.*

Proof: Let all agents possess the same preference ordering P, and without loss of generality, let the preference ordering be $\{a, b, c, d, ...\}$. Consider the first round of the algorithm. Since the mechanism is strategyproof, no agent can do better than to pick their best alternative, and so all agents pick a. The mechanism assigns a to individual 1. In the next round, all agents pick b, and the mechanism assigns b to agent 2. The rounds go on until all agents have received an alternative. So the agents have been allotted $\{a, b, c...\}$.

It is easy to see that a serial dictatorship with the same fixed priority $\{1, 2, ...\}$ will produce the same allocation.

Thus the equivalence is established. ■

However, EA does not assume ownership. Thus the TTCA proof involving the necessary presence of cycles at each stage may not apply.

In any round of the algorithm, consider the case where two agents, say, point to the same alternative. The TTCA would resolve this tie by awarding the alternative to the agent that belongs to a cycle. Naturally, because of ownership, both may not belong to a cycle. Thus one agent is awarded the alternative, and the other is not.

In the case of EA, ties are broken by means of the fixed priority. This is similar to a 'first come, first served' approach. The advantage is that even agents with unpopular endowments may be awarded their preferred alternatives, and are not excluded from the mechanism. The disadvantage is the appeal to a fixed priority of agents, which is implicitly hierarchical. Unless the list is permuted regularly (or indeed occasionally), this is not an appealing property.

5 INCLUSIVITY

Note that it is easy to show that an exclusive, efficient and group-strategyproof mechanism must give at least one agent his best alternative. Instead, we show by means of an example that this is not true in the case of inclusivity.

LEMMA **2** *An inclusive, group-strategyproof and efficient mechanism may not give any agent his best alternative.*

Proof: Suppose there are three individuals $\{1, 2, 3\}$ and four alternatives $\{a, b, c, d\}$. Suppose that individual preferences are given according to the table:

Table 1: A preference profile - 3 agents, 4 alternatives

Agent 1	Agent 2	Agent 3
a	c	d
b	b	b
c	d	a
d	a	c

It is clear that the set of feasible allocations are the ones that assign the same element to all three agents. Otherwise the inclusivity constraint is violated for at least one agent. So there are four such allocations: $\{a, a, a\}, \{b, b, b\}, \{c, c, c\}$, and $\{d, d, d\}$. Of these, it is easy to show that all except the second allocation are not group-strategyproof. Any two of the three agents may mutually agree upon some other alternative that they both prefer to the assigned alternative. (e.g., in the case of $\{a, a, a\}$, agents 2 and 3 may agree on alternative d, which they both prefer to a, and declare it their best. Their respective inclusivity constraints are satisfied, whereas agent 1's is violated. Efficiency will demand that they are allocated d.) On the other hand, $\{b, b, b\}$ is group-strategyproof. It is also efficient. It is thus the unique outcome of any inclusive, efficient and group-strategyproof mechanism. But no agent gets his best alternative. ■

We extend this result to another lemma.

LEMMA **3** *When the number of agents equals the number of alternatives and is an odd number, there is no inclusive, efficient and group-strategyproof mechanism.*

Proof: We give a sketch.

Consider the case where $\#N = \#X = 3$, and consider the purely cyclic preference profile. As argued above, any feasible allocation must give the same alternative to each agent. But no such allocation is group-strategyproof in this case.

Suppose now that $\#N = \#X = 5$. Now, any feasible allocation must award only one or two alternatives. One in the case of unanimity and two if preferences are split, with three agents getting one alternative and two getting the other. No other allocation is feasible. Again, consider the purely cyclic preference profile. None of the above feasible allocations is group-strategyproof.

We may extend this argument to any odd n. Thus the lemma is established. ■