Pension, Possible Phaseout and Endogenous Fertility in General Equilibrium*

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Abstract

The rich literature on Pay-As-You-Go (PAYG) type pensions provides a notion that when pension return is dominated by the market return, generally it is impossible to phase pension out without hurting any generation. We show that PAYG pensions can indeed be phased out in a much richer framework where fertility is endogenous and general equilibrium effects are present. Interestingly, the factor that helps us to phase the pension out in a Pareto way is hidden in the structure of PAYG pension itself. Individualistic agents fail to recognize the benefits of their fertility decision on these programs and, therefore, end up in an allocation that is strictly dominated by the allocations that internalize this externality. Exploiting this positive externality, competitive economy can improve its allocations and can reach the planner's steady-state in finite time where each generation secures as much utility as in the competitive equilibrium. Clearly, it is possible to transition in a Pareto way to an economy either with no pension or with pensions whose return is not dominated by market return.

Keywords: Endogenous fertility, Pension, Externality, Phase out of pension, Efficiency

*We are grateful to the editor, an associate editor, and two referees for their invaluable inputs that have helped us improving the paper significantly. We would also like to thank Joydeep Bhattacharya, Shankha Chakraborty, David de la Croix, Mausumi Das, Dirk Krueger and Morten Ravn for many helpful comments and suggestions.

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1 Introduction

Benefits of having a healthy number of children are enjoyed by any society if there is a publicly funded contributory system such as the Pay-As-You-Go (PAYG) pensions since it broadens income and hence the pension tax base. An individual agent however fails to internalize this benefit into account. An individual's decision to have children is driven solely by the direct utility she enjoys from her children, whereas they also have a positive externality on the society. This particular form of externality was recognized quite some time ago and is now well-established in the pension-related literature.[1] This social benefit of children can lead to an extreme form of free-riding behavior where individuals keep themselves away from investing in their own children but avail the benefits of having children through a PAYG type pension system.

In fact a severe budgetary pressure that PAYG type pensions are facing is mainly due to a demographic shift in the developed countries, typically the home of PAYG. A fall in fertility has coincided with the increase in size of the pension system (Boldrin, De Nardi and Jones (2015)), raising concerns over its feasibility. From a theoretical perspective too, the celebrated Aaron - Samuelson results confirm that the PAYG pension regime is welfare reducing if the economy is dynamically efficient (Aaron (1966)) where PAYG return is dominated by the market return.[2] There are some other concerns including behavioral ones that also sometimes go against this age-old instrument.[3] All these create an impression that, given the present state of affairs, a

[1] For an up-to-date discussion of the fiscal externality due to children see Barnett et al. (2018).

[2] According to Abel et al. (1989), U.S. and other OECD countries are dynamically efficient. Mankiw (1995) mentions: "... excessive capital accumulation is not a practical concern for policy-makers. Actual economies appear to have less capital than the Golden Rule level." Barbie, Hagedorn and Kaul (2004) present a test criterion based on Zilcha (1991) and robust evidence that the U.S. economy is dynamically efficient.

[3] Admittedly pension programs have many critiques, some of which are purely on philosophical grounds. Possibly the most important one is a myopia or present bias in consumption. Individuals differ in their tastes and the government may not be the best judge of what is in their best interest (see Friedman (1962), Feldstein (2005)). A paternalistic intervention like public pension based on the value judgement may increase old age consumption (for a review of the literature dealing with the rationale for social security via its effect on savings see de la Croix and Michel (2002)). Social security in the presence of time inconsistency have been discussed extensively (see, for example, İmrohoroğlu, İmrohoroğlu and Joines (2003) and Caliendo (2011)). Andersen and Bhattacharya (2011) revisit the role played by myopia in generating a rationale for PAYG pension in dynamically efficient economies. Also, provision of old age benefits distorts retirement behavior and the tax that is imposed on the working population may distort labor supply (see Feldstein (1985)).

PAYG may not be desirable on its own unless it serves some other purpose.[4] Since for a plethora of reasons sustainability of PAYG type pensions has been a matter of concern for the last few decades, many countries are pursuing reforms and re-evaluating the generosity of this age-old program. Even the extreme form of it, the elimination of the program, has also been discussed extensively in the literature as well as in the policy circle. The literature on phasing out PAYG pensions or moving to a fully funded one from PAYG type pensions is very rich. Possibly the most important issue in this literature is whether all these reforms can be carried out in a Pareto way. The broad conclusion is that it is generally difficult to compensate the first generation of pensioners for the loss incurred without making at least one later generation worse off.[5]

[4] Apart from shortsightedness as one of the main motives (also see, for example, Kotlikoff (1987) and Kaplow(2015)), a benevolent planner can have numerous important reasons behind justifying a social security system. Some of them are income redistribution (see Diamond (1977)), risk sharing between or within generations (see Enders and Lapan (1982), Smith (1982) and Sinn (2004), among others), repairing the annuity markets (see, for example, Diamond (1977) and Feldstein (1990) among others). Krueger and Kubler (2006) analyzed the role of unfunded pensions in the presence of idiosyncratic risks when financial markets are incomplete. Political economy issues and sustainability of social security have also been analyzed extensively (see, for example, Browning (1975), Lambertini and Azariadis (1998), Conesa and Krueger (1999), Cooley and Soares (1999), Azariadis and Galasso (2002), Lancia and Russo (2016), Bishnu and Wang (2017), Ono and Uchida (2016) and, for a detailed review of earlier literature, see Galasso and Profeta (2002)). One more crucial reason can be the need for a balance between two differently directed intergenerational goods, mainly education and pensions. This rich literature started with Pogue and Sgontz (1977), Becker and Murphy (1988), and further enriched by Kaganovich and Zilcha (1999), Boldrin and Montes (2005), and very recently contributed by Docquier, Paddison and Pestieau (2007), Bishnu (2013), Wang (2014) among others (education and pension combination has also been used to explain, for example, growth as in Zhang (1995) and inequality as in Glomm and Kaganovich (2003)).

[5] This issue has been analyzed in many different frameworks. For example, Fenge (1995) uses a setup where agents are not liquidity constrained and, since a move from a PAYG to an actuarially fair fully funded system has no behavioral impact, there is no case for Pareto improvement. Breyer (1989) also has a similar result, while in a different context Rangel (1997) and Kotlikoff (2002) reach the same conclusion (also see, for example, Breyer and Straub (1993), Miles (1999), and Sinn (2000)). A detailed discussion on this can be found in Lindbeck and Persson (2003) and for related issues see Barr and Diamond (2006). Cremer and Pestieau (2000) argue that economic and demographic factors play a relatively small role in the old-age crisis, rather political factors are far more crucial. Cipriani and Markis (2012) focus on endogenous longevity that interacts with social security system. In that environment, they show that if the economy is dynamically efficient, PAYG pensions must be sufficiently low to ensure positive economic growth. Further, a transition to a funded social security system will promote growth, and can thereby take place by fully compensating the losers. Very recently Andersen et. al. (2021) have shown that a Pareto improving transition from PAYG to a fully funded system is feasible when agents have present bias. A point to note here is that our analysis assumes that the labor supply is inelastic. Breyer and Straub (1993) (also discussed in Blake (2006)) show that an unfunded system replaced by a fully funded one leads to an intergenerational Pareto improvement but a necessary condition for this improvement is that the labor supply is distorted.

Using a general equilibrium framework, in this paper we present a novel mechanism that exploits the well accepted inefficiency due to externality, spawning from having children in a contributory public pension system as discussed above, for pension reforms in a Pareto way.[6,7] By reform we mean moving from a competitive equilibrium to an equilibrium that the planner would like to have as it internalizes the externalities. Overall, in this paper, we follow three important steps to present our results. First, we characterize the planner's best allocation under endogenous fertility where inefficiency is immediate in the presence of externalities but the planner internalizes it. We find similar criteria regarding the welfare improving role of PAYG pensions in terms of relative returns from PAYG vis-a-vis the market when compared to the case of exogenous fertility where there is no such externality. Second, we show that by exploiting this positive externality of having children in a PAYG pension system, the competitive economy can improve its allocations and finally reaches the planner's equilibrium in finite time. Importantly, throughout the process, each generation secures as much utility as in the competitive equilibrium. Third, PAYG pensions can be completely phased out without hurting any agent in any generation. We clearly specify the instruments needed under various scenarios to decentralize the planner's optimum.

Thus, in a nutshell, the present paper attempts to present a reform of PAYG and, if possible, a complete phaseout of pensions, when fertility is endogenous. Importantly, the model captures the general equilibrium effects where by capitalizing on an inefficiency that arises from the PAYG pensions itself, it can be phased out and,

Andersen and Bhattacharya (2013)) investigate the matter further and come to the conclusion that under the sufficient condition that the old be no less risk-averse than the young, the classic Aaron - Samuelson result can be extended to an economy with endogenous labor supply. While our analysis assumes that labor supply is inelastic, the standard efficiency concepts also changes in our setup where fertility is endogenous. Further, especially when fertility is endogenous, labor supply is definitely affected through various channels including the time cost of raising children. We however do not focus on that aspect in this paper.

[6] A partial equilibrium setup is relatively simple to deal with and all the main results hold. These results are available on request.

[7] Here we must specify that 'Pareto criterion' to compare allocations in terms of individual welfare applies only to fixed populations. However, in this analysis, we deal with endogenous fertility and therefore the standard 'Pareto' as in under exogenous fertility is not directly applicable. Here by 'Pareto criterion' we refer to the utility of a representative agent across two scenarios, that is, the same agent if born under two different scenarios, bypassing the issue of different population sizes across the allocations. This is one of the efficiency concepts used in the endogenous population literature and has been discussed in detail in section 3 below.

during the entire journey, no agent from any generation is worse off. Apart from a rich set of works (some of which we have cited above), two recent papers that are somewhat similar to this work in the spirit of phasing out PAYG pensions are Andersen and Bhattacharya (2017) and Bishnu et al. (2021). Unlike the present paper, both of them have exogenous fertility as well as exogenous factor prices. Also, both of them have education in their setup. While the first paper uses human capital externalities to phase out pensions starting from the complete market allocation (the best possible allocation when a perfect credit market to borrow funds for education is present), the second paper characterizes the optimal path of phasing out pensions starting from an incomplete market using only the market inefficiency. We have verified that our main mechanism in this paper is clearly valid in a framework where education is also present.[8]

In what follows we explain the basic mechanism that is at work in our paper. First consider our approach of dealing with the externality. Given the positive externality of children in a PAYG regime, we know and formally show that any generation can be made better off if this positive externality can be internalized. Internalizing the benefits of fertility reduces the effective cost of having children. This leads to a better allocation than the competitive equilibrium with PAYG.

We then derive the planner's optimum which consists of a path that the planner would ideally want to follow and finally the steady-state thereafter (we call it the *O Steady-state*). These allocations of the planner internalize the externalities and incorporate all the general equilibrium effects. We impose a constraint that rules out the possibility of having a transfer from the retired to the working class, an event that is typically not observed. In this framework, we observe that, similar to the exogenous fertility case, the planner allocations at the steady-state either reach the scenario where the market and PAYG returns are the same (the 'golden rule' as in exogenous fertility case) with a positive PAYG pension, or, if the economy has a higher market return than PAYG pensions, that is, PAYG return is dominated by the market return

[8] A unified treatment of externalities associated with both fertility and human capital accumulation within pay-as-you-go pension systems has been presented in Cremer et. al. (2011). However, their focus is not on the phaseout of PAYG pensions, rather on how these externalities interact with each other and how one must use the combination of child and education subsidies to internalize both the externalities.

(‘dynamically efficient’ as in exogenous fertility case), PAYG pension support should be zero. This needs no further explanations but to say that PAYG is not welfare improving when it is return dominated as is standard in the literature on pensions with exogenous fertility. Since we rule out the possibility of having a transfer from the retired to the working class by imposing a non-negative resource constraint, the constrained planner may be prevented from achieving the ‘golden rule’ for certain preference and production parameters. This corresponds to the second possible optimum where pension is zero.

While this is an additional observation, our main finding is that a Pareto dominating transition to this O Steady-state is clearly possible in finite periods. That is, starting from the competitive PAYG steady state, we follow a path that guarantees each generation at least the utility that the competitive equilibrium provides, and, in finite time, reaches the O steady-state. Interestingly, we can exploit the inefficiency attached to PAYG pension itself and phase it out in a Pareto way when PAYG return is market dominated (as in a dynamically efficient economy under an exogenous fertility setup). The intuition behind this result is presented in the following paragraph.

After taking over the economy from PAYG steady state, the planner starts reallocating resources that increases income and hence the tax base because of internalizing the inefficiencies attached to externalities resulting in higher fertility. With enhanced resources, the planner can increase the utility of the present generation from the utility under PAYG steady state. This increase in current generation’s utility allows the planner to reduce its pension benefits while ensuring that the utility does not fall below the utility under PAYG steady state. Therefore, while satisfying the Pareto criterion of not making any generation worse off than under the PAYG pension system, the planner can decrease this generation’s pension benefits and hence the next generation’s pension tax. The planner can again reduce this generation’s pension benefits and hence the following generation’s pension tax. As the next generation faces a lower pension tax, the planner can keep on decreasing subsequent generations’ pension taxes. This process continues till the optimal steady state for the planner is reached. We also show that, throughout the transition, capital per worker keeps increasing. However, the impact on capital at the beginning and end periods of the transition path is analytically ambiguous; it depends on the features of the economy.

Since capital per worker determines the factor prices, we can accordingly trace the trajectories of the factor prices.

Given the above results, it remains to show how to decentralize this planner's solution so that the above outcome is implemented in a competitive PAYG equilibrium. Since the competitive PAYG equilibrium cannot internalize the inefficiencies associated with endogenous fertility, to provide incentives to enhance the fertility rate, we find that a subsidy on child care is needed. The subsidy scheme makes the current generation better off as it has more children and hence higher pensions and utility. This gives us the ground to start the pensions reform program of declining pensions over generations as discussed in details in the last paragraph. When returns from PAYG pensions and the markets are the same as in the 'golden rule', pension should perpetually be accompanied by a child care subsidy as appears in van Groezen, Leers and Meijdam (2003). Schoonbroodt and Tertilt (2014) also point out that the PAYG pensions alone cannot guarantee an efficient outcome in the presence of endogenous fertility which is due to the same reason that the costs and benefits of producing children remain unaligned. To achieve an efficient outcome, government transfers need to be tied to fertility choice. However, in the complete phaseout situation, as we have characterized above when PAYG return is dominated by the market return, an increase in fertility is not at all encouraged whereas further capital accumulation (and hence lower return to capital) is needed for the economy to move closer to the 'golden rule'. This is accomplished through a child care tax accompanied by a savings subsidy.

We summarize our results as follows. Harnessing the positive externality of children in a PAYG regime, the government can implement a pension reform in a Pareto way. At the end of the transition, the economy reaches its *optimal steady-state*. We characterize this optimal steady-state and show that there exists a path to the optimum such that the transition as well as the steady-state dominate the current steady-state with PAYG pensions. That is, under the pension reform, no generation has less utility than the alternate scenario where this reform was not undertaken. At the optimal steady-state, either the economy achieves the 'golden rule' with the endogenous return on capital R equalling the rate of return from pension (equivalent to the fertility rate), or there is 'under-accumulation' where PAYG is return dominated by the market. In the first case PAYG pension is positive and must be accompanied by a child care subsidy. In the later case where PAYG pension hits its zero lower bound and is therefore completely phased out, we show that the optimal steady-state must be supported by a subsidy on savings and a tax on child care.

Thus, one way to interpret our results is that apart from other issues related to the sustainability and reform of PAYG pensions, our analysis broadly corrects a somewhat double policy error associated with introducing PAYG pensions from the efficiency perspective. First, when PAYG pension is return dominated by the market, a situation represented by the standard dynamically efficient economies, there is no long run welfare gain through pensions and second, it creates an externality in the process when fertility choice is endogenous. Thus, in some sense, our theoretical analysis exploits the second error to correct the first one, but more importantly, does it in a Pareto way. Our proposed child subsidy as a part of the correcting mechanism goes well with the observed child-friendly policies in some countries. We would like to mention here that the phase out of pensions in our paper does not follow an 'inverted U' pattern as is observed in Andersen and Bhattacharya (2017) or Bishnu et al. (2021) representing the rise and fall of pensions over time. In other words, both the rise and fall of pensions may not be rationalized in the present analysis if just the fertility externality is used to phase pensions out.

We have tried our best to keep the model as simple as possible so that the main mechanism does not get overlooked when additional things are incorporated in the model. For example, we have verified that a multi-period model instead of only three periods or a model that has human capital as a choice variable clearly do not change the main findings of the paper.

The rest of the paper is organized as follows. Section 2 sets up the model. In section 3 we characterize the PAYG competitive equilibrium at the steady-state. Section 4 shows suboptimality of the PAYG competitive equilibrium. Section 5 characterizes the O steady-state while section 6 demonstrates a Pareto-improving transition from PAYG competitive equilibrium to the O steady-state. In section 7 we decentralize the planner's optimal steady-state and its transition. Section 8 concludes. All the proofs are presented in the Appendix.

2 The model

We consider an overlapping generations economy where agents live for three periods. They are young in the first period, middle-aged in the second and old in the third. An agent is born in the first period. She earns wage in the second period, consumes in that period and saves for her old age. She also decides to have children in this second period. Finally, she consumes the returns from her investment in the third period. An agent derives utility from her consumption in the middle age and consumption in the old age. For simplicity, we assume that the agents do not consume anything when young. The agent also derives utility from the number of her children. For notation, we identify a generation by the period when it is in middle age. That is, if an agent was born in period $t-1$ and is of middle age in period t, we call her a generation t agent.

We assume that the utility of a generation t agent is given by

$$u(c_t^m) + \beta u(c_{t+1}^o) + v(n_t),$$

where c_t^m and c_{t+1}^o are the agent's consumption in her middle age and old age respectively. The agent discounts her utility from consumption in old age by β where $\beta \in (0,1)$. The utility from consumption is given by the function $u(.)$ which is assumed to be strictly increasing and concave, that is, $u'(.) > 0$ and $u''(.) < 0$. It also satisfies Inada conditions, that is, $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. Since fertility is an important issue, especially for the analysis of sustainability of PAYG type pensions, we model fertility as endogenous. An agent's number of children is denoted by n_t. The utility an agent derives from her children is denoted by $v(.)$. We assume that $v(.)$ also is strictly increasing, concave and satisfies Inada conditions.[9]

The production function in period t, $f(s_t, n_{t-1})$, follows constant returns to scale technology with respect to the factors – capital accumulated through the savings of generation $t-1$ agents, s_t, and labor available in period t, n_{t-1}. Further, it satisfies Inada conditions along with the standard concavity assumptions. We also assume

[9] This formulation of utility is fairly general as well as simple. In a model with human capital, the last term of the utility may be modified to capture quality - quantity trade off (see for example, de la Croix and Doepke (2003), Doepke (2004)).

that capital gets fully depreciated after production. Factor markets being competitive, equilibrium factor prices are given by their marginal products: $R_t(s_t, n_{t-1}) = f_1(s_t, n_{t-1})$ and $w_t(s_t, n_{t-1}) = f_2(s_t, n_{t-1})$, where f_i represents marginal productivity of the ith factor of production.

2.1 The PAYG Regime

There is a government administered pension program which is of PAYG type where an agent in her middle age pays a proportion of her income while working to support pension for the old. In return, she receives pension support in her old age. The PAYG scheme requires paying a proportional pension tax τ_t at time t. A generation t agent pays this pension tax τ_t of the earning w_t in her middle age and receives pension support p_{t+1} in $t+1$ when old. The agent supplies labor inelastically and earns a wage w_t. Raising each child has a constant cost q and an agent who has n_t children bears a cost of qn_t. She also saves for her old age which is denoted by s_{t+1}. Thus, for this generation t agent, the middle age budget constraint is given by

$$c_t^m + qn_t + s_{t+1} = (1 - \tau_t)w_t.$$

In the old age the agent survives on her savings in the middle age s_{t+1} which earns a gross interest rate R_{t+1}, as well as pension support p_{t+1} from the government. Thus, the agent's old age budget constraint is given by

$$c_{t+1}^o = s_{t+1}R_{t+1} + p_{t+1}.$$

We assume that the government's balance budget condition for the pension program holds in every period. In period $t+1$ it funds pension to the old, p_{t+1}, by taxing the earnings of n_t middle-aged of generation $t+1$ at the rate τ_{t+1}. Thus the government's balanced budget constraint in period $t+1$ is given by

$$p_{t+1} = \tau_{t+1}n_t w_{t+1}.$$

3 The PAYG Equilibrium at the Steady-state

With a proportional PAYG pension tax τ_t, a generation t agent in this economy solves the following problem taking the factor prices as given:

$$\begin{aligned}
&\max_{\{s_{t+1},n_t\}} u(c_t^m) + \beta u(c_{t+1}^o) + v(n_t) \\
&\quad \text{subject to} \\
&\quad\; c_t^m = (1-\tau_t)w_t - qn_t - s_{t+1}, \\
&\quad\; c_{t+1}^o = s_{t+1}R_{t+1} + p_{t+1}.
\end{aligned} \tag{A}$$

The government's budget balancing ensures $p_{t+1} = w_{t+1}(s_{t+1}, n_t)n_t\tau_{t+1}$. But individuals act atomistically and do not take the government's pension budget constraint into account while solving their problem. The change in tax base to fund the pension due to one individual's fertility choice will be small and imperceptible to the individual. Therefore, parents take into account only the direct utility they enjoy from the quantity of their children and ignore the pension benefits they receive as a result of investing in the number of their children.

Under the PAYG regime at the steady-state, with proportional pension tax τ in each period, let per-capita savings and per-capita fertility be denoted by s^{PG} and n^{PG} respectively, while the factor prices be denoted by w^{PG} and R^{PG}. The PAYG steady-state equilibrium allocation and prices must satisfy the following conditions:

$$\begin{aligned}
&u'(w^{PG}(1-\tau) - qn^{PG} - s^{PG}) = \beta R^{PG} u'(s^{PG}R^{PG} + n^{PG}w^{PG}\tau), \\
&qu'(w^{PG}(1-\tau) - qn^{PG} - s^{PG}) = v'(n^{PG}), \\
&w^{PG} = f_2(s^{PG}, n^{PG}), \\
&R^{PG} = f_1(s^{PG}, n^{PG}).
\end{aligned} \tag{1}$$

The first equation is the standard Euler equation where an atomistic agent ignores the general equilibrium effects of her choices. The second equation has an agent equalizing the marginal costs of a child with its marginal benefits. Here, along with the general equilibrium effects, the pension externality generated by a child is also ignored by atomistic agents. The third and the fourth conditions characterize the factor prices in general equilibrium. Let us denote the steady-state welfare associated with the PAYG regime by U^{PG}.

After characterizing the PAYG equilibrium at the steady-state above, in the rest of the paper we proceed to demonstrate how we can reform PAYG pensions to enable a transition of the economy to the long-run optimum in a Pareto way ensuring that no generation is worse off. In our Pareto comparison we are concerned with the utility of a representative agent across two scenarios, bypassing different population sizes across them. This is one of the efficiency concept used in the endogenous population literature. Other efficiency concepts take into account the population size in allocations.

As mentioned in footnote 7, the concept of standard Pareto criteria under exogenous fertility setup may not be easily extended to a framework where population is endogenous since it is typically designed for a given set of population. The literature therefore tries to find out a suitable criteria to represent efficient allocation under the endogenous fertility setup. Michel and Wigniolle (2007) rank allocations using a notion called Representative-Consumer efficiency, in short RC-efficiency. This efficiency criteria refers to an allocation where no other allocation exists that guarantees a higher level of utility for all generations with a strict improvement for at least one generation involved in the problem. This efficiency concept does not take into account population size in different allocations. One allocation is ranked above another if it generates higher utility for representative agents in all generations, even though it may come at the cost of a smaller population size. To address this issue, Michel and Wigniolle (2007) propose another efficiency criterion which they call Children for Representative-Consumers efficiency, in short CRC-efficiency. According to this criterion, an allocation dominates another only when it has at least an equal number of children. In our paper, we use the utility of the representative agent to compare allocations, that is, precisely the RC-efficiency. In fact, our optimality formulation obeys CRC-efficiency which is also expected because by construction, a RC-efficient allocation is also CRC-efficient.

Golosov, Jones and Tertilt (2007) present efficiency in a very general framework where agents can also be heterogeneous. They present two different notions of efficiency, namely $\mathcal{A}$-efficiency and $\mathcal{P}$-efficiency. In the $\mathcal{A}$-efficiency concept, efficiency is defined through taking into account utility of of agents alive in all allocations. The notion of $\mathcal{P}$-efficiency, however, requires assigning utility to unborn agents. When agents are homogeneous as in our setup, $\mathcal{A}$-efficiency and RC-efficiency are the same. Another such study is by Conde-Ruiz, Giménez and Pérez-Nievas (2010) which ranks allocations exclusively on preferences of those agents who are actually born and calls it Millian efficiency. As pointed out in a recent study by de la Croix and Doepke (2021), the main difficulty in this analysis is that the utility of not being born cannot be determined by introspection. They argue that in a belief based world, utility of unborn can be assessed by agents who are alive if it is assumed that the world has a fixed supply of souls who reincarnate in different human bodies time to time. Thus, unlike our representation which focuses just on the welfare of a representative agent, they consider the issue of optimal welfare from the point of view of a soul where, along with the future incarnation utility, what matters is the waiting time of the soul's incarnation.

A brief road-map would be helpful at this stage. First, we establish the sub-optimal fertility choice result in a competitive equilibrium when agents do not take into consideration the effect of their chosen fertility when an instrument like PAYG is present. Second, we focus on the planner's optimal allocation where, by construction, this externality is not present or, equivalently, the planner completely internalizes the externality. We refer to the steady-state with the planner's optimal allocation as the O *steady-state*. Third, we demonstrate how, starting from the PAYG steady-state economy that faces the externality issue, the sub-optimality can be meaningfully exploited to construct a transition path to reach the O steady-state. It is ensured that in this entire transition process no generation is worse off compared to the PAYG steady-state. Finally, we show that both the transition and the O steady-state can be decentralized using appropriate policy instruments.

4 Sub-optimality of the PAYG Equilibrium

Now we demonstrate that the PAYG steady-state has individuals making sub-optimal choices as the pension externality of children is ignored by the individuals. Our aim is to show that given the availability of resources, an improvement over the PAYG steady state is clearly possible. For any particular generation, we keep the resources available in their middle age at $e^{PG} \equiv w^{PG}(1-\tau)$ which also acknowledges the fact that a

PAYG type pension, funded by a proportional tax, is present. In this hypothetical problem, an individual internalizes the effect of her chosen fertility when a PAYG type pension is present. Additionally, two requirements need to be satisfied. First, the allocation should obey the fact that total expenditure should not exceed the resources available at the middle age, e^{PG}. Second, total output should not be lower than consumption of old plus the total net wage given to the middle-aged agents where each agent secures e^{PG}. Formally, the agent solves the following optimization problem:

$$\begin{aligned} \max_{\{s,n\}} \quad & u(c^m) + \beta u(c^o) + v(n) \\ & \text{subject to} \\ & \quad c^m + qn + s \le e^{PG}, \\ & \quad ne^{PG} + c^o \le f(s,n). \end{aligned} \tag{B}$$

where $e^{PG} \equiv w^{PG}(1-\tau)$ is the net income of the middle-aged agents. So given that the economy is at the steady-state with each generation receiving e^{PG} and leaving e^{PG} for the next generation as in the PAYG steady-state, this problem solves for the maximum utility any generation can get. As discussed above, while the first constraint guarantees that the total expenditure in middle age is less than the available resources e^{PG}, the second one ensures that the total resources made available to all the middle-aged agents along with the old-age consumption should not exceed the total output produced. Note that this is a concave maximization problem and the PAYG steady-state allocation is in the constraint set as it satisfies both the constraints.

Now we show that the PAYG steady-state allocation is sub-optimal as the first order conditions do not match. The first-order conditions for the above problem with respect to s and n are

$$\begin{aligned} & u'(c^m) = \beta f_1(s,n) u'(c^o), \\ & \left(q + \frac{e^{PG} - f_2(s,n)}{f_1(s,n)}\right) u'(c^m) = v'(n). \end{aligned} \tag{2}$$

While the Euler condition of intertemporal consumption is the same, the effective cost of children now includes the present value of surplus, $e^{PG} - f_2(s,n)$, generated by a child. This particular effect is ignored in the individual agent's optimization problem. Thus while the marginal cost of raising kids in terms of utility, $qu'(c^m)$, is still the same, the marginal benefit is not only the direct utility gain of having an additional one, $v'(n)$, but also an additional utility gain due to pensions discounted back to present value $(\frac{f_2(s,n)-e^{PG}}{f_1(s,n)})u'(c^m)$. This additional income gain through pension (and hence utility) is because the agent now internalizes the effect of her choice of fertility. Therefore, equivalently, the effective cost of an additional child is its direct cost (q) minus the present value of pension extracted from her in the next period. Since the effective cost is less in this optimal solution, it is intuitive that the optimal allocation should have a higher number of children than the PAYG steady-state allocation, that is, $n^{opt} > n^{PG}$, where superscript *opt* denotes the optimal choices fixing the resources at the PAYG steady-state equilibrium level e^{PG}. This is indeed the case and we have the following proposition.

Proposition 1. *Fertility under the PAYG competitive equilibrium at the steady state is sub-optimal. Fixing the resources available to both the current as well as next generation middle-aged same at the PAYG equilibrium steady-state level, the current middle-aged can guarantee a higher level of utility than the PAYG equilibrium by choosing an optimal allocation $n^{opt} > n^{PG}$. Thus, the PAYG steady-state is dominated by another allocation which leaves the same resources for all subsequent generations and increases fertility and utility for one generation.*

Proof. See Appendix A.1.

5 Characterizing the (Planner's) O Steady-state

To characterize the optimal steady-state desired by the planner, we aim to maximize the utility of the representative agent subject to resource constraint. Another constraint is also imposed to ensure that net transfers to the old cannot be negative, that is, we guarantee the possibility of having a non-negative pension.[10] For tractability, we assume a Cobb-Douglas functional form for the production function given by $f(s, n) = s^\alpha n^{1-\alpha}$. However, for the rest of the analysis, to represent the production function we use $f(s, n)$ instead of this exact Cobb-Douglas form for notational simplicity.

[10]We make this assumption in line with the empirical observation that transfers are typically from the middle-aged to the old. Removing this assumption does not affect the results in Proposition 4, however, it does affect the optimal steady-state itself. In the absence of the additional constraint of non-negative transfers, the optimal steady-sate would satisfy the 'golden rule' with $R = n$. Moreover, it would be implemented by either a positive or a negative pension. Hence, implementation-wise, instead of a savings subsidy, a negative pension may be needed.

Thus, formally, the O steady-state solves the following problem:

$$\begin{aligned} &\max_{\{c^m, c^o, s, n\}} u(c^m) + \beta u(c^o) + v(n) \\ &\text{subject to} \\ &\quad n(c^m + qn + s) + c^o \leq f(s, n), \\ &\quad n(c^m + qn + s) \leq f(s, n)(1 - \alpha). \end{aligned} \tag{C}$$

The first constraint is the resource constraint which says that in any period, the total expenditure by all the middle-aged agents plus consumption of the old cannot be more than the total output produced. The second constraint says that the aggregate expenditure of the middle-aged is less than their total wage income, and thus it rules out the possibility of any non-negative transfers from old to middle age. Note that the feasible set is not convex. So it is not necessary that an allocation satisfying the first-order conditions is optimal. In fact, an allocation satisfying the first-order conditions may not even exist. We discuss this issue now.

For this purpose, let us denote the total resources available to a middle-aged agent by e. This could be used for her consumption, expenditure on children and savings for the future. Then, for a given e, the steady-state problem (C) can be rewritten as

$$\begin{aligned} &V(e) \equiv \max_{\{s, n\}} u(c^m) + \beta u(c^o) + v(n) \\ &\text{subject to} \\ &\quad c^m + qn + s \leq e, \\ &\quad ne + c^o \leq f(s, n), \\ &\quad ne \leq f(s, n)(1 - \alpha). \end{aligned} \tag{C$'$}$$

This is a standard concave programming problem and the optimal allocation for a given e is characterized by the first-order conditions. $V(e)$ corresponds to the maximum welfare for a given e. Then, in order to find a solution to (C), it suffices to find an $e \in (0, \infty)$ that maximizes $V(e)$. However, it may well be the case that under some parameter values, utility is increasing in e. Then there may not exist an optimal steady-state e^{opt}. Therefore, we assume the existence of such an optimal e, e^{opt}, characterized by the first-order condition $V'(e^{opt}) = 0$.

Assumption 1. *There exists a unique $e \equiv e^{opt} \in (0, \infty)$ which satisfies $V'(e) = 0$ and* *maximizes* $V(e)$.

There are two issues regarding the existence of an optimal steady-state. First, it may be the case that, for all e, $V'(e) > 0$ and steady-state welfare is always increasing in resources e. This is the case of no allocation satisfying the first-order conditions. Assumption 1 rules out this case. Second, there is the potential multiplicity of local maxima. There can be multiple e's satisfying $V'(e) = 0$. Alternatively, in problem (C), multiple allocations can satisfy the first-order conditions as the constraint set is not convex. Thus, the first-order conditions are necessary for the existence of the optimal steady-state, but not sufficient. Theoretically, we cannot rule out any of the two issues. That necessitates Assumption 1.

In Appendix A.4, we show both the existence and sufficiency of the first-order conditions for log utility under certain parameter restrictions. It requires elasticity of production with respect to capital, α, to be low enough. Otherwise, higher capital always implies higher utility and *optimal* steady-state capital stock is unbounded. When the share of capital in output is large enough, that is, the production function is close to linear in capital, a planner who is concerned with only the utility of a representative agent can always increase steady-state level of resources by lowering fertility. Just as generally there is no optimal steady-state level of capital with AK-type production function, there will not be an optimal steady-state in an economy with endogenous fertility choice and a Cobb-Douglas production function with a very high capital share. Thus, the existence of the optimal steady-state as well as it being the unique allocation satisfying first-order conditions for optimality is not always a given, and, we need to assume that. Other papers have also shown sufficiency of the first-order conditions under certain assumptions on functional forms in their framework, see for example, Dávila (2018) and Abio et al.(2004).

For problem (C′), let the Lagrange multipliers for the non-negative transfers to old be μ_1. Since the first two constraints hold with equality, substituting for c^m and c^o from these constraints the first-order conditions with respect to s and n respectively are

$$u'(c^m) = \beta u'(c^o) f_1(s,n) + \mu_1 f_1(s,n)(1-\alpha),$$
$$v'(n) = \left(q + \frac{e - f_2(s,n)}{f_1(s,n)} + \frac{\mu_1 e \alpha}{u'(c^m)}\right) u'(c^m). \tag{3(a)}$$

In addition, we need the envelope condition corresponding to $V'(e^{opt}) = 0$:

$$u'(c^m) = (\beta u'(c^o) + \mu_1) n. \tag{3(b)}$$

Together, equations (3(a)) and (3(b)), are the first-order conditions of the original problem (C) and characterize the optimal resources e^{opt} along with allocation (c^m, c^o, s, n). By assumption 1, the first-order conditions are also sufficient to characterize the optimal solution. Moreover, the solution to (C) gives a strictly higher utility than the PAYG competitive equilibrium allocation at the steady-state since that allocation is feasible under (C) but is suboptimal as shown in the previous section. This discussion leads us to the following proposition.

Proposition 2. *A solution to the O steady-state problem (C) exists and is characterized by the first-order conditions* (3(a)) *and* (3(b)) *along with the budget constraints. This allocation dominates the PAYG steady-state allocation.*

The first equation in (3(a)) gives the first-order condition with respect to capital. If $\mu_1 = 0$, it is the standard Euler equation. However, in case the non-negative pension constraint binds, the value of capital increases as it helps to relax the non-negative pension constraint. This is what the second term on the right-hand side of the equation reflects. The second equation equates the marginal benefits of an additional child with its costs. The benefit of a child is the direct utility gain. The costs are the direct cost q and the present value of the net resource $e - f_2(s, n)$ which must be provided to the additional child. We can also call it capital dilution since one additional child means more capital needs to be saved to maintain the same level of resources. The last term in the brackets in the right-hand side of the second equation in (3(a)) represents the additional cost of a child when the non-negative transfers to the old constraint binds. In that case, the value of a child decreases as an additional child tightens the non-negative transfers constraint.

Now consider the envelope condition corresponding to $V'(e^{opt}) = 0$ given by $u'(c^m) = (\beta u'(c^o) + \mu_1)n$. If $\mu_1 = 0$, that is, if the non-negative pension constraint does not bind, then, using the Euler equation, we get

$$n = \frac{u'(c^m)}{\beta u'(c^o)} = f_1(s, n) = R.$$

This is the condition associated with the maximum welfare level in the O steady-state, that is, the 'golden rule' level of capital. However, if $\mu_1 > 0$, we have $R = f_1(s, n) > n$.[11] The intuition is simple. We need some inter-generational transfer to achieve the golden rule.

[11] It follows from simple algebra. When $\mu_1 > 0$, combining once again with the first equation in (3(a)), we get the following from (3(b)):
$u'(c^m) - (\beta u'(c^o) + \mu_1)n = 0 \Rightarrow (\beta u'(c^o) + \mu_1)(f_1(s, n) - n) = \mu_1 f_1(s, n)\alpha > 0 \Rightarrow f_1(s, n) > n.$

If that transfer is pension, we achieve that. However, if the transfer required is from the old to the middle-aged, non-negative pension constraint prevents that. In that case, the golden rule is not achieved. Moreover, when $\mu_1 > 0$, the ratio of marginal utilities is more than $R = f_1(s, n)$, as can be seen from the first equation in (3(a)). Thus, the planner must intervene to ensure that an individual gets a higher return than the market when $\mu_1 > 0$. The rationale for this also follows from the inability to achieve the golden rule through inter-generational transfers. The planner needs to transfer resources from the old to the middle-aged in order to guarantee a higher capital and achieve the golden rule. But it cannot go below zero pension. When this direct transfer is not allowed, an indirect way to do that is through some incentives on saving that can guarantee a higher level of capital. Another instrument that is needed here is a child care subsidy in case of positive transfers to old to ensure that the first-order condition with respect to n also holds. We discuss the decentralized implementation of both the optimum steady-state and the transition in detail later. The above discussion is summarised in the following proposition.

Proposition 3. *At the O steady-state, two outcomes are possible: either $R = n$ and there is a positive PAYG pension, or $R > n$ but PAYG pension is zero.*

So far we have characterized the O steady-state. Depending on parameters, it either achieves the golden rule level of capital defined by $R = f_1(s, n) = n$ along with non-negative pensions, or stops at $R = f_1(s, n) > n$ with zero pensions.[12] Importantly, we observe that the planner's optimality and the associated concept of 'golden rule' under endogenous fertility choice is the same as the scenario where fertility is exogenous. Specifically, when return on PAYG n is dominated by the market return R, that is, $R > n$, PAYG type pension is not recommended by the planner as in the standard *dynamic efficiency* situation under exogenous fertility. On the other hand, when the golden rule is achieved and therefore the returns are the same, that is, $R = n$, a PAYG is no longer welfare reducing.

Starting with a PAYG regime with resources available to the middle-aged given by $e^{PG} \equiv w^{PG}(1 - \tau)$, our aim now is to reach this optimal level of resources e^{opt} and the corresponding allocation in a Pareto-way. Here again, by Pareto way we mean that the transition as well as the steady-state dominate the PAYG steady-state. That is, each generation must have at least as much utility as in the PAYG steady-state. To achieve such a transition, we exploit the inefficiency of the PAYG system itself.

[12]For log utility, it can be shown that if weight β on c^o is greater than some threshold, the O steady-state has pension and golden rule. For β below that threshold, pension is 0 and $R > n$.

6 A Pareto-dominating Transition to the O Steady-state

In this section, we show the existence of a transition path from the PAYG steady-state to the O steady-state such that the transition path dominates the PAYG steady-state. For this purpose, consider that the economy is at the PAYG steady-state in period t and the government-engineered transition to the O steady-state starts in this specific period t. Recall that in PAYG steady-state, the resources available to each generation is e^{PG}. Hence, in the PAYG steady-state in period t, $e_t = e^{PG}$ and, without any further government intervention, $e_j = e_t$, $\forall j \geq t$. We want to show that if the planner intervenes in this PAYG steady-state economy in period t, it can generate a sequence $e^{PG} = e_t < e_{t+1} < e_{t+2} < ...e_{t+T+1} = e^{opt} = e_{t+T+2} = e_{t+T+3} = ...$ such that the utility of any generation $j \geq t$ is at least U^{PG}. Given a path of resources, the planner's problem is to choose per-capita savings s_{j+1} and fertility n_j for generation j, who receives resources e_j and leaves behind resources e_{j+1} to each of the n_j children in the next generation, as follows.

$$
\begin{aligned}
W(e_j, e_{j+1}) \equiv & \max_{\{s_{j+1}, n_j\}} u(c^m) + \beta u(c^o) + v(n_j) \\
& \text{subject to} \\
& \quad c^m + qn_j + s_{j+1} \leq e_j, \\
& \quad n_j e_{j+1} + c^o \leq f(s_{j+1}, n_j), \\
& \quad n_j e_{j+1} \leq f(s_{j+1}, n_j)(1-\alpha).
\end{aligned} \tag{D}
$$

Recall that in section 4 we have shown that the PAYG steady-state is suboptimal. Each generation inherits resources $e^{PG} = w^{PG}(1-\tau)$ and leaves behind the same amount of resources for the next generation. But it does not do so optimally as it ignores the pension benefits of a child, that is, $U^{PG} < W(e^{PG}, e^{PG})$. In Figure 1 the curve represents the maximum level of utility that a generation can achieve if it inherits resources e and leaves the same amount of resources for the next generation, $V(e) = W(e, e)$. It represents the efficient frontier and $V(e)$ is maximized at the resource level e^{opt}. In Figure 1 the point (e^{PG}, U^{PG}) lies below this efficient frontier illustrating inefficiency of the PAYG equilibrium. Taking that into account, the current generation's utility can be increased beyond the PAYG steady-state level, U^{PG}. At the PAYG equilibrium at the steady-state in period t, $e_t = e_{t+1} = e^{PG}$, and, by definition of $W(e_t, e_{t+1})$, we have $W(e_t, e_{t+1}) > U^{PG}$ as the externality associated with the children is taken care of. Now, since $W(e_t, e_{t+1})$ is increasing in its first argument and decreasing in its second argument,

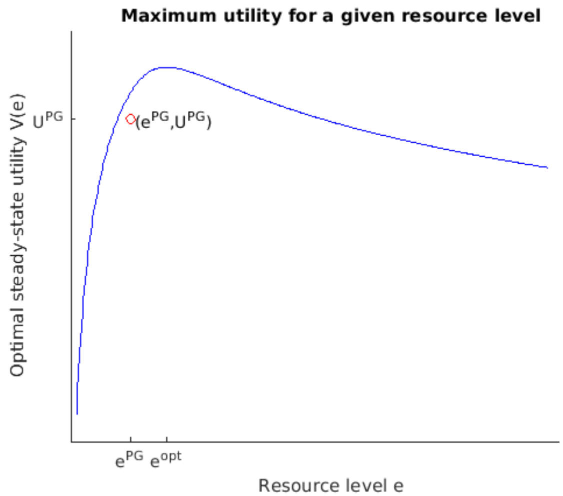

Figure 1. Steady-state Utility

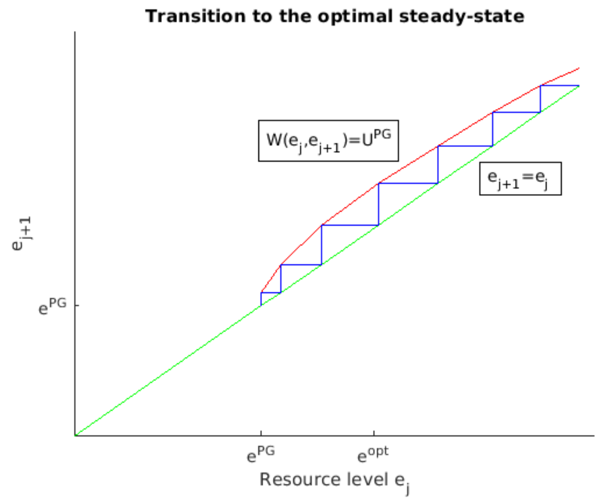

Figure 2. Transition to the O *Steady-state*

e_{t+1} can be increased till $W(e_t, e_{t+1}) = U^{PG}$. Thus, the next generation experiences a resource (disposable income) gain while the current generation is no worse off. Now generation $t+1$ has $e_{t+1} > e^{PG}$, so e_{t+2} can be increased from the e^{PG} level to maintain the equality $W(e_{t+1}, e_{t+2}) = U^{PG}$. Moreover, with $W(e_t, e_{t+1}) = W(e_{t+1}, e_{t+2})$ and $e_t < e_{t+1}$, it follows that $e_{t+1} < e_{t+2}$. Thus, iterating forward from the PAYG steady-state level at time t, resources will move in sequence $e^{PG} = e_t < e_{t+1} < e_{t+2} < ...$ such that $W(e_t, e_{t+1}) = W(e_{t+1}, e_{t+2}) = W(e_{t+2}, e_{t+3}) = ... = U^{PG}$. To reach the steady-state welfare maximizing resource level e^{opt}, we need to show that this sequence reaches e^{opt} in finite time. Let $\hat{e} < e^{opt}$ be such that $W(\hat{e}, e^{opt}) = U^{PG}$. Then we need to show that the sequence reaches $\hat{e}$ in finite time as reaching $\hat{e}$ implies reaching e^{opt} in the next period. We establish this in the following proposition.

Proposition 4. *There exists a Pareto-way of attaining the optimal allocation in finite periods. Formally, the sequence $\{e_j\}$ defined above reaches $\hat{e}$ in finite time T, that is, $e_{t+T} \geq \hat{e}$. Moreover, the transition ensures representative agent in each generation gets as much utility as in the PAYG equilibrium at the steady-state.*

Proof. See Appendix A.2.

The appendix has the detailed proof. Here we provide a sketch of the argument. The increasing sequence $\{e_j\}_{j \geq t}$ which lies on the *indifference curve* $W(e_j, e_{j+1}) = U^{PG}$ must either reach e^{opt} in finite time or converge to some $e \leq e^{opt}$. The second case is equivalent to saying that the indifference curve $W(e_j, e_{j+1}) = U^{PG}$ (in red color in Figure 2) is intersecting the 45 degree line $e_{j+1} = e_j$ (in green color) at some $e \leq e^{opt}$. We rule out this second case in the appendix, where we show that this indifference curve $W(e_j, e_{j+1}) = U^{PG}$ has a slope greater than one for any $e_j \in (e^{PG}, e^{opt})$. Thus, it will always be above the 45 degree line for any $e_j \in (e^{PG}, e^{opt})$ and the cascading sequence $\{e_j\}_{j \geq t}$ (in blue color in Figure 2) will surpass e^{opt} in a finite number of periods. Thus, Proposition 4 establishes that it is indeed possible for the planner to devise a transition plan that leads the economy to the O steady-state in finite time while ensuring that each generation enjoys as much utility that it would have gotten had the economy continued to be in the PAYG steady-state. Of course, this is a specific plan where the utility of every generation is pegged to the benchmark level U^{PG}. There can be alternative plans where some of the utility gains is distributed to the initial generations. In those plans, achieving the O steady-state will be delayed.

6.1 Dynamics of capital and factor prices

In this subsection, we characterize the path of capital stock. Recall that, there are three distinct phases after the transition to the O steady-state begins. First, in period t, the planner handles the externality and $W(e_t, e_{t+1}) = U^{PG}$ with $e_{t+1} > e_t = e^{PG}$. Second, from period $t+1$ to $t+T$, $W(e_j, e_{j+1}) = U^{PG}$ for $j \in \{t+1, t+2, ..., t+T\}$ with $e_j < e_{j+1}$. Finally, in period $t+T+1$, the economy reaches the O steady-state and then stays there forever. In the following proposition, we establish that the capital per worker increases through the transition. However, whether capital per worker increases or decreases at the beginning and end periods of the phaseout – periods t and $t+T+1$, respectively – depends on the particular features of the economy. For exposition, let k_j be the capital per worker in period j. It is the savings of generation $j-1$ divided by their fertility. Moreover, let k^{PG} and k^{opt} be the PAYG and O steady-state levels of capital per worker, respectively. Then, the PAYG steady-state has capital per worker $k^{PG} = k_t = k_{t-1} = k_{t-2} = ...$, and the planner initiating the transition has a path of capital per worker $k_{t+1}, k_{t+2}, ..., k_{t+T+1}, k_{t+T+2}, ...$. With the O steady-state reached in period $t+T+1$, we have $k_{t+T+2} = k_{t+T+3} = ... = k^{opt}$.

Proposition 5. *During the economy's transition from the PAYG steady-state in period t to the O steady-state in period $t+T+1$, the capital per worker increases monotonically except at the beginning and end periods of the transition, periods t and $t+T+1$, respectively. Formally, $k_{t+1} < k_{t+2} < ... < k_{t+T+1}$. However, whether $k^{PG} = k_t < k_{t+1}$ and $k_{t+T+1} < k_{t+T+2} = k^{opt}$ hold or not is ambiguous.*

Proof. For the proof of k_j increasing throughout the transition process and an illustration of the ambiguity in period $t + T + 1$, see Appendix A.3.

For an intuition of the increase in capital per worker throughout the transition, observe that both the resources available for a generation and resources to be left for the next generation keep increasing. Thus, each generation gets more resources. This increases both savings and fertility. But it must also leave more resources for each of its child making children costlier. Hence the planner's choice for any generation must shift towards more savings and less fertility which increases capital per-worker.

The impact of the planner internalizing the externality on capital per worker in period t is ambiguous. In the PAYG steady-state, fertility is suboptimal. In period t, internalizing that externality implies less savings and more fertility. However, this increase in utility is offset by increasing the resources available to the next generation, that is, $e_{t+1} > e_t = e^{PG}$. Through this second channel, children become costlier and a planner should reallocate to more savings and less fertility. Hence, there are two effects working in opposite directions. So it is not clear which way the ratio of savings to fertility, that is, capital per worker, will move. Now consider the end period of the transition process, period $t + T + 1$, when the increasing sequence $e_t, e_{t+1}, e_{t+2}, ...$ reaches $e_{t+T+1} = e^{opt}$. Compared to the previous period where the problem was $W(e_{t+T}, e^{opt})$, the current generation has more resources e^{opt} and their problem is $W(e^{opt}, e^{opt})$. So the $t + T + 1$ generation gets more resources than their predecessor, but they also have to leave the same amount of resources to the next generation. More resources in the young age imply more savings and more fertility, as fertility is a normal good. But what happens to capital per worker, which is savings divided by fertility, depends on the relative changes in savings and fertility and therefore, is not certain in terms of its direction. In our formulation in the appendix, it depends on how elastic the demand for fertility is compared to consumption. If the utility of an agent is relatively inelastic in consumption and elastic in fertility, an increase in resources implies savings would rise more than fertility to ensure consumption smoothing between two periods. This leads to an increase in capital per-worker.

This nature of the path for per capita capital along with the assumption of the production function determine the path of factor prices since they are the marginal products of the respective factors of production. At the beginning and end periods of the transition, factor prices may increase or decrease depending on the specification of the economy. However, throughout the transition period, wage increases and the return on capital decreases monotonically.

7 Decentralizing Optimal Allocation

In the previous sections we discussed optimality in terms of a planner's desired allocation. In this section we discuss one possible decentralized implementation of the optimal allocations for both the O steady-state and the transition. We show that when there is a transfer from middle-aged to old, that is, a PAYG type pension, the government needs a childcare subsidy. On the other hand, when the non-negative pension constraint binds, the instruments needed are a childcare tax and a savings subsidy.

The O steady-state does not need a separate discussion as $V(e^{opt})$ in (C′) can be written as $W(e^{opt}, e^{opt})$ in (D). In other words, the planner's problem in transition, $W(e_j, e_{j+1})$, is more general and encompasses the steady-state $V(e)$ as a special case: $W(e, e) = V(e)$. We first characterize the optimal allocation for the planner's problem in transition, problem (D).

For any (e_j, e_{j+1}), with μ_1 being the Lagrange multiplier associated with the non-negative transfers to old, the first-order conditions for (D) are given by

$$u'(c^m) = \beta f_1(s,n) u'(c^o) \left(1 + \frac{\mu_1(1-\alpha)}{\beta u'(c^o)}\right),$$

$$v'(n) = \left(q + \frac{e_{j+1} - f_2(s,n)}{f_1(s,n)} + \frac{\mu_1 e_{j+1} \alpha}{u'(c^m)}\right) u'(c^m). \tag{4}$$

These first-order conditions are very similar to the first-order conditions for problem (C′) given by equation (3(a)), and, in fact, are reduced to the same conditions when $e_j = e_{j+1} = e^{opt}$.

These conditions can be replicated in a decentralized way. To see that, consider the following problem of an individual generation j agent facing a proportional labor income tax τ_j to fund pensions, a child care subsidy of $\phi_{n,j}$ per child, a savings subsidy of $\phi_{s,j}$ per unit, and a proportional tax $\tau_{n,j}$ to finance these subsidies:

$$\max_{\{s,n\}} u(c^m) + \beta u(c^o) + v(n)$$

subject to

$$c^m + (q - \phi_{n,j})n + s(1 - \phi_{s,j}) \leq w_j(1 - \tau_j - \tau_{n,j}), \tag{E}$$

$$c^o \leq sR_{j+1} + p_{j+1}.$$

An individual takes the taxes and subsidies as given and the government's budget balance requires $\phi_{n,j}n + \phi_{s,j}s = w_j\tau_{n,j}$ and $p_{j+1} = nw_{j+1}\tau_{j+1}$. The subsidy on children reduces the marginal cost of child care while the savings subsidy encourages savings. Thus, the first-order conditions change. The solution to this problem is given by the following first-order conditions:

$$(1-\phi_{s,j})u'(c^m) = \beta R_{j+1}u'(c^o),$$
$$(q-\phi_{n,j})u'(c^m) = v'(n). \tag{5}$$

Comparing the two sets of first-order conditions, we find that an appropriately designed child care subsidy along with a savings subsidy, financed by a proportional tax[13], can mimic the first-order conditions of the planner. The subsidies on savings and child care are given by

$$\phi_{s,j} = \frac{\mu_1(1-\alpha)}{\beta u'(c^o) + \mu_1(1-\alpha)} \geq 0,$$
$$\phi_{n,j} = \frac{f_2(s,n) - e_{j+1}}{f_1(s,n)} - \frac{\mu_1 e_{j+1}\alpha}{u'(c^m)}.$$

There are two possible cases. Either the non-negativity constraint on transfers to old binds or it does not bind. When the non-negative pension constraint does not bind, we have $\mu_1 = 0$, so that $\phi_{s,j} = 0$. That is, when there is a positive pension transfer, there is no need for a savings subsidy. Moreover, $f_2(s,n) > e_{j+1}$ which in turn implies that $\phi_{n,j} > 0$. Thus, a positive pension is always accompanied by a positive child care subsidy. On the other hand, when the non-negative pension constraint binds, we have $f_2(s,n) = e_{j+1}$ and $\mu_1 > 0$, implying that the savings subsidy is positive. However, in this case, the sign of $\phi_{n,j}$ is negative. Hence, childcare is subsidized when the old gets pension and is taxed when transfers to the old are zero. Thus, the optimal public policies involve a package of either a positive PAYG pension and a childcare subsidy, or zero pension accompanied by a subsidy on savings and a tax on childcare. Comparing with equation (3(a)), we see that we need this combination of subsidies and taxes even at the steady-state to achieve the O steady-state allocation which corresponds to $e_j = e_{j+1} = e^{opt}$.

The following proposition summarizes the above discussion on the policy instruments required to implement the optimal allocations.

Proposition 6 *The O steady-state as well as the transition to it starting from a PAYG*

[13]Instead of a proportional tax, an alternate implementation can use government borrowing through bonds to finance subsidies.

steady-state can be implemented in a decentralized way. When a PAYG pension is present, a child care subsidy should accompany it. However, in the absence of a PAYG pension, a subsidy on savings and a child care tax are recommended.

An important observation is worth mentioning here. The above results somewhat reconfirms the *"Siamese twins"* results of van Grozen et al. (2003). In a model with fertility and pension, they prove the interdependence between childcare subsidy and pension in the presence of a market failure, precisely when externalities in public pension via fertility is ignored by the competitive equilibrium. On the other hand, when pension is zero and $\mu_1 > 0$, we need a child care tax as well as a subsidy on savings. The subsidy on savings is needed only when the non-negative pension constraint binds with $R > n$ and the golden rule is not achieved. It can be interpreted as a second-best tool to incentivize savings when the first-best option of transfer from the old to the middle-aged is not allowed. That means when the market provides a higher return than that of PAYG, more capital and thus lower R is needed for the economy to move towards the 'golden rule'. In fact the 'golden rule' could have been achieved if transfers from the old to the middle-aged were allowed in our model. In the absence of that agents are to be given a subsidy on savings which acts as a second-best instrument. In this case, a tax on childcare is also needed to reduce fertility and hence the transfer burden to young. Intuitively, when there is underaccumulation of capital, the planner would rather incentivize investment in capital instead of fertility.

8 Conclusion

For quite some time the PAYG pension system has been facing criticisms, mainly due to the demographic shift. In this paper, motivated by the budgetary concerns of PAYG pensions, we construct a general equilibrium model with endogenous fertility. Typically the competitive equilibrium ignores the benefits of externality associated with endogenous choice of fertility and therefore the allocations under the PAYG competitive setup are sub-optimal. We characterize the optimal steady-state allocations of a planner and find that in an endogenous fertility framework too, PAYG pensions are optimal only if they are not return-dominated by the market. When PAYG pensions optimally exist, as in the case of 'golden rule' in our analysis, it must be accompanied by a childcare subsidy. PAYG pensions on its own cannot generate the efficient outcome in this scenario when PAYG return is not dominated by the market return. On the other hand, when PAYG return is dominated by the market, starting from the PAYG competitive equilibrium, it is possible to reach the optimal steady-state where pension is zero in finite periods and this can be achieved without hurting any generation. Phasing pensions out without making any generation worse off has been proven to be a challenging task in the literature. In this paper we find a novel way out of PAYG pension when it is return dominated. An inefficiency within the pension system itself, an externality that is well acknowledged in the literature but somehow overlooked till date, has been exploited for the purpose of phasing pensions out.

A Proofs

A.1 Proof of Proposition 1

Proof. The first-order conditions characterizing the optimal are

$$u'(c^m) = \beta f_1(s,n)u'(c^o),$$
$$\left(q + \frac{e^{PG} - f_2(s,n)}{f_1(s,n)}\right)u'(c^m) = v'(n).$$

The PAYG steady-state has a positive pension tax. Since the disposable income of workers is less than their wage in presence of a pension tax, $e^{PG} < f_2(s^{PG}, n^{PG})$ and the PAYG steady-state first-order conditions do not match with the first-order conditions under the optimal one. Thus, the PAYG allocation is sub-optimal in the sense that there is some other allocation which satisfies the first-order conditions and gives a higher utility to the current middle-aged.

The next step is to show that the optimal allocation has $n^{opt} > n^{PG}$. Define $k = \frac{s}{n}$. We prove by ruling out the case that $n^{opt} \leq n^{PG}$. Note that $n^{opt} = n^{PG}$ is not possible as that implies $k^{opt} = k^{PG}$ by the Euler condition. This in turn means PAYG allocation matches the optimal allocation which cannot be true as the first-order conditions are different. Next consider the case $n^{opt} < n^{PG}$ and suppose that it holds. Now either $k^{opt} \leq k^{PG}$ or $k^{opt} > k^{PG}$. We rule both cases out.

First, consider the possibility of $k^{opt} \leq k^{PG}$. If this holds, then $n^{opt} < n^{PG}$ and $k^{opt} \leq k^{PG}$ imply

$$c^{m,opt} = e^{PG} - (q + k^{opt})n^{opt} \geq c^{m,PG} = e^{PG} - (q + k^{PG})n^{PG},$$

$$c^{o,opt} = n^{opt}(f(k^{opt},1) - e^{PG}) < n^{PG}(f(k^{PG},1) - e^{PG}) = c^{o,PG}.$$

But then, $u'(c^m) = \beta f_1(s,n)u'(c^o) = \beta f_1(k,1)u'(c^o)$ cannot hold for both the PAYG steady-state and optimal allocation as $k^{opt} \leq k^{PG}$ and $c^{o,opt} < c^{o,PG}$ imply

$$\beta f_1(k^{opt},1)u'(c^{o,opt}) > \beta f_1(k^{PG},1)u'(c^{o,PG}) \text{ [by concavity of u and f]}$$

$$\Rightarrow u'(c^{m,opt}) > u'(c^{m,PG}) \Rightarrow c^{m,opt} < c^{m,PG}.$$

This contradicts $c^{m,opt} \geq c^{m,PG}$ and hence $k^{opt} \leq k^{PG}$ is ruled out.

Now consider the other case $k^{opt} > k^{PG}$. Comparing the first-order conditions for fertility for the optimal and PAYG steady-states we get

$$u'(c^{m,opt}) = \frac{v'(n^{opt})}{q + \frac{e^{PG} - f_2(k^{opt},1)}{f_1(k^{opt},1)}} > \frac{v'(n^{PG})}{q} = u'(c^{m,PG}) \text{ [as } n^{opt} < n^{PG} \text{ and } e^{PG} < f_2(k^{PG},1) < f_2(k^{opt},1)]$$

$$\Rightarrow u'(c^{m,opt}) > u'(c^{m,PG}) \Rightarrow c^{m,opt} < c^{m,PG}.$$

On the other hand, using the Euler conditions for both the cases, we get

$$u'(c^{o,opt}) = \frac{u'(c^{m,opt})}{\beta f_1(k^{opt},1)} > \frac{u'(c^{m,PG})}{\beta f_1(k^{PG},1)} = u'(c^{o,PG}) \text{ [as } c^{m,opt} < c^{m,PG} \text{ and } k^{opt} > k^{PG}]$$

$$\Rightarrow c^{o,opt} < c^{o,PG}.$$

Thus, for $k^{opt} > k^{PG}$, we have got consumption in both periods and fertility less in the optimal case as compared to the PAYG steady-state. This, however, is not possible as the optimal must give a higher utility than the PAYG steady-state.

Hence both $k^{opt} > k^{PG}$ and $k^{opt} \leq k^{PG}$ are ruled out. So $n^{opt} \leq n^{PG}$ is not possible. ■

A.2 Proof of Proposition 4

Proof. Suppose not, that is, $e_j \leq \hat{e} \ \forall j \geq t$. Then the increasing sequence is bounded above by $\hat{e}$ and must converge to $\tilde{e} \leq \hat{e}$. Remember that the sequence $\{e_j\}_{j=t}^{\infty}$ is an increasing sequence and satisfies $W(x,y) = U^{PG}$ with $x = e_j$ and $y = e_{j+1} \ \forall j \geq t$. By taking limits,

$$W(e_j, e_{j+1}) = U^{PG} \Rightarrow W(\tilde{e}, \tilde{e}) = U^{PG}.$$

Moreover, $W(x,y) = U^{PG}$ gives y as a function of x. This graph of y as a function of x is above the 45 degree line $y = x$ for $x < \tilde{e}$ since it contains points $\{(e_j),(e_{j+1})\}_{j=t}^{\infty}$ with $e_j < e_{j+1} < \tilde{e}$. So, for any $x < \tilde{e}$, $e_j < x < e_{j+1}$ for some $j \geq t$. Then

$$W(x,y) = W(e_j, e_{j+1}) = U^{PG} \text{ and } x > e_j \Rightarrow y > e_{j+1} > x.$$

The graph crosses the 45 degree line at $x = \tilde{e}$. So, the slope $\frac{\partial y}{\partial x} < 1$ at $\tilde{e}$. Let the three multipliers associated with the three constraints in (D) be λ_1, λ_2, and μ_1. By envelope condition which we explicitly derive later, totally differentiating $W(x,y) = U^{PG}$ gives

$$\frac{\partial y}{\partial x} = \frac{\lambda_1(x,y)}{n(x,y)(\lambda_2(x,y) + \mu_1(x,y))}.$$

At $\tilde{e}$,

$$1 > \frac{\partial y}{\partial x} \Rightarrow \lambda_1(\tilde{e},\tilde{e}) < n(\tilde{e},\tilde{e})(\lambda_2(\tilde{e},\tilde{e}) + \mu_1(\tilde{e},\tilde{e})).$$

Now, recall the function $V(e)$ from (C′) which is just a special case of $W(e,e')$ from (D) when $e = e'$. For $\tilde{e} < e^{opt}$, V is increasing in e and reaches maximum at e^{opt}. So, $V'(\tilde{e}) > 0$. Since, $V(e) = W(e,e)$, by applying envelope condition in (D), $V'(e) = \lambda_1 - n(\lambda_2 + \mu)$. Hence,

$$V'(\tilde{e}) > 0 \Rightarrow \lambda_1(\tilde{e},\tilde{e}) > n(\tilde{e},\tilde{e})(\lambda_2(\tilde{e},\tilde{e}) + \mu_1(\tilde{e},\tilde{e})).$$

Thus there is a contradiction between the last two inequalities and the economy reaches $\hat{e}$ and e^{opt} in finite time.

A formal derivation of envelope conditions is given below. First we derive $\frac{\partial y}{\partial x} = \frac{\lambda_1}{n(\lambda_2+\mu_1)}$. For that,

$$W(x,y) = U^{PG} \Rightarrow u(x - qn - s) + \beta u(f(s,n) - ny) + v(n) = U^{PG}. \tag{A11}$$

Total differentiating the above gives us

$$u'(c^m)(dx - qdn - ds) + \beta u'(c^o)(f_1(s,n)ds + f_2(s,n)dn - ydn - ndy) + v'(n)dn = 0.$$

Substituting $\lambda_1 = u'(c^m)$ and $\lambda_2 = \beta u'(c^o)$, we get

$$\lambda_1 (dx - qdn - ds) + \lambda_2(f_1(s,n)ds + f_2(s,n)dn - ydn - ndy) + v'(n)dn = 0$$

$$\Rightarrow (\lambda_2 f_1(s,n) - \lambda_1)ds + \lambda_1 dx - n\lambda_2 dy + (v'(n) + \lambda_2(f_2(s,n) - y) - \lambda_1 q)dn = 0.$$

Further, substituting first order conditions from (4), we get

$$-\mu_1 f_1(s,n)(1-\alpha)ds + \lambda_1 dx - \mu_1(f_2(s,n)(1-\alpha) - y)dn = n\lambda_2 dy$$

$$\Rightarrow \lambda_1 dx - \mu_1[(f_2(s,n)(1-\alpha) - y)dn + f_1(s,n)(1-\alpha)ds] = n\lambda_2 dy.$$

If non-negative pension constraint does not bind, $\mu_1 = 0$ and $\frac{\partial y}{\partial x} = \frac{\lambda_1}{n\lambda_2} = \frac{\lambda_1}{n(\lambda_2+\mu_1)}$. Otherwise, $f(s,n)(1-\alpha) = ny$ and totally differentiating gives us

$$(f_2(s,n)(1-\alpha) - y)dn + f_1(s,n)(1-\alpha)ds = ndy.$$

Substituting this expression in previous equation again gives us $\frac{\partial y}{\partial x} = \frac{\lambda_1}{n(\lambda_2+\mu_1)}$.

The derivation of second envelope condition $V'(e) = \lambda_1 - n(\lambda_2 + \mu_1)$ is similar. Since $V(e) = W(e,e)$, differentiating it with respect to e is the same as setting $x = y$ and hence $dx = dy$ in the left hand side of (A11) and differentiating that with respect to x and y. Then the same algebra follows.

The last statement in the proposition is satisfied by construction as $W(e_j, e_{j+1}) = U^{PG}$ $\forall j \geq 0$. ■

A.3 Proof of Proposition 5

There are two parts in the proof. The first part of this section shows that through the transition to the O steady-state, the increasing sequence of resources which keeps the utility of each generation at the PAYG level implies an increasing sequence of capital per-worker. The second part of this section illustrates that what happens to the capital stock in the very last period, when the transition gets completed and economy reaches the O steady-state, is ambiguous.

A.3.1 Rising path of capital throughout the transition

Proof. Formally, we want to show that if $W(e_j, e_{j+1}) = W(e_{j+1}, e_{j+2}) = U^{PG}$, then $k_{j+1} < k_{j+2}$. Alternatively, if $W(e,e') = U^{PG}$ and k is the associated capital, then a small increase in (e,e') to $(e+\Delta, e'+\Delta')$ keeping $W(e+\Delta, e'+\Delta') = U^{PG}$ increases the capital. There are 2 possible cases depending on whether the non-negative pension constraint holds for $W(e,e')$ or not.

First, suppose the non-negative pension constraint holds. Then, the resource available to the workers are just their wages and $(1-\alpha)k^{\alpha} = e'$. Then, increasing e' increases the capital k. Now, consider the case of non-negative pension constraint not holding. Then the allocations (n, k, c^m, c^o) for any (e,e') solve the first-order conditions for the problem (D) and are given by

$$
\begin{aligned}
&u'(c^m) = \beta\alpha k^{\alpha-1}u'(c^o),\\
&qu'(c^m) = v'(n) + \beta\left(k^\alpha(1-\alpha) - e'\right)u'(c^o) \Leftrightarrow \beta\left(q\alpha k^{\alpha-1} + e' - k^\alpha(1-\alpha)\right)u'(c^o) = v'(n),\\
&u(c^m) + \beta u(c^o) + v(n) = U^{PG},\\
&c^m = e - (q+k)n,\\
&c^o = n(k^\alpha - e').
\end{aligned}
$$

The first equation is the Euler condition while the second equation is the first-order condition for fertility. Third equation keeps the utility pegged at the PAYG steady-state level. The last two equations are the feasibility conditions. We want to show that if (e, e') increase and the allocation still generates the utility U^{PG}, capital k must increase. Suppose not, i.e., for $(e+\Delta, e'+\Delta')$, with $\Delta, \Delta' > 0$, the capital k decreases or stays same. Then, the following cases can be ruled out.

- k decreases or stays the same, and c^m increases or stays the same. Then, from the Euler condition, c^o increases or stays the same. Since, total utility is pegged to U^{PG}, n must decrease or stays the same. But with $c^o = n(k^\alpha - e')$ and e' increasing, this is not possible. Hence, k decreasing or staying same along with c^m increasing or staying same is ruled out.

- k decreases or stays the same, and c^m decreases. Then, from the resource constraint $c^m + (q+k)n = e$ and the fact that e increases, n must increase. Subsequently, from the first-order condition of n, c^o must increase. Then, again from the first-order condition of n, with k decreasing and n, c^o and e' increasing, $u'(c^m)$ must decrease, that is, c^m must increase. This contradicts the assumption that c^m decreases.

The two cases above rule out all the exhaustive cases for k decreasing or staying same. Hence, k must increase. ■

A.3.2 Ambiguity in period $t+T+1$: an example

Now we illustrate why we cannot comment on whether capital will increase or decrease in the very last period of the transition. Recall that the increasing sequence of resources reaches e^{opt} in period $t+T+1$, that is, $e_{t+T} < e_{t+T+1} = e^{opt}$. Thereafter, it stays at e^{opt} forever. Thus, the planner's problem is $W(e_{t+T}, e^{opt})$ for generation $t+T$ and $W(e^{opt}, e^{opt})$ for subsequent generations. We show that, depending on functional form, an increase in resources inherited may imply all the cases – an increase, decrease or the same level of capital per worker.

Formally, we need to show that the optimal k for the problem $W(e, e')$ can either increase, decrease or stay constant with an increase in e keeping e' constant. Let $u(c) = \frac{(c)^{1-\sigma}}{1-\sigma}$ and $v(n) = \gamma \log(n)$ where $\sigma > 0$. Then, assuming that the non-negative pension constraint does not bind, the first-order conditions are

$$(e - (q + k)n)^{-\sigma} = \beta\alpha k^{\alpha-1} \left(n(k^{\alpha} - e')\right)^{-\sigma},$$

$$\gamma(k^{\alpha} - e')^{\sigma} = \beta \left(q\alpha k^{\alpha-1} + e' - k^{\alpha}(1 - \alpha)\right) n^{1-\sigma}.$$

In the special case of $\sigma = 1$, the second equation above is reduced to

$$\gamma(k^{\alpha} - e') = \beta \left(q\alpha k^{\alpha-1} + e' - k^{\alpha}(1 - \alpha)\right).$$

Hence, when both $u(c)$ and $v(n)$ are logarithmic, capital per-worker k stays the same when resources inherited by a generation e increase. From the Euler equation, which is the first of the two first-order conditions above, it follows that n increases when e increases. Thus, for log utility, an increase in e keeping e' the same implies fertility n increases and k stays constant. Now consider when $\sigma \neq 1$ but is either slightly less than 1 or slightly larger than 1. Formally, consider $\sigma = 1 + \kappa$ or $1 - \kappa$ where κ is arbitrarily small and positive. Rewriting the first-order condition for fertility yields

$$\gamma(k^{\alpha} - e')^{\sigma} = \beta \left(q\alpha k^{\alpha-1} + e' - k^{\alpha}(1 - \alpha)\right) n^{1-\sigma}.$$

The left-hand side in the equation above is increasing in k while the right-hand side is decreasing in k. It follows that k is monotonically increasing in $n^{1-\sigma}$. By a continuity argument, if n increases when e increases for $\sigma = 1$, the same must hold for $\sigma = 1 + \kappa$ or $1 - \kappa$. But this will impact k differently depending on whether $\sigma > 1$ or $\sigma < 1$.

- If $\sigma = 1 + \kappa > 1$, then an increase in e implies an increase in n, that is, a decrease in $n^{1-\sigma}$. As a consequence, k decreases in response to an increase in e.

- If $\sigma = 1 - \kappa < 1$, then an increase in e implies an increase in n, that is, an increase in $n^{1-\sigma}$. As a consequence, k increases in response to an increase in e.

Summarizing, when both $u(c)$ and $v(n)$ have log functional form, an increase in e in problem $W(e, e')$ has no impact on capital per-worker. If $u(c)$ has an elasticity of substitution slightly larger than 1, an increase in e reduces k. On the other hand, if utility is less elastic in consumption than in fertility, an increase in e increases k.

A.4 Sufficiency of first-order conditions: the case of logarithmic utility

We consider utility with the following functional form: $\log(c^m) + \beta\log(c^o) + \gamma\log(n)$. We need to show that first-order conditions are sufficient to characterize the solution to (C) even though the budget set is not convex. In other words, there is a unique e satisfying $V'(e) = 0$ for $V(e)$ defined in (C′). We assume $\alpha < \frac{\gamma+\beta}{1+2\beta+\gamma}$ throughout this appendix. This condition ensures that the elasticity of output with respect to capital is not too high. Otherwise, welfare will be increasing in capital and there is no optimal steady-state as steady-state. We prove the sufficiency in two steps. First, we prove that for a variant of (C′) where only resource constraints are present, there is a unique e satisfying $V'(e) = 0$. Using this result from first step, we prove the uniqueness of e satisfying $V'(e) = 0$ for (C′) in the second step.

Step 1: Sufficiency of first order conditions when only resource constraints are present

Proof. For unconstrained version, for any given e, problem is given by following.

$$
\begin{aligned}
&V(e) \equiv \max_{\{s,n\}} \log(c^m) + \beta\log(c^o) + \gamma\log(n) \\
&\quad \text{subject to} \\
&\qquad c^m + qn + s \le e, \\
&\qquad ne + c^o \le f(s,n).
\end{aligned}
$$

Then for any e, the optimum s and n are determined by

$$\frac{1}{c^m} = \frac{\beta f_1(s,n)}{c^o},$$

$$\frac{\gamma}{n} = \frac{q}{c^m} + \frac{\beta e}{c^o} - \frac{\beta f_2(s,n)}{c^o}.$$

Here constant returns to scale of f ensure $R = f_1(s,n) = f_1(k,1)$ where $k = \frac{s}{n}$ is per-capita capital. Substituting c^m, c^o from resource constraints gives us

$$\frac{1}{e-(q+k)n} = \frac{\beta f_1(k,1)}{n(f(k,1)-e)}, \tag{A12(a)}$$

$$\frac{\gamma}{n} = \frac{\beta(qf_1(k,1)+e-f_2(k,1))}{n(f(k,1)-e)}. \tag{A12(b)}$$

Note that First order condition (A12(b)) implies

$$\frac{\gamma}{\beta}(f(k,1)-e) = qf_1(k,1) + e - f_2(k,1)$$

$$\Rightarrow \frac{\gamma}{\beta}f(k,1) - qf_1(k,1) + f_2(k,1) = \frac{(\beta+\gamma)e}{\beta}.$$

Left hand side is increasing in k. Hence, k is increasing in e. Further rearranging the terms of this equation gives us

$$f(k,1) = \frac{q\beta}{\gamma}f_1(k,1) - \frac{\beta}{\gamma}f_2(k,1) + \frac{\beta+\gamma}{\gamma}e.$$

With a Cobb-Douglas production function, this becomes

$$k^\alpha = \frac{q\beta\alpha}{\gamma}k^{\alpha-1} - \frac{\beta(1-\alpha)}{\gamma}k^\alpha + \frac{\beta+\gamma}{\gamma}e \tag{A13}$$

$$\Rightarrow (\gamma + \beta(1-\alpha))k = q\beta\alpha + (\beta+\gamma)ek^{1-\alpha}.$$

Now First order condition (A12(a)) implies

$$\beta f_1(k,1)e = n\left[(q+k)\beta f_1(k,1) + f(k,1) - e\right]$$

$$\Rightarrow \frac{n}{R} = \frac{n}{f_1(k,1)} = \frac{\beta e}{(q+k)\beta f_1(k,1) + f(k,1) - e}$$

$$\Rightarrow \frac{n}{R} = \frac{\beta e}{q\beta\alpha k^{\alpha-1} + (1+\beta\alpha)k^\alpha - e}.$$

Substituting k^α from (A13) in terms of $k^{\alpha-1}$ and e makes $\frac{n}{R}$ of type $\frac{e}{ak^{\alpha-1}+be}$ where $a > 0$. Since, $\alpha < 1$ and k is increasing in e, $\frac{n}{R}$ is of type $\frac{e}{ah(e)+be}$ where $h(e)$ is positive and decreasing in e. Derivative of $\frac{n}{R}$ with respect to e is $\frac{ah(e)-eah'(e)}{(ah(e)+be)^2} > 0$. So, $\frac{n}{R}$ is increasing in e. Moreover, when $e \to \infty$, $k \to \infty$ and $\frac{n}{R} \to \frac{1}{b}$ where $b = \frac{(1+\beta\alpha)(\beta+\gamma)}{\gamma+\beta(1-\alpha)} - 1$. Hence,

$$\lim_{e\to\infty}\frac{n}{R} > 1 \Leftrightarrow b < 1 \Leftrightarrow \alpha < \frac{\gamma+\beta}{1+2\beta+\gamma}.$$

By assumption, this condition is satisfied. Hence, $\frac{n}{R}$ is increasing in e and equals 1 at some threshold $\hat{e}$. With λ_1 and λ_2 being the Lagrangian multipliers for the two constraints in the problem defining $V(e)$, envelope condition implies

$$V'(e) = \lambda_1 - \lambda_2 n(e) = \beta u'(c^o(e))(R(e) - n(e)).$$

So, there is a cutoff $\hat{e}$ at which $V'(e) = 0 \Leftrightarrow \frac{n}{R} = 1$. Below that $V'(e) > 0$ and above that $V'(e) < 0$. Thus there is a unique e satisfying $V'(e) = 0$. ■

Step 2: Sufficiency of first order conditions for (C)

Proof. We need to show the sufficiency of first-order conditions, that is, there is a unique e such that $V'(e) = 0$ where $V(e)$ is defined by (C′). We have already proved in step 1 that there is exactly one e where $V'(e) = 0$ when non-negative pension constraint is not binding. In this step, first, we show there is at most one e where $V'(e) = 0$ when non-negative pension constraint binds. Then, using this result, and the result in step 1, we show there is exactly one e satisfying $V'(e) = 0$ and the corresponding allocation uniquely solves (C).

With non-negative transfer constraint $e = k^\alpha(1-\alpha)$ binding in an interval, problem (C′) can be rewritten in that interval as

$$V(k) \equiv \max_{\{n\}} \log(k^\alpha(1-\alpha) - (q+k)n) + \beta \log(nk^\alpha \alpha) + \gamma \log(n).$$

We converted the problem in e to problem in k as $e = k^\alpha(1-\alpha)$. $V'(k) = V'(e)e'(k)$ and both the derivatives have same sign as $e'(k) > 0$. The first order condition with respect to n gives

$$\frac{q+k}{k^\alpha(1-\alpha) - (q+k)n} = \frac{\beta+\gamma}{n}$$

$$\Rightarrow n = \frac{\beta+\gamma}{1+\beta+\gamma}\frac{k^\alpha(1-\alpha)}{q+k}.$$

Putting this value of n in objective,

$$V(k) = \alpha(1+2\beta+\gamma)\log(k) - (\beta+\gamma)\log(q+k)$$

$$\Rightarrow V'(k) = \frac{\alpha(1+2\beta+\gamma)}{k} - \frac{\beta+\gamma}{q+k}$$

$$\Rightarrow V'(k) = \frac{\alpha(1+2\beta+\gamma)q + (\alpha(1+\beta) - (1-\alpha)(\beta+\gamma))k}{k(q+k)}.$$

Since we have assumed $\alpha < \frac{\gamma+\beta}{1+2\beta+\gamma} \Leftrightarrow \alpha(1+\beta) - (1-\alpha)(\beta+\gamma) < 0$, $V'(k)$ is initially positive and then becomes negative after some $\hat{k}$. Let the corresponding e be e_1. Hence, $V'(e) > (<) 0$ for $e < (>) e_1$ when non-negative pension constraint is binding.

So far, we have shown that for each of the two cases of non-negative pension constraint binding or not binding, $\lim_{e\to 0} V'(e) > 0$, $\lim_{e\to\infty} V'(e) < 0$ and there is at max one e satisfying $V'(e) = 0$. Thus, combining these two cases, there is at least one e satisfying $V'(e) = 0$ and, at max, there can be two e_1, e_2 satisfying $V'(e_1) = V'(e_2) = 0$. Now, we rule out this case of two e's, namely e_1 and e_2 with $e_1 < e_2$ satisfying $V'(e) = 0$. In this case, we have $V'(e_3) < 0$ for e_3 slightly greater than e_1 and e_4 slightly smaller than e_2. This follows from $V'(e) > (<)0$ for $e > (<)e_i$ for $i \in 1, 2$. Then $V'(e_4) > 0$ and $V'(e_3) < 0$ means there is some e in (e_1, e_2) such that $V'(e) = 0$. Thus, there are three e's satisfying $V'(e) = 0$ which is a contradiction as we established earlier that at most there can be two e's satisfying $V'(e) = 0$. Hence, the third case of $V'(e_1) = V'(e_2) = 0$ is ruled out. This leads us to our conclusion that for problem (C′), there is exactly one e satisfying $V'(e) = 0$ and that e satisfies all first-order conditions. Thus, first-order conditions are sufficient to characterize the optimal steady-state. Hence the proof. ■

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