Skill Heterogeneity, and Housing in a Neoclassical Growth Model*
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Abstract

Growth theory has a well established tradition dealing with transitional dynamics, multiple steady states, poverty traps, etc, but very little of this has been employed to simulate the medium to long run effects of natural disasters. This paper is a first step in this direction. In particular we build upon two major effects of a disaster that are not captured in a traditional two factor, single good growth model- the destruction of housing stock and the alteration of skill composition of the labor force. We construct a dynamic general equilibrium model with skill heterogeneity and different production sectors. Agents are also differentiated by their initial housing stock and their utility is defined over a non-durable good in addition to housing. Further, production of both involves adjustment costs. In this paper we study some of the steady state properties of such a model and calibrate it to US Data.

Keywords: Multi Sector Models, Natural Disasters, Residential Investment, Adjustment Costs, Human Capital.

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1 Introduction

The possibility of reconstruction and growth of an area affected by a large scale natural disaster ultimately rests on the decisions of thousands of individuals – whether they wish to return, the availability of housing, and on

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entrepreneurs' decisions to reinvest in the region. These decisions themselves are further based on the resources (both human and physical), incentives, opportunities, and the risk factors that present themselves to every agent. Moreover, both individuals and firms are heterogenous units even within their respective groups. For example, individuals may be skilled or unskilled which then leads to different constraints and choices. Entrepreneurs or firms are spread over different industries, each of which have their own "input-mix", risk levels, and demand, amongst various other distinguishing factors. Clearly any respectable model of economic recovery must cover at least some of these key aspects. Dynamic general equilibrium (DGE) models where consumers and producers make optimizing decisions in the face of various constraints, and also uncertainty, provide an appealing framework to answer some of these questions. However DGE models, and in particular, standard growth models, have usually focussed on transitional paths that are almost exclusively guided by the dynamics of a capital-labor ratio. This is clearly inadequate for some of the complexity associated with recovery from a natural disaster shock- in particular the role of housing, skill differentials, speed of adjustment, borrowing constraints, etc. While the possible list is endless with increasing complexity arising from multiple interdependencies, in this paper, we try to take a small step forward by considering the implications of some of these variables. In particular, we consider a two sector neo-classical growth model with some of the key features that would be necessary to even begin understanding natural disasters. We model the production side as being characterized by one sector that produces a non-durable good which is used towards consumption and investment. The other sector produces "housing". Both sectors in the economy use capital, skilled labor and unskilled labor in varying intensities. To capture the sluggishness in investment in response to unfavorable shocks, we include adjustment costs for capital in both sectors. From the consumer side, we have two types of consumers based on their wage earnings- skilled labor and unskilled labor. Both groups have similar preferences over the non durable consumption good and housing stock. Both groups have independent housing stock endowments. To keep matters simple we assume exogenous and identical population growth rates for each group. Introducing these simple characteristics, as one will see, already introduces substantial interdependecies. Nevertheless, we can replicate some of the steady state properties of the US economy.[1]

[1] However, we make no attempt to replicate rising wage inequality in the US.

2 Model

At the current stage, we frame the problem in continuous time since that allows for greater analytical tractability. On the consumer side, we assume that the economy is populated by a continuum of consumers, all of whom have infinite horizons, and are initially distinguished by their skill levels. In particular, individuals can be high-skilled (S) or low-skilled (L). The empirical counterparts to these are usually college-graduates versus high-school graduates or production workers versus non production workers. Both types of individuals derive utility from a homogeneous non-durable consumption good and housing. The role of housing is particularly important when modelling recovery from a disaster. The utility function (u) follows standard properties in the literature and preferences are homogeneous. In each period, individuals derive labor income (w), interest income from ownership of capital (r), and dividends from profit making firms (π). This income is allocated between consumption (C_1), housing investments (C_2), and savings for the future.

2.1 The Consumer's Problem.

In this economy, we have two types of consumers, skilled and unskilled $(j = S, L)$. We assume that at time period zero, the initial stock of each type is specified exogenously. Since we do not model the actual choice of human capital accumulation, it is reasonable to assume that the rate of growth of both groups $(n_j)$are the same. Endogenizing the choice or the actual investment process is left for future extension. The consumer maximize their discounted utility which is defined over per capita consumption of the non durable good (c-good 1) and housing stock (h- good 2),

$$\int_0^\infty u\left(c_{1jt}, h_{jt}\right) e^{-(\rho - n_j)t} dt$$

We also need to further specialize the utility function. Based on the housing and business cycle literature, we adopt a Cobb-Douglas embedded within a CRRA formulation,

$$u\left(c_{1jt}, h_{jt}\right) = \left(c_{1jt}^{\gamma} h_{jt}^{1-\gamma}\right)^{1-\sigma}$$

This formulation is consistent with the observation that residential investment in housing as a share of total expenditure does not show any trend behaviour (For example, David and Heathcote (IER,yy) note that the share has not changed between the Consumer Expenditure Surveys of 1984 and 2005).

For an individual of type j we can write the asset use condition as,

$$\dot{Z}_j = \dot{A}_j + (p_2 \dot{H}_j) + C_{1j} + r_h p_2 H_j$$

where the price of the non durable good is normalized to 1, the (relative) price of housing is p_2, H refers to the housing stock, $(p_2 \dot{H})$ is the change in the amount of housing stock held, and r_h is the rental rate on per unit of housing. $\dot{A}_j$ refers to net change in financial assets held by households of type j. In this economy all financial assets held by households are the equivalent of bonds held by firms. In terms of sources of asset changes, for each type j, we have interest income from assets, dividend income from firms, rental payments and capital gains on existing housing stock and wage income,[2]

$$\dot{Z}_j = r_t A_{jt} + \Pi_{1j}(t) + \Pi_{2j}(t) + \left(r_h + \frac{\dot{p}_2}{p_2} \right) p_2 H + w_{jt}(j_{1t} + j_{2t})$$

where Π_{ij} $(i = 1.2)$ refers to dividend income for consumer type j from sector i.

Combining both these equations and noting that since households rent to themselves, we are left with a standard dynamic budget constraint,

$$\dot{A}_j = r_t A_{jt} + \Pi_{1j}(t) + \Pi_{2j}(t) + w_{jt}(j_{1t} + j_{2t}) - p_2 \dot{H} - C_{1j}$$

In per capita (of type j) terms

$$\dot{a}_{jt} = (r_t - n_{jt}) a_{jt} + \pi_{1j}(t) + \pi_{2j}(t) + w_{jt} - c_{1jt} - p_{2t} \left(\dot{h} + n_j h \right)$$

where $\frac{\dot{H}}{L_j} = \dot{h} + n_j h$

Housing stock evolves according to,

$$\dot{H}_{jt} = C_{2jt} - \delta_h H_{jt}$$

where C_{2jt} is the investment in housing by group j and δ_h is the depreciation rate for housing. In per capita (of each group) terms we can write this as,

[2] The asset use and source conditions are based on expressions used by Brito and Perreira (JUE, 2002).

$$\dot{h}_{jt} = c_{2jt} - (\delta_h + n_{jt})\, h_{jt} \tag{1}$$

The Current value Hamiltonian for the consumer is given by,

$Z = \frac{\left(c_{1jt}^{\gamma} h_{jt}^{1-\gamma}\right)^{1-\sigma}}{1-\sigma} + \lambda_j \left[(r_t - n_{jt})\, a_{jt} + \pi_{1j}(t) + \pi_{2j}(t) + w_{jt} - c_{1jt} - p_{2t}(c_{2jt} - \delta_h h_{jt})\right] + \varphi_j \left[c_{2jt} - (\delta_h + n_{jt})\, h_{jt}\right]$

where λ_j is the shadow value of household assets (note that this can differ across skill groups) and φ_{jt} is the shadow value of housing stock.

The FOC's are

$$\gamma \frac{\left(c_{1jt}^{\gamma} h_{jt}^{1-\gamma}\right)^{1-\sigma}}{c_{1jt}} = \lambda_j \tag{2}$$

$$p_t = \frac{\varphi_j}{\lambda_j} \tag{3}$$

$$\Rightarrow \frac{\dot{p}_{2t}}{p_{2t}} = \frac{\dot{\varphi}_j}{\varphi_j} - \frac{\dot{\lambda}_j}{\lambda_j} \tag{4}$$

There are two other conditions on costate variables:

$$\dot{\lambda}_j = -(r_t - \rho)\, \lambda_j \tag{5}$$

$$(1-\gamma) \frac{\left(c_{1jt}^{\gamma} h_{jt}^{1-\gamma}\right)^{1-\sigma}}{h_{jjt}} = \rho \varphi_j - \dot{\varphi}_j \tag{6}$$

Now going back to eqs(2) and 5 and differentiating with respect to time,

$$\Rightarrow -(r_t - \rho) = (\gamma(1-\sigma) - 1)\frac{\dot{c}_{1j}}{c_{1j}} + (1-\gamma)(1-\sigma)\frac{\dot{h}_j}{h_j} \tag{7}$$

This is, of course, the two good counterpart of the standard consumption growth solution in growth models $\left(\frac{\dot{c}}{c} = \frac{1}{\sigma}(r_t - \rho)\right)$

Finally using all four FOC's, we also get,

$$\frac{C_{1jt}}{p_{2t} H_{jt}} = \frac{\gamma}{(1-\gamma)} \left(r_t - \frac{\dot{p}_{2t}}{p_{2t}}\right) \tag{8}$$

i.e.the ratio of spending on non durable goods relative to household wealth is positively correlated with the interest rate and negatively correlated with the rate of inflation of housing. In the balanced growth path, both these variables would be constant and thus this ratio would be constant as well.

Therefore, the rate of growth of housing stock is

$$\frac{\dot{H}_j}{H_j} = \frac{\dot{C}_{1j}}{C_{1j}} - \frac{\dot{p}_2}{p_2} - \frac{\partial \ln\left(r_t - \frac{\dot{p}_2}{p_2}\right)}{\partial t}$$

Substituting this in 7

$$\frac{\dot{C}_{1j}}{C_{1j}} = \frac{(r_t - \rho)}{\sigma} - \frac{(1-\gamma)(1-\sigma)}{\sigma}\left(\frac{\dot{p}_2}{p_2} + \frac{\partial \ln\left(r_t - \frac{\dot{p}_2}{p_2}\right)}{\partial t}\right) + n_j$$

Therefore the solution for C_{1t} along the balanced growth path is,

$$C_{1t} = C_{10} \exp\left[\int_0^t \left(\frac{(r_s - \rho)}{\sigma} + n_j - \frac{(1-\gamma)(1-\sigma)}{\sigma}\frac{\dot{p}_2}{p_2}\right) ds\right]$$

With some further substitutions, we can then also show that, the rate of growth of housing stock is given by,

$$H_{jt} = H_{j0} \exp\left[\int_0^t \left(\frac{(r_t - \rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n_j\right) ds\right] \tag{9}$$

Furthermore,

$$\frac{C_{2jt}}{H_{jt}} = \frac{(r_t - \rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\left(\frac{\dot{p}_2}{p_2} + \frac{\partial \ln\left(r_t - \frac{\dot{p}_2}{p_2}\right)}{\partial t}\right) + n_j + \delta_h \tag{10}$$

In the balanced growth path,

$$\frac{C_{2jt}}{H_{jt}} = \frac{(r_t - \rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n_j + \delta_h \tag{11}$$

Therefore both consumption of the non durable good and housing investment can be tied to down to initial housing stock.

We also have a transversality condition,

$$\lim_{t\to\infty} a_{jt}\lambda_j(t) \exp\left[-\int_0^t (\rho - n_{js})\, ds\right] = 0$$

and a no-ponzi game condition,

$$\lim_{t\to\infty} a_{jt} \exp\left[-\int_0^t (r_s - n_{js})\, ds\right] \geq 0$$

We have a second transversality condition

$$\lim_{t\to\infty} h_{jt}\varphi(t) \exp\left[-\int_0^t (\rho - n_{js})\, ds\right] = 0$$

From the first order condition (5), $\dot{\lambda}_j = -(r_t - \rho)\lambda_j$ we can solve for λ_j[3]

$\lambda_j(t) = \lambda_j(0)\exp\left(-\int_0^t (r_t - \rho)\, ds\right)$.This in turn can be substituted into the transversality condition,

$\lim_{t\to\infty} \lambda_j(0)\exp\left[-\int_0^t (r_t - \rho)\, ds\right]\exp\left[-\int_0^t (\rho - n_{js})\, ds\right] a_{jt} = 0 \Rightarrow$

$$\lim_{t\to\infty}\left\{\exp\left[-\int_0^t (r_t - n_{js})\, ds\right] a_{jt}\right\} = 0$$

2.1.1 Relative consumption expenditures

Recall equation (8)

$$\frac{C_{1jt}}{p_{2t}H_{jt}} = \frac{\gamma}{(1-\gamma)}\left(r_t - \frac{\dot{p}_{2t}}{p_{2t}}\right)$$

and equation (10) in the balanced growth path,

$$\frac{C_{2jt}}{H_{jt}} = \frac{(r_t - \rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\left(\frac{\dot{p}_2}{p_2} + \frac{\partial \ln\left(r_t - \frac{\dot{p}_2}{p_2}\right)}{\partial t}\right) + n_j + \delta_h \tag{12}$$

$$\Rightarrow \frac{p_{2t}C_{2jt}}{p_{2t}H_{jt}} = \frac{(r_t - \rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n_j + \delta_h \tag{13}$$

Going from this to the relative consumption spending values.

[3] For constant r: $e^{(r_t-\rho)t}\left(\dot{\lambda}_j + (r_t - \rho)\lambda_j\right) = 0 \Rightarrow \int e^{(r_t-\rho)t}\left(\dot{\lambda}_j + (r_t-\rho)\lambda_j\right) = \int 0 \Rightarrow \int \frac{\partial e^{(r_t-\rho)t}\lambda_j}{\partial t} = \int \frac{\partial \text{cons tan t}}{\partial t} \Rightarrow e^{(r_t-\rho)t}\lambda_j = c \Rightarrow \lambda_j = e^{-(r_t-\rho)t}c \Rightarrow \lambda_j(0) = c \Rightarrow \lambda_j(t) = \lambda_j(0)\exp\left(-\int_0^t (r_t-\rho)\, ds\right)$

$$\frac{C_{1jt}}{p_{2t}C_{2jt}} = \frac{\frac{\gamma}{(1-\gamma)}\left(r_t - \frac{\dot{p}_{2t}}{p_{2t}}\right)}{\frac{(r_t-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n_j + \delta_h} \tag{14}$$

Therefore as long as population growth rates are the same, the relative aggregate expenditure shares within the two groups will be the same (which is to be expected assuming identical preferences).

2.1.2 Consumption of Relative Skill Groups

Here we show that the consumption expenditures of non durable goods by skilled labor relative to unskilled labor is tied to initial housing wealth,

In the Balanced Growth Path, we have seen that,

$C_{1jt} = \frac{\gamma}{(1-\gamma)}p_{20}H_{j0}\left(r_0 - \frac{\dot{p}_{20}}{p_{20}}\right)\exp\left[\left(\frac{(r-\rho)}{\sigma} + n_j - \frac{(1-\gamma)(1-\sigma)}{\sigma}\frac{\dot{p}_2}{p_2}\right)t\right]$

This is true for both groups.

Therefore relative consumption of groups for good 1 can be written as

$$\frac{C_{1Lt}}{C_{1St}} = \frac{\frac{\gamma}{(1-\gamma)}p_{20}H_{L0}\left(r_0 - \frac{\dot{p}_{20}}{p_{20}}\right)\exp\left[\left(\frac{(r-\rho)}{\sigma} + n_L - \frac{(1-\gamma)(1-\sigma)}{\sigma}\frac{\dot{p}_2}{p_2}\right)t\right]}{\frac{\gamma}{(1-\gamma)}p_{20}H_{S0}\left(r_0 - \frac{\dot{p}_{20}}{p_{20}}\right)\exp\left[\left(\frac{(r-\rho)}{\sigma} + n_S - \frac{(1-\gamma)(1-\sigma)}{\sigma}\frac{\dot{p}_2}{p_2}\right)t\right]}$$

$$\Rightarrow \frac{C_{1Lt}}{C_{1St}} = \frac{H_{L0}\exp\left[n_L t\right]}{H_{S0}\exp\left[n_S t\right]}$$

It is completely tied down by initial housing stock. Furthermore, this means that since consumption of the second good is a strict share of consumption of the first good, this would lead to the relative shares also being a strict function of initial housing stock.

$$\frac{C_{2Lt}}{C_{2St}} = \frac{\frac{\frac{(r_t-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n_L + \delta_h}{\frac{\gamma}{(1-\gamma)}\left(r_t - \frac{\dot{p}_{2t}}{p_{2t}}\right)}C_{1Lt}}{\frac{\frac{(r_t-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n_S + \delta_h}{\frac{\gamma}{(1-\gamma)}\left(r_t - \frac{\dot{p}_{2t}}{p_{2t}}\right)}C_{1St}}$$

$$\Rightarrow \frac{C_{2Lt}}{C_{2St}} = \frac{H_{L0}}{H_{S0}},$$

assuming equal population growth rates across groups.[4] Interestingly the model implies that relative consumption of the two groups have nothing to do with relative wages along the balanced growth path. Instead initial housing stock matters. This is actually not as counter-intuitive as it may initially

[4] By now, an obvious question arises as to what would happen if population growth rates were not similar. Ths simple answer to that is a balanced growth path is not well defined and instead one has to face asymptotical behaviour along the so-called balanced growth path.

seen. This is because transversality conditions as depicted in the appendix indicate that initial housing stock and initial physical capital stock cannot be completely independent of each other in this model because of national accounting considerations. As a result the growth of capital stock (and in turn, output and wages) are tied to the growth of housing stock, which in turn is a function of inflation rates, interest rates and other parameters in the model. Thus ultimately one can reduce the relative consumption to relative housing endowments.

2.2 Production

The production functions of both sectors are Cobb-Douglas. However the sectors can have different factor intentisities and different labor augmenting technology levels (and growth rates).[5] For sector 1,

$$Y_1 = K_1^{\alpha_1} S_1^{\beta_1} \left(E_1 L_1\right)^{1-\alpha_1-\beta_1}$$

The growth rate of unskilled-labor augmenting technology, E_1 is x_1. Total investment expenditures associated with any investment is given by the adjustment cost function

$$I_1\left(t\right)\left[1+\phi\left(\frac{I_1}{K_1}\right)\right]$$

Following Barro and Sala-i-Martin (2006?) and Baxter (Restat 1996), we assume that the adjustment cost function exhibits the following properties,

$$\phi\left(\frac{I_1}{K_1}\right) = \frac{b_1}{2}\frac{I_1}{K_1}$$
$$\phi'\left(\frac{I_1}{K_1}\right) = \frac{b_1}{2}$$

The firm Current Value Hamiltonian is:

$J = K_1^{\alpha_1} S_1^{\beta_1} \left(E_1 L_1\right)^{1-\alpha_1-\beta_1} - I_1\left(t\right)\left[1+\phi\left(\frac{I_1}{K_1}\right)\right] - w_L L_1 - w_S H_1 + q_1[I_1 - \delta_1 K_1]$

We can write the production function in terms of effective units

$$y_1 = \hat{k}_1^{\alpha_1} \hat{s}_1^{\beta_1},$$

where $\hat{k}_1 = K_1/E_1L_1$ and $\hat{s}_1 = S_1/E_1L_1$.

[5] Therefore, the function is reminscient of Mankiw, Romer and Weil (QJE 1992)

We have the three first order conditions
$\frac{\partial J}{\partial L} = 0 \Rightarrow \frac{\partial F}{\partial L} = w_L \Rightarrow$

$$(1 - \alpha_1 - \beta_1)\, \hat{k}_1^{\alpha_1} \hat{s}_1^{\beta_1} E_1 = w_L$$

$\frac{\partial F}{\partial S} = w_S \Rightarrow$

$$\beta_1 \hat{k}_1^{\alpha_1} \hat{s}_1^{\beta_1 - 1} = w_S$$

$\frac{\partial J}{\partial I} = 0 \Rightarrow -\left[1 + \phi\left(\frac{I_1}{K_1}\right)\right] - \frac{I_1(t)}{K_1}\phi'\left(\frac{I_1}{K_1}\right) + q_1 = 0$

$$\begin{aligned} q_1 - 1 &= \frac{I_1(t)}{K_1} b_1 \\ \Rightarrow \frac{I_1}{K_1} \equiv \frac{\hat{\imath}_1}{\hat{k}_1} &= \frac{q_1 - 1}{b} \end{aligned} \tag{15}$$

where $\hat{\imath}_1 = I_1 / E_1 L_1$.
We have the condition on the co-state variables,
$\frac{\partial J}{\partial K} = r q_1 - \dot{q}_1$

$$\Rightarrow \alpha_1 \hat{k}_1^{\alpha_1 - 1} \hat{s}_1^{\beta_1} + \frac{(q_1 - 1)^2}{2 b_1} = (r + \delta_1)\, q_1 - \dot{q}_1 \tag{16}$$

Finally the capital accumulation equation can also be written in efficiency ajdusted terms ,

$$\begin{aligned} \frac{d\hat{k}_1}{dt} &= \hat{\imath}_1 - (\delta_1 + n_1 + x_1)\, \hat{k}_1 \\ \Rightarrow \frac{d\hat{k}_1}{dt} &= \hat{\imath}_1 - (\delta_1 + n_1 + x_1)\, \hat{k}_1 \end{aligned} \tag{17}$$

Turning to the second sector (housing), everything is analogous except note that the price of the output in this sector is p_{2t}.

$$Y_2 = K_2^{\alpha_2} S_2^{\beta_2} (E_2 L_2)^{1 - \alpha_2 - \beta_2}$$

The second firm's current value Hamiltonian is given by,
$J = p_2 K_2^{\alpha_2} S_2^{\beta_2} (E_2 L_2)^{1 - \alpha_2 - \beta_2} - I_2 \left[1 + \phi\left(\frac{I_2}{K_2}\right)\right] - w_L L_2 - w_S S_2 + q_2 [I_2 - \delta_2 K_2]$

FOC's will give us the following

$$p_2 (1-\alpha_2-\beta_2)\, \hat{k}_2^{\alpha_2} \hat{s}_2^{\beta_2} E_2 = w_L$$

$$p_2 \beta_2 \hat{k}_2^{\alpha_2} \hat{s}_2^{\beta_2 - 1} = w_S \tag{18}$$

$$q_2 - 1 = \frac{I_2(t)}{K_2} \phi' \left(\frac{I_2}{K_2}\right) + \phi \left(\frac{I_2}{K_2}\right) \Rightarrow \frac{I_2}{K_2} \equiv \frac{\hat{\imath}_2}{\hat{k}_2} = \frac{q_2 - 1}{b_2}$$

$$p_2 \alpha_2 \hat{k}_2^{\alpha_2 - 1} \hat{s}_2^{\beta_2} + \frac{(q_2-1)^2}{2b_2} = (r+\delta_2)\, q_2 - \dot{q}_2 \tag{19}$$

$$\frac{d\hat{k}_2}{dt} = \hat{\imath}_2 - (\delta_2 + n_2 + x_2)\, \hat{k}_2$$

Finally for firms in both sectors, we have the transversality condition,

$$\lim_{t \to \infty} q_i(t) K_i(t) \exp \left[- \int_0^t r(s)\, ds \right] = 0$$

2.2.1 Description of bonds

Unlike standard growth models, once adjustment costs are involved, one needs to provide a proper description of the financing of investment. Here we follow, Abel and Blanchard (1983), where bonds are issued by firms to finance only net new investment

Therefore firm 1,

$$\dot{B}_1 = I_1 \left(1 + \phi\left(\frac{I_1}{K_1}\right)\right) - \delta_1 K_1 (1 + \phi(\delta_1)) \tag{20}$$

and dividends (not profits)[6]

$$\Pi_1 = Y_1 - w_L L_1 - w_S S_1 - r B_1 - \delta_1 K_1 (1 + \phi(\delta_1)) \tag{21}$$

and similarly for firm 2:

$$\dot{B}_2 = I_2 \left(1 + \phi\left(\frac{I_2}{K_2}\right)\right) - \delta_2 K_2 (1 + \phi(\delta_2)) \tag{22}$$

$$\Pi_2 = p_2 Y_2 - w_L L_2 - w_S S_2 - r B_2 - \delta_2 K_2 (1 + \phi(\delta_2)) \tag{23}$$

This is all in terms of the numeraire good.

Therefore ,

[6] This is also the net cash flow. Following Abel and Blanchard, this is all used as dividends.

$$\dot{B}_1 + \dot{B}_2$$
$$= I_1 \left(1 + \phi\left(\frac{I_1}{K_1}\right)\right) - \delta_1 K_1 \left(1 + \phi\left(\delta_1\right)\right) + I_2 \left(1 + \phi\left(\frac{I_2}{K_2}\right)\right) - \delta_2 K_2 \left(1 + \phi\left(\delta_2\right)\right)$$

With respect to the Hamiltonian the firm note the farm is maximizing its discounted net cash flows, which for firm 1 is,

$$V_1(0) = \int_0^\infty \left(K_1^{\alpha_1} S_1^{\beta_1} \left(E_1 L_1\right)^{1-\alpha_1-\beta_1} - I_1(t)\left[1 + \phi\left(\frac{I_1}{K_1}\right)\right] - w_L L_1 - w_S H_1\right) \exp\left[-\int_0^t r(s)\,ds\right] dt \tag{24}$$

It can be then shown that (see appendix),

$$q(0)\,K_1(0) = V_1(0) \tag{25}$$

Furthermore, it can also be shown that,

$$V_1(0) = B_1(0) + \int_0^\infty \Pi_1(t) \exp\left[-\int_0^t r(s)\,ds\right] dt$$

This in turn give us

$$q(0)\,K_1(0) = V_1(0) = B_1(0) + \int_0^\infty \Pi_1(t) \exp\left[-\int_0^t r(s)\,ds\right] dt \tag{26}$$

Remark 1 *Therefore the value of the firm is equal to the current value of capital stock (from theory), and from an accounting perspective is equal to current bonds outstanding plus PDV of all future net cash flows.*

Similarly for sector 2

$$q(0)\,K_2(0) = V_2(0) = B_2(0) + \int_0^\infty \Pi_2(t) \exp\left[-\int_0^t r(s)\,ds\right] dt \tag{27}$$

All of these are of course standard results.

2.2.2 Factor Price Equalization

Having presented the basic profit maximization problem, we now discuss some factor market conditions that must hold at all times.

First of all note that since q is the shadow value of investment goods and both types of firms are in a perfectly competitive environment,

$$q_1 = q_2 = q$$

This in turn implies,

$$\begin{aligned} &\Rightarrow b_2 \frac{\hat{\imath}_2}{\hat{k}_2} = b_1 \frac{\hat{\imath}_1}{\hat{k}_1} \\ &\Rightarrow b_2 \frac{I_2}{k_2} = b_1 \frac{I_1}{k_1} \end{aligned}$$

Since there is only one interest rate, the condition that q equalize across sectors also gives us the following arbitrage condition,

$$\Rightarrow p_2 \alpha_2 \hat{k}_2^{\alpha_2 - 1} \hat{s}_2^{\beta_2} + \frac{(q-1)^2}{2b_2} - \delta_2 q = \alpha_1 \hat{k}_1^{\alpha_1 - 1} \hat{s}_1^{\beta_1} + \frac{(q-1)^2}{2b_1} - \delta_1 q \qquad (28)$$

In addition to q, wages in both sectors must equalize. For unskilled wages,

$$p_2 \times MPL_2 = p_1 \times MPL_1 = w_L$$

$$p_2 \left(1 - \alpha_2 - \beta_2\right) \hat{k}_2^{\alpha_2} \hat{s}_2^{\beta_2} E_{2,0} e^{x_2 t} = \left(1 - \alpha_1 - \beta_1\right) \hat{k}_1^{\alpha_1} \hat{s}_1^{\beta_1} E_{1,0} e^{x_1 t}, \qquad (29)$$

and skilled wages:

$$\begin{aligned} p_2 \times MPS_2 &= p_1 \times MPS_1 = w_S \\ \Rightarrow \beta_1 \hat{k}_1^{\alpha_1} \hat{s}_1^{\beta_1 - 1} &= p_2 \beta_2 \hat{k}_2^{\alpha_2} \hat{s}_2^{\beta_2 - 1} \qquad (30) \end{aligned}$$

Taking the ratio of the two wages, we get,

$$\frac{\beta_1 \hat{k}_1^{\alpha_1} \hat{s}_1^{\beta_1 - 1}}{\left(1 - \alpha_1 - \beta_1\right) \hat{k}_1^{\alpha_1} \hat{s}_1^{\beta_1} E_{1,0} e^{x_1 t}} = \frac{p_2 \beta_2 \hat{k}_2^{\alpha_2} \hat{s}_2^{\beta_2 - 1}}{p_2 \left(1 - \alpha_2 - \beta_2\right) \hat{k}_2^{\alpha_2} \hat{s}_2^{\beta_2} E_{2,0} e^{x_2 t}}$$

$$\frac{\beta_1}{\left(1 - \alpha_1 - \beta_1\right) s_1} = \frac{\beta_2}{\left(1 - \alpha_2 - \beta_2\right) s_2} \qquad (31)$$

where $s_1 = S_1/L_1$. So this implies that "skill intensity" growth is the same across both sectors,

$$\frac{\dot{s}_1}{s_1} = \frac{\dot{s}_2}{s_2}$$
$$\Rightarrow \frac{\dot{S}_1}{S_1} - \frac{\dot{L}_1}{L_1} = \frac{\dot{S}_2}{S_2} - \frac{\dot{L}_2}{L_2} \equiv \kappa$$

where κ denotes the growth rate of skill intensity on both sectors.

3 The Balanced Growth Path

3.1 Aggregate conditions

In this economy aggregate GDP is

$$Y_1 + p_{2t}Y_{2t} = C_{1L} + C_{1S} + p_{2t}C_{2Lt} + p_{2t}C_{2St} + I_1\left(1 + \phi\left(\frac{I_2}{K_2}\right)\right) + I_2\left(1 + \phi\left(\frac{I_2}{K_2}\right)\right)$$

Further the individual sector conditions must hold,

$$C_1 + C_2 \leq Y_1$$
$$C_2 \leq Y_2$$

We also have the following aggregate labor market conditions that must hold at every time period,

$$\begin{aligned} L_1 + L_2 &= L \\ S_1 + S_2 &= S \end{aligned}$$

3.2 Description of the Balanced Growth Path

The most sensible depiction of a stationary economy is one where at least all of the following hold

1. Output per worker growth is constant (since that is what most growth models, if not all, assume) in both sectors. Here we will take this as Y/L growth is constant in each sector.

2. Inflation rate is fixed (likely to be tied down by technology and other production function parameters).

3. Interest rate is fixed

4. q is constant $\left(\frac{\dot{q}}{q}=0\right)$

5. Some key ratios such as capital output ratios, $\frac{K_1}{Y_1}$ and $\frac{K_2}{p_2 Y_2}$ should be fixed. Also investment rates. but that will be guaranteed if the capital output ratios are fixed.

6. Consumption spending shares should be fixed: $\frac{C_{1j}}{p_2 C_{2j}}$ $(j=L,S)$ (or at least $\frac{C_{1j}}{p_2 \dot{H}_{2j}}$).

7. GDP shares of the two sectors be fixed.

   We turn next to solving the model and showing that these properties are easily satisfied. To achieve this, we first solve for some key growth rates, in particular the inflation rates and the rates of growth of capital intensities in both sectors. In particular, we establish that the inflation rate will be fixed along the BGP. Having calculated these variables, we next derive some of the key ration listed above. Finally, we derive the interest rate in the economy which ties the growth rates of variable from the consumption side and the production side.

3.2.1 Features inherited from the model (ie without relying on BGP)

There are some growth rates that are already implicit in the factor market equilibrium conditions. First, one of the key properties that we would like to see in the BGP is constant growth rates of output per worker in both sectors.,

$$\frac{\dot{y}}{y}=\frac{\dot{y}_{1t}}{y_{1t}}=\frac{\dot{p}_{2t}}{p_{2t}}+\frac{\dot{y}_{2t}}{y_{2t}}$$

We can rewrite the latter equality as:

$$\Rightarrow \alpha_1\frac{\dot{k}_1}{k_1}+\beta_1\kappa+(1-\alpha_1-\beta_1)\,x_1=\frac{\dot{p}_{2t}}{p_{2t}}+\alpha_2\frac{\dot{k}_2}{k_2}+\beta_2\kappa+(1-\alpha_2-\beta_2)\,x_2 \quad (32)$$

However, this is the same as the growth version of the wage equalization condition (**??**) and thus not a "steady state" condition.Therefore the requirement that growth rate of output per unskilled worker in each sector be equal is actually not a steady state condition but a perpetual condition. From hindsight,this should have been obvious: with cobb douglas technology we know that marginal product of labor is a linear function of Y/L.

2 From earlier discussions we also know that

$$\frac{\dot{s}_1}{s_1} = \frac{\dot{s}_2}{s_2} = \kappa$$
$$\Rightarrow \frac{\dot{S}_1}{S_1} - \frac{\dot{L}_1}{L_1} = \frac{\dot{S}_2}{S_2} - \frac{\dot{L}_2}{L_2} = \kappa$$

A corollary,

$$\Rightarrow \frac{(Y_1 \dot{/} S_1)}{Y_1/S_1} = \frac{(Y_2 \dot{/} S_2)}{Y_2/S_2}$$

3.2.2 Steady state growth of p, q, and capital per worker in each sector.

Consider the interest rate condition from any sector. In particular consider sector 2, equation (19)

$$p_2 f'_{\hat{k}_2}\left(\hat{k}_2, \hat{s}_2\right) + \frac{(q-1)^2}{2b_2} = (r_t + \delta_2)\, q - \dot{q}$$

At the steady state $\frac{\dot{q}}{q} = 0$. In this case $\frac{p_2 \alpha_2 \hat{k}_2^{\alpha_2 - 1} \hat{s}_2^{\beta_2}}{q}$ will be a constant Therefore,

$$\frac{\dot{p}_2}{p_2} + \frac{(\alpha_2 - 1)\frac{d\hat{k}_2}{dt}}{\hat{k}_2} + \beta_2 \frac{\frac{d\hat{s}_2}{dt}}{\hat{s}_2} = 0$$

$$\Rightarrow \frac{\dot{p}_2}{p_2} = (1 - \alpha_2)\frac{\dot{k}_2}{k_2} - (1 - \alpha_2 - \beta_2)\, x_2 - \beta_2 \kappa \tag{33}$$

Therefore the rate of inflation is positively tied to rate of growth of capital per worker in sector two, negatively tied to the rate of technological change in sector two and negatively tied to the rate of growth of skilled intensity in sector two.

By similar reasoning, for sector 1,

$$\Rightarrow \frac{\dot{k}_1}{k_1} = \frac{(1 - \alpha_1 - \beta_1)}{(1 - \alpha_1)} x_1 + \frac{\beta_1}{(1 - \alpha_1)} \kappa \tag{34}$$

We can now use this in equation (32) and making appropriate substitutions. Now using this equation and substituting in equation (33)

$$\frac{\dot{k}_2}{k_2} = \frac{(1 - \alpha_1 - \beta_1)}{(1 - \alpha_1)} x_1 + \frac{\beta_1}{(1 - \alpha_1)} \kappa \tag{35}$$

Interestingly this suggests that capital per unskilled worker growth is the same in both sectors and is completely driven by technological progress in sector 1.

We can use these to solve for the inflation rate,[7]

$$\frac{\dot{p}_2}{p_2} = \frac{(1-\alpha_2)(1-\alpha_1-\beta_1)}{(1-\alpha_1)} x_1 - (1-\alpha_2-\beta_2) x_2 + \frac{(1-\alpha_2)\beta_1 - (1-\alpha_1)\beta_2}{(1-\alpha_1)} \kappa \tag{36}$$

$$\frac{\dot{p}_2}{p_2} = \tilde{p} \text{ for short} \tag{37}$$

Therefore the rate of inflation can be viewed as a weighted difference between the two rates of technological change and the rate of growth of skill intensities. The former makes a lot of sense - in a simple model, normally the rate of growth of prices would be inversely related to the rate of growth of technology. Here too, note that the rate of growth of technology in sector one (non-durable goods) leads to a higher relative price for residential investment while technological progress in the housing sector reduces the relative price. However, an increase in the share of skilled labor, β_1, increases the relative price (it works very much like technological progress in the first sector). Finally, note that we can also write the above as,

$$p_2(t) = p(0) \exp^{\left(\frac{(1-\alpha_2)(1-\alpha_1-\beta_1)}{(1-\alpha_1)} x_1 - (1-\alpha_2-\beta_2) x_2 + \frac{(1-\alpha_2)\beta_1 - (1-\alpha_1)\beta_2}{(1-\alpha_1)} \kappa\right) t} \tag{38}$$

Using equation (34) we can also show that the growth rate of $\hat{k}_1$,the capital per "effective unskilled worker",

$$\frac{\frac{d\hat{k}_1}{dt}}{\hat{k}_1} = \frac{\beta_1}{(1-\alpha_1)} (\kappa - x_1) \tag{39}$$

So this means capital per effective worker will no longer be fixed at the steady state. Solving the above differential equation given $K_1(0) = K_0, E_1(0) = E_0$

$$\hat{k}_1 = \frac{K_1(0)}{E_1(0) L_1(0)} \exp\left(\frac{\beta_1}{(1-\alpha_1)} (\kappa - x_1) t\right)$$

with $L_1(0)$ being endogenous.

[7] The implicit assumption here is that κ is constant and it is not obvious that it should be so. Later on we will show that not only is κ constant but equal to zero.

Calculating rate of growth of capital per effective worker in the second sector

$$\Rightarrow \frac{\frac{d\hat{k}_2}{dt}}{\hat{k}_2} = (x_1 - x_2) + \frac{\beta_1}{(1-\alpha_1)}(\kappa - x_1)$$

So capital per efficiency unit could actually be falling over time in sector 2.

Solving for steady state q Now capital per effective worker growth is given from equation (17)

$\frac{d\hat{k}_1}{dt} = \hat{\imath}_1 - (\delta_1 + n_1 + x_1)\hat{k}_1$

So this gives us

$\frac{\frac{d\hat{k}_1}{dt}}{\hat{k}_1} = \frac{\hat{\imath}_1}{\hat{k}_1} - (\delta_1 + n_1 + x_1)$

Using the fact that $1\frac{}{\hat{k}_1}$=q-1$\frac{}{b_1}$ and the expression for capital per effective worker growth (39) we have,

$$\frac{q-1}{b_1} = (\delta_1 + n_1 + x_1) + \frac{\beta_1}{(1-\alpha_1)}(\kappa - x_1) \tag{40}$$

For sector 2

$$\frac{q-1}{b_2} = (\delta_2 + n_2 + x_1) + \frac{\beta_1}{(1-\alpha_1)}(\kappa - x_1) \tag{41}$$

This would imply the following parameter restriction,

$$\frac{b_1}{b_2} = \frac{(\delta_2 + n_2 + x_1) + \frac{\beta_1}{(1-\alpha_1)}(\kappa - x_1)}{(\delta_1 + n_1 + x_1) + \frac{\beta_1}{(1-\alpha_1)}(\kappa - x_1)}$$

Now that we have all this, it is easy to calculate steady state q

$$q = 1 + b_1\left(\frac{\beta_1}{(1-\alpha_1)}(\kappa - x_1) + (\delta_1 + n_1 + x_1)\right) \tag{42}$$

3.3 Key Ratios in the Economy

3.3.1 Relative Investment Shares

Investment goods are a part of the output of the first sector. Therefore to calculate the overall investment share of GDP in the economy, we need to calculate investment good production as a share of sector 1 GDP first,

$$\frac{I}{Y_1} = \frac{I_1(t)\left[1+\phi\left(\frac{I_1}{K_1}\right)\right] + I_2(t)\left[1+\phi\left(\frac{I_2}{K_2}\right)\right]}{Y_1}$$

Note that we already know that $\frac{K_1/L_1}{K_2/L_2}$ is a constant (from equations (34) and (35)).

$$\frac{\dot{k}_1}{k_1} = \frac{\dot{k}_2}{k_2} = \frac{(1-\alpha_1-\beta_1)}{(1-\alpha_1)}x_1 + \frac{\beta_1}{(1-\alpha_1)}\kappa$$

Recall that in the steady state
$\frac{\alpha_1 \hat{k}_1^{\alpha_1-1}\hat{s}_1^{\beta_1}}{q} + \frac{(q_1-1)^2}{2qb_1} = (r_t+\delta_1)$
and
$\frac{p_2\alpha_2 \hat{k}_2^{\alpha_2-1}\hat{s}_2^{\beta_2}}{q} + \frac{(q-1)^2}{2qb_2} = (r_t+\delta_2)$
The capital output ratio in sector 1 is of course
$\frac{\hat{k}_1}{\hat{y}_1} = \hat{k}_1^{\alpha_1-1}\hat{s}_1^{\beta_1}$
Notice that from the first arbitrage equation this will be a constant. Therefore

$$\frac{\hat{k}_1}{\hat{y}_1} = \frac{1}{\hat{k}_1^{\alpha_1-1}\hat{s}_1^{\beta_1}} \tag{43}$$

$$\Rightarrow \frac{\hat{k}_1}{\hat{y}_1} = \left(\frac{(r_t+\delta_1)q}{\alpha_1} - \frac{(q_1-1)^2}{2\alpha_1 b_1}\right)^{-1} \tag{44}$$

Similarly, Capital Output Ratio in the second sector $\frac{\hat{k}_2}{p_2\hat{y}_2}$,

$$\frac{\hat{k}_2}{p_2\hat{y}_2} = \left(\frac{(r_t+\delta_2)q}{\alpha_2} - \frac{(q-1)^2}{2\alpha_2 b_2}\right)^{-1}$$

But this is also constant from the second arbitrage condition above. Since we have the relative capital output shares, we can also figure out investment-output shares in each sector. Note we know from equation (15) that,

$$\frac{\hat{\imath}_1}{\hat{k}_1} = \frac{q-1}{b_1}$$

and similarly,

$$\frac{\hat{\imath}_2}{\hat{k}_2} = \frac{q-1}{b_2}$$

Therefore, we can also calculate the investment output ratios which would be

$$\frac{I_1}{Y_1} = \frac{q-1}{b_1}\left(\frac{(r_t+\delta_1)\,q}{\alpha_1} - \frac{(q_1-1)^2}{2\alpha_1 b_1}\right)^{-1} \tag{45}$$

and,

$$\frac{I_2}{p_2 Y_2} = \frac{q-1}{b_2}\left(\frac{(r_t+\delta_2)\,q}{\alpha_2} - \frac{(q-1)^2}{2\alpha_2 b_2}\right)^{-1} \tag{46}$$

Thus the key parameter distinguishing the investment gdp ratio in the two sectors are the share of capital, adjustment cost function parameter and the rate of depreciation.

3.3.2 GDP composition in the economy

Having calculated investment GDP shares, we now calculate the consumption to GDP shares which in turn will enable us to calculate the relative GDP shares of the two sectors. At every point in time, the following equation must hold for sector 1,

$Y_1 = C_{1L} + C_{1S} + I_1\left(1+\phi\left(\frac{I_1}{K_1}\right)\right) + I_2\left(1+\phi\left(\frac{I_2}{K_2}\right)\right)$

$\Rightarrow 1 = \frac{C_{1L}}{Y_1} + \frac{C_{1S}}{Y_1} + \frac{I_1\left(1+\phi\left(\frac{I_1}{K_1}\right)\right)}{Y_1} + \frac{I_2\left(1+\phi\left(\frac{I_2}{K_2}\right)\right)}{p_{2t}Y_{2t}}\left(\frac{p_{2t}Y_{2t}}{Y_1}\right)$

But we know that $p_{2t}Y_{2t} = p_{2t}C_{2Lt} + p_{2t}C_{2St}$

$$\Rightarrow 1 = \frac{C_{1L}}{Y_1} + \frac{C_{1S}}{Y_1} + \frac{I_1\left(1+\phi\left(\frac{I_1}{K_1}\right)\right)}{Y_1} + \frac{I_2\left(1+\phi\left(\frac{I_2}{K_2}\right)\right)}{p_{2t}Y_{2t}}\left(\frac{p_{2t}C_{2Lt}+p_{2t}C_{2St}}{Y_1}\right) \tag{47}$$

Further we already know that

$$\frac{C_{1jt}}{p_{2t}C_{2jt}} = \frac{\frac{\gamma}{(1-\gamma)}\left(r_t - \frac{\dot{p}_{2t}}{p_{2t}}\right)}{\frac{(r_t-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n + \delta_h}$$

Therefore we have

$$p_{2t}C_{2Lt} = \frac{\frac{(r_t-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n + \delta_h}{\frac{\gamma}{(1-\gamma)}\left(r_t - \frac{\dot{p}_{2t}}{p_{2t}}\right)} C_{1Lt}$$

and

$$p_{2t}C_{2St} = \frac{\frac{(r_t-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n + \delta_h}{\frac{\gamma}{(1-\gamma)}\left(r_t - \frac{\dot{p}_{2t}}{p_{2t}}\right)} C_{1St}$$

This gives us,

$$\begin{aligned} \frac{p_{2t}C_{2Lt}}{Y_1} &= \zeta\frac{C_{1Lt}}{Y_1}, \\ \frac{p_{2t}C_{2St}}{Y_1} &= \zeta\frac{C_{1St}}{Y_1} \end{aligned}$$

where $\zeta = \frac{\frac{(r_t-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n + \delta_h}{\frac{\gamma}{(1-\gamma)}\left(r_t - \frac{\dot{p}_{2t}}{p_{2t}}\right)}$. This means that investment in residential stock relative to GDP in sector 1 is given by,

$$\frac{p_{2t}C_{2t}}{Y_1} = \zeta\left(\frac{C_{1Lt}}{Y_1} + \frac{C_{1St}}{Y_1}\right) \tag{48}$$

Substituting this in equation (47)

$$\left(\frac{C_1}{Y_1}\right) = \frac{1 - \frac{I_1\left(1+\phi\left(\frac{I_1}{K_1}\right)\right)}{Y_1}}{\left(1 + \zeta\frac{I_2\left(1+\phi\left(\frac{I_2}{K_2}\right)\right)}{p_{2t}Y_{2t}}\right)} \tag{49}$$

Since output for the second sector can only be devoted to residential investment (i.e $p_{2t}C_{2t} = p_{2t}Y_{2t}$), we already have our GDP shares,

$$\frac{p_{2t}C_{2t}}{Y_1} = \frac{\zeta C_1}{Y_1} \tag{50}$$

3.3.3 Labor Shares

Having derived the GDP shares, we can now easily calculate labor shares in the economy as well. Since wages are equalized across two sectors, implying through cobb douglas,

$$\frac{L_2}{L_1} = \frac{(1-\alpha_2-\beta_2)\,p_2Y_2}{(1-\alpha_1-\beta_1)\,Y_1}$$

$$\Rightarrow \frac{L_2}{L_1} = \frac{(1-\alpha_2-\beta_2)}{(1-\alpha_1-\beta_1)}\frac{\zeta C_1}{Y_1} \tag{51}$$

For Skilled Labor, we also have

$$\frac{S_2}{S_1} = \frac{\beta_2}{\beta_1} \frac{\zeta C_1}{Y_1} \tag{52}$$

Together these imply

$$\frac{\dot{S}_1}{S_1} = \frac{\dot{S}_2}{S_2}$$

This obviously implies that

$$\frac{\dot{S}_1}{S_1} = \frac{\dot{S}_2}{S_2} = n = \frac{\dot{L}_1}{L_1} = \frac{\dot{L}_2}{L_2}$$

This also means that

$$\kappa = 0,$$

thereby simplifying some of the earlier calculations for inflation rates, etc.

3.4 Matching Constant Growth Rates

1. **Rate of growth of Aggregate GDP,** Y :

   Recall from the discussion surrounding equation (32)

$$\frac{\dot{y}}{y} = \frac{\dot{y}_{1t}}{y_{1t}} = \frac{\dot{p}_{2t}}{p_{2t}} + \frac{\dot{y}_{2t}}{y_{2t}}$$

This would mean then that,

$\Rightarrow \alpha_1 \frac{\dot{k}_1}{k_1} + \beta_1 \frac{\dot{s}_1}{s_1} + (1-\alpha_1-\beta_1)\, x_1 = \frac{\dot{p}_{2t}}{p_{2t}} + \alpha_2 \frac{\dot{k}_2}{k_2} + \beta_2 \frac{\dot{s}_2}{s_2} + (1-\alpha_2-\beta_2)\, x_2$

and we had shown that

$\frac{\dot{k}_1}{k_1} = \frac{\dot{k}_2}{k_2} = \frac{(1-\alpha_1-\beta_1)}{(1-\alpha_1)} x_1 + \frac{\beta_1}{(1-\alpha_1)} \kappa$

Substituting these into the above expression and also noting that $\kappa = 0$, we now have an expression for output per worker growth,

$$\frac{\dot{y}_{1t}}{y_{1t}} = \frac{\dot{p}_{2t}}{p_{2t}} + \frac{\dot{y}_{2t}}{y_{2t}} = \frac{(1-\alpha_1-\beta_1)}{(1-\alpha_1)} x_1$$

Therefore, rate of growth of aggregate GDP per worker is also $\frac{(1-\alpha_1-\beta_1)}{(1-\alpha_1)}x_1$. Therefore the rate of growth of Aggregate GDP is

$$\frac{\dot{Y}_t}{Y_t} = \frac{\dot{Y}_{1t}}{Y_{1t}} = \frac{\dot{p}_2}{p_2} + \frac{\dot{Y}_{2t}}{Y_{2t}} = \frac{(1-\alpha_1-\beta_1)}{(1-\alpha_1)}x_1 + n \tag{53}$$

$$\frac{\dot{Y}_{1t}}{Y_{1t}} = \frac{(1-\alpha_1-\beta_1)}{(1-\alpha_1)}x_1 + n \tag{54}$$

2 **Rate of Growth of Non Durable Consumption,** $C_1 = c_{1L}L + c_{1s}S$

From equation (**??**) we know that consumption per individual in either group will be same, $\frac{\dot{c}_{1j}}{c_{1j}} = -\frac{(1-\sigma)(1-\gamma)}{\sigma}\tilde{p} + \frac{(r_t-\rho)}{\sigma}$. Thus,

$$\frac{\dot{C}}{C} = \frac{(r_t-\rho)}{\sigma} - \frac{(1-\gamma)(1-\sigma)}{\sigma}\frac{\dot{p}_2}{p_2} + n \tag{55}$$

3 **Rate of growth of Investment in sector 1,** $I_1 \left(= \hat{\imath}_1. * E_1. * L_1\right),$

This follows from the fact that the investment GDP ratio is constant.

$$\frac{\dot{I}_1}{I_1} = \frac{(1-\alpha_1-\beta_1)}{(1-\alpha_1)}x_1 + n = \frac{\dot{Y}_{1t}}{Y_{1t}}$$

4 **Rate of growth of Investment in sector 2,**

$$\frac{\dot{I}_2}{I_2} = \frac{(1-\alpha_1-\beta_1)}{(1-\alpha_1)}x_1 + n$$

5 **Solving for the interest rate:**

Recall that in the BGP, C_2 and H grow at the same rate (11). This means that,

$$\frac{\dot{p}_2}{p_2} + \frac{\dot{C}_2}{C_2} = \frac{\dot{p}_2}{p_2} + \frac{\dot{Y}_{2t}}{Y_{2t}}$$

$$\begin{aligned} \Rightarrow \quad & \left(\frac{(r_t-\rho)}{\sigma} - \left(\frac{(1-\sigma)(1-\gamma)}{\sigma}\right)\frac{\dot{p}_2}{p_2}\right) \\ = \quad & \frac{\alpha_2(1-\alpha_1-\beta_1)}{(1-\alpha_1)}x_1 + (1-\alpha_2-\beta_2)x_2 + n \end{aligned}$$

We can use the above condition to back out the interest rate:

$$
\begin{aligned}
r &= \rho + \\
&\sigma\left(\frac{\alpha_2\left(1-\alpha_1-\beta_1\right)}{\left(1-\alpha_1\right)}x_1+\left(1-\alpha_2-\beta_2\right)x_2+n+\left(\frac{\left(1-\sigma\right)\left(1-\gamma\right)}{\sigma}\right)\frac{\dot{p}_2}{p_2}\right)
\end{aligned} \tag{56}
$$

Once the interest rate is solved, one can go back and solve for all the ratios. It is easy to see that this algorithm can be handled sequentially. The interest rate itself does not depend on any of the ratios. The only endogenous variable is the inflation rate, which in turn is a function of other parameters.

4 Simulations

Though the model at this stage is still somewhat preliminary. Nevertheless, we would like to see how closely it can replicate some of the key data for the US macroeconomy. To get hold of parameter values, we rely on the business cycle literature that now has quite a body of work on housing related issues.[8] Davis and Heathcote (2005) consider a model which has two final goods sectors. One produces consumption and investment goods and another produces housing services. The latter is produced using structures and land and hence is different from ours. Final goods are in turn produced from manufactures, structures, and services in a cobb-douglas production function. Finally each of these are produced using capital and labor in a cobb-douglas production function. Clearly our model is somewhat similar though we do not have the middle layer of production and we abstract from land. On the other hand, we have adjustment costs and also two different types of labor. Though their paper is focussed on business cycle issues, since they also produce some steady state results, we rely on their parameter estimates as much as feasible. Their paper also lists the averages for key ratios for 1948-2001 and we try to assess the success of our model by comparing them to those calculations. Their assumptions for various values are listed in table 1 along with our assumptions.[9] As one can see, we

[8] For example, see Iacoviello, Matteo (AER 2005), Silos (2007), Ortalo-Magne, Francois, and Sven Rady (RESTUD 2007), Morris and Heathcote (IER 2005) among various other papers.

[9] Note that their paper has government, while ours does not. Therefore, our numbers are a rescaling of their estimates after factoring out government shares.

stick to most of their assumptions except for housing depreciation. Their estimates of 1.4% per year was too low and generated an excessive housing stock in our model. Furthermore, their estimates seem to be compartively lower than Greenwood and Hercowitz (1991, estimate is 8%), Silos (JEDC 2007, estimate is 04.3%) and Iacoviello (2005, estimate is 5%). We finally adopted the numbers by Silos. In addition to assumptions regarding these parameter, we also need values for parameters in the production function. Here most of the assumptions are dictated by available data from various national account and other government sources. In particular the capital shares and the rate of productivity growth are listed in Davis and Heathcote (2005, Table 3). However, we also need estimates for skilled labor share and adjustment costs. The skilled labor share is more difficult to calculate. Here, we use Weil (Economic Growth, 2008) where they estimate that the share of skilled labor in developing country is 2/3 of all labor costs. We use this number for the consumption and investment goods sector. For the housing sector, we assume that skilled labor is about 1/3 of all labor costs since it is well known that this sector is one of the least-skilled sectors in the economy. Finally, we need information on initial endowments and initial capital stock and technology levels. The initial endowment is chosen based on CPS data and the supply of skilled workers is equal to the number of adults over age 25 who have a college degree and half of those who went to college but did not complete college. The supply of unskilled workers is those who have a high school degree or less and the remaining half of college incompletes (as is common in the literature, See Goldin and Katz (2007), Unel (2008).).[10] Finally, $K(0)$ is chosen to be 2.2. This is chosen to get realistic numbers for the capital output ratio.[11] The initial level of technologies do not matter for any of our calibration and hence are assumed to be 1.

[10] Note that most of the labor literature weighs these pure numbers by relative wages so that 1 person in skilled labor is not the equivalent of 1 person in unskilled labor. Our model is much more simple and incorporating this is left as a future task.

[11] Note that we are calculating ratios and not actual levels and hence K(0) of 2.2. is obviously not to be taken seriously as a level estimate.

Table 1A
Parameter Assumptions

	C&E	Source
γ	0.92	Davis and Heathcote (2005)
σ	2	Davis and Heathcote (2005)
ρ	0.05	Davis and Heathcote (2005)
δ_h	0.043	Silos (2007)
δ_k	0.09	Silos (2007)
n	0.0101	Davis and Heathcote (2005)

Table 1B
Production Technologies

	Sector 1	Sector 2
α (Capital share)	0.25*	0.13*
β (Skilled Labor Share)	0.49*	0.28*
b (Adjustment Cost Function)	0.03	0.04
x (Productivity Growth)	2%*	-.27%*

Table 1C
Endowments

L/S	0.9
$K(0)$	4

Table 2 lists some of the values for various endogenous variables in our model and compares them with the data. The actual data numbers come from Davis and Heathcote as well. As is obvious our model does a reasonable job in terms of almost all variables. To remind the readers, the expenditure and investment shares of GDP are based on non-government GDP shares since we do not model government.

Table 2
Model vs. Data

	Data	Model
r	6	6.3
$\frac{p}{p}$	0.01	0.007
C_1/GDP	0.778	0.774
p_2C_2/GDP	0.057	0.059
$(I_1+I_2)/GDP$	0.164	0.157
$(K_1+K_2)\,GDP$	1.83	1.81
p_2H/GDP	1.20	1.21
w_s/w_L	1.91	2.19

Overall, the model does a good job of replicating the key variables. Obviously as one can see the wage premium is over predicted. However, we believe that this only indicates that the human capital sector is not well characterized (investment decisions, efficiency units) etc. Furthermore, one needs to collect better information on shares of skilled and unskilled labor which may make the model result closer to the data. Finally, we end with one note of skepticism regarding the data itself. While we already noted the large discrepancies in the business cycle literature on housing and physical capital depreciation rates, the same is also true for housing capital stock as a share of GDP. For example, Davis and Heathcote estimate that housing capital as a share of all GDP is about 100% (which translates in to our estimate of 120% of non Government GDP). However Silos (2007) estimates the ratio to be 170% of all GDP. The differences could be due to time periods. The former looks at 1948-2001 while the latter looks at 1963-2003. Obviously this requires further investigation.

5 Conclusion

What we have presented here is a first small step in a more ambitopus project that aims to provide a structure for understanding convergence issues after a "natural disaster" strikes an economy. Such a shock can be modelled as affecting the endowments of four key variables- capital stock, housing stock, skilled labor and unskilled labor shares. Clearly any model that aims to estimate the rate of convergence needs to deal with transition dynamics which is our most obvious next step. Further no economy is closed and hence we need to also add a degree of openness into the model. To achieve both of these, we need to think about modelling migration and also the degree of capital flows. Finally, we also need to think of multiple equilibria.

APPENDIX

The relationship between bonds, PDV of cash flows, and transversality conditions. Note that firms of both types seek to maximize the PDV of net cash flows.

Thus for Firm 1,

$$V_1 = \int_0^\infty \left(K_1^{\alpha_1} S_1^{\beta_1} (E_1 L_1)^{1-\alpha_1-\beta_1} - I_1(t) \left[1 + \phi\left(\frac{I_1}{K_1}\right)\right] - w_L L_1 - w_S H_1 \right) \exp\left[-\int_0^t r(s)\, ds\right] dt \quad (57)$$

and the transversality condition is $\lim_{t\to\infty} q(t) K_1(t) \exp\left[-\int_0^t r(s)\, ds\right] = 0$.

Now we can rewrite this in efficiency terms,
$\lim_{t\to\infty} q(t) \hat{k}_1(t) \exp\left[-\int_0^t (r(s) - n_{L1}(s) - x_1)\, ds\right] = 0$

Remark 2 *As we have seen $\hat{k}_1(t)$ is not constant but grows at the rate $\frac{\beta_1}{(1-\alpha_1)}(\kappa - x_1)$. Therefore, $\exp\left[-\int_0^t (r(s) - n_{L1}(s) - x_1)\, ds\right]$ has to increase at a faster rate to make sure the integal goes to zero.*

Now the next step is to show that the standard result $qK_1 = V_1$ holds here as well.

Notice that
$\frac{d(qK)}{dt} = \dot{q}K_1 + q\dot{K}_1$

Using the following equations $\alpha_1 \hat{k}_1^{\alpha_1-1} \hat{s}_1^{\beta_1} + \frac{(q_1-1)^2}{2b_1} = (r(t) + \delta_1) q_1 - \dot{q}_1; q_1 - 1 = \frac{I_1(t)}{K_1} b_1$, and $\dot{K}_1 = I_1 - \delta_1 K_1$

$\frac{d(qK)}{dt} = \dot{q}K_1 + q\dot{K}_1 = \left((r(t) + \delta_1) q_1 - \alpha_1 \hat{k}_1^{\alpha_1-1} \hat{s}_1^{\beta_1} - \frac{(q_1-1)^2}{2b_1}\right) K_1 + q(I_1 - \delta_1 K_1)$

$\frac{d(qK)}{dt} = \dot{q}K_1 + q\dot{K}_1 = \left(r(t) q_1 - \alpha_1 \hat{k}_1^{\alpha_1-1} \hat{s}_1^{\beta_1} - \frac{(q_1-1)^2}{2b_1}\right) K_1 + qI_1$

$\frac{d(qK)}{dt} = r(t) q_1 K_1 - \alpha_1 K_1^{\alpha_1-1} S_1^{\beta_1} (E_1 L_1)^{1-\alpha_1-\beta_1} K_1 + I_1 \left[1 + \phi\left(\frac{I_1}{K_1}\right)\right]$

$\alpha_1 K_1^{\alpha_1-1} S_1^{\beta_1} (E_1 L_1)^{1-\alpha_1-\beta_1} K_1 = \alpha_1 Y_1 = Y - w_L L - w_s S$[12]

$\frac{d(qK)}{dt} = r(t) q_1 K_1 - \left(Y - w_L L - w_s S - I_1 \left[1 + \phi\left(\frac{I_1}{K_1}\right)\right]\right)$

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[12] (This can be easily shown by noting that $Y - w_L L - w_s S$
$= K_1^{\alpha_1} S_1^{\beta_1} (E_1 L_1)^{1-\alpha_1-\beta_1} - (1 - \alpha_1 - \beta_1) K_1^{\alpha_1} S_1^{\beta_1} E_1^{1-\alpha_1-\beta_1} L_1^{-\alpha_1-\beta_1} L_1 - \beta_1 K_1^{\alpha_1} S_1^{\beta_1-1} (E_1 L_1)^{1-\alpha_1-\beta_1} S_1$
$= K_1^{\alpha_1} S_1^{\beta_1} (E_1 L_1)^{1-\alpha_1-\beta_1} [1 - (1 - \alpha_1 - \beta_1) - \beta_1] = \alpha_1 K_1^{\alpha_1} S_1^{\beta_1} (E_1 L_1)^{1-\alpha_1-\beta_1} = \alpha_1 Y_1$

therefore, we have
$\frac{d(qK)}{dt} = r(t) q_1 K_1 - $ (current net cashflow)
$\frac{d(qK)}{dt} = r(t) q_1 K_1 - F(t)$
This is a problem of the following form
$\dot{y}(t) - r(t) y(t) = -F(t)$
Multiplying both sides by $\exp\left[\int_0^t r(s)\,ds\right]$ and iIntegrating,
$\int (\dot{y}(t) - r(t) y(t)) \exp\left[-\int_0^t r(s)\,ds\right] dt = \int \exp\left[-\int_0^t r(s)\,ds\right] F(t)\,dt$
If we integrate this from zero to infinity,
$\int_0^\infty (\dot{y}(t) - r(t) y(t)) \exp\left[-\int_0^t r(s)\,ds\right] dt = \int_0^\infty \exp\left[-\int_0^t r(s)\,ds\right] F(t)\,dt$
In our particular example, the RHS is nothing but $V_1(0)$.

The left hand side is $\left[y(t) \exp\left[-\int_0^t r(s)\,ds\right]\right]_0^\infty = y(0)$ since the infinite limit will be zero by the transversality condition. Therefore, we have
$y(0) = V(0)$

But again in our particular case $y(0) = q(0) K(0)$, and therefore we have

$$q(0) K_1(0) = V_1(0) \tag{58}$$

Remark 3 *Next we use the Bond and dividend equation to get an alternative expression fo $V(0)$*

We know from equation (57) that,
$V_1(0) = \int_0^\infty \left(K_1^{\alpha_1} S_1^{\beta_1} (E_1 L_1)^{1-\alpha_1-\beta_1} - I_1(t)\left[1 + \phi\left(\frac{I_1}{K_1}\right)\right] - w_L L_1 - w_S S_1\right) \exp\left[-\int_0^t r(s)\,ds\right] dt$
We now substiute into this expressions for Bonds and Dividends.
Recall from equations (20) and (21)

$$\dot{B}_1 = I_1 \left(1 + \phi\left(\frac{I_1}{K_1}\right)\right) - \delta_1 K_1 (1 + \phi(\delta_1)) \tag{59}$$
$$\Pi_1 = Y_1 - w_L L_1 - w_S S_1 - r B_1 - \delta_1 K_1 (1 + \phi(\delta_1)) \tag{60}$$

This allows us to write down the value of the firm as
$V_1(0) = \int_0^\infty \left(K_1^{\alpha_1} S_1^{\beta_1} (E_1 L_1)^{1-\alpha_1-\beta_1} - I_1(t)\left[1 + \phi\left(\frac{I_1}{K_1}\right)\right] - w_L L_1 - w_S S_1\right) \exp\left[-\int_0^t r(s)\,ds\right] dt$
$V_1(0) = \int_0^\infty \begin{pmatrix} Y - w_L L_1 - w_S S_1 - r B_1 - \delta_1 K_1 (1 + \phi(\delta_1)) + \ldots \\ \ldots + r B_1 + \delta_1 K_1 (1 + \phi(\delta_1)) - I_1(t)\left[1 + \phi\left(\frac{I_1}{K_1}\right)\right] \end{pmatrix} \exp\left[-\int_0^t r(s)\,ds\right] dt$
$V_1(0) = \int_0^\infty \begin{pmatrix} \Pi_1 + \ldots \\ \ldots + r B_1 - \dot{B}_1 \end{pmatrix} \exp\left[-\int_0^t r(s)\,ds\right] dt$

$V_1(0) = \int_0^\infty \left(\Pi_1 + +rB_1 - \dot{B}_1\right) \exp\left[-\int_0^t r(s)\,ds\right] dt$
$V_1(0) = \int_0^\infty \Pi_1 \exp\left[-\int_0^t r(s)\,ds\right] dt - \int_0^\infty \left(\dot{B}_1 - rB_1\right) \exp\left[-\int_0^t r(s)\,ds\right] dt$
$V_1(0) = \int_0^\infty \Pi_1 \exp\left[-\int_0^t r(s)\,ds\right] dt - \int_0^\infty \left(\dot{B}_1 - rB_1\right) \exp\left[-\int_0^t r(s)\,ds\right] dt$

The second integral in the Right hand side is $\left[B_1(t) \exp\left[-\int_0^t r(s)\,ds\right]\right]_0^\infty = B_1(0)$

$$V_1(0) = B_1(0) + \int_0^\infty \Pi_1(t) \exp\left[-\int_0^t r(s)\,ds\right] dt$$

This in turn give us

$$q(0)\,K_1(0) = V_1(0) = B_1(0) + \int_0^\infty \Pi_1(t) \exp\left[-\int_0^t r(s)\,ds\right] dt \qquad (61)$$

Remark 4 *Therefore the value of the firm is equal to the current value of capital stock (from theory), and from an accounting perspective is equal to current bonds outstanding plus PDV of all future net cash flows.*

Similarly for sector 2

$$q(0)\,K_2(0) = V_2(0) = B_2(0) + \int_0^\infty \Pi_2(t) \exp\left[-\int_0^t r(s)\,ds\right] dt \qquad (62)$$

.0.1 The Consumption Function, Initial Conditions and Transversality Conditions

To be able to solve the entire model numerically, not only do we need first order conditions but also the initial price level, shadow values and also the relationship between initial household assets, firm bonds and physical capital stocks. This requires special attention to the transversality conditions and their role in the intertemporal budget constraint.

Note that the budget constraint for individual type j is
$\dot{A}_j = r_t A_{jt} + \Pi_{1j}(t) + \Pi_{2j}(t) + w_{jt} j_t - C_{1jt} - p_{2t}\dot{H}_{2jt}$

Rewriting this in per capita terms where population growth of indivdiual j is n_{jt} :
$\dot{a}_{jt} = (r_t - n_{jt})\,a_{jt} + \pi_{1j}(t) + \pi_{2j}(t) + w_{jt} - c_{1jt} - p_{2t}\left(\dot{h} + n_j h\right)$

Figuring out the consumption Function What we need to do is to tie down initial consumption to present discounted value of assets and wages. Once we have that recall we have a well defined formula for pdv of assets and wages from the production side (equation (61) and 62). This will allow us to "close" the initial condition conditions

Recall:$C_{1j0} = \frac{\gamma}{(1-\gamma)} p_{20} H_{j0} \left(r_0 - \frac{\dot{p}_{20}}{p_{20}} \right)$

Therefore initial consumption stock is tied to housing stock.

Moroever, Since $\gamma c_{1jt}^{\gamma(1-\sigma)-1} h_{jt}^{(1-\gamma)(1-\sigma)} - \lambda_{jt} = 0$

$\Rightarrow \gamma \left(\frac{C_{1j0}}{L_{j0}} \right)^{\gamma(1-\sigma)-1} \left(\frac{H_{j0}}{L_{j0}} \right)^{(1-\gamma)(1-\sigma)} = \lambda_{j0}$

But we also know that

$C_{1j0} = \frac{\gamma}{(1-\gamma)} p_{20} H_{j0} \left(r_0 - \frac{\dot{p}_{20}}{p_{20}} \right)$

Therefore, this helps pin down the initial asset price to the housing price,

$$\lambda_{j0} = \gamma \left(\frac{\gamma}{(1-\gamma)} p_{20} \left(r_0 - \frac{\dot{p}_{20}}{p_{20}} \right) \right)^{\gamma(1-\sigma)-1} \left(\frac{H_{j0}}{L_{j0}} \right)^{-\sigma} \tag{63}$$

Remark 5 *This also clearly indicates that the shadow value for assets will be different for the two consumer groups.*

What about initial ϕ_0? If we have backed out λ_0 and back out initial p_0, we should be able to do that as well.

.0.2 Intertemporal Budget Constraints and Transversality Conditions

We first use the budget constraint:

$\dot{A}_j = r_t A_{jt} + \Pi_{1j}(t) + \Pi_{2j}(t) + w_{jt} j_t - C_{1jt} - p_{2t} \dot{H}_{jt}$

Integrating this over time,

$A_{jT} \left(\exp \left[- \int_0^T r_t ds \right] \right) + \int_0^T \left(C_{1jt} + p_{2t} \dot{H}_{jt} \right) \exp \left[- \int_0^t r_t ds \right] dt$

$= A(0) + \int_0^T (\Pi_{1j}(t) + \Pi_{2j}(t) + w_{jt} j_t) \exp \left[- \int_0^t r_t ds \right] dt$

Consider the case when $T \to \infty$,then by the transversality condition $A_{jT} \left(\exp \left[- \int_0^T r_t ds \right] \right) = 0$

$\Rightarrow \int_0^\infty \left(C_{1jt} + p_{2t} \dot{H}_{jt} \right) \exp \left[- \int_0^t r_t ds \right] dt = A_j(0) + \int_0^\infty (\Pi_{1j}(t) + \Pi_{2j}(t) + w_{jt} j_t) \exp \left[- \int_0^t r_t ds \right] dt$

After repeated substitutions for the left hand side,

$$\frac{\gamma}{(1-\gamma)} p_{20} H_{j0} \left(r - \frac{\dot{p}_2}{p_2}\right) \frac{1}{\left(\frac{(r-\rho)}{\sigma} + n_j - \frac{(1-\gamma)(1-\sigma)}{\sigma}\frac{\dot{p}_2}{p_2} - r\right)}$$
$$+ \left(\frac{(r-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n_j\right) p_{20} H_{j0} \frac{1}{\left(\tilde{p} + \frac{(r-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\tilde{p} + n_j - r\right)}$$
$$= A_j(0) + \int_0^\infty (\Pi_{1j}(t) + \Pi_{2j}(t) + w_{jt} j_t) \exp[-rt]\, dt$$

However, we still need the initial prices, the real interest rate and the rate of inflation. To achieve this, next we show the relationship between initial bonds and housing stock.

Tying Down Transversality Conditions

Proposition 6 $A_S(0) + A_L(0) = B_1(0) + B_2(0)$

Proof. First of all note that,

$\sum_j \left(A_j(0) + \int_0^\infty (\Pi_{1j}(t) + \Pi_{2j}(t) + w_{jt} j_t) \exp\left[-\int_0^t r(s)\,ds\right] dt\right) = A_S(0) +$

$A_L(0) + \int_0^\infty \Pi_1(t) \exp\left[-\int_0^t r(s)\,ds\right] dt$

$+ \int_0^\infty \Pi_2(t) \exp\left[-\int_0^t r(s)\,ds\right] dt + \int_0^\infty (w_{s1t} S_{1t} + w_{l1t} L_{1t}) \exp\left[-\int_0^t r(s)\,ds\right] dt +$

$\int_0^\infty (w_{s2t} S_{2t} + w_{l2t} L_{2t}) \exp\left[-\int_0^t r(s)\,ds\right] dt$

Now consider the workers' PDV of dividend flows and wage flows

$\sum_j \left(\int_0^\infty (\Pi_{1j}(t) + \Pi_{2j}(t) + w_{jt} j_t) \exp\left[-\int_0^t r(s)\,ds\right] dt\right)$.

By Definition this has to be equal to the sum of PDV of dividend flows of firms and wages paid by firms.

$= \int_0^\infty \Pi_1(t) + \Pi_2(t) + \int_0^\infty (w_{s1t} S_{1t} + w_{l1t} L_{1t}) \exp\left[-\int_0^t r(s)\,ds\right] dt$

$+ \int_0^\infty (w_{s2t} S_{2t} + w_{l2t} L_{2t}) \exp\left[-\int_0^t r(s)\,ds\right] dt$

From the definition of dividends, we know that this means

$\int_0^\infty (\Pi_1(t) + \Pi_2(t)) \exp\left[-\int_0^t r(s)\,ds\right] dt$

$\equiv \sum_{1,2} \int_0^\infty (Y_i - w_L L_i - w_S S_i - r B_i - \delta_i K_i (1 + \phi(\delta_i))) \exp\left[-\int_0^t r(s)\,ds\right] dt$

Therefore the sum of dividends for workers plus wages in PDV is

$\sum_j \left(\int_0^\infty (\Pi_{1j}(t) + \Pi_{2j}(t) + w_{jt} j_t) \exp\left[-\int_0^t r(s)\,ds\right] dt\right)$

$= \int_0^\infty (Y - r B_1 - \delta_1 K_1 (1 + \phi(\delta_1))) \exp\left[-\int_0^t r(s)\,ds\right] dt$

However from the definition of Bonds we also know that,
$\int_0^\infty \left(\dot{B}_1 - rB_1\right) \exp\left[-\int_0^t r(s)\, ds\right] dt = B_1(0)$.
Further that, $\dot{B}_1 = I_1 \left(1 + \phi\left(\frac{I_1}{K_1}\right)\right) - \delta_1 K_1 (1 + \phi(\delta_1))$
$\Rightarrow \int_0^\infty I_1 \left(1 + \phi\left(\frac{I_1}{K_1}\right)\right) \exp\left[-\int_0^t r(s)\, ds\right] dt$
$- \int_0^\infty (\delta_1 K_1 (1 + \phi(\delta_1)) + rB_1) \exp\left[-\int_0^t r(s)\, ds\right] dt = B_1(0)$
Therefore,
$\int_0^\infty (Y_1 - rB_1 - \delta_1 K_1 (1 + \phi(\delta_1))) \exp\left[-\int_0^t r(s)\, ds\right] dt$
$= \int_0^\infty Y_1 \exp\left[-\int_0^t r(s)\, ds\right] dt - \int_0^\infty (rB_1 + \delta_1 K_1 (1 + \phi(\delta_1))) \exp\left[-\int_0^t r(s)\, ds\right] dt$
$= \int_0^\infty Y_1 \exp\left[-\int_0^t r(s)\, ds\right] dt - \int_0^\infty I_1 \left(1 + \phi\left(\frac{I_1}{K_1}\right)\right) \exp\left[-\int_0^t r(s)\, ds\right] dt - B_1(0)$

Similarly for sector 2.

All this means that the PDV of workers' income and endowments
$\sum_j \left(A_j(0) + \int_0^\infty (\Pi_{1j}(t) + \Pi_{2j}(t) + w_{jt} j_t) \exp\left[-\int_0^t r(s)\, ds\right] dt\right)$
$= \sum_j A_j(0) - \sum_i B_i(0) + \sum_i \int_0^\infty \left(p_i Y_i - I_i \left(1 + \phi\left(\frac{I_i}{K_i}\right)\right)\right) \exp\left[-\int_0^t r(s)\, ds\right] dt$

However note that the left hand side of the workers intermporal budget constraint is the PDV of consumption.

Since the sum of consumption plus investment must equal GDP at every point in time, it must be the case that

$$\sum_j A_j(0) = \sum_i B_i(0)$$

■

Since we now know that

$$\sum_j \frac{\gamma}{(1-\gamma)} p_{20} H_{j0} \left(r - \frac{\dot{p}_2}{p_2}\right) \frac{1}{\left(\frac{(r-\rho)}{\sigma} + n_j - \frac{(1-\gamma)(1-\sigma)}{\sigma} \frac{\dot{p}_2}{p_2} - r\right)}$$
$$+ \left(\frac{(r-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right) \frac{\dot{p}_2}{p_2} + n_j\right) p_{20} H_{j0} \frac{1}{\left(\tilde{p} + \frac{(r-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right) \tilde{p} + n_j - r\right)}$$
$$= \sum_i \int_0^\infty \left(p_i Y_i - I_i \left(1 + \phi\left(\frac{I_i}{K_i}\right)\right)\right) \exp[-rt]\, dt$$

Focussing on the Right hand side,
$\int_0^\infty \left(Y_1 - I_1 \left(1 + \phi\left(\frac{I_1}{K_1}\right)\right)\right) \exp[-rt]\, dt + \int_0^\infty \left(p_2 Y_2 - I_2 \left(1 + \phi\left(\frac{I_2}{K_2}\right)\right)\right) \exp[-rt]\, dt$

$\Leftrightarrow \int_0^\infty \left(K_1^{\alpha_1} S_1^{\beta_1} (E_1 L_1)^{1-\alpha_1-\beta_1} - K_1 \left(\frac{q-1}{b_1}\right) \left(1 + \frac{q-1}{2}\right)\right) \exp\left[-rt\right] dt$

$+ \int_0^\infty \left(p_2 K_2^{\alpha_2} S_2^{\beta_2} (E_2 L_2)^{1-\alpha_2-\beta_2} - K_2 \left(\frac{q-1}{b_2}\right) \left(1 + \frac{q-1}{2}\right)\right) \exp\left[-rt\right] dt$

Now K_1 and K_2 are exogenous. Growth rates will be calculated later but for the time being let us say that $K_1(t) = K_1(0) e^{g_{k1} t}$ and $K_2(t) = K_2(0) e^{g_{k2} t}$ and

$S_1(t) = S_1(0) e^{nt}$ and $S_2(t) = S_2(0) e^{nt}$

$L_1(t) = L_1(0) e^{nt}$ and $L_2(t) = L_2(0) e^{nt}$

where $S_1(0) + S_2(0) = S_2(0)$ and

$L_1(0) + L_2(0) = L(0)$

Applying these and again after repeated substitution, we can show that the solution for the initial price is:

$$\sum_j \frac{\gamma}{(1-\gamma)} p_{20} H_{j0} \left(r - \frac{\dot{p}_2}{p_2}\right) \frac{1}{\left(\frac{(r-\rho)}{\sigma} + n - \frac{(1-\gamma)(1-\sigma)}{\sigma}\frac{\dot{p}_2}{p_2} - r\right)}$$

$$+\left(\frac{(r-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2} + n\right) p_{20} H_{j0} \frac{1}{\left(\tilde{p} + \frac{(r-\rho)}{\sigma} - \left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\tilde{p} + n - r\right)}$$

$$= \left(\frac{-1}{\left(\frac{(1-\alpha_1-\beta_1)x_1 + (1-\alpha_1)n}{(1-\alpha_1)}\right) - r}\right) \sum_i \left(p(0) Y(0) - K_i(0) \left(\frac{q-1}{b_2}\right)\left(1 + \frac{q-1}{2}\right)\right)$$

where $Y(0)$ is an outcome of labor market clearing and initial capital endowments across sectors.

We are now left with an endogenous interest rate but that can be solved without depending on anything here.

As equation (**??**) suggests there is a tight relationship between initial prices, initial housing stocks and asset allocations. Transversality and initial conditions are thus important to solving the model.

.0.3 Some Thoughts on $p_2(0)$, and on the interdependence of $\mathbf{K}(0)$ and $H(0)$

Consider the two conditions,

$Y_1 = C_1 + I_1 \left(1 + \phi\left(\frac{I_2}{K_2}\right)\right) + I_2 \left(1 + \phi\left(\frac{I_2}{K_2}\right)\right)$

$p_{2t} Y_{2t} = p_{2t} C_{2t}$

Starting with the second condition,

$C_{20} = Y_{20}$

However this implies from equation (11).

$\left(\frac{(r_t-\rho)}{\sigma}-\left(\frac{(1-\gamma+\gamma\sigma)}{\sigma}\right)\frac{\dot{p}_2}{p_2}+n_j+\delta_h\right)H_0=K_{20}^{\alpha_2}S_{20}^{\beta_2}\left(E_{20}L_{20}\right)^{1-\alpha_2-\beta_2}$

Now note that S_{20}, L_{20} are solved endogenously along the balanced growth path through equations (51) and (52) which in turn depend upon GDP shares. However, GDP share is also completely parameterized (well they are functions of interest rates etc but that in turn is parameterized). Now there is no reason why E_{20} should be endogenous, since it is technology. This implies that H_0 and K_{20} must satisfy the above equation. Now, technically K_0 ought to be exogenous. Therefore what this really means is that if we defin

e $K_{20}=\varkappa K\,(0<\varkappa<1)$ then $\varkappa$ needs to satisfy the above condition given any exogenous H_0.

Now moving to the first equation,

Note that C_1/Y_1 is completely parameterized by equation (49). Let's call this ratio ψ. Therefore we have, using equation (8),

$\frac{\gamma}{(1-\gamma)}\left(r_t-\frac{\dot{p}_{2t}}{p_{2t}}\right)p_{20}H_{j0}=K_{10}^{\alpha_1}S_{10}^{\beta_1}\left(E_{10}L_{10}\right)^{1-\alpha_1-\beta_1}$

$\Rightarrow\frac{\gamma}{(1-\gamma)}\left(r_t-\frac{\dot{p}_{2t}}{p_{2t}}\right)p_{20}H_{j0}=\left((1-\varkappa)K_0\right)^{\alpha_1}S_{10}^{\beta_1}\left(E_{10}L_{10}\right)^{1-\alpha_1-\beta_1}$

Now here the only unknown is p_{20}. Thus this equation helps pin down the relative price in time period zero.