On the growth effects of North-South trade: the role of institutional quality*

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Abstract

By using a model of intra-industry trade and endogenous growth, we show that intensified trade and competition from the developing countries lowers the rate of growth of the integrated economy if the quality of the institution protecting the intellectual property rights in the developed countries is below a threshold value.

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1 Introduction

Whether trade integration with low-wage developing countries (the South) is beneficial to growth in industrialized countries (the North) has been a issue of considerable interest in recent times. The formal academic literature on this issue emphasizes that trade increases competition from foreign producers and greater market opportunities abroad. This leads to beneficial productivity gain due to the entry-exit of good and bad firms. In the 'new' growth

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theories[1] there have been persuasive intellectual argument supporting the proposition that trade integration is beneficial to growth for both the North and South. A large body of empirical literature documented trade as an explanation for variations in per-capita income across countries. However the results of this literature are not always robust to controlling for the quality of institutions and geography[2]. This leads to the question of how institutional quality affect the rate of growth in an open economy.

In this paper we explore this issue in a standard R&D based two country endogenous growth model. We extend the basic Helpman (1993) model by introducing imperfect intellectual property rights protection in the North. Each innovation in the North is granted a patent (or, if granted, its right is enforced) with probability q. We interpret the value of q as the quality (or, strength) of the Northern property rights protecting institution. Each Northern monopoly firm faces a constant hazard rate of imitation from the South. Following Arnold (2002), we interpret a higher risk of Southern imitation as *"intensified competition from the South, which may be due to looser international enforcement of intellectual property rights, higher integration into the world economy, or a higher level of education of Southern workers"*. With this structure, we show that more integrated North-South trade lowers the rate of Northern product innovation and hence the rate of growth of the world economy if the quality of the Northern institution falls short of a threshold value.

2 The Model

There are two countries in the world - North and South that are engaged by free trade in differentiated products. North only innovates and South only imitates. There is complete lack of the enforcement of IPRs in the South. This enables South to imitate the Northern goods. For simplicity we assume imitation is costless in South. Because of lower factor costs, production then moves to the South.

We assume that each firm in the North is granted a patent for its innovation with a probability $q \in [0, 1]$. q measures the degree of IPR enforcement in North[3]. With probability $(1 - q)$ the patent is not enforced in the North and the good become free to be produced by the local producers in the North under perfect competition. A higher value of q represents better quality of the Northern IPR protecting institution. When $q = 1$, the quality of

[1] See, for example, Romer (1986), Lucas (1988), Grossman and Helpman (1991) and Barro and Sala-i-Martin (1995), among others. Grossman and Helpman (1991), however, have models where openness affects growth negatively.

[2] Dollar (1992), Sachs and Warner (1995), Frankel and Romer (1999), Wacziarg and Welch (2003), among others, find evidence that trade enhances economic performance. Rodriguez and Rodrik (2001) and Rodrik et al. (2004) show that in many cases the results break down when variables representing institutions and geography are included in the analysis. Bhattacharyya et al. (2009) consider the potential complementarities between trade openness and institutional quality. They observe that in order for a country to benefit from trade, its institutional quality needs to be above a certain threshold level.

[3] Alternatively, we might interpret q as being the fraction of the country's territory in which the patent is enforced. This interpretation is similar to that in Grossman and Lai (2004) - see footnote 3, pp-1638). Also, our interpretation of q as the quality of the Northern IPR protecting institution is similar to Eicher and Penalosa (2008) (see p-242, section 2.4 of their paper).

the Northern institution is perfect and our model becomes identical to the Helpman (1993) model.

At each point in time, there are n_N (n_S) goods being produced in the North (South). In North, products with unenforced patent continue to be produced under perfect competition. Products with enforced patents in the North are produced under monopolistic competition until they are imitated by the South. For simplicity, we assume that South do not target those Northern products that are not granted patent in the North[4]. We represent n_M as the number of varieties in the North that are perfectly protected under patent and n_C as those varieties that are either not awarded a patent or whose patents are not enforced. We have $n_N = n_M + n_C$ and $n = n_N + n_S$ where n_S represents the number of varieties produced in the South. We assume that imitation is costless in South. In each infinitesimal time interval dt, a fraction mdt of the n_M Northern products becomes producible in the South. Thus the number of products in the South grows according to $\dot{n_S} = mn_M$. Imitated products are competitively produced in the South. The expected value of a Northern patent is given by qv_M, where v_M is the value of a patent - calculated as the sum of future stream of profits until it gets imitated by the South.

Labour is the only factor of production. One unit of each of the final good can be produced by one unit of labour anywhere in the world economy. In North, labour is used in the R&D sector and in the production sector. In South, it is used by the production sector. Labour is intersectorally mobile North but internationally immobile. Let L_r denotes employment in the R&D sector in the North which invent new goods according to the production function $\dot{n} = \frac{nL_r}{a}$ $(with\ a > 0)$[5]. So, at each point in time, there are $\dot{n}$ new products entering the economy out of which only q fraction is granted patent. So, the number of monopoly products in the North evolves according to

$$\dot{n_M} = q\dot{n} - \dot{n_S}\ ; \tag{1}$$

where $\dot{n_S}$ is defined earlier. The number of competitively produced products in the North grows according to

$$\dot{n_C} = (1-q)\dot{n}\ . \tag{2}$$

Identical individuals in both North and South choose consumption and saving so as to maximize $W = \int_0^\infty e^{-\rho t} logU(t)dt$ where t is time and $\rho > 0$ is the discount rate. The instantaneous utility takes the form of a constant elasticity of substitution (CES) given by $U = \left(\int_0^n x(z)^\alpha dz\right)^{\frac{1}{\alpha}}$ where $0 < \alpha < 1$. Maximization of the intertemporal utility function of a representative Northern households requires that his expenditure E_N is chosen such that

$$\frac{\dot{E_N}}{E_N} = r - \rho\ ; \tag{3}$$

[4]When both types of firm in the North faces the same imitation risk, more imitation will raise the rate of innovation in North (as in Helpman (1993)). In this paper we focus on the case where only the monopoly firms in the North are targeted.

[5]Note the presence of scale effect in the R&D sector production in the North.

where $r > 0$ is the interest rate[6]. Since consumers in the North and in the South are characterized by the same CES utility function, the demand curve for the differentiated product i is given by $x_i = E\frac{p_i^{-\varepsilon}}{\int_0^n p_j^{1-\varepsilon}dj}$. Here the elasticity of demand is denoted by $\varepsilon(= \frac{1}{1-\alpha} > 1)$, world-wide expenditure is denoted by $E(= E_N + E_S)$ and p_i represents the price of ith variety of the differentiated good. Following Helpman (1993), we assume that financial capital does not flow between the two regions.

Northern monopolists maximize profit by setting price that is a constant markup over the marginal cost: $p_M = \frac{w_N}{\alpha}$, where w_i $(i = N, S)$ is the wage rate in the ith region. They earn a profit $\pi_M = (1-\alpha)p_M x_M$. All other competitively produced products in the North are priced according to their marginal cost, so that: $p_C = w_N$. The price of the imitated Southern product is given by $p_S = w_S$. We shall derive a condition and assume it to hold under which $w_N > w_S$, so that wage rate in the South is always lower than in North. Northern labour market equilibrium condition can be written as

$$L_N = L_r + n_M x_M + n_C x_C; \tag{4}$$

where L_N, $n_M x_M$, $n_C x_C$ and L_r stand for the Northern labour endowment, labour employed in the monopoly production sector, labour employed in the competitive production sector and labour employed in the R&D sector in the North. In the South, labours are only employed in the production sector. So

$$L_S = n_S x_S; \tag{5}$$

is the labour market equilibrium condition in the South. In this paper we only focus on the steady state equilibrium properties of this economy. The following equality is satisfied in the steady state equilibrium.

$$\frac{\dot{n}}{n} = \frac{\dot{n_M}}{n_M} = \frac{\dot{n_C}}{n_C} = \frac{\dot{n_S}}{n_S} = g. \tag{6}$$

We normalise by setting $\frac{\dot{E_N}}{E_N} = \frac{\dot{n}}{n}$[7]. So, equation (3) implies that $r = \rho + g$. The value of a representative Northern monopoly firm (v_M) is defined as the discounted present value of its profits over the infinite time horizon. Also, at time t, a monopolist's probability of remaining a monopolist at time $\tau \geq t$ is given by the expression $e^{-m(\tau-t)}$. So we get $v_M(t) = \int_t^\infty e^{-(r+m)(\tau-t)}\pi_M(\tau)d\tau$. In the steady state, $\pi_M(t)$ is constant over time. Thus the value of a Northern monopoly firm is given by $v_M = \frac{\pi_M}{r+m}$. In the free entry equilibrium with ongoing innovation, the expected value of patent is equal to the cost of its development. So, we have

$$qv_M = \frac{w_N a_N}{n}. \tag{7}$$

From equations (1) and (2) and using the steady state equilibrium condition given by equation (6), we obtain

$$\frac{n_M}{n} = \frac{qg}{m+g}; \tag{8}$$

[6]Since there are no R&D in the South, Southern people do not save and they do not solve any intertemporal choice problem.

[7]This is equivalent to setting the expected value of a Northern monopoly firm to unity at each date (see Lai (1998, pp-137)).

and

$$\frac{n_C}{n_M} = (1-q)\frac{m+g}{qg} \ . \tag{9}$$

Using the static demand function for differentiated goods, the relative sale of a Northern competitive good over monopoly good can be given by

$$\frac{x_C}{x_M} = \left(\frac{p_C}{p_M}\right)^{-\varepsilon} = \alpha^{-\varepsilon}. \tag{10}$$

Using equations (9) and (10) and Northern labour market equilibrium condition (i.e., equation (4)), we obtain

$$n_M x_M = \frac{L_N - a_N g}{\left(1 + \alpha^{-\varepsilon}(1-q)\frac{m+g}{qg}\right)}. \tag{11}$$

Using equations (7), (8), (11) and a bit reformulation we obtain[8]

$$\frac{\frac{1-\alpha}{\alpha}\left(\frac{L_N}{a_N} - g\right)}{\rho + m + g} - \frac{g}{m+g} = \frac{1-q}{q}\alpha^{-\varepsilon}. \tag{12}$$

Equation (12) is the only equation with g as unknown. The LHS of the above equation is clearly a decreasing function of g meeting the vertical axis at a value $\frac{1-\alpha}{\alpha}\frac{L_N}{a_N}\frac{1}{\rho+m}$. The RHS does not depend on g. This implies that unique equilibrium exists with $g > 0$ if $\frac{1-\alpha}{\alpha}\frac{L_N}{a_N}\frac{1}{\rho+m} > \frac{1-q}{q}\alpha^{-\varepsilon}$ is satisfied (see figure 1 below). This is formally stated in the following proposition:

Proposition 1. *Unique steady state equilibrium with ongoing rate of innovation exists if* $q > \frac{1}{1+\frac{1-\alpha}{\alpha}\alpha^{\varepsilon}\frac{L_N}{a_N}\frac{1}{\rho+m}}$.

The RHS of the inequality in proposition 1 serves as the minimum quality of the Northern institution below which incentive for innovation cease to exist in North. When $m = 0$, North survives as a closed economy. Then proposition 1 also implies that, for g to be positive, the minimum quality of the Northern institution has to be higher in an open economy compared to its value under autarky. The expression for the North-South relative wage can be given by

$$\frac{w_N}{w_S} = \alpha\left[\frac{1-\alpha}{\alpha a_N}\frac{L_S}{\frac{\rho}{\left(1+\frac{g}{m}\right)} + m}\right]^{1-\alpha} . \tag{13}$$

Note that, $w_N > w_S$ can always be restored for the size of the South to be sufficiently large[9]. We now report some comparative steady state results of this model.

[8] Derivation of equation (12) is shown in Appendix A

[9] One sufficient condition can be given by $L_S > \alpha^{-\frac{\alpha}{1-\alpha}}\frac{a_N}{1-\alpha}(\rho + m)$. See Appendix B for the derivation of equation (13).

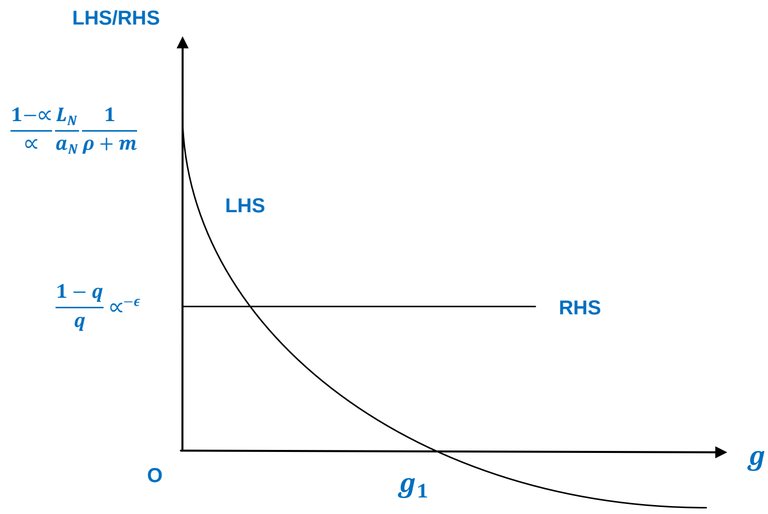

Figure 1: Existence of equilibrium

3 Comparative steady state results

3.1 Institutional quality

An increase in q represents an improvement in the institutional quality of the North. From equation (12), an increase in q shifts the RHS curve in figure 1 downward. The LHS curve remains unaffected. So, g is increased in the new steady state equilibrium. When $q = 1$, then the RHS curve coincides with the horizontal axis g attains its maximum value denoted by $g = g_1$ in figure 1. This is the equilibrium rate of growth in the Helpman (1993) model[10]. Also from equation (13), an increase in q raises g and this in-turn raises $\frac{w_N}{w_S}$. We summarize these results in the following proposition.

Proposition 2. *An improvement in the quality of the Northern institution raises the equilibrium balanced rate of growth of the world economy and raises the North-South relative wage.*

Better quality of the Northern institution raises the share of products produced under monopolistic competition in the North. This lowers the size of the aggregate manufacturing employment there. Full employment then requires that additional workers will be relocated from manufacturing to R&D. This raises the rate of innovation in the North. A higher rate of growth raises the share of the products produced in the North compared to the South. This raises the relative demand for labour in North and raises Northern relative wage.

[10]Note that when $q = 1$, our equation (12) is exactly the same as equation (29) in the Helpman (1993) model (page-1261).

3.2 Intensified intra-industry trade

Here we find the effect of m on g. An increase in m can be seen as intensified competition from the South. Helpman (1993) interpreted it as a lax of IPR protection in the South. From equation (12) we obtain

$$\frac{dg}{dm} > (=) < \ 0 \qquad iff \quad q > (=) < \frac{1}{1+\frac{\rho g \alpha^{\varepsilon}}{(m+g)^2}}.$$

When $q = 1$, we always have the situation where $\frac{dg}{dm} > 0$. By continuity, we should expect $\frac{dg}{dm} > 0$ for q sufficiently close to unity. One sufficient condition for $\frac{dg}{dm} < 0$ can be given by $q \leq \frac{1}{1+\frac{\rho\alpha^{\varepsilon}}{4m}}$. From proposition 1 we know that the value of q has to be greater than a threshold value for the rate of growth to be positive. Then a sufficient condition for $\frac{dg}{dm} < 0$ can be given by $\frac{1}{1+\frac{1-\alpha}{\alpha}\alpha^{\varepsilon}\frac{L_N}{a_N}\frac{1}{\rho+m}} < q \leq \frac{1}{1+\frac{\rho\alpha^{\varepsilon}}{4m}}$. This range is nonempty for $m > \frac{\rho}{\frac{1-\alpha}{\alpha}\frac{L_N}{a_N}\frac{4}{\rho}-1}$[11]. Also, $q > \frac{1}{1+\frac{\rho\alpha^{\varepsilon}}{4m}}$ can be treated as a necessary condition for $\frac{dg}{dm} > 0$.

To find out the effect of m on the North-South relative wage, equation (13) reveals that relative wage depends on $\frac{g}{m}$ and m. When $\frac{dg}{dm} < 0$, the ratio $\frac{g}{m}$ decreases as m goes up. Then equation (13) implies that $\frac{w_N}{w_S}$ should fall due to an increase in m. In the other case, when $\frac{dg}{dm} > 0$, we can use equation (12) to verify that $\frac{g}{m}$ goes down[12]. So in this case too, an increase in m should lower the relative wage. We can summarize the above findings in the following proposition:

Proposition 3. *Intensified North-South intra-industry trade ($m \uparrow$) lowers the rate of growth (or, product innovation) if the quality of the Northern institution (q) falls below a threshold value given by $\frac{1}{1+\frac{\rho\alpha^{\varepsilon}}{4m}}$. However an increase in m always lowers the Northern terms-of-trade.*

The intuition of these results can be given as follows. An increase in m raises both the profit rate ($\frac{\pi_M}{v_M}$) and the cost of capital ($r + m$) of a Northern innovating firm. When $q = 1$ (as in Helpman (1993) model), the effect of m on the profit rate dominates its effect on the cost of capital. So we see an increase in m raises the Northern rate of innovation. Since this is true for $q = 1$, we expect this to be true for q close to unity. When $q \neq 1$, the marginal effect of m on the profit rate depends monotonically on the parameter q. This is because, now the profit rate varies positively with the size of the institutionally dependent sector in the North. There is a value of q at which the marginal effect of m on the profit rate is exactly counter balanced by its effect on the cost of capital. So, the rate of innovation should be invariant at this value of q. For any q below that threshold value, we expect that trade integration lowers the rate of product development in the North. Also, higher value of m raises the share of firms in the South compared to that in the North. This raises the relative demand for the Southern labour. So, the North-South relative wage is always decreased due to an increase in m.

[11] We assume that the value of m satisfies this condition. Detail derivations are done in Appendix C.

[12] If an increase in m raises g, the first term in the LHS of (12) goes down. Then the second term must also go down since the RHS of (12) does not depend on m. This happens only when $\frac{g}{m}$ goes down.

4 Conclusion

We introduce imperfect enforcement in the protection of intellectual property rights in the North in a standard R&D based endogenous growth model as like Helpman (1993). This imperfection is captured by an exogenous probability ($q \in [0,\ 1]$) of granting patent in the North, so that an innovator in the North can obtain a patent or can enforce her patent rights in court in case of infringement with probability q. We interpret the value of q as the quality (or the strength) of the Northern property rights protecting institution. We show that the rate of innovation is a positive monotonic function of the quality of the Northern institution and that the balanced rate of growth of the integrated world economy depends crucially on the quality of the Northern institution. In particular, we find a threshold value of the institutional quality parameter q, below which intensified intra-industry trade (or higher threat of Southern imitation or lax of intellectual property rights protection in the South) may lower the rate of Northern innovation. However, trade intensification redistributes the world distribution of income in favour of the South. These results are interesting in view of the recent empirical literature where institutional quality plays an important role in reaping the benefits of trade integration.

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Appendix A: **Derivation of equation** (12)

We have $\pi_M = \frac{1-\alpha}{\alpha} w_N x_M$ and $v_M = \frac{\pi_M}{r+m}$. Then equation (7) implies

$$\begin{aligned} q\frac{\frac{1-\alpha}{\alpha} w_N x_M}{r+m} &= \frac{w_N a_N}{n}; \\ or, \quad q\frac{1-\alpha}{\alpha}(n_M x_M)\frac{n}{n_M} &= a_N(r+m); \end{aligned}$$

Using equations (8) and (11), the above expression can be written as

$$\frac{1-\alpha}{\alpha}\frac{m+g}{g}\left(\frac{L_N - a_N g}{1+\alpha^{-\varepsilon}(1-q)\frac{m+g}{qg}}\right) = a_N(r+m). \tag{A.1}$$

Using equations (3) and (6) and due to our normalisation of $\frac{\dot{E_N}}{E_N} = \frac{\dot{n}}{n}$, we get $r = \rho + g$. Then equation (A.1) can be written as

$$\begin{aligned} \frac{1-\alpha}{\alpha}\left(\frac{L_N - a_N g}{\frac{g}{m+g} + \alpha^{-\varepsilon}\frac{1-q}{q}}\right) &= a_N(\rho+m+g); \\ \Rightarrow \quad \frac{\frac{1-\alpha}{\alpha a_N}(L_N - a_N g)}{\rho+m+g} &= \frac{g}{m+g} + \alpha^{-\varepsilon}\frac{1-q}{q}; \\ \Rightarrow \quad \frac{\frac{1-\alpha}{\alpha}\left(\frac{L_N}{a_N} - g\right)}{\rho+m+g} - \frac{g}{m+g} &= \alpha^{-\varepsilon}\frac{1-q}{q}. \end{aligned}$$

This last expression is equation (12) in the text.

Appendix B: **Derivation of equation** (13) **[North-South relative wage]**

$$\begin{aligned} \frac{w_N}{w_S} = \frac{\alpha\left(\frac{w_N}{\alpha}\right)}{p_S} = \alpha\frac{p_M}{p_S} &= \alpha\left(\frac{x_S}{x_M}\right)^{1-\alpha}; \quad \text{[Using static demand } x_i = E\frac{p_i^{-\varepsilon}}{\int_0^n p_j^{1-\varepsilon}dj} \text{ and } \varepsilon = \frac{1}{1-\alpha}\text{]} \\ &= \alpha\left[\frac{L_S}{n_S}\,\frac{n_M\left(1+\frac{1-q}{q}\frac{m+g}{g}\alpha^{-\varepsilon}\right)}{L_N - a_N g}\right]^{1-\alpha}; \text{[Using equations (5) and (11)]} \\ &= \alpha\left[L_S\,\frac{n_M}{n_S}\,\frac{\left(1+\frac{1-q}{q}\frac{m+g}{g}\alpha^{-\varepsilon}\right)}{L_N - a_N g}\right]^{1-\alpha}; \\ &= \alpha\left[L_S\,\frac{1-\alpha}{\alpha a_N}\,\frac{m+g}{m(\rho+m+g)}\right]^{1-\alpha}; \text{[Using (A.1) and } \frac{n_M}{n_S} = \frac{\frac{\dot{n_S}}{m}}{n_S} = \frac{g}{m}\text{]} \\ &= \alpha\left[\frac{1-\alpha}{\alpha a_N}\,\frac{L_S}{m\left(\frac{\rho}{m+g}+1\right)}\right]^{1-\alpha} = \alpha\left[\frac{1-\alpha}{\alpha a_N}\,\frac{L_S}{\frac{\rho}{\left(1+\frac{g}{m}\right)}+m}\right]^{1-\alpha}. \end{aligned}$$

This is equation (13) in the text. Clearly, $w_N > w_S$ can be restored if $\frac{1-\alpha}{\alpha a_N} L_S > \left(\frac{1}{\alpha}\right)^{\frac{1}{1-\alpha}} \left(\frac{\rho}{1+\frac{g}{m}} + m\right)$. Since the RHS of this inequality is always less than $\left(\frac{1}{\alpha}\right)^{\frac{1}{1-\alpha}} (\rho + m)$, one sufficient condition for $w_N > w_S$ can be given by $L_S > \alpha^{-\frac{\alpha}{1-\alpha}} \frac{a_N}{1-\alpha}(\rho + m)$.

Appendix C: Sign of $\frac{dg}{dm}$

Multiplying both sides of equation (12) by $(\rho + m + g)$, we obtain

$$\frac{1-\alpha}{\alpha} \frac{L_N}{a_N} - \frac{g}{\alpha} - \frac{\rho g}{m+g} = \frac{1-q}{q} \alpha^{-\varepsilon} (\rho + m + g).$$

Here g is the only variable. Taking derivative w.r.t. m and arranging the terms we obtain

$$\frac{dg}{dm} \left[\underbrace{-\frac{1}{\alpha} - \frac{\rho}{m+g} + \frac{\rho g}{(m+g)^2} - \frac{1-q}{q} \alpha^{-\varepsilon}}_{term\ 1} \right] = \underbrace{\frac{1-q}{q} \alpha^{-\varepsilon} - \frac{\rho g}{(m+g)^2}}_{term\ 2}.$$

$Term$ 1 is clearly **negative** since $\frac{\rho g}{(m+g)^2} < \frac{\rho}{m+g}$. So the sign of $\frac{dg}{dm}$ depends only on the sign of $term$ 2. So we get

$$\begin{aligned} \frac{dg}{dm} &> (=) < \quad 0 \quad iff \quad \left(\frac{1-q}{q} \alpha^{-\varepsilon} - \frac{\rho g}{(m+g)^2} \right) < (=) > 0; \\ or, \quad \frac{dg}{dm} &> (=) < \quad 0 \quad iff \quad q > (=) < \frac{1}{1 + \frac{\rho g \alpha^{\varepsilon}}{(m+g)^2}}. \end{aligned}$$

We are interested in a situation when $\frac{dg}{dm} < 0$. This happens if $q < \frac{1}{1+\frac{\rho g \alpha^{\varepsilon}}{(m+g)^2}}$. The expression $\frac{g}{(m+g)^2}$ attains a maximum at $g = m$; so that

$$\begin{aligned} \frac{g}{(m+g)^2} \leq \frac{m}{(m+m)^2} &= \frac{1}{4m}; \\ or, \quad \frac{\rho \alpha^{\varepsilon} g}{(m+g)^2} &\leq \frac{\rho \alpha^{\varepsilon}}{4m}; \quad \text{[Multiplying both sides by } \rho\alpha^{\varepsilon}\text{]} \\ or, \quad 1 + \frac{\rho \alpha^{\varepsilon} g}{(m+g)^2} &\leq 1 + \frac{\rho \alpha^{\varepsilon}}{4m}; \quad \text{[Adding 1 to both sides]} \\ or, \quad \frac{1}{1 + \frac{\rho \alpha^{\varepsilon} g}{(m+g)^2}} &\geq \frac{1}{1 + \frac{\rho \alpha^{\varepsilon}}{4m}}. \quad \text{[taking ratios]} \end{aligned}$$

Then one sufficient condition for $\frac{dg}{dm} < 0$ can be given by $q \leq \frac{1}{1+\frac{\rho\alpha^{\varepsilon}}{4m}}$. From Proposition 1, for g to be positive we must have $q > \frac{1}{1+\frac{1-\alpha}{\alpha}\alpha^{\varepsilon}\frac{L_N}{a_N}\frac{1}{\rho+m}}$. Then the sufficient condition for $\frac{dq}{dg} < 0$ is given by $\frac{1}{1+\frac{1-\alpha}{\alpha}\alpha^{\varepsilon}\frac{L_N}{a_N}\frac{1}{\rho+m}} < q \leq \frac{1}{1+\frac{\rho\alpha^{\varepsilon}}{4m}}$. Clearly this range is nonempty for $m > \frac{\rho}{\frac{1-\alpha}{\alpha}\frac{L_N}{a_N}\frac{4}{\rho}-1}$.