# ON TESTING CONDITIONAL MOMENT RESTRICTIONS*

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**Abstract**

This article provides a unified approach to specification testing of econometric models defined through conditional moment restrictions. The article reviews alternative methodologies in the verge of the state of the arts and discusses new applications and developments to models of particular interest in econometrics practice. We focus our attention on omnibus tests,

*This article is dedicated to the memory of Mita Das. When Mita visited Universidad Carlos III de Madrid in the summer of 2006 she was involved in different projects related to development, dealing with quantifying the "stigma" effect in families with AIDS patients, forest degradation in the Himalayas and child labour. The econometric models behind these projects were simultaneous equations nonlinear in variables models, which often include equations modelling discrete choice decisions. She was particularly concerned with a proper estimation and model checking. During her visit, we discussed application of specification testing techniques I have worked on during the last few years to the models she was interested in. These discussions were of great intellectual stimulus to me and they led to improve my lectures on this topic. This article is in a great part based on the discussions we had and the notes I wrote for that occasion.

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able to detect alternative specifications in any direction, i.e. in the direction of non parametric alternatives. We first discuss specification tests of parametric conditional distribution functions. The joint distribution is estimated semiparametrically, imposing the parametrically specified conditional distribution, and tests statistics are based on functionals of the empirical process consisting on the difference between the semiparametric estimator and the empirical joint distribution function, its pure nonparametric counterpart. Tests for conditional moment restrictions on functions indexed by parameters are motivated in the same way. Now, a nonparametric estimator of the integral of the conditional moment is the empirical process forming a basis of the tests. Related to these tests are those designed to testing conditional moment restrictions using a smooth estimator of the conditional moment, which are discussed in a separate section. These tests are related to significance tests of nonparametric models and specification tests of semiparametric models. In the two last sections we discuss applications to particular econometric models. A section is devoted to models of interest in microeconometrics, like quantile regression, discrete choice, censored regression and count data. Another section is devoted to specification testing in time series models, paying attention to testing serial independence, the martingale hypothesis, testing lack of autocorrelation and conditional symmetry.

**Keywords**: Specification tests; Conditional distributions; Conditional moments; Empirical processes; Smoothers.

## 1. INTRODUCTION

A popular methodology in economics applied research consists of developing behavioral equations of economic agents defined as the first order optimization solution in an optimization problem, eg. an utility maximization subject to budget constraints or the minimization of a cost function subject to a given production technology. The equations are usually represented in terms of some $q-valued$ vector of variables, $Z$, and are indexed by a vector of parameters, $\theta$, taking values in a restricted subset of $\mathbb{R}^p$, $\Theta$ say, i.e. $\theta \in \Theta \subset \mathbb{R}^p$. Let $U_\theta : \mathbb{R}^q \to \mathbb{R}^m$ be the $m$ functions, which describe the equations for some particular parameter value, i.e.

$$U_{\theta_0}(Z) = 0 \text{ for some } \theta_0 \in \Theta. \tag{1}$$

Usually $Z = (Y', X')$, where $Y$ is a $q_1 - valued$ vector of endogenous variables caused by the $q_2 - valued$ vector of exogenous variables $X$.

The econometric modeling in this context consists of introducing the random sampling by considering $Z$ as a random variable with an unknown joint distribution, $F$ say, and the set of unknown parameters $\theta_0$ are assumed to be identified by means of the set of conditional moment restrictions

$$\mathbb{E}\left[\left(U_{\theta_0}(Z) \otimes I_\ell\right) vec\left(\gamma(X)\right)\right] = 0 \text{ for some } \theta_0 \in \Theta, \tag{2}$$

where $\otimes$ stands for Kronecker's product, $I_\ell$ is the $\ell \times \ell$ identity matrix, and $\gamma$ is a $\ell \times m$ matrix of functions in $\mathbb{R}^q$; each column of $\gamma(X)$ is the $\ell - valued$ vector of instrumental variables, or instruments, for the corresponding component of $U_{\theta_0}(Z)$. Thus, there are $m\ell^2$ moment restrictions - some of the components of $\gamma(X)$ may be zero.

A somehow more natural approach consists of assuming that $\theta_0$ is such that (1) is satisfied in average for each possible value of $X$. That is, it is assumed that

$$\mathbb{E}\left[U_{\theta_0}(Z) \middle| X = x\right] = 0 \text{ for some } \theta_0 \in \Theta \text{ and all possible } x \in \mathbb{R}^{q_2}, \tag{3}$$

which means that an infinite number of moment conditions like (2) are satisfied, one for each possible function of $X$. Notice that (3) is in fact an identification restriction on the parameters $\theta \in \Theta$. A number of tests have been developed for testing

(2), which is necessary, but no sufficient for (3), e.g. Newey (1985a, b), Tauchen (1985) or Wooldridge (1990). These tests were called $M$ tests. Sargan's tests for overidentified restrictions, Sargan (1958, 1959), can also be classified within this class. Testing the infinite moment restrictions in (3) involves using nonparametric procedures. The parameter vector $\theta_0$ can be identified in different ways by imposing alternative moment restrictions on $U_{\theta_0}(Z)$, from the full specification of the conditional distribution of $Y$ given $X$ to much more less restrictive assumptions, like (3), conditional quantile restrictions, conditional symmetry, etc. Of course, the interpretation of the parameters depends very much on the identification restrictions done.

This article offers a unified approach to exiting omnibus tests on conditional moment restrictions, which can be usually represented as (3), able to eventually detect any nonparametric alternative. These specific tests are essential prior to interpreting the parameters or performing any statistical inference.

The rest of the article is organized as follows. In the next Section we present tests for the conditional distribution function of $Y$ given $X$, $F_{Y|X}$ say, and therefore for the conditional distribution of $U_{\theta_0}(Z)$. Given a family of conditional distributions functions indexed by a vector of parameters $\vartheta \in \Xi$, $\mathcal{F} = \left\{F_{Y|X,\vartheta} : \vartheta \in \Xi\right\}$, a correct specification means that $F_{Y|X} \in \mathcal{F}$. Notice that $\theta = g(\vartheta)$ for a suitable $g : \Xi \to \Theta$, and when the specification is correct and $F_{Y|X} = F_{Y|X,\vartheta_0}$ for some $\vartheta_0 \in \Xi$ and $\theta_0 = g(\vartheta_0)$. The tests are based on a comparison between an unrestricted estimator of the nonparametric joint distribution function,

$$F(y,x) = \int_{-\infty}^{x} F_{Y|X}(y|\,\bar{x})\, F_X(d\bar{x}),$$

where $F_X$ is the marginal distribution of $X$, and its semiparametric counterpart, imposing the parametrically specified conditional distribution, $F_{\vartheta_0}(y,x)$, with

$$F_{\vartheta}(y,x)=\int_{-\infty}^{x}F_{Y|X,\vartheta}(y|\,\bar{x})\,F_X(d\bar{x}), \tag{4}$$

where, for a $s-valued$ vector $w=(w_1,...,w_s)'$, $\int_{-\infty}^{w}\cdot d\bar{w}=\int_{-\infty}^{w_1}\cdots\int_{-\infty}^{w_s}\cdot dw_1\ldots dw_s$.

Specification tests are based on suitable functionals of an estimator of $F_{\vartheta}-F$. These tests form a basis for motivating omnibus tests of (3) in Section 3 and other conditional restrictions in subsequent sections. Tests of (3) are based on functionals of estimators of the semiparametric function,

$$\begin{aligned} T_{\theta}(x) &= \int_{-\infty}^{x}\mathbb{E}\left[U_{\theta}(Z)|\,X=\bar{x}\right]F_X(d\bar{x}) \\ &= \int_{\mathbb{R}^{q_1}}\int_{-\infty}^{x}U_{\theta}(\bar{y},\bar{x})\,F_{Y|X}(d\bar{y}|\,\bar{x})\,F_X(d\bar{x}) \\ &= \int_{\mathbb{R}^{q_1}}\int_{-\infty}^{x}U_{\theta}(\bar{y},\bar{x})\,F(d\bar{y},d\bar{x}) \\ &= \mathbb{E}\left[U_{\theta}(Z)\,1_{\{X\le x\}}\right], \end{aligned} \tag{5}$$

which is known as integrated regression function when $U_{\theta}$ are the residuals of a regression function. Section 4 presents tests based on smooth estimators of non-parametric curves. First, we discuss alternative tests to those presented in Section 3, which use, as test statistics, estimators of

$$S_{\theta_0}=\int_{\mathbb{R}^{q_2}}f_X(\bar{x})\,\mathbb{E}\left[U_{\theta_0}(Z)|\,X=\bar{x}\right]T_{\theta_0}(d\bar{x}), \tag{6}$$

where, henceforth, $f_V$ is the probability density function of a generic random variable $V$. Since the conditional expectation is of nonparametric nature, it is estimated using smoothers, typically kernels. This contrasts with tests presented in Section 3, whose test statistics are functionals of estimators of $T_{\theta_0}$, without resorting to use smoothers. In Section 4, we also discuss testing hypothesis involving restrictions on nonparametric curves, like significance tests in nonparametric models, and specification tests of semiparametric models. The next sections are devoted to applications of the methodology presented in previous sections to different specification tests. Section 5 is devoted to problems of particular interest in microeconometrics, like quantile regression and models with discrete and limited dependent variables. Section 6 discusses applications to time series modeling, which include testing serial independence and the martingale hypothesis.

Along the article, regularity conditions are not discussed and, in particular, the required smoothness of underlying nonparametric functions is given by granted.

## 2. TESTING THE SPECIFICATION OF THE CONDITIONAL DISTRIBUTION

Let $\mathcal{F} = \left\{F_{Y|X,\vartheta} : \vartheta \in \Xi\right\}$ be a family of conditional distribution functions of the scalar random variable $Y$ given the $q_2 - valued$ vector of random variables $X$, e.g. the Gaussian family. The null hypothesis of correct specification can be formally written as

$$H_0 : F_{Y|X} \in \mathcal{F}.$$

The alternative hypothesis, $H_1$, is the negation of the null. The null hypothesis can also be alternatively written as,

$$H_0 : F_{\vartheta_0} - F = 0 \text{ for some } \vartheta_0 \in \Xi,$$

where $F_\vartheta$ was defined in (4).

### 2.1. Test statistic.

The tests statistics for $H_0$ are based on the classical proposals for omnibus specification testing of a marginal distribution, introduced by Cramér (1928), von Mises (1931), Kolmogorov (1933), Glivenko (1933), Cantelli (1933) and Smirnov (1937). The idea consists of comparing the sample distribution function and the specification under the null. This idea has been extended to nonparametric testing of the specification of many other functions, like the conditional distribution.

Let $\hat{F}_{Xn}$ be the empirical distribution of $X$, i.e. $\hat{F}_{Xn}(x) = n^{-1}\sum_{i=1}^{n} 1_{\{X_i \leq x\}}$, $1_{\{A\}}$ the indicator function of the event $A$ and for any $s-valued$ vector $w = (w_1, ..., w_s)'$, $\{w \leq \bar{w}\} = \{w_1 \leq \bar{w}_1, \ldots, w_s \leq \bar{w}_s\}$. Given a random sample of $Z$, $\{Z_i\}_{i=1}^{n}$ say, with $Z_i = (Y_i, X_i')'$, a natural estimator of $F_{\vartheta}(z)$ is

$$\begin{aligned} \hat{F}_{\vartheta n}(z) &= \int_{-\infty}^{x} F_{Y|X,\vartheta}(y\,|\bar{x})\,\hat{F}_{Xn}(d\bar{x}) \\ &= \frac{1}{n}\sum_{i=1}^{n} F_{Y|X,\vartheta}(y\,|X_i)\,1_{\{X_i\leq x\}}, \end{aligned}$$

where $z = (y, x')'$. The nonparametric distribution $F$ can also be estimated, without imposing the parametric specification under the null, by the empirical distribution $\hat{F}_n(z) = n^{-1}\sum_{i=1}^{n} 1_{\{X_i\leq x\}}1_{\{Y_i\leq y\}}$. Andrews (1998) suggested testing $H_0$ using as test statistics functionals of the empirical process

$$\xi_{\vartheta n} = \sqrt{n}\left(\hat{F}_{\vartheta n} - \hat{F}_n\right). \tag{7}$$

For each $\vartheta \in \Xi$, $\xi_{\vartheta n}$ are random step functions in $\mathbb{R}^q$ taking values in $\mathcal{D}[\mathbb{R}^q]$, the space of functions continuous from above, with limits form below in the sense of Bickel and Wichura (1971).

An estimator of the unknown parameter $\vartheta_0$ is needed in order to implement the test. While the asymptotic distribution function of the standard empirical process without estimated parameters is asymptotically distribution free, when parameters are estimated, the asymptotic distribution depends on these estimated parameters, as shown by Durbin (1973).

Given a particular $\sqrt{n} - consistent$ estimator of $\vartheta_0$, $\hat{\vartheta}_n$ say, the conditional maximum likelihood estimator is obviously a good option, test statistics are suitable functionals of $\xi_{\hat{\vartheta}_n n}$. Consider the continuous functional $\rho : \mathcal{D}[\mathbb{R}^q] \longmapsto \mathbb{R}^+$

such that, given two functions $f_1, f_2 \in \mathcal{D}[\mathbb{R}^q]$ with $|f_1(z)| \leq |f_2(z)|$ for each $z \in \mathbb{R}^q$, $\rho(f_1) \leq \rho(f_2)$.The test statistic is $\rho\left(\xi_{\hat{\vartheta}_n n}\right)$. A leading proposal for $\rho$ is the Kolmogorov-Smirnov criteria, $\rho(f) = \sup_{z \in \mathbb{R}^q} |f(z)|$, the corresponding test statistic is

$$K_n = \sup_{z \in \mathbb{R}^q} |\xi_{\vartheta n}(z)|.$$

Andrews (1998), in this context, suggested the computationally more economical statistic $\max_{0 \leq i \leq n} \left|\xi_{\hat{\vartheta}_n n}(Z_i)\right|$. Another alternative is the Cramér-von Mises criteria, $\rho(f) = \int_{\mathbb{R}^q} f(z)^2 \mu(dz)$, for some given measure $\mu$. A reasonable choice is the empirical distribution of the data $\hat{F}_n$, which corresponds, in this context, to the statistic

$$C_n = \int_{\mathbb{R}^q} \xi_{\hat{\vartheta}_n n}(z)^2 \hat{F}_n(dz) = \frac{1}{n} \sum_{i=1}^{n} \xi_{\hat{\vartheta}_n n}(Z_i)^2.$$

The null hypothesis $H_0$ is rejected at the $(1-\alpha) - level$ of significance when $\phi^{\alpha}_{\hat{\vartheta}_n n} = 1_{\left\{\rho\left(\xi_{\hat{\vartheta}_n n}\right) > c_\alpha\right\}}$ takes the value one. The critical value $c_\alpha$ must satisfy that $\lim_{n\to\infty} \mathbb{E}\left(\phi^{\alpha}_{\hat{\vartheta}_n n}\right) \leq \alpha$ under $H_0$, in order to control the type I error. However, such critical values cannot be obtained because, as it happens in the simplest case, when testing the specification of a marginal distribution, the asymptotic distribution of $\rho\left(\xi_{\hat{\vartheta}_n n}\right)$ under $H_0$ depends on the unknown features of $F$, i.e. $\vartheta_0$ and $F_X$.

The idea of comparing integrated nonparametric curves has been developed for testing different conditional moment restrictions, as we shall discuss later. The implementation usually requires the assistance of the bootstrap, in the lines described in the next Subsection, designed for each specific problem.

### 2.2. Bootstrap approximations.

Andrews (1997) suggested to approximate the critical values with the assistance of a parametric bootstrap. Consider the bootstrap resample $\{Y_i^*, X_i\}_{i=1}^n$, where $Y_i^*$ is randomly generated, using a numerical random number generator, with distribution

$F_{Y|X,\hat{\vartheta}_n}(\cdot|\, X_i)$. Then, the bootstrap test statistic is $\rho(\xi_n^*)$, where

$$\xi_n^* = \sqrt{n}\left(\hat{F}_{\hat{\vartheta}_n^* n} - \hat{F}_n^*\right),$$

and $\hat{F}_n^*$ is the sample distribution of $\{Y_i^*, X_i\}_{i=1}^n$. The bootstrap critical value is the $(1-\alpha)-th$ quantile of the conditional distribution of $\rho(\xi_n^*)$ given the observed sample, i.e.

$$c_{n\alpha}^* = \inf\{c : \mathbb{P}[\rho(\xi_n^*) \leq c|\, \{Y_i, X_i\}_{i=1}^n] \geq 1-\alpha\}.$$

The bootstrap test consists of using $c_{n\alpha^*}$ rather than $c_\alpha$, and it is described by the binary random variable $\phi_{\hat{\theta}_n n}^{\alpha *} = 1_{\{\rho(\xi_{\hat{\vartheta}_n n}) > c_{n\alpha}^*\}}$. Andrews (1997) showed, under weak regularity conditions, that,

$$\lim_{n\to\infty} \mathbb{E}\left(\phi_{\hat{\vartheta}_n n}^{\alpha *}\right) \begin{cases} \leq \alpha \text{ under } H_0 \\ = 1 \text{ under } H_1 \end{cases}, \tag{8}$$

which justifies to use the procedure in practice. If (8) does not hold, we say that the bootstrap test is inconsistent. The bootstrap critical value $c_{n\alpha}^*$ may be difficult to calculate in practice, but it can be approximated, as accurately as desired, by Montecarlo, i.e. by generating a large number of bootstrap resamples, $B$ say, and for each resample we compute the corresponding bootstrap empirical process, $\xi_n^{(b)*}$, and the corresponding test statistic $\rho\left(\xi_n^{(b)*}\right)$, $b = 1, ..., B$. The Montecarlo estimator of $c_{n\alpha}^*$ is

$$\hat{c}_{n\alpha}^* = \inf\left\{c : \frac{1}{B}\sum_{b=1}^{B} 1_{\left\{\rho\left(\xi_n^{(b)*}\right) < c\right\}} \geq 1-\alpha\right\}$$

The bigger $B$, the better the accuracy of $\hat{c}_{n\alpha}^*$ approximating $c_{n\alpha}^*$.

This methodology is applied to implementing many other test procedures designed to test other conditional moment restrictions based on resampling schemes designed for each specific problem.

Delgado and Stute (2008) have proposed to implement the test by substituting $\xi_{\hat{\vartheta}_n n}$ by a martingale transformation in the lines suggested by Khamaladze (1981, 1988 and 1993) for the standard empirical process with estimated parameters, which converges to a standard Kiefer's process. So, the corresponding critical values of the test can be tabulated without the assistance of the bootstrap. However, this procedure is computationally more demanding than implementing the parametric bootstrap. However, the bootstrap approximation applied to the martingale transform results in important level accuracy improvements.

### 2.3. Smooth alternatives.

It is worth mentioning that we can use smooth alternatives to $\hat{F}_{\vartheta n}$ and $\hat{F}_n$ in (7). Let $k$ be a kernel function, typically a probability density, and

$$\mathbb{K}_\ell(t) = \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_\ell} \prod_{j=1}^{\ell} k(t_j)\, dt_j, \ t = (t_1, ..., t_\ell)'.$$

The smooth versions of $\hat{F}_{\vartheta n}$ and $\hat{F}_n$ are

$$\tilde{F}_{\vartheta n}(z) = \frac{1}{n}\sum_{i=1}^{n} F_{Y|X,\vartheta}(y\,|X_i)\, \mathbb{K}_{q2}\left(\frac{x - X_i}{h}\right)$$

and

$$\tilde{F}_n(z) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{K}_1\left(\frac{y - Y_i}{h}\right) \mathbb{K}_{q2}\left(\frac{x - X_i}{h}\right),$$

respectively. That is, we just substitute indicator functions by integrated kernels. The smooth version of $\xi_{\vartheta n}$ is

$$\tilde{\xi}_{\vartheta n}(z) = \sqrt{n}\left(\tilde{F}_{\vartheta n} - \tilde{F}_n\right)(z),$$

and $\rho\left(\tilde{\xi}_{\vartheta n}\right)$ is the smoothed version of $\rho(\xi_{\vartheta n})$. Applying the result in van der Vaart (1994), $\rho\left(\tilde{\xi}_{\vartheta n}\right) = \rho(\xi_{\vartheta n}) + o_p\left(n^{-1/2}\right)$, which means that the two tests are asymptotically equivalent.

In this section we have implicitly considered that $Y$ is a continuous variable. However, $Y$ is discrete or censored in important econometric models, e.g. in discrete choice models and in count data models. Specification tests in these situations are discussed in Section 5, once tests for conditional moment restrictions are presented.

## 3. TESTING CONDITIONAL MOMENT RESTRICTIONS.

Henceforth, we consider $U_\theta$ scalar. In order to test the hypothesis

$$H_0 : \mathbb{E}\left[\left. U_{\theta_0}(Z) \right| X = x\right] = 0 \text{ for some } \theta_0 \in \Theta \text{ and all possible } x, \tag{9}$$

in the direction of nonparametric alternatives $H_1$, we can use the natural estimator of $T_\theta$ in $(5)$,

$$\begin{aligned} T_{n\theta}(x) &= \int_{\mathbb{R}^{q_1}} \int_{-\infty}^{x} U_\theta(\bar{y}, \bar{x}) F_n(d\bar{y}, d\bar{x}) \\ &= \frac{1}{n} \sum_{i=1}^{n} U_\theta(Z_i) 1_{\{X_i \leq x\}}, \end{aligned}$$

and proceed like in the previous section for testing (9) using the empirical process $\tau_{n\theta}(x) = \sqrt{n} T_{\theta n}(x)$. Conceptually, $\tau_{n\theta}$ and $\xi_{n\theta}$ are identical and tests can be performed applying the same methodology. Given a $\sqrt{n}-$*consistent* estimator of $\theta_0$, $\hat{\theta}_n$ say, and a functional $\rho : \mathcal{D}\left[\mathbb{R}^{q_2}\right] \longmapsto \mathbb{R}^+$, with the described properties in Subsection 2.1., the test statistic is $\rho\left(\tau_{n\hat{\theta}_n}\right)$ and $H_0$ is rejected at the $\alpha$ significance level when the binary random variable $\eta^\alpha_{\hat{\theta}_n} = 1_{\left\{\rho\left(\tau_{n\hat{\theta}_n}\right) > d_\alpha\right\}}$ takes the value one. The critical values, $d_\alpha$ are unknown, because the asymptotic distribution of $\rho\left(\tau_{n\hat{\theta}_n}\right)$ depends on the unknown parameters $\theta_0$ and other unknown features of the data generating process. See e.g. Bierens (1982, 1990), Bierens and Ploberger (1997), Stute (1997), and Delgado, Domínguez and Lavergne (2006). In particular, it depends on the conditional variance of $U_{\theta_0}(Z)$ given $X$.

Stute, González-Manteiga and Presedo (1998) proposed a bootstrap procedure designed to respect the conditional moment relation between $U_{\theta_0}(Z)$ and $X$ up to the second order, using the wild bootstrap resampling proposed by Wu (1986). Let $\{V_i^3\}_{i=1}^n$ be numerical randomly generated *iid* variables with mean zero and variance one. It is usually chosen distributions with $\mathbb{E}(V_i) = 1$ in the context of regression models in order to respect also the third order moments relation, see Liu

(1998). The choice of

$$V_i = \begin{cases} -\frac{\sqrt{5}-1}{2} \text{ with probability } \frac{\sqrt{5}+1}{2\sqrt{5}} \\ \frac{\sqrt{5}+1}{2} \text{ with probability } \frac{\sqrt{5}-1}{2\sqrt{5}} \end{cases}, \ i = 1, .., n$$

proposed by Mammen (1993) has been very popular. The bootstrap sample is $\{Y_i^*, X_i\}_{i=1}^n$, where $Y_i^*$ is the solution to the equation $U_{\hat{\theta}_n}(Y_i^*, X_i) = U_{\hat{\theta}_n}(Y_i, X_i) V_i$, $i = 1, ..., n$. Bootstrap critical values are

$$d_{n\alpha}^* = \inf \left\{ d : \mathbb{P}\left( \rho\left(\tau_{n\hat{\theta}_n}^*\right) \leq d \middle| \{Y_i, X_i\}_{i=1}^n \right) \geq 1 - \alpha \right\}.$$

These critical values are approximated by Monte Carlo, as explained in Subsection 2.2. Stute, González-Manteiga and Presedo (1998) showed that the resulting Bootstrap test is consistent.

These tests can also be applied to two sample tests for the equality between regression curves, or other conditional moments, as proposed by Delgado (1993) and Ferreira and Stute (2004).

## 4. TESTS BASED ON SMOOTHERS

Härdle and Mammen (1993), in the context of specification testing of a regression model, proposed to use as test statistics functionals of the difference between a parametric and a nonparametric fit. This idea has been very popular in the econometric literature, see Zheng (1996), Li and Wang (1998), Wang (2000, 2001), Horowitz and Spokoiny (2001) or Guerre and Lavergne (2005), and it has been also used for significance testing in conditional models, e.g. Delgado and Stengos (1993), Fan and Li (1996), Chen and Fan (1999), Lavergne and Vuong (2000) or Delgado and González-Manteiga (2001), amongst others.

The tests are based on $U - statistics$ and this methodology has been applied to testing different restrictions on nonparametric curves, from testing conditional independence to specification testing of semiparametric models

#### 4.1. Tests of conditional moment restrictions using smoothers.

Henceforth, given a sample $\{V_i\}_{i=1}^n$ from the p-valued vector of random variables $V$, the product kernel estimator of the nonparametric probability density function $f_V$ is

$$\tilde{f}_{Vn}(v) = \frac{1}{n}\sum_{j=1}^{n}\prod_{\ell=1}^{p} k_h(V_{\ell i} - v_\ell),$$

where $k_h(\cdot) = h^{-1}k(\cdot/h)$, $v = (v_1, ..., v_p)'$ and $V_i = (V_{1i}, ..., V_{pi})'$, and $k : \mathbb{R} \to \mathbb{R}$ is a kernel function, typically a univariate symmetric probability density. The Nadaraya-Watson kernel estimator of $\mathbb{E}[U_{\theta_0}(Z)|\, X = x]$ is

$$\tilde{m}_n(x) = \frac{1}{n\tilde{f}_{Xn}(x)}\sum_{i=1}^{n} U_{\hat{\theta}_n}(Z_i)\prod_{\ell=1}^{q_2} k_h(X_{\ell i} - x_\ell).$$

Thus, $S_{\theta_0}$ in (6) can be estimated by

$$\begin{aligned}\hat{S}_{\hat{\theta}_n n} &= \int_{\mathbb{R}^{q_1}}\int_{\mathbb{R}^{q_2}} \tilde{f}_{Xn}(x)\,\tilde{m}_n(x)\,T_{n\hat{\theta}_n}(dx)\\ &= \frac{1}{n}\sum_{i=1}^{n}\tilde{f}_{Xn}(X_i)\,\tilde{m}_n(X_i)\,U_{\hat{\theta}_n}(Z_i).\end{aligned}$$

Using kernel weights, with symmetric kernels,

$$\begin{aligned}\hat{S}_{\hat{\theta}_n n} &= \begin{pmatrix} n \\ 2 \end{pmatrix}^{-1}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} H_{n\hat{\theta}_n}(Z_i, Z_j)\\ &\quad + \frac{k(0)}{n^2 h^{q_2}}\sum_{i=1}^{n} U_{\hat{\theta}_n}(Z_i)^2\end{aligned}$$

is a degenerate $U - statistic$ with kernel

$$H_{\theta n}(z_1, z_2) = U_\theta(z_1)\,U_\theta(z_2)\prod_{\ell=1}^{q_2} k(x_{\ell 1} - x_{\ell 2}),\ \ z_j = \left(y_j, x_j'\right)',$$

which is symmetric with respect to its arguments and has zero mean. The test statistics is

$$\hat{s}_{\hat{\theta}_n n} = \frac{\hat{S}_{\hat{\theta}_n n}}{\sqrt{2\widehat{Var}_n\left(H_{\theta_0 n}(Z_1, Z_2)\right)}},$$

where

$$\widehat{Var}_n\left(H_{\theta_0 n}\left(Z_1, Z_2\right)\right) = \frac{1}{n\left(n-1\right)h^{q_2}}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{\ell=1}^{q_2} k\left(X_{\ell i}-X_{\ell j}\right) \times U_{\hat{\theta}_n}\left(Z_i\right)U_{\hat{\theta}_n}\left(Z_j\right)$$

is a consistent estimator of $Var\left(H_{\theta_0 n}\left(Z_1, Z_2\right)\right)$. Under $H_0$ and suitable regularity conditions, dealing with the smoothness of underlying nonparametric functions,

$$\hat{s}_{\hat{\theta}_n n} \xrightarrow{d} N\left(0,1\right) \text{ as } h + \frac{1}{nh^{q_2}} \to 0.$$

Since $\hat{s}_{\hat{\theta}_n n}$ is a positive magnitude under $H_0$, it is performed a one-sided test using the critical values of the standard normal. The asymptotic normal approximation can be very poor in practice and too sensitive to the choice of the smoothing parameter. However, bootstrap approximations are more accurate and less sensitive to the bandwidth choice, e.g. applying the wild bootstrap resampling described in the previous section as suggested by Delgado, Domínguez and Lavergne (2005).

Power comparisons between the tests presented in Section 3 are tricky. These tests are unable to detect nonparametric local alternatives converging to the null at the rate $n^{-1/2}$. However, they are able to detect peak local alternatives, of the type introduced by Rosenblatt (1975), which are not detected by the tests presented in Section 3.

Testing conditional restrictions on nonparametric functions usually involve using smoothers. We discuss below tests of conditional independence and specification tests of semiparametric models.

### 4.2. Testing conditional independence

Consider the null hypothesis,

$$H_0 : F_{Y|X} = F_{Y|X^{(1)}},$$

where $X^{(1)}$ is a $q_2^{(1)}-dimensional$ subset of $X$, i.e. $X = \left(X^{(1)\prime}, X^{(2)}\right)^{\prime}$. That is, $Y$ is conditional independent of $X^{(2)}$ given $X^{(1)}$. The joint distribution $F$ can be expressed under the restriction in the null hypothesis as

$$F^{(1)}(z) = \int_{-\infty}^{x^{(1)}} F_{Y|X^{(1)}}\left(y|\, \bar{x}^{(1)}\right) F_X\left(d\bar{x}^{(1)}, \bar{x}^{(2)}\right), \ z = \left(y, x^{(1)\prime}, x^{(2)\prime}\right)^{\prime},$$

which can be estimated by

$$\begin{aligned}
\tilde{F}_n^{(1)}(z) &= \int_{-\infty}^{x^{(1)}} \tilde{F}_{Y|X^{(1)}n}\left(y|\, \bar{x}^{(1)}\right) \hat{F}_{Xn}\left(d\bar{x}^{(1)}, \bar{x}^{(2)}\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} \tilde{F}_{Y|X^{(1)}n}\left(y|\, X_i^{(1)}\right) 1_{\left\{X_i^{(1)} \le x^{(1)}\right\}} 1_{\left\{X_i^{(2)} \le x^{(2)}\right\}}, \ z = \left(y, x^{(1)\prime}, x^{(2)\prime}\right)^{\prime},
\end{aligned}$$

where $\tilde{F}_{Y|X^{(1)}n}$ is a kernel estimator of $F_{Y|X^{(1)}}$. Delgado and González-Manteiga (2001) characterize $H_0$ as

$$H_0 : \int_0^{x^{(1)}} f_{X^{(1)}}\left(\bar{x}^{(1)}\right)\left(F - F^{(1)}\right)\left(y, d\bar{x}^{(1)}, x^{(2)}\right) = 0,$$

where $f_{X^{(1)}}$ is the probability density of $X^{(1)}$. They suggested to test $H_0$ using as test statistics functionals of the empirical process

$$\tilde{\xi}_n(z) = \sqrt{n}\int_0^{x^{(1)}} \tilde{f}_{X^{(1)}n}\left(\bar{x}^{(1)}\right)\left(\hat{F}_n - \hat{F}_n^{(1)}\right)\left(y, d\bar{x}^{(1)}, x^{(2)}\right), \ z = \left(y, x^{(1)\prime}, x^{(2)\prime}\right)^{\prime},$$

where $\tilde{f}_{X^{(1)}n}$ is a kernel estimator of $f_{X^{(1)}}$ and $\tilde{F}_{Y|X^{(1)}n}$, in $\hat{F}_n^{(1)}$, is a kernel estimator of $\tilde{F}_{Y|X^{(1)}n}$. Notice that

$$\begin{aligned}
\tilde{\xi}_n(z) &= \frac{1}{n^{3/2}}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(1_{\{Y_i \le y\}} 1_{\left\{X_i^{(1)} \le x^{(1)}\right\}} - 1_{\{Y_j \le y\}} 1_{\left\{X_j^{(1)} \le x^{(1)}\right\}}\right) \\
&\quad \times 1_{\left\{X_i^{(2)} \le x^{(2)}\right\}} \prod_{\ell=1}^{q_2^{(1)}} k_h\left(X_{\ell i}^{(1)} - X_{\ell j}^{(1)}\right), \ z = \left(y, x^{(1)\prime}, x^{(2)\prime}\right)^{\prime},
\end{aligned}$$

is a $U-process$. Delgado and González-Manteiga (2001) showed that $\tilde{\xi}_n$ converges to a Gaussian process with an unknown covariance function depending on $f_{X^{(1)}}$ and $F_{Y|X^{(1)}}$. Given a suitable functional and its corresponding test statistic, the critical values are approximated with the assistance of the bootstrap. The bootstrap resample is $\{(Y_i^*, X_i)\}_{i=1}^n$, where $Y_i^*$ is the solution to

$$U_i^* = \tilde{F}_{Y|X^{(1)}n}\left(Y_i^* \middle| X_i^{(1)}\right), \ i = 1, ...n,$$

and $\{U_i^*\}_{i=1}^n$ are randomly generated with an uniform distribution in the interval $[0,1]$. The bootstrap critical values are approximated by Monte Carlo.

A nonparametric significance test for marginal effects, where the null hypothesis is

$$H_0 : \mathbb{E}\left(U_{\theta_0}(Z) \middle| X\right) = \mathbb{E}\left(U_{\theta_0}(Z) \middle| X^{(1)}\right) \text{ for some } \theta_0 \in \Theta$$

can be performed using the empirical process

$$\begin{aligned} \tilde{\tau}_n(x) &= \int_{\mathbb{R}} \int_{-\infty}^{x} U_{\hat{\theta}_n}(\bar{y}, \bar{x}) \tilde{\xi}_n(d\bar{y}, d\bar{x}) \\ &= \frac{1}{n^{3/2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( U_{\hat{\theta}_n}(Z_i) 1_{\left\{X_i^{(1)} \le x^{(1)}\right\}} \right. \\ &\quad \left. - U_{\hat{\theta}_n}(Z_j) 1_{\left\{X_j^{(1)} \le x^{(1)}\right\}} \right) 1_{\left\{X_i^{(2)} \le x^{(2)}\right\}} \prod_{\ell=1}^{q_2^{(1)}} k_h\left(X_{\ell i}^{(1)} - X_{\ell j}^{(1)}\right). \end{aligned}$$

The bootstrap test is performed using the wild resample $\{(Y_i^*, X_i)\}_{i=1}^n$, where $Y_i^*$ solves the equation,

$$U_{\hat{\theta}_n}(Y_i^*, X_i) = \hat{m}_n^{(1)}\left(X_i^{(1)}\right) + V_i\left[U_{\hat{\theta}_n}(Y_i, X_i) - \hat{m}_n^{(1)}\left(X_i^{(1)}\right)\right],$$

where $\hat{m}_n^{(1)}(\cdot) = n^{-1}\sum_{j=1}^n U_{\hat{\theta}_n}(Z_i) \prod_{\ell=1}^{q_2^{(1)}} k_h\left(\cdot - X_{\ell j}^{(1)}\right)$ and $\{V_i\}_{i=1}^n$ are randomly generated variables with mean zero and variance one.

### 4.3. Specification testing of semiparametric models.

Consider the null hypothesis for the specification of the partially linear model

$$H_0 : \mathbb{E}\left(Y \middle| X\right) = X^{(1)\prime}\theta_0 + g\left(X^{(2)}\right) \text{ for some } \theta_0 \in \Theta,$$

where $g$ is a nonparametric function. The null hypothesis can be alternatively written as

$$H_0 : \mathbb{E}\left( R_Y \middle| X \right) = R'_{X^{(1)}}\theta_0 \text{ for some } \theta_0 \in \Theta,$$

where $R_Y = Y - \mathbb{E}\left( Y \middle| X^{(2)} \right)$ and $R_{X^{(1)}} = X^{(1)} - \mathbb{E}\left( X^{(1)} \middle| X^{(2)} \right)$ are nonparametric residuals. If these residuals were observed for each data point, $\{R_{Yi}, R_{X^{(1)}i}\}_{i=1}^{n}$, we could proceed as in Section 3. In order to implement the test, the residuals are estimated using kernel smoothers, say $\left\{ \tilde{R}_{Yni}, \tilde{R}_{X^{(1)}ni} \right\}_{i=1}^{n}$ and $\theta_0$ is estimated using a semiparametric estimator, $\hat{\theta}_n$ say; e.g. Robinson (1988). Test statistics are functionals of the $U - process$ $\tilde{\tau}_n(x) = \sqrt{n}\tilde{T}_{n\hat{\theta}_n}(x)$ with

$$\begin{aligned} \tilde{T}_{n\theta}(x) &= \frac{1}{n}\sum_{i=1}^{n} \tilde{f}_{X^{(1)}n}\left( X_i^{(1)} \right)\left( \tilde{R}_{Yni} - \tilde{R}'_{X^{(1)}ni}\hat{\theta}_n \right) 1_{\left\{ \tilde{R}_{X^{(1)}ni} \leq x \right\}} \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[ (Y_i - Y_j) - \left( X_i^{(1)} - X_j^{(1)} \right)' \theta \right] \prod_{\ell=1}^{q_2^{(2)}} k_h\left( X_{\ell i}^{(2)} - X_{\ell j}^{(2)} \right) \end{aligned}$$

where $\tilde{f}_{X^{(1)}n}$ is the density estimator of the density of $f_{X^{(1)}}$. Delgado and González-Manteiga (2001) showed that this is a non-degenerated $U - process$ with a case dependent limiting distribution. Since the critical values are unknown, they can be approximated using wild bootstrap resamples $\{Y_i^*, X_i\}_{i=1}^{n}$, where

$$Y_i^* = \tilde{R}'_{X^{(1)}ni}\hat{\theta}_n + \tilde{m}_{ni}^{(2)} + V_i\left( Y_i - \tilde{R}'_{X^{(1)}ni}\hat{\theta}_n - \tilde{m}_{ni}^{(2)} \right),$$

$\tilde{m}_{ni}^{(2)} = Y_i - \tilde{R}_{Yni}$ and $\{V_i\}_{i=1}^{n}$ are randomly generated variables with mean zero and variance one.

A similar idea has been presented for testing the specification of index models. The null hypothesis is in this case,

$$H_0 : \mathbb{E}\left( Y \middle| X \right) = G\left( X'\theta_0 \right) \text{ for some } \theta_0 \in \Theta,$$

where $G$ is a nonparametric function. This model has been considered in the econometrics literature by Powell, Stock and Stoker (1989) and Robinson (1989) amongst others. After noticing that in this case $\mathbb{E}\left(Y|\,X\right) = \mathbb{E}\left(Y|\,X'\theta_0\right)$ with probability one under $H_0$, and given a suitable estimator of $\theta_0$, $\hat{\theta}_n$ say, $G_i = G\left(X_i'\theta_0\right)$ can be estimated by the kernel estimator,

$$\tilde{G}_{ni} = \frac{1}{n\tilde{f}_{X'\hat{\theta}_n}\left(X_i'\hat{\theta}_n\right)} \sum_{j\neq i}^{n} Y_j \prod_{\ell=1}^{q_2} k_h\left(\hat{\theta}_n'\left(X_i - X_j\right)\right).$$

Now, the $U-process$

$$\tilde{\tau}_n\left(x\right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(Y_i - \tilde{G}_{ni}\right) 1_{\{X_i \leq x\}}$$

forms a basis for the test. The critical values of the resulting tests are approximated by bootstrap using the resample $\{Y_i^*, X_i\}_{i=1}^n$ where $Y_i^* = \tilde{G}_{ni} + V_i\left(Y_i - \tilde{G}_{ni}\right)$, $i = 1, ..., n$ where $\{V_i\}_{i=1}^n$ are random numbers generated from a random variable with mean zero and variance one. Related tests for the specification of index models have been proposed by Stute and Zhu (2005).

## 5. SOME APPLICATIONS TO MICROECONOMETRIC MODELS.

In this section we discuss the application of omnibus testing methodology presented in previous sections, to some popular microeconometric models. We pay attention to quantile regression and regression models with discrete or limited dependent variables.

### 5.1. Quantile regression models.

Consider the quantile regression function $m : \mathbb{R}^{q_2} \times [0,1]$, such that

$$F_{Y|X}\left(m\left(x,\alpha\right)|\,x\right) = \alpha \text{ for each } \alpha \in (0,1) \text{ and } x \in \mathbb{R}^{q_2}.$$

The specification test consists of testing the hypothesis

$$H_0 : m\left(x,\alpha\right) = x'\beta_0\left(\alpha\right) \text{ for each } \alpha \in (0,1) \qquad (10)$$
$$\text{and some function } \beta_0 : [0,1] \to \mathbb{R}^{q_2}.$$

Let $\hat{\beta}_n$ be a suitable estimator of the function $\beta_0$. See e.g. Koenker and Bassett (1978). A test can be performed using as test statistics continuous functionals of

$$\tau_n(x,\alpha) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( 1_{\left\{ Y_i \le X_i' \hat{\beta}_n(\alpha) \right\}} - \alpha \right) 1_{\{X_i \le x\}}. \tag{11}$$

In order to implement the test with the assistance of the bootstrap, we realize that under $H_0$, $Y_i = X_i'\beta_0\left(F_{Y|X}(Y_i|\, X_i)\right)$, $i \geq 1$. Thus, taking into account that under $H_0$ $\left\{F_{Y|X}(Y_i|\, X_i)\right\}_{i=1}^n$ are independent and identically distributed as a uniform in $[0,1]$ and independent of $\{X_i\}_{i=1}^n$, a bootstrap assisted test can be performed using the resample $\{Y_i^*, X_i\}_{i=1}^n$, where $Y_i = X_i'\beta_0(U_i^*)$ and $\{U_i^*\}_{i=1}^n$ are random numbers generated according to a uniform in $[0,1]$. The critical values are approximated by Monte Carlo, as explained in Subsection 2.2.

Domínguez (1998), Koul and Stute (1999) and Bierens and Ginther (2007) have studied the asymptotic properties of the test. Zheng (1998) proposed the smoothed based counterpart.

### 5.2. Discrete choice and count data models.

Suppose that $Y$ is a discrete random variable taking integer values with

$$\mathbb{P}\{Y = k|\, X = x\} = \phi(k|\cdot),\ k \in \mathbb{N},$$

i.e. $\phi$ is a conditional probability function. Given a family of probability functions indexed by a vector of parameters $\mathcal{P} = \{\phi_\vartheta : \sum_{k=0}^{\infty} \phi_\vartheta(k|\cdot) = 1,\ \phi_\vartheta \geq 0,\ \vartheta \in \Xi\}$, consider the hypothesis,

$$H_0 : \phi \in \mathcal{P}.$$

Popular families in econometrics research are the Poisson and Negative Binomial, see e.g. Cameron and Trivedi (1998). Of course, discrete choice models belongs to this class with $k = 0, 1$.

Given a suitable estimator of $\vartheta_0$, $\hat{\vartheta}_n$ say, e.g. the $ML$, we can estimate $p(x,k) = \mathbb{P}(X \leq x, Y = k)$ for each $k \in \mathbb{N}$ by its fully nonparametric sample analog

$$\hat{p}(x,k) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{Y_i = k\}} 1_{\{X_i \leq x\}}, \; k \in \mathbb{N},$$

and by its semiparametric counterpart imposing the specification under the null $p_{\hat{\vartheta}_n}(x,k)$, with

$$p_{\vartheta}(x,k) = \frac{1}{n}\sum_{i=1}^{n} \phi_{\vartheta}(k|\, X_i)\, 1_{\{X_i \leq x\}}, \; k \in \mathbb{N}.$$

For each $k \in \mathbb{Z}$, we can consider the empirical process

$$\xi_{n\vartheta}(x,k) = \sqrt{n}\left(\hat{p}_n(x,k) - p_{\vartheta}(x,k)\right).$$

Test statistics are functionals of the empirical process

$$\epsilon_{n,N}(x) = \sum_{k=0}^{N} \xi_{n\hat{\vartheta}_n}(x,k)^2,$$

where $N$ is a fairly large integer. Usually, in economics practice, count data observations take few large values with very small sample frequencies. When $\sum_{k=0}^{M} \phi_{\vartheta}(k|\,\cdot) = 1$, with $M$ small, e.g. in discrete choice models $M = 2$, we take $N = M - 1$. The empirical process $\epsilon_{n,N}$ has a cumbersome limiting distribution under the null hypothesis. Bootstrap assisted tests are implemented using resamples $\{Y_i^*, X_i\}_{i=1}^n$, where $Y_i^*$ are random numbers generated according to a discrete random variable distribution with probability function $\phi_{\hat{\vartheta}_n}(\cdot|\, X_i)$, $i = 1, ..., n$. Bootstrap critical values are approximated by Monte Carlo.

Alvarez and Delgado (2002) have implemented these specification tests for Poisson and Negative Binomial models in the context of dental care demand.

### 5.3. Censored models

Consider the censored model

$$Y = \max\{0, X'\beta_0 + U\},$$

where $U$ is the error term of the underlying latent variable regression model. In the Tobit model is assumed that $U$ is independent of $X$ and distributed according to a normal random variable centered at zero and with unknown variance $\sigma_0^2$, which prevent for the presence of conditional heteroskedasticity and other higher order conditional moment heterogeneity. This hypothesis can be tested using the test presented in Section 2. The bootstrap resample in this case is $\{Y_i^*, X_i\}_{i=1}^n$, where $Y_i^* = \max\{0, X_i'\beta_n + U_i^*\}$, where $\beta_n$ is the $ML$ estimator and $\{U_i^*\}_{i=1}^n$ are generated according to a normal with mean zero and variance $\sigma_n^2$, where $\sigma_n^2$ is the maximum likelihood estimator of $\sigma_0^2$.

The $ML$ estimator is inconsistent under misspecification of the underlying conditional distribution function, including heteroskedasticity. There is a large literature on the estimation of censored models in the presence of a conditional distribution of unknown parametric form, robust to heteroskedasticity. Powell (1984, 1986) suggested censored regression quantile estimators, consistent under weaker assumptions. The main specification assumptions are stated as follow.

Let $Y^\dagger$ be the underlying unobserved latent variable, such that $Y = Y^\dagger 1_{\{Y^\dagger > 0\}}$, and consider its conditional quantile function $m^\dagger$, such that

$$F_{Y^\dagger|X}\left(m^\dagger(x,\alpha)\middle| x\right) = \alpha \text{ for each } \alpha \in (0,1) \text{ and } x \in \mathbb{R}^{q_2}.$$

The hypothesis of correct specification is similar to (10), i.e.

$$H_0 : m^\dagger(x,\alpha) = x'\beta_0(\alpha) \text{ for some function } \beta_0 : [0,1] \to \mathbb{R}^{q_2}.$$

The problem is that $Y^\dagger$ is not observable and we cannot apply the test discussed in subsection 5.1. However, we can take advantage of the fact that under $H_0$, the conditional quantile function of $Y$ given $X$ is

$$H_0 : m(x,\alpha) = \max\{0, x'\beta_0(\alpha)\} \text{ for each } \alpha \in (0,1) \tag{12}$$

Using this, Powell (1984, 1986) extended the estimator of Koenker and Bassett (1978) to these circumstances. This estimator, $\hat{\beta}_n(\alpha)$ say, is squared root consistent for each $\alpha \in (0,1)$ when the specification in (12) is correct. This hypothesis can be tested using the procedure discussed in Subsection 5.1., based on the empirical process in (11) applied to this specification, i.e.

$$\tau_n(x,\alpha) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( 1_{\left\{ Y_i \leq \max\left(0, X_i' \hat{\beta}_n(\alpha)\right)\right\}} - \alpha \right) 1_{\{X_i \leq x\}}.$$

The critical values are approximated using the bootstrap procedure discussed in Subsection 5.1. Now, we use a bootstrap resample $\{Y_i^*, X_i\}_{i=1}^n$ with $Y_i^* = \max\left(0, Y_i^{\dagger *}\right)$, where $Y_i^{\dagger *} = X_i' \beta_0(U_i^*)$ and $\{U_i^*\}_{i=1}^n$ are random numbers generated according to a uniform in $[0,1]$. Related to this test is that proposed by Nikabadze and Stute (1997) for regression models with random censorship.

## 6. SOME APPLICATIONS TO TIME SERIES MODEL SPECIFICATION

In this section, we present testing restrictions in models dealing with strictly stationary time series data. Along this Section it is assumed that the underlying time series process is strict stationary. First, we discuss the generalization of the tests presented in Section 3 to specification testing of Markovian time series. Then we discuss testing serial independence, testing the martingale hypothesis and testing conditional symmetry.

### 6.1. Specification testing of Markovian time series.

Consider an univariate strictly stationary time series process $\{Z_t\}_{t \in \mathbb{Z}}$. There are different hypothesis interesting to test. For instance,

$$H_0 : \mathbb{E}\left( U_{\theta_0}(Z_t) \middle| \{Z_j\}_{j=s}^{t-1} \right) = 0 \text{ for some } \theta_0 \in \Theta.$$

Given observations $\{Z_t\}_{t=1}^{n}$ and a suitable $\sqrt{n}-consistent$ estimator of $\theta_0$, $\hat{\theta}_n$ say, tests are based on the empirical process

$$\tau_n(z) = \frac{1}{\sqrt{n}} \sum_{t=s+1}^{n} U_{\hat{\theta}_n}(Z_t) \prod_{\ell=1}^{s} 1_{\{Z_{t-\ell} \leq z_\ell\}}.$$

Koul and Stute (1998) have studied the asymptotic properties of this empirical process under very week serial dependence regularity conditions. They propose to use the martingale part of this empirical process, which is asymptotically distribution free, for performing omnibus tests of $H_0$. The test can also be implemented with the assistance of a wild bootstrap.

A large number of existing time series tests can be developed applying the methodology presented in preceeding sections.

### 6.2. Testing serial independence.

Another interesting hypothesis is that $\zeta_t = U_{\theta_0}(Z_t)$ and $\zeta_{t-s} = U_{\theta_0}(Z_{t-s})$ are independent, with $\theta_0$ known at the moment, which can be expressed as

$$H_0 : \gamma_\zeta(s, u_1, u_2) = 0 \text{ for a given } s \in \mathbb{N},\ (u_1, u_2) \in \mathbb{R}^2.$$

where, given a stationary time series process $\{\zeta_i\}_{i=1}^{n}$,

$$\gamma_\zeta(s, u_1, u_2) = G_\zeta(s, u_1, u_2) - G_\zeta(0, u_1, \infty)\, G_\zeta(0, \infty, u_2) \tag{13}$$

is the generalized autocorrelation function, with

$$G_\zeta(s, u_1, u_2) = \mathbb{E}\left(1_{\{\zeta_t \leq u_1\}} 1_{\{\zeta_{t-s} \leq u_2\}}\right).$$

The methodology presented along this article can also be applied to this case, extending the tests proposed by Hoeffding (1948) and Blum, Kiefer and Rosenblatt (1961) for the independence of two random variables using *iid* observations. See Delgado (1999) for a review. The sample analog of $G_\zeta(s, u_1, u_2)$ is

$$\hat{G}_{\zeta n}(s, u_1, u_2) = \frac{1}{n} \sum_{t=s+1}^{n} 1_{\{\zeta_t \leq u_1\}} 1_{\{\zeta_{t-s} \leq u_2\}},$$

and the corresponding estimator of $\gamma_\zeta(s, u_1, u_2)$ is,

$$\hat{\gamma}_{\zeta n}(s, u_1, u_2) = \hat{G}_{\zeta n}(s, u_1, u_2) - \hat{G}_{\zeta n}(0, u_1, \infty)\,\hat{G}_{\zeta n}(0, \infty, u_2)$$

Then, tests statistics are continuous functionals of the empirical process $\xi_{\zeta n} = \sqrt{n}\hat{\gamma}_{\zeta n}$.

Skaug and Tjøstheim (1993) proposed this test based on $\xi_{\zeta n}$ for testing $H_0$. They show that the limiting process under the null share the distribution of the classical Hoeffding-Blum-Kiefer-Rosenblatt test, which has been tabulated for the main criteria. Delgado (1996) considers the extension to testing serial $p-independence$ (independence at $p$ lags) is still distribution-free. Delgado and Mora (1998) showed that estimating parameters does not have effect on the asymptotic pivotal distribution of the empirical process.

Though asymptotic tests are possible in this case, the asymptotic distribution of the test statistics can also be approximated with the assistance of the bootstrap by using as resample $\{\zeta_t\}_{t=1}^n$ a random permutation of $\{\zeta_t^*\}_{t=1}^n$. When parameters are estimated, the test statistics are not distribution free and the bootstrap assisted test is sorely needed.

The hypothesis of pairwise independence

$$H_0 : \zeta_t \text{ independent of } \zeta_{t-s} \text{ for all } s \in \mathbb{Z}$$

can also be tested by resorting to the generalized spectral density. This hypothesis can be expressed in terms of $\gamma_\zeta$ evaluated at all the lags. Let us consider the generalized spectral density

$$h_\zeta(\lambda, u_1, u_2) = \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\gamma_\zeta(s, u_1, u_2)\exp(-ij\lambda)\,,\ \ \lambda \in [-\pi, \pi]\,.$$

Pairwise serial independence tests are based on estimates of the generalized spectral distribution

$$\begin{aligned} H_\zeta(\omega,u_1,u_2) &= 2\int_0^{\omega\pi} h_\zeta\left(\bar{\lambda},u_1,u_2\right)d\bar{\lambda} \\ &= \gamma_\zeta(0,u_1,u_2)\,\lambda + 2\sum_{s=1}^{\infty}\gamma_\zeta(s,u_1,u_2)\frac{\sin(s\pi\lambda)}{s\pi}, \omega\in[0,1]. \end{aligned}$$

Under $H_0$, $H_\zeta(\omega,u_1,u_2) = \omega\cdot\gamma_\zeta(0,u_1,u_2)$, which forms a basis for the test. Hong (2000) proposed, in the context of simple hypothesis, the estimator,

$$\begin{aligned} \hat{H}_{\zeta n}(\omega,u_1,u_2) &= \hat{\gamma}_{\zeta n}(0,u_1,u_2)\,\omega \\ &\quad +2\sum_{s=1}^{n-1}\left(1-\frac{s}{n}\right)^{1/2}\hat{\gamma}_{\zeta n}(s,u_1,u_2)\frac{\sin(\omega s\pi)}{s\pi},\ \omega\in[0,1]. \end{aligned}$$

The restricted estimator under $H_0$ is

$$\hat{H}^o_{\zeta n}(\omega,u_1,u_2) = \hat{\gamma}_{\zeta n}(0,u_1,u_2)\,\omega,\ \omega\in[0,1].$$

Test statistics are fuctionals of the empirical process $\bar{\xi}_{\zeta n}$, where,

$$\bar{\xi}_{\zeta n} = \sqrt{\frac{n}{2}}\left(\hat{H}_{\zeta n} - \hat{H}^o_{\zeta n}\right). \tag{14}$$

This empirical process is asymptotically distribution-free when parameters are known. The asymptotic distribution is the product of independent Brownian-Bridges in $[0,1]$. However, the distribution is case dependent when the parameters are estimated. In this case, the critical values of the resulting tests can be approximated with the assitance of a bootstrap procedure using as resample the random permutation of the original sample, as explained before. The spectral representation of $\bar{\xi}_{\zeta n}$,

$$\bar{\xi}_{\zeta n}(\omega,u_1,u_2) = \sqrt{\frac{2}{n}}\sum_{j=1}^{n-1}\left(1-\frac{j}{n}\right)^{1/2}\hat{\gamma}_{\zeta n}(j,u_1,u_2)\frac{\sin(j\pi\omega)}{j\pi},\ \omega\in[0,1]$$

is useful for interpreting the resulting tests. The Crámer von Mises criterium provides the statistic

$$\begin{aligned} CM_n &= \int_0^1 \int_{\mathbb{R}^2} \bar{\xi}_{\zeta n}(\omega, u_1, u_2)^2 \, d\lambda G_{\zeta n}(0, du_1, \infty) \, G_{\zeta n}(0, du_2, \infty) \\ &= \frac{1}{n} \sum_{j=1}^{n-1} (n-j) \frac{\hat{\sigma}_{\zeta n}^2(j)}{(\pi j)^2}, \end{aligned}$$

with

$$\begin{aligned} \hat{\sigma}_{\zeta n}^2(j) &= \int_{\mathbb{R}^2} \hat{\gamma}_{\zeta n}(j, u_1, u_2)^2 \, G_{\zeta n}(0, du_1, \infty) \, G_{\zeta n}(0, \infty, du_2) \\ &= \frac{1}{n} \sum_{t=1}^{n} \sum_{i=1}^{n} \hat{\gamma}_{\zeta n}(j, Z_i, Z_t)^2 . \end{aligned}$$

That is these tests are taking into account all the pairwise dependences amongst the different lags. A bootstrap test is performed using as resample a random permutation of $\{Z_i\}_{i=1}^n$ .

### 6.3. Testing the martingale hypothesis.

Another interesting hypothesis is the martingale hypothesis,

$$H_0 : \mathbb{E}\left( U_{\theta_0}(Z_t) \middle| \{Z_j\}_{j=-\infty}^{t-1} \right) = \mathbb{E}\left( U_{\theta_0}(Z_t) \right), \text{ for some } \theta_0 \in \Theta.$$

This hypothesis is hard to test because the conditioning set is infinite dimensional. However, we can test the slightly less ambitious restriction

$$H_0 : \mathbb{E}\left( U_{\theta_0}(Z_t) \middle| Z_{t-s} \right) = \mathbb{E}\left( U_{\theta_0}(Z_t) \right) \text{ for each } s < t, \text{ some } \theta_0 \in \Theta.$$

Notice that $H_0$ can be alternatively expressed as

$$H_0 : \mathbb{E}\left( U_{\theta_0}(Z_t) \, 1_{\{Z_{t-s} \leq z\}} \right) = \mathbb{E}\left( U_{\theta_0}(Z_t) \right) \text{ for each } s < t, \text{ some } \theta_0 \in \Theta,$$

which in turns can be expressed in terms of the generalized spectral distribution, by noticing that, under $H_0$

$$\mathbb{C}ov\left( U_{\theta_0}(Z_t), 1_{\{Z_{t-s} \leq z\}} \right) = \int_{-\infty}^{\infty} U_{\theta_0}(\bar{z}) \, \gamma_Z(s, d\bar{z}, z),$$

where $\gamma_Z$ was defined in (13) with $\zeta$ in place of $Z$. Then, given a suitable estimator of $\theta_0$, $\hat{\theta}_n$ say, tests statistics of the martingale hypothesis can be based on the two parameter empirical process

$$\begin{aligned}\bar{\tau}_{\theta n} &= \int_{-\infty}^{\infty} U_{\theta}\left(\bar{z}\right) \bar{\xi}_{Zn}\left(\omega, d\bar{z}, z\right) \\ &= \sqrt{\frac{2}{n}} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right)^{1/2} \tilde{\gamma}_{\theta n}\left(j, z\right) \frac{\sin\left(j\pi\omega\right)}{j\pi},\end{aligned}$$

with

$$\tilde{\gamma}_{\theta n}\left(j, z\right) = \frac{1}{n} \sum_{i=j+1}^{n} \left[U_{\theta}\left(Z_i\right) - \bar{U}_{\theta n}\right] 1_{\{Z_{i-j} \leq z\}}$$

and $\bar{U}_{\theta n} = n^{-1} \sum_{i=1}^{n} U_{\theta}\left(Z_i\right)$. Escanciano and Velasco (2006a) have studied tests based on this empirical process. They have shown that $\bar{\tau}_{\theta_0 n}$, suitably scaled, converges to a Brownian Motion under $H_0$, and the corresponding test statistics can be tabulated. However, when parameters are estimated, the tests are not asymptotically distribution free, but critical values can be approximated with the assistance of a wild bootstrap. Related tests are those proposed by Escanciano and Velasco (2006b) and Escanciano (2007).

### 6.4. Testing lack of autocorrelation.

The classical omnibus tests for lack of autocorrelation proposed by Bartlett (1955) can also be expressed in terms of the empirical process $\bar{\xi}_{\zeta n}$ discussed in (14). Suppose we are interested in testing

$$H_0 : \mathbb{C}ov\left(U_{\theta_0}\left(Z_i\right), U_{\theta_0}\left(Z_{i-s}\right)\right) = 0 \text{ for all } s \in \mathbb{Z} \text{ and some } \theta_0 \in \Theta.$$

When parameters are known under $H_0$, The lack of autocorrelation between $\zeta_i = U_{\theta_0}\left(Z_i\right)$ and $\zeta_{i-s} = U_{\theta_0}\left(Z_{i-s}\right)$ for all $s \in \mathbb{Z}$ can be carried out using the Bartlett's

$T_p - process\ T_n(\omega) = \tilde{\xi}_{\zeta n}(\omega) \big/ \hat{\gamma}_{\zeta n}(0)$ with

$$\begin{aligned}\tilde{\xi}_{\zeta n}(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_1 u_2 \bar{\xi}_{\zeta n}(\omega, du_1, du_2) \\ &= \sqrt{\frac{2}{n}} \sum_{j=1}^{n-1} \left(1 - \frac{j}{n}\right)^{1/2} \hat{\gamma}_{\zeta n}(j) \frac{\sin(j\pi\omega)}{j\pi},\end{aligned}$$

where $\hat{\gamma}_{\zeta n}(j) = n^{-1} \sum_{i=j+1}^{n} \left(\zeta_i - \bar{\zeta}_n\right)\left(\zeta_{i+1} - \bar{\zeta}_n\right)$ is the sample autocorrelation function of $\{\zeta_i\}_{i=1}^{n}$. The empirical process $\tilde{\xi}_{\zeta n}$ converges to the standard Brownian Bridge, see e.g. Anderson (1993), and the asymptotic critical values are tabulated. However, when parameters are estimated the resulting tests are not distribution free and critical values can be approximated using bootstrap methods, as suggested by Chen and Romano (2002) or implementing a suitable asymptotically distribution free transformation, as suggested by Delgado, Hidalgo and Velasco (2005).

### 6.5. Conditional symmetry

Testing symmetry of the conditional distribution of residuals $U_{\theta_0}(Z_t)$ around zero is interesting in many applications, see Bai and Ng (2001). Let the information set at time $t$ be $\mathcal{I}_t = \{(Y_{s-1}, X_s),\ t - m + 1 \leq s \leq t\}$. The hypothesis to test is

$$H_0 : F_{U_{\theta_0}(Z_t)|\mathcal{I}_t}(\cdot|\, \ell) \in \mathcal{S} \text{ for each } \ell \in \mathbb{R}^{mq},$$

where $\mathcal{S} = \{G : G(u) = 1 - G(-u)\}$ is the class of symmetric distributions around zero.

Given a suitable estimator of $\theta_0$, $\hat{\theta}_n$ say, test are constructed using the sample analog of the joint distribution of $(U_{\theta_0}(Z_t), \mathcal{I}_t)$, $G_{\theta_0}$ say,

$$\hat{G}_n(u, \ell) = \frac{1}{n} \sum_{t=m+1}^{n} 1_{\left\{U_{\hat{\theta}_n}(Z_t) \leq u\right\}} \prod_{j=1}^{m} 1_{\{Y_{t-j} \leq y_j\}} 1_{\{X_{t-j+1} \leq x_j\}},$$

$\ell = (y_1, ..., y_m, x_1', ..., x_m')'$. Tests are functionals of the empirical process

$$\xi_n(u, x) = \sqrt{n}\left[\hat{G}_n(u, \ell) - \hat{G}_n(\infty, \ell) + \hat{G}_n(u, \ell)(-\infty, x)\right],$$

which extends the empirical process proposed by Butler (1969) for testing simple symmetry hypothesis, when parameters are known, which in turn is a variation of the empirical process introduced by Smirnov (1947).

Delgado and Escanciano (2007) showed, in a context allowing serial dependence, that the limiting distribution of $\xi_n$ depends on $\theta_0$ and other features of $G_{\theta_0}$. They proposed a bootstrap assisted test using a resample $\{Y_i^*, X_i\}_{i=1}^n$, where $U_{\hat{\theta}_n}(Y_i^*, X_i) = U_{\hat{\theta}_n}(Y_i, X_i) V_i$, and $\{V_i\}_{i=1}^n$ are Rademacher's random variables, i.e. $\mathbb{P}(V_i = 1) = \mathbb{P}(V_i = -1) = 0.5$.

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