Principal-Agent Assignment: Implications for Incentives and Income Distribution in Tenancy Relationships☆

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Abstract

I analyze a problem of assigning heterogeneous agents (tenants) to heterogeneous principals (landlords), where partnerships are subject to moral hazard in effort choice. The agents differ in wealth endowment and the principals differ in land quality. When the liability of each agent is limited by his initial wealth, a share contract is typically incentive compatible. A pure rent contract, on the other hand, is optimal in the absence of incentive problems. In a Walrasian equilibrium of the economy, wealthier agents work in more productive lands following a positively assortative matching pattern since higher wealth has greater effect in high-productivity lands. Agent's share of the match output is in general non-monotone with respect to initial wealth. If wealth is more unequally distributed than land quality, then the equilibrium share (of the agents) is a monotonically increasing function of wealth. Under symmetric information, all agents earn the same expected wage, and hence no income inequality is observed in equilibrium. When incentive problems are important, wealthier agents earn higher wages, and the income inequality decreases if the agents are more heterogeneous than the principals.

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1. Introduction

While a plethora of writings on the theory of sharecropping have stressed the role of the agent's wealth endowment in determining his output share in a tenancy relationship, the roles of land quality and outside option have been paid little attention. It has been argued that share tenancy emerges as an incentive device when the agent's liability is limited by his initial wealth (e.g. Shetty, 1988; Laffont and Matoussi, 1995; Ray and Singh, 2001), and wealth has a positive effect on the agent's output share because higher wealth implies the possibility of greater rent extraction by the principal without weakening incentives.[1] Rao (1971) and Braido (2008), among few others, have emphasized the role of land quality in share contracts.[2] On the other hand, the role of agent's outside option in determining his share is also important. Banerjee, Gertler, and Ghatak (2002) argued that higher outside options following the introduction of *Operation Barga*, the land reform act of 1978 in the Indian state of West Bengal, had significant favorable effects on the output share and productivity of the sharecroppers. In the present paper I propose a unified framework that analyzes the joint effects of land and wealth heterogeneities on the tenancy contracts through endogenous outside option.

Most of the theoretical works on share tenancy employ variants of the partial equilibrium agency model (e.g. Grossman and Hart, 1983) where a principal (landlord) of given characteristics leases her land to an agent (tenant) of given characteristics, and offers a tenancy contract that consists of a fixed rent component and a given share of output. The optimal contract determines the incentive structure of the final payoff to the agent. In such models, the level of earning of the agent is determined entirely by his exogenously given outside option. Endogenous determination of the agent's outside option thus calls for a general equilibrium framework. The present paper starts with this motivation. It extends Sattinger's (1979) 'differential rent' model to a situation where agents are assigned to principals, and each principal-agent relationship is subject to limited liability and moral hazard in effort choice. In particular, I consider a model where principals are heterogeneous with respect to the quality or productivity of the lands they own, and agents differ in wealth endowment. In an equilibrium, each principal of a given type chooses optimally an agent by maximizing her residual profits, taking the expected wages as given. A Walrasian equilibrium implies that each agent must receive an expected wage equal to his outside option, defined as the maximum of the payoffs that could be obtained by switching to alternative partnerships. As wealthier agents and more productive principals have absolute advantages in any partnership, equilibrium wage and profit are increasing respectively in wealth and land quality. Optimal choice of agents by the principals also implies that wealthier agents work in high-productivity lands following a positively assortative matching pattern because higher wealth has greater (marginal) effect in more productive lands. The equilibrium relationship between the output shares and initial wealths of the agents determines the equilibrium share function. The equilibrium output share of an agent depends on his initial wealth, on the productivity of the land he cultivates through the equilibrium matching function, and on his outside option via the equilibrium wage function. A principal can extract more surplus from a wealthy agent in the form of fixed rent without affecting incentives, and hence higher wealth implies greater output share.

[1] Basu (1992), Sengupta (1997), and Ghatak and Pandey (2000) have been important contributions to the literature which argue that share tenancy emerges due to limited liability even if the agent may have zero wealth. There is also a large literature which claims that sharecropping emerges because of pure risk-sharing motives (e.g. Stiglitz, 1974; Newbery, 1977), or as an incentive device under moral hazard (e.g. Eswaran and Kotwal, 1985).

[2] Rao (1971), using farm management data from India, shows that the quality of land has explained 90% variations in contracts offered to the share croppers. Braido (2008) argues that typically lower quality lands are leased out to the share tenants.

In a more productive land, less incentive is required in order to induce a given effort level, and hence lower shares are associated with high-quality lands. Finally, higher outside option, which implies greater incentives to shirk, implies higher output share. Because of these two opposing effects the output shares of the agents are in general non-monotone with respect to initial wealth. It is shown that when wealth is more heterogeneously distributed relative to land quality, the positive effects dampen the negative effect of land quality on the output shares of the agents, and the equilibrium share thus increases with wealths.

As the levels of earnings of the agents are endogenously determined, the equilibrium of the model has interesting implications for earnings inequality. First, a Walrasian equilibrium implies that the outside option of each agent is determined endogenously, which in turn implies the endogenous determination of the agents' bargaining power. Second, I show that if the distribution of wealth is more disperse relative to the distribution of land quality, then the equilibrium wage function is concave. A concave wage function implies a lower income inequality since the wage differential decreases as the wealth levels of the agents go up. A convex wage function, on the other hand, increases wage inequality. Finally, similar to Sattinger's (1979) assignment model, the present work also implies that the final distribution of equilibrium wages is skewed to the right relative to the distribution of wealths. With heterogeneous lands, agents with greater wealth are assigned to more productive lands which boosts their expected incomes above what they would be earning if all lands were identical.

The present paper contributes to the recent literature on the problems of assigning agents to principals in environments characterized by informational asymmetries. In particular, in the context of share tenancy, this paper is related to Ghatak and Karaivanov (2010) who analyze a model of partnership based on double-sided moral hazard, and show that partnerships may not emerge in equilibrium if the individuals differ in terms of degrees of absolute advantage in accomplishing specific tasks and the matching is endogenous. They also found conditions under which a matching may be assortative or non-monotone. One major contribution of their work is that sharecropping may emerge as a consequence of endogenous matching between principals and agents. In an important contribution to this literature, Ackerberg and Botticini (2002) have analyzed the landlord-tenant contracts in renaissance Tuscany, and showed that contracts are influenced in a significant way by the endogenous nature matching between the landlords and tenants. Chakraborty and Citanna (2005) show, in a model of occupational choice, that less wealth-constrained individuals choose to take up projects in which incentive problems are more important due to endogenous sorting effects. Unlike most of the recent contributions, e.g. the present model, Chakraborty and Citanna's (2005) paper is a model of one-sided matching. It is worth mentioning that, while the current paper generalizes the popular assignment models (e.g. Sattinger, 1975, 1979) by considering situations in which matches are subject to moral hazard in a Walrasian equilibrium framework, most of the aforementioned papers consider partnership formation as a cooperative matching game, and employ stability as the solution concept. Serfes's (2005) is an assignment model that analyzes the trade-off between risk-sharing and incentives, and shows a non-monotone relationship between risk and incentive. Legros and Newman (2007) propose a sufficient condition, called the *generalized difference condition*, under which stable allocations exhibit assortative matching when the two-sided matching induces a non-transferable utility (a concave Pareto frontier) game. Similar conditions are obtained in Lemma 1(b).

2. A model of principal-agent assignment

2.1. Description

Consider an economy with a continuum $[0, 1]$ of heterogeneous risk-neutral principals and a continuum $[0, 1]$ of heterogeneous risk-neutral agents. The positive real numbers $\lambda \in \Lambda \equiv [\lambda_{min}, \lambda_{max}]$ and $\omega \in \Omega \equiv [\omega_{min}, \omega_{max}]$ denote the 'qualities' of the principals and the agents, respectively. Quality or type of an individual may be interpreted as productivity, efficiency, wealth, etc. which would influence final payoffs. For example, higher values of λ could imply more productive ("better") principals. The distributions of qualities are exogenous to the model. Let $G(\lambda)$ be the cumulative distribution of λ, which denotes the fraction of principals with qualities lower than λ, and $g(\lambda)$ be the corresponding density function. Similarly, let $F(\omega)$ be the cumulative distribution of ω with the corresponding density $f(\omega)$. I denote by $\xi \equiv (F, G)$ the principal-agent economy.

Principals and agents are assigned to each other to form partnerships or matches. Each individual in a given match has to take a set of actions which are inputs to the final match output. Some of these actions such as effort, investment decision may not be publicly verifiable which induce incentive problems in each partnership. This section extends the 'differential rents' model of Sattinger (1979) to situations where matches are subject to incentive problems.[3] Individuals of identical quality will be perfect substitutes, and hence only qualities and not the names matter. The principal-agent assignment can be described by a one-to-one correspondence $l : \Omega \longrightarrow \Lambda$ or its inverse $w \equiv l^{-1}$. Therefore if $\lambda = l(\omega)$, or equivalently $\omega = w(\lambda)$, then (λ, ω) denotes a typical match or partnership.

In a match (λ, ω), the principal and agent write a binding contract $c(\lambda, \omega)$ which specifies the way the total match-surplus is to be divided between them. Let the surplus be given by $\phi(\lambda, \omega, u(\omega))$ where $u(\omega)$ is the expected wage of the agent.[4] The expected profit of the principal is thus given by:

$$v(\lambda) = \phi(\lambda, \omega, u(\omega)) - u(\omega). \tag{1}$$

I make the following assumptions on $\phi(\lambda, \omega, u(\omega))$:

1. $\phi(\lambda, \omega, u(\omega))$ is twice differentiable, where ϕ_i denotes the partial derivative of ϕ with respect to the i-th argument, and ϕ_{ij} denotes the cross partial derivative with respect to the i-th and j-th arguments;
2. $\phi_i > 0$, for $i = 1, 2$, and $\phi_3 \in [0, 1)$: the match-surplus is strictly increasing in the qualities of the principal and the agent, and increasing in the agent's wage;
3. $\phi_{22} \leq 0$ and $\phi_{33} \leq 0$: concavity of the surplus function with respect to the quality and wage of the agent.

In a Walrasian equilibrium, each type λ principal picks an agent with quality ω in order to maximize her

[3] Edmans, Gabaix, and Landier (2009), and Dam (2010) also build on the assignment model presented in Sattinger (1979) to incorporate incentive problems.

[4] The match-surplus depends on the qualities of the principal and the agent, and on the agent's wage.Sattinger's (1979) model is an assignment model with transferable utilities due to the absence of any incentive problems. It is well-known that, in the presence of incentive problems in a principal-agent relationship, optimality of contracts is not defined by total surplus maximization since how much surplus would be produced depends on how it is distributed within the match. Consequently, incentive problems give rise to non-transferabilities which implies that the match-surplus also depends on the expected wage of the agent. In an assignment model similar to Sattinger (1979), the surplus function will be given by $\phi(\lambda, \omega)$.

expected profit, i.e., each type λ principal solves

$$v(\lambda) = \max_{\omega'} \{\phi(\lambda,\, \omega',\, u(\omega')) - u(\omega')\}, \tag{P}$$

taking the wages $u(\omega)$ as given. Therefore, an equilibrium consists of an assignment rule l or w, and the vectors of profits $(v(\lambda))_{\lambda \in \Lambda}$ and wages $(u(\omega))_{\omega \in \Omega}$ such that

(a) $w(\lambda) = \text{argmax}_\omega \{\phi(\lambda,\, \omega,\, u(\omega)) - u(\omega)\}$ for each λ: each principal chooses her partner optimally;
(b) If $[\lambda_1,\, \lambda_2] = l([\omega_1,\, \omega_2])$ for any subintervals $[\omega_1,\, \omega_2]$ of Ω and $[\lambda_1,\, \lambda_2]$ of Λ, then it must be the case that $G(\lambda_2) - G(\lambda_1) = F(\omega_2) - F(\omega_1)$: market clearing.

The first condition implies that the principal-agent assignment is optimal. The second is a measure consistency requirement which says that if an interval of agent-types $[\omega_1,\, \omega_2]$ is mapped by the rule l into an interval of principal-types $[\lambda_1, \lambda_2]$, then these two sets cannot have different measures. This is the standard market clearing condition for each type.

2.2. Equilibrium

The equilibrium wages $u(\omega)$ and profits $v(\lambda)$ are determined by solving the maximization problem (P) of each type λ principal. While solving (P), a principal must guarantee the agent his outside option. An agent's outside option is the maximum payoff he can obtain by switching to other matches. In a Walrasian equilibrium, the expected wage of a type ω agent must be equal to his outside option. A wage offer less than the outside option will not be accepted. On the other hand, if the wage of an agent is strictly higher than his outside option, then the principal can lower her offer a bit and still the offer will be accepted by the agent, thereby contradicting the notion of equilibrium. The first-order condition of the maximization problem (P) is given by:

$$\begin{gathered} \phi_2(\lambda,\, \omega,\, u(\omega)) + [\phi_3(\lambda,\, \omega,\, u(\omega)) - 1]u'(\omega) = 0, \\ \text{i.e.,}\;\; u'(\omega) = \frac{\phi_2(\lambda,\, \omega,\, u(\omega))}{1 - \phi_3(\lambda,\, \omega,\, u(\omega))} \quad \text{for}\;\; \lambda = l(\omega). \end{gathered} \tag{FOC}$$

It follows from the Envelope theorem that

$$v'(\lambda) = \phi_1(\lambda,\, \omega,\, u(\omega)) \quad \text{for}\;\; \lambda = l(\omega). \tag{E}$$

Given the assumptions on the match-surplus function, the above two expressions are positive. In every match, an agent's wage and a principal's profit are paid according to their contributions to the match-surplus. For example, the marginal wage of a type ω agent is equal to his marginal contribution to the total surplus $\phi(\lambda,\, \omega,\, u(\omega))$. The equilibrium wages $u(\omega)$ and profits $v(\lambda)$ are found by solving the differential equations (FOC) and (E) for each match $(\lambda,\, \omega)$.

Suppose that the equilibrium assignment is given by $\lambda = l(\omega)$. The next step is to determine the sign of $l'(\omega)$. The following definition introduces the notion of assortative or monotone matching.

Definition 1 *If $l'(\omega) \geq (\leq)\, 0$, then the assignment is said to be positively (negatively) assortative.*

Whether the equilibrium matching is assortative is determined by the second-order conditions of the maximization problem (P). In Appendix A it is shown that this second-order condition is satisfied if and only if

$$[\phi_{21}(l(\omega),\, \omega,\, u(\omega)) + \phi_{31}(l(\omega),\, \omega,\, u(\omega))u'(\omega)]l'(\omega) \geq 0. \qquad \text{(SOC)}$$

It follows from the above inequality that if $\phi_{21}(\lambda,\, \omega,\, u(\omega))$ and $\phi_{31}(\lambda,\, \omega,\, u(\omega))$ are both positive (negative), then $l'(\omega) \geq (\leq)\, 0$. The following lemma summarizes the above findings.

Lemma 1 *Let $\lambda = l(\omega)$ be an equilibrium principal-agent assignment, and $u(\omega)$ and $v(\lambda)$ be the associated equilibrium wages and profits, respectively.*

(a) *The equilibrium wages and profits are increasing functions of the qualities of the agents and the principals, respectively;*
(b) *If $\phi_{21}(\lambda,\, \omega,\, u(\omega)) \geq (\leq)\, 0$ and $\phi_{31}(\lambda,\, \omega,\, u(\omega)) \geq (\leq)\, 0$ for all $(\lambda,\, \omega,\, u(\omega))$, then the equilibrium assignment is positively (negatively) assortative.*

The proofs of the above assertions and those of the subsequent ones are relegated to the Appendix. The above lemma extends the results of Sattinger (1979) to environments with non-transferable utility. Similar results have also been proved by Legros and Newman (2007). High-quality individuals have absolute advantages in any partnership because $\phi_1 > 0$ and $\phi_2 > 0$, i.e., the match surplus is increasing in λ and ω. Therefore, "better" individuals consume higher expected payoffs in an equilibrium, which is the assertion of Part (a) of the above lemma. Also, two individuals of the same quality must obtain same expected payoffs. Thus, the equilibrium satisfies 'equal treatment of equals' property. The equilibrium wage and profit functions are similar to the 'hedonic prices' in Rosen (1974). An important difference between the aforementioned works and the present paper is that I introduce non-transferability in an assignment model.

An assortative matching is a consequence of complementarity or substitutability between the principal- and agent-qualities. Consider the case of a positively assortative assignment. Complementarity implies that high-quality agents have comparative advantages over the low-quality ones in matches involving high-quality principals. This sort of comparative advantage determines that a better agents must be assigned better principals. In the context of non-transferable utilities, the complementarity has two aspects. First, $\phi_{21} \geq 0$ implies that a high-quality principal and a high-quality agent together produce higher aggregate surplus. This is the usual 'type-type' complementarity, which also determines positive sorting in the standard assignment models (e.g. Rosen, 1974; Sattinger, 1979). Second, $\phi_{31} \geq 0$ implies that it is (marginally) less costly for a high-type principal to transfer surplus to a high-type agent. This 'type-payoff' complementarity reinforces the reasons under which an equilibrium induces a positively assortative matching.[5]

[5]Notice the difference between the above optimality conditions and those in an assignment model with transferable utilities. In Sattinger's (1979) differential rents model, the first-order condition associated with the principal's optimal choice is given by:

$$u'(\omega) = \frac{\partial \phi(\lambda,\, \omega)}{\partial \omega}.$$

And the second-order condition is given by:

$$\frac{\partial^2 \phi(\lambda,\, \omega)}{\partial \lambda \partial \omega}\, l'(\omega) \geq 0.$$

3. Application: tenancy contracts under limited liability

This section considers optimal share-tenancy contracts between risk-neutral principals (landlords) and risk-neutral agents (tenants) in an attempt to identify a situation to which the results of Lemma 1 apply. Principals own a plot of land apiece which can be leased out to a sharecropper, the agent. Agents are heterogeneous with respect to their initial wealth $\omega \in \Omega \subset \mathbb{R}_{++}$, which is uniformly distributed. Here initial wealth represents the type or quality of an agent. On the other hand, $\lambda \in \Lambda \subset (0, 1)$ represents the productivity of lands, which is also uniformly distributed. I assume that $\lambda^2/8 \leq \omega$ for all (λ, ω). I further assume that there is no alternative markets for land and labor services. Hence, an unused plot of land generates zero profit to its owner, and an unemployed agent consumes his wealth endowment. A land with quality λ produces a stochastic output which is given by:

$$\tilde{y} = \begin{cases} 1 & \text{with probability } \lambda e, \\ 0 & \text{with probability } 1 - \lambda e, \end{cases}$$

where $e \in [0, 1]$ is the non-verifiable effort exerted by an agent. The cost effort is an increasing and convex function $\psi(e)$. For simplicity, I assume that $\psi(e) = e^2/2$.

A tenancy contract for an arbitrary match (λ, ω) is a vector $c(\lambda, \omega) = (\alpha(\lambda, \omega), R(\lambda, \omega))$ where α is the agent's share of output, and R is the fixed rental payment made to the principal. I restrict attention to the class of contracts for which $R \geq 0$ and $\alpha \in [0, 1]$. If $\alpha = 1$ and $R > 0$, then the contract is a *pure rent contract*. A contract with $\alpha < 1$ and $R > 0$, on the other hand, is referred to as a *share contract*.[6] Given $c(\lambda, \omega)$, the expected payoffs of the principals and the agents are respectively given by:

$$V(c(\lambda, \omega)) = \lambda e(\lambda, \omega)[1 - \alpha(\lambda, \omega)] + R(\lambda, \omega), \tag{2}$$

$$U(c(\lambda, \omega)) = \lambda e(\lambda, \omega)\alpha(\lambda, \omega) - R(\lambda, \omega) - \frac{[e(\lambda, \omega)]^2}{2}. \tag{3}$$

Within a principal-agent relationship (λ, ω), the principal therefore solves the following maximization problem.

$$\max_{c(\lambda, \omega)} \ \lambda e(\lambda, \omega)[1 - \alpha(\lambda, \omega)] + R(\lambda, \omega) \tag{M}$$

$$\text{subject to} \quad \lambda e(\lambda, \omega)\alpha(\lambda, \omega) - R(\lambda, \omega) - \frac{[e(\lambda, \omega)]^2}{2} = u(\omega), \tag{PC}$$

$$e(\lambda, \omega) = \text{argmax}_e \left\{ \lambda e \alpha(\lambda, \omega) - R(\lambda, \omega) - \frac{e^2}{2} \right\}, \tag{IC}$$

$$R(\lambda, \omega) \leq \omega. \tag{LL}$$

The first constraint is the participation constraint of the agent. In the context of a Walrasian equilibrium, this constraint requires a bit more attention. In a standard agency model (e.g. Ray and Singh, 2001), a given principal-agent relationship is treated as an isolated entity, and the participation constraint of the

[6] I do not address the issue of existence of share contracts. See Sengupta (1997), and Ray and Singh (2001) for a discussion on its existence, and why one can restrict attention to the values of α in the interval $[0, 1]$.

agent is given by:

$$\lambda e(\lambda,\,\omega)\alpha(\lambda,\,\omega) - R(\lambda,\,\omega) - \frac{[e(\lambda,\,\omega)]^2}{2} \geq \bar{u}, \tag{4}$$

where $\bar{u}$ is the agent's *outside option*, which is defined as the maximum of the expected wages that the agent may earn by switching to alternative matches. Therefore, a contract $c(\lambda,\,\omega)$ must guarantee the agent at least his outside option. Unlike the agency models that involve only one principal and one agent, the outside option of an agent is no more exogenous since it depends on the contract offers by the other principals. First notice that in a Walrasian equilibrium an agent's expected wage must be equal to his outside option, otherwise another agent of the same type may bid his wage down to $\bar{u}$. Second, the participation constraint of the type ω must be binding. If it is not the case, then the principal can lower the agent's wage a bit, and the contract will still be accepted which contradicts the definition of an equilibrium. Therefore, I replace $\bar{u}$ by $u(\omega)$, and write the participation constraint with equality. The second constraint is the incentive compatibility constraint, which says that the agent will choose the effort level that maximizes his expected wage. The last one is the limited liability constraint which guarantees non-negative final income to the agent even when the match output is zero. In a principal-agent relationship, if the agent's limited liability constraint does not bind at the optimum, provision of incentives is not costly for the principal, and hence the *first-best* effort level can be implemented. This is equivalent to the situation where the principal could enforce any level of effort she liked. When the limited liability constraint is binding, it is typically costly for the principal to provide incentives, and only the *second-best* effort can be implemented. Define by:

$$\Gamma := \{(\lambda,\,\omega) \mid \lambda^2/8 \leq \omega + u(\omega) \leq \lambda^2/2\} \subset S = \Lambda \times \Omega.$$

It is easy to show that if $(\lambda,\,\omega) \in S \setminus \Gamma$, then the limited liability constraint (LL) does not bind, and the optimal contract and effort are at their first-best levels. On the other hand, if $(\lambda,\,\omega) \in \Gamma$, then the limited liability constraint binds, and only the second-best contract and effort are implemented. The optimal output share $\alpha^*(\lambda,\,\omega,\,u(\omega))$ of the agent, rental payment $R^*(\lambda,\,\omega,\,u(\omega))$, and effort $e^*(\lambda,\,\omega,\,u(\omega))$ are described in Table 1.

Table 1: Optimal contract, effort and surplus of an arbitrary match $(\lambda,\,\omega)$

	First-best	Second-best
Agent's output share	1	$(1/\lambda)\sqrt{2(\omega + u(\omega))}$
Rental payment	$(\lambda^2/2) - u(\omega)$	ω
Effort	λ	$\sqrt{2(\omega + u(\omega))}$

The optimal contract terms and effort, in general, depend on the productivity of the land, the wealth of the agent and the agent's wage. I omit the standard analysis of the optimal contract. Under the first-best, the agent gets the entire share of output, and pays a fixed rent in both states of the nature for leasing out the land. Therefore, the optimal tenancy contract is a pure rent contract. The optimal effort is at its highest level, and does not depend on the agent's wealth and his wage. When the limited liability constraint binds, the second-best contracts are implemented, and the agent obtains an output share strictly lower than 1 but higher than 1/2. It is increasing in ω and $u(\omega)$, but decreases with λ. The optimal effort is lower than its first-best level, which is increasing in ω and $u(\omega)$, but constant with respect to λ.

Given the optimal contracts and effort, it is now easy to compute the match surplus, which equals the expected match output $\lambda e(\lambda,\, \omega)$ minus the effort cost $\psi(e(\lambda,\, \omega))$.

Lemma 2 *Consider an arbitrary match $(\lambda,\, \omega)$.*

(a) *When $(\lambda,\, \omega) \in S \setminus \Gamma$, i.e., when the first-best contract and effort are implemented, the match surplus is given by:*

$$\phi(\lambda,\, \omega,\, u(\omega)) = \frac{\lambda^2}{2},$$

with $\phi_1(\lambda,\, \omega,\, u(\omega)) \in (0,\, 1)$ and $\phi_2(\lambda,\, \omega,\, u(\omega)) = \phi_3(\lambda,\, \omega,\, u(\omega)) = 0$;

(b) *When $(\lambda,\, \omega) \in \Gamma$, i.e., when the second-best contract and effort are implemented, the match surplus is given by:*

$$\phi(\lambda,\, \omega,\, u(\omega)) = \lambda\sqrt{2(\omega + u(\omega))} - \omega - u(\omega),$$

with $\phi_i(\lambda,\, \omega,\, u(\omega)) \in (0,\, 1)$ for $i = 1,\, 2,\, 3$.

In the first-best situation, when the agent gets the entire share of output, the match surplus takes its maximum value. Under a share contract (second-best), match surplus is lower because there is loss of efficiency due to informational asymmety.[7]

3.1. The equilibrium wages, profits and assignment

As the value of the match surplus $\phi(\lambda,\, \omega,\, u(\omega))$ is known for each arbitrary match, the next step is to determine the equilibrium wage $u(\omega)$ and profit $v(\lambda)$ functions, which are the equilibrium relationships between the expected payoffs of the individuals and their types. First, the marginal functions are determined using equations (FOC) and (E). Hence one should check whether the conditions for Lemma 1(a) are satisfied in this context.

$$\begin{aligned} v'(\lambda) &= \phi_1(\lambda,\, \omega,\, u(\omega)) \in (0,\, 1), && (5) \\ u'(\omega) &= \frac{\phi_2(\lambda,\, \omega,\, u(\omega))}{1 - \phi_3(\lambda,\, \omega,\, u(\omega))} \geq 0. && (6) \end{aligned}$$

The following proposition describes the marginal profit and wage functions.

Proposition 1 *The equilibrium profit $v(\lambda)$ is an increasing function of land-productivity, and the equilibrium wage $u(\omega)$ is an increasing function of the initial wealth.*

When the first-best contracts are implemented in all matches, the match surplus is independent of ω as the limited liability constraints do not bind, i.e., the initial wealth of each agent does not enter into the contract offered to him. The fixed rental payment does not appear in the surplus because it is a transfer

[7] It can be shown that, for $\omega + u(\omega) < \lambda^2/8$, the agent's limited liability constraint binds, but the participation constraint does not. I ignore this situation as the optimal contracts must be parts of a Walrasian equilibrium.

from the agent to the principal. Therefore, the surplus is also independent of $u(\omega)$. As a consequence, one has $\phi_2 = \phi_3 = 0$. Since ϕ for each match depends only on the productivity of land, the marginal surplus is higher for better principals, and hence they receive higher equilibrium profits. As all agents receive the entire share of output and exert the first-best effort, their wages are the same irrespective of the wealth levels. When the second-best contracts are implemented in all matches, it is the case that $\phi_1 > 0$ and $\phi_2 > 0$, i.e., owners of more productive lands and wealthier agents have absolute advantages in producing surplus in any match. Hence, high-quality principals obtain greater equilibrium profits, and wealthier agents consume higher wages in equilibrium. Therefore, the equilibrium wage is a constant function under the first-best, whereas in the second-best situation, the equilibrium wage is a stricly increasing function of wealth. Also notice that under second-best, for given wages and for each type λ principal, her expected profit, $\phi(\lambda,\, \omega,\, u(\omega)) - u(\omega)$, is increasing in wealth. This is the well-known *tenancy ladder* phenomenon, i.e., wealthier agents are always preferred to the less wealthy ones.

Prior to determining the equilibrium matching pattern, notice that an Walrasian equilibrium implies full employment, i.e., no agent is unemployed and no land is left idle. To see this, suppose in an equilibrium there is one agent of a given type ω is unemployed. Then there must be one principal with her plot of land uncultivated. Suppose that the idle plot is of a given productivity λ. In this situation the unemployed agent consumes ω and the principal consumes zero profit. Then the principal can offer a contract to the agent, which consists of $\alpha = 1/2$ and $R = 0$. This contract satisfies the limited liability constraint and generates a profit equal to $\lambda^2/4 > 0$ to the principal, and a gross expected payoff $\lambda^2/8 + \omega > \omega$ to the agent. This contradicts the definition of Walrasian equilibrium.

Lemma 1(b) provides sufficient conditions for assortative matching. In order to determine an equilibrium assignment, it is thus sufficient to check the signs of ϕ_{21} and ϕ_{31} in the present context. From Lemma 2(a), it is immediate to show that $\phi_{21} = \phi_{31} = 0$ if the first-best contracts are implemented. Under the second-best, on the other hand,

$$\phi_{21}(\lambda,\, \omega,\, u(\omega)) = \phi_{31}(\lambda,\, \omega,\, u(\omega)) = \frac{1}{\sqrt{2(\omega + u(\omega))}} > 0. \tag{7}$$

Therefore the equilibrium matching pattern follows from Lemma 1(b), which is described in the following proposition.

Proposition 2 *Let $\lambda = l(\omega)$ be an equilibrium assignment.*

(a) *If the first-best contracts are implemented in all matches, then any matching pattern is consistent with an equilibrium;*
(b) *If the second-best contracts are implemented in every match, then the equilibrium assignment is positively assortative, i.e., wealthier agents cultivate more productive lands.*

If $\phi_{21} = \phi_{31} = 0$, $l'(\omega)$ can have any sign so that the second-order condition (SOC) is satisfied, and hence any matching pattern is consistent with an equilibrium. To see this, consider the aggregate surplus of the economy, which is given by:

$$\int_{\lambda_{min}}^{\lambda_{max}} \int_{\omega_{min}}^{\omega_{max}} \phi(\lambda,\, \omega,\, u(\omega))d\lambda\, d\omega = \frac{1}{2}\int_{\lambda_{min}}^{\lambda_{max}} \lambda^2\, d\lambda.$$

Since the above expression is independent of ω, the aggregate surplus is maximized for any matching pattern. When the limited liability constraints bind in all matches, the equilibrium contracts depend on the wealth endowment. As ϕ_{21} and ϕ_{31} are both strictly positive, higher wealth has greater impact on the match-surplus when combined with more productive land. In other words, wealthier agents have comparative advantages in matches consisting of more productive principal, and hence the unique equilibrium matching is positively assortative.

If in a Walrasian equilibrium a given land-quality λ is assigned to a given wealth level ω, then a positively assortative matching implies

$$\begin{aligned} F(\omega) - F(\omega_{min}) &= G(\lambda) - G(\lambda_{min}), \\ \text{or,} \quad l(\omega) &= \lambda_{min} + (\Delta\,\lambda/\Delta\,\omega)(\omega - \omega_{min}), \end{aligned}$$

where $\Delta\,\lambda \equiv \lambda_{max} - \lambda_{min}$ and $\Delta\,\omega \equiv \omega_{max} - \omega_{min}$. Notice that the slope $l'(\omega)$ is an increasing function of the relative dispersion $\Delta\,\lambda/\Delta\,\omega$. If the lands are homogeneous, i.e., $\Delta\lambda = 0$, then the matching function is horizontal. On the other hand, homogeneity of the agents implies a vertical matching function.

As the equilibrium matching function is known, it is now possible to write down the expression for marginal wage $u'(\omega)$. Using Lemma 2 in equation (6), it is easy to show that

$$u'(\omega) = \frac{1 - \alpha^*(\lambda,\, \omega,\, u(\omega))}{2\alpha^*(\lambda,\, \omega,\, u(\omega)) - 1} \tag{8}$$

In the first-best situation $\alpha^* = 1$, i.e., each tenant gets the entire share of the match-output, and hence $u'(\omega) = 0$. When the second-best contracts are implemented, substituting for α^* from 3 and $\lambda = l(\omega) = \lambda_{min} + (\Delta\,\lambda/\Delta\,\omega)(\omega - \omega_{min})$ in the above expression, one gets the following marginal wage function.

$$u'(\omega) = \begin{cases} 0 & \text{if the first-best contracts are implemented}, \\ \frac{\lambda_{min} + (\Delta\,\lambda/\Delta\,\omega)(\omega - \omega_{min}) - \sqrt{2(\omega + u(\omega))}}{2\sqrt{2(\omega + u(\omega))} - \lambda_{min} - (\Delta\,\lambda/\Delta\,\omega)(\omega - \omega_{min})} & \text{if the second-best contracts are implemented.} \end{cases} \tag{9}$$

The levels of equilibrium wages are found by solving the differential equation (9). Notice that if the first-best contracts are implemented in all matches, then $u(\omega)$ is a constant function. In order to solve the second equation in (9), first one needs to determined the associated constant of integration. In an Walrasian equilibrium, the last agent employed must be an agent with the lowest wealth ω_{min}, which gives the boundary condition for solving (9). It is easy to check that the constant of integration must be equal to $u(\omega_{min})$. Since no principals other than the ones of type λ_{min} would hire the least wealthy agents, they will be pushed to their *reservation wage*, the minimum wage rate at which an agent is not willing to work, which must be equal to ω_{min}. Unfortunately, it is not possible to explicitly solve the equilibrium wage function when the second-best contracts are implemented. In Subsection 3.3, I will analyze more properties of the wage function.

3.2. Equilibrium contracts

The principal objective of this subsection is to analyze behavior of the equilibrium contracts and effort with respect to initial wealth. In a partial equilibrium setup where a principal-agent pair is treated in isolation, the contract for the pair in general depends on three parameters: the productivity of the land, the wealth of the agent and the outside option of the agent. For instance, the optimal output share of the agent in an arbitrary pair is given by $\alpha^*(\lambda, \omega, u(\omega))$. To analyze the behavior of α in this context with respect to ω, one must determine the sign of $\partial\alpha^*/\partial\omega$. If $(\lambda, \omega) \in \Gamma$, the optimal share (of the type ω agent) α is an increasing function of the agent's wealth, i.e., the partial derivative of α with respect to ω is positive (e.g. Ray and Singh, 2001, Proposition 3). In this subsection I show that the monotonicity of the agents' output share with respect to initial wealth may not hold in a principal-agent market with two-sided heterogeneity, and look for sufficient conditions under which the equilibrium share is increasing in wealth.

Notice that in an equilibrium with the first-best contracts neither the optimal output share α nor the effort e depends on the initial wealth. I therefore focus only on the equilibrium with second-best contracts. The initial wealth ω, apart from directly influencing the incentive compatible contracts and effort, affects the optimal values in two indirect ways: through the matching and the wage functions. Thus in equilibrium, the contracts and effort are solely functions of ω, which are given by:

$$\alpha(\omega) = \alpha^*(l(\omega), \omega, u(\omega)) = \frac{1}{l(\omega)}\sqrt{2(\omega + u(\omega))}, \tag{10}$$

$$R(\omega) = R^*(l(\omega), \omega, u(\omega)) = \omega, \tag{11}$$

$$e(\omega) = e^*(l(\omega), \omega, u(\omega)) = \sqrt{2(\omega + u(\omega))}. \tag{12}$$

First notice from equations (11) and (12) that the equilibrium rent and effort are strictly increasing functions of ω because $R'(\omega) = 1$ and

$$e'(\omega) = \frac{\partial e^*}{\partial \lambda}\, l'(\omega) + \frac{\partial e^*}{\partial \omega} + \frac{\partial e^*}{\partial u}\, u'(\omega) = \frac{1 + u'(\omega)}{e(\omega)} > 0.$$

Now consider the equilibrium share function $\alpha(\omega)$. Differentiation of (10) with respect to ω gives

$$\alpha'(\omega) = \frac{\partial \alpha^*}{\partial \lambda}\, l'(\omega) + \frac{\partial \alpha^*}{\partial \omega} + \frac{\partial \alpha^*}{\partial u}\, u'(\omega) = \frac{\alpha(\omega)}{l(\omega)}\left[\frac{1 + u'(\omega)}{l(\omega)\alpha^2(\omega)} - l'(\omega)\right]. \tag{13}$$

Therefore, the sign of $\alpha'(\omega)$ depends on that of $(1 + u'(\omega))/(l(\omega)\alpha^2(\omega)) - l'(\omega)$, which is a condition on the endogenous variables of the model. The equilibrium share function $\alpha(\omega)$ is, in general, non-monotone. Under the second-best contracts, it is easy to show that

$$\frac{1 + u'(\omega)}{l(\omega)\alpha^2(\omega)} = \frac{1}{l(\omega)\alpha^2(\omega)}\left(1 + \frac{1 - \alpha(\omega)}{2\alpha(\omega) - 1}\right) = \frac{1}{l(\omega)\alpha(\omega)[2\alpha(\omega) - 1]} \geq 1$$

since each term of the denominator is less than 1. Therefore, a sufficient condition for $\alpha(\omega)$ to be increasing in ω is that $l'(\omega) = \Delta\,\lambda/\Delta\,\omega < 1$, i.e., the matching function is sufficiently flat. Now, a sufficiently flat matching function results in when $\Delta\,\lambda < \Delta\,\omega$, i.e., wealth is more unequally distributed than land-productivity. The following proposition summarizes the above findings.

Proposition 3 *Let $e(\omega)$, $R(\omega)$ and $\alpha(\omega)$ be the equilibrium effort, rent and share functions, respectively. Then in an Walrasian equilibrium,*

(a) *effort and rent are increasing functions of ω;*

(b) *the share function is in general non-monotone. If wealth is more unequally distributed than land-productivity, then output share is an increasing function of ω.*

As I have mentioned earlier, there are three effects that determine the behavior of incentives with respect to wealth. The first one is the *matching effect*. In the current model, land-productivity works as a substitute for incentive as the agent who cultivates a high-productivity land requires lower incentive. Therefore, α tends to be lower in a match involving the owner of a high-quality land. The second effect is the *wealth effect*. Since the fixed rent component is higher for higher wealth (because $R(\omega) = \omega$), a wealthier agent is given a higher share of output in order to have the same level of incentive. Therefore, agent's output share and effort increase with initial wealth. The aforementioned effects are also present in a partial equilibrium setup consisting of one principal and one agent. The third effect is the *outside option effect*, which emerges because of wage is determined endogenously in a general equilibrium model. A higher outside option turn implies greater bargaining power for the agents. Since the agent has little incentives to work hard, the principal has to offer higher share of the match output. Therefor higher expected wage has favorable impact on effort and output share. As equilibrium effort and rent do not depend on the land-productivity, they increase with wealth. Because of the aforementioned two opposing effects, the share function may be non-monotone.[8]

The sufficient condition in Proposition 3(b) is not hard to understand. When the agents are more heterogeneously distributed than the principals, the negative effect of land-quality on incentives is dampened by the positive wealth and outside option effects. Therefore, higher output share for the agent is associated with higher initial wealth. It is also easy to see that if the distribution of wealth is too "tight" relative to the distribution of quality of the land, then a less wealthy agent receives a greater output share compared to the wealthier agents.

3.3. Wage inequality

In Subsection 3.1 it has been mentioned that it was not possible to have an explicit analytical solution to the second differential equation in (9), i.e., it was not possible to determine the level of equilibrium wage when the second-best contracts are implemented. Yet it is possible to analyze the shape of the equilibrium wage function. In this subsection I relate the shape of the wage function to the nature of wage/income inequality in equilibrium. To this end, the first task is to determine under what conditions the equilibrium wage is either a concave or a convex function of initial wealth. Therefore I will study the sign of $u''(\omega)$, the second derivative of the wage function which is found by differentiating the wage equation (9). When the first-best contracts are implemented, the wage function is flat since $u'(\omega) = 0$. From equation (8), it is easy to check that

$$\operatorname{sgn}[u''(\omega)] = -\operatorname{sgn}[\alpha'(\omega)]. \tag{14}$$

Therefore following Proposition 3(b), it is possible to relate the shape of the wage function to the initial distributions of types.

Proposition 4 *Let $u(\omega)$ be the equilibrium wage function.*

(a) *If the first-best contracts are implemented in all matches, then the equilibrium wage is a constant function of the initial wealth;*

[8] Serfes (2005) also establishes a non-monotone share function. The main difference of the current model with that of Serfes (2005) is that the latter assumes zero outside option for each agent, whereas outside option is endogenous in the present paper. Besley and Ghatak (2005) identify the matching and outside option effects that determine contracts between principals and agents.

(b) *Suppose that the second-best contracts are implemented in all matches. If wealth is more unequally distributed than land-productivity, then the equilibrium wage function is concave. On the other hand, if the distribution of wealth is too tight relative to that of productivity of the lands, then the wage function is convex.*

A constant wage function implies that the wage differential associated with any two wealth levels is zero for all levels of wealth. In other words, when the first-best contracts are implemented, there is no wage inequality among the agents. The non-trivial case emerges when the second-best contracts are implemented in all matches. In this case the wage function may be non-linear. The above proposition provides sufficient conditions under which the equilibrium wage function is either concave or convex. A concave wage function reduces wage inequality since the wage differential $u'(\omega)$ goes down as the wealth level increases. This situation occurs when the matching function is relatively flat, i.e., the principals are relatively more homogeneous than the agents. A convex wage function, on the other hand, raises the wages of the wealthier agents relative to the less-wealthy ones, and hence increases the wage inequality.

Proposition 4 relates the initial distributions of types to the changes in wage inequality, but does not provide any conclusions regarding the final distribution of wage. A standard result in assignment models (e.g. Sattinger, 1979; Kremer, 1993; Teulings, 1995) is that the distribution of wage is positively skewed relative to the initial distribution of wealth. The same is true in the present context, which is implied by the condition stated in Lemma 1(b).[9]

First, notice that the distribution of wage would have the same shape as the wealth distribution if all agents have chosen to work in the lands with identical productivity. With both-sided heterogeneity, in an equilibrium, wealthier agents are assigned to lands with greater productivity. Therefore, the distribution of equilibrium wages will never resemble the distribution of wealth. Differentiation of (6) with respect to λ gives

$$\frac{\partial u'(\omega)}{\partial \lambda} = \frac{1}{(1-\phi_3)^2}[(1-\phi_3)\phi_{21} + \phi_2\phi_{31}]. \tag{15}$$

Hence, $\phi_{21} > 0$ and $\phi_{31} > 0$, the sufficient conditions for a positively assortative matching, also imply that the above derivative is positive. This implies that a positive sorting enhances the wages of the agents above what they would have earned by working for the principals of the same quality. Therefore, the distribution of equilibrium wages will be positively skewed compared to the distribution of wealth.

4. Conclusions

Incentive contracts may be quite different in a market with many heterogenous principals and agents as opposed to the contracts for an isolated principal-agent partnership. In the equilibrium, individual contracts are influenced by the two-sided heterogeneity via principal-agent matching. In this paper, I have developed a simple assignment model of incentive contracting between principals and agents. Agents who differ in their wealth endowment are assigned to lands differing in productivity. In a Walrasian equilibrium of the market, wealthier agents work in more productive lands since they have comparative advantages in high-quality lands. Optimal tenancy contracts are share contracts when incentive problems

[9] The proof of this assertion is easily adapted from Sattinger (1975) and Teulings (1995), which is presented in Appendix G.

are important in all matches. It is shown that when wealth is more heterogeneously distributed relative to land-productivity, higher output shares for the agents are associated with higher initial wealths, although share is in general non-monotone in wealth. It is also shown that if the distribution of wealth is relatively more disperse than that of land-productivity, then the equilibrium wage function is concave in wealth implying a reduction in income inequality. Moreover, because of positively assortative matching the distribution of equilibrium wage is positively skewed relative to the distribution of initial wealth.

In the present model the first-best contracts may not be implemented due to informational asymmetries. In particular, the market failure stems from the fact that, in the presence of limited liability, less wealthy agents cannot be expected to exert high effort, as they cannot be forced to share losses with the principals in the event of failure. An important assumption in the paper is the fact that the relationship between a principal and an agent lasts only for one period. Possibly, such a relationship usually involves dynamic considerations too, which in turn implies some degree of relaxation on the limited liability constraint, and the conclusions of the current paper may alter. Aghion and Bolton (1997) consider a model of income inequality and analyze the trickle down effects of wealth accumulation. They show that high capital accumulation induces an invariant wealth distribution, and redistribution of wealth from rich to poor enhances the productive efficiency of the economy. Mookherjee and Ray (2002) analyze a dynamic model of equilibrium short period credit contracts assuming that the bargaining power is exogenously distributed between the lenders (principals) and the borrowers (agents). When lenders have all the bargaining power, less wealthy borrowers have no incentive to save and poverty traps emerge. On the other hand, if the borrowers have all the bargaining power, income inequality reduces due to strong incentives for savings. An important difference between the model of Mookherjee and Ray (2002) and that of mine is that in the present model the bargaining power of each agent is endogenously determined via endogenous outside option. Land-quality, on the other hand, may also vary in a dynamic model due to technological changes. Ray (2005) considers a dynamic landlord-tenant relationship where the tenant has to make land-specific investments in order to maintain the land-quality, and shows that share tenancy arises because of this sort of multi-task. Extension of the present model to a dynamic model of principal-agent market, which incorporates the above-mentioned features, would be an interesting research agenda for the future.

Appendix A. Proof of Lemma 1

The first part of the lemma immediately follows from the assumptions on $\phi(\lambda,\, \omega,\, u(\omega))$. Given that ϕ is twice-continuously differentiable, one has $\phi_{ij} = \phi_{ji}$. The second-order condition is given by:

$$\begin{aligned} & \frac{\partial^2 \phi}{\partial \omega^2} = [\phi_{22} + \phi_{32} u'(\omega)] + [\phi_{32} + \phi_{33} u'(\omega)] u'(\omega) + (1 - \phi_3) u''(\omega) \leq 0, \\ \implies \; & [\phi_{22} + \phi_{32} u'(\omega)] + [\phi_{32} + \phi_{33} u'(\omega)] u'(\omega) \leq (1 - \phi_3) u''(\omega). \end{aligned} \tag{A.1}$$

Differentiating (FOC) with respect to ω, one gets

$$u''(\omega) = \frac{1}{(1-\phi_3)^2} \left[(1-\phi_3) \frac{\partial \phi_2}{\partial \omega} + \phi_2 \frac{\partial \phi_3}{\partial \omega} \right] = \frac{1}{1-\phi_3} \left[\frac{\partial \phi_2}{\partial \omega} + u'(\omega) \frac{\partial \phi_3}{\partial \omega} \right].$$

Now,

$$\left.\frac{\partial \phi_2}{\partial \omega}\right|_{\lambda=l(\omega)} = \phi_{21} l'(\omega) + \phi_{22} + \phi_{32} u'(\omega),$$
$$\left.\frac{\partial \phi_3}{\partial \omega}\right|_{\lambda=l(\omega)} = \phi_{31} l'(\omega) + \phi_{32} + \phi_{33} u'(\omega).$$

Therefore, $(1-\phi_3)u''(\omega)$ evaluated at $\lambda = l(\omega)$ is given by:

$$(1-\phi_3)u''(\omega) = [\phi_{21} + \phi_{31}u'(\omega)] + [\phi_{22} + \phi_{32}u'(\omega)] + [\phi_{32} + \phi_{33}u'(\omega)]u'(\omega). \tag{A.2}$$

Substituting for $(1-\phi_3)u''(\omega)$ from the above expression into (A.1), the second-order condition reduces to:

$$[\phi_{21} + \phi_{31}u'(\omega)]l'(\omega) \geq 0. \tag{A.3}$$

From the above inequality, it follows that $l'(\omega) \geq (\leq)\, 0$ if and only if $\phi_{21} + \phi_{31}u'(\omega) \geq (\leq)\, 0$. Therefore, (SOC) is a necessary and sufficient condition for monotone matching. Clearly, $\phi_{21} \geq (\leq)\, 0$ and $\phi_{31} \geq (\leq)\, 0$ are sufficient conditions for monotone assignment. ■

Appendix B. Proof of Lemma 2

For an arbitrary match (λ, ω), the surplus is given by:

$$\phi(\lambda, \omega, u(\omega)) = \lambda e^*(\lambda, \omega, u(\omega)) - \frac{[e^*(\lambda, \omega, u(\omega))]^2}{2}.$$

Substituting for the values of $e^*(\lambda, \omega, u(\omega))$, both under the first- and second-best, from Table 1 one gets

$$\phi(\lambda, \omega, u(\omega)) = \begin{cases} \frac{\lambda^2}{2} & \text{if the first-best contracts are implemented,} \\ \lambda\sqrt{2(\omega + u(\omega))} - \omega - u(\omega) & \text{if the second-best contracts are implemented.} \end{cases}$$

Differentiating the first equation with respect to λ, ω and $u(\omega)$, one gets

$$\phi_1(\lambda, \omega, u(\omega)) = \lambda \in (0, 1),$$
$$\phi_2(\lambda, \omega, u(\omega)) = 0,$$
$$\phi_3(\lambda, \omega, u(\omega)) = 0.$$

Differentiating the second equation with respect to λ, ω and $u(\omega)$, one gets

$$\phi_1(\lambda, \omega, u(\omega)) = \sqrt{2(\omega + u(\omega))} = e^*(\lambda, \omega, u(\omega)) \in (0, 1),$$
$$\phi_2(\lambda, \omega, u(\omega)) = \frac{\lambda}{\sqrt{2(\omega + u(\omega))}} - 1 = \frac{1}{\alpha^*(\lambda, \omega, u(\omega))} - 1 \in (0, 1),$$
$$\phi_3(\lambda, \omega, u(\omega)) = \frac{\lambda}{\sqrt{2(\omega + u(\omega))}} - 1 = \frac{1}{\alpha^*(\lambda, \omega, u(\omega))} - 1 \in (0, 1).$$

The last two expressions lie in $(0,\, 1)$ because $\alpha^*(\lambda,\, \omega,\, u(\omega)) > 1/2$. ■

Appendix C. Proof of Proposition 1

Lemma 2 describes the match-surplus function $\phi(\lambda,\, \omega,\, u(\omega))$ for an arbitrary match $(\lambda,\, \omega)$. When the first-best contracts are implemented, one has $\phi_1(\lambda,\, \omega,\, u(\omega)) = \lambda \in (0,\, 1)$, $\phi_2(\lambda,\, \omega,\, u(\omega)) = 0$ and $\phi_3(\lambda,\, \omega,\, u(\omega)) = 0$. Therefore, $v'(\lambda) = \lambda \in (0,\, 1)$ and $u'(\omega) = 0$. Now consider the second-best contracts. From Lemma 2, one gets

$$\frac{\phi_2(\lambda,\, \omega,\, u(\omega))}{1 - \phi_3(\lambda,\, \omega,\, u(\omega))} = \frac{\lambda - \sqrt{2(\omega + u(\omega))}}{2\sqrt{2(\omega + u(\omega))} - \lambda} = \frac{1 - \alpha^*}{2\alpha^* - 1} > 0$$

since $\alpha^* \in (1/2,\, 1)$. Therefore, $v'(\lambda) \in (0,\, 1)$ and $u'(\omega) > 0$. ■

Appendix D. Proof of Proposition 2

To determine the equilibrium matching pattern one only needs to check the signs of ϕ_{21} and ϕ_{31}, and then apply Lemma 1(b). When the first-best contracts are implemented, from the proof of the previous lemma, one has $\phi_2 = \phi_3 = 0$. Therefore, $\phi_{21} = \phi_{31} = 0$. Hence, $l'(\omega)$ may have any sign in order to satisfy the condition stated in Lemma 1(b). Now consider the second-best contracts. From the proof of Proposition 1 it follows that

$$\phi_{21}(\lambda,\, \omega,\, u(\omega)) = \phi_{31}(\lambda,\, \omega,\, u(\omega)) = \frac{1}{\sqrt{2(\omega + u(\omega))}} > 0.$$

Therefore, $l'(\omega)$ must be positive in order that the second-order condition (SOC) is satisfied. ■

Appendix E. Proof of Proposition 3

Here I consider only the second-best contracts. From Table 1 it follows that

$$\frac{\partial e^*}{\partial \lambda} = 0, \quad \frac{\partial e^*}{\partial \omega} = \frac{\partial e^*}{\partial u(\omega)} = \frac{1}{\sqrt{2(\omega + u(\omega))}} > 0.$$

Therefore,

$$e'(\omega) = \frac{1 + u'(\omega)}{\sqrt{2(\omega + u(\omega))}} > 0.$$

Given that $R(\omega) = \omega$, $R'(\omega) = 1 > 0$. From the expression of α^* in Table 1, one has

$$\frac{\partial \alpha^*}{\partial \lambda} = -\frac{\sqrt{2(\omega + u(\omega))}}{l^2(\omega)} < 0, \quad \frac{\partial \alpha^*}{\partial \omega} = \frac{\partial \alpha^*}{\partial u(\omega)} = \frac{1}{l(\omega)\sqrt{2(\omega + u(\omega))}} > 0.$$

Therefore,

$$\alpha'(\omega) = \frac{\alpha(\omega)}{l(\omega)} \left[\frac{1 + u'(\omega)}{l(\omega)\alpha^2(\omega)} - l'(\omega) \right].$$

Notice that equation (8) implies that

$$\frac{1+u'(\omega)}{l(\omega)\alpha^2(\omega)} = \frac{1}{l(\omega)\alpha(\omega)[2\alpha(\omega)-1]} \geq 1.$$

Therefore,

$$\mathrm{sgn}[\alpha'(\omega)] = \mathrm{sgn}\left[\frac{1}{l(\omega)\alpha(\omega)[2\alpha(\omega)-1]} - \frac{\Delta\,\lambda}{\Delta\,\omega}\right],$$

and the proof follows. ■

Appendix F. Proof of Proposition 4

When the first-best contracts are implemented, the differential equation for wage is given by $u'(\omega) = 0$ which implies that $u(\omega)$ is a constant function. Under the second-best contracts,

$$u'(\omega) = \frac{1-\alpha(\omega)}{2\alpha(\omega)-1}.$$

The above equation implies that

$$u''(\omega) = -\frac{\alpha(\omega)}{[2\alpha(\omega)-1]^2} \quad \Longleftrightarrow \quad \mathrm{sgn}[u''(\omega)] = -\,\mathrm{sgn}[\alpha'(\omega)],$$

and hence the result follows. ■

Appendix G. Skewness of the income distribution

Consider two arbitrary levels of wealth ω_1 and ω_2 with $\omega_2 > \omega_1$, and suppose that, in a Walrasian equilibrium, both types choose to work in the same quality $\bar{\lambda}$ of lands. Since these assignments are optimal one must have

$$u(\omega_2) - u(\omega_1) = \phi(\bar{\lambda},\,\omega_2,\,u(\omega_2)) - \phi(\bar{\lambda},\,\omega_1,\,u(\omega_1)). \tag{G.1}$$

Following Sattinger (1975), it is easy to show that if $\phi_{21} = \phi_{31} = 0$, e.g. under first-best contracts, then the distribution of wage and wealth will have the same shape.[10] Take an arbitrary wealth level $\bar{\omega}$ and let $\bar{\lambda} = l(\bar{\omega})$ be the corresponding land quality. Consider a distribution of wage defined by $u^*(\bar{\omega}) = u(\bar{\omega})$ and

$$u^*(\omega) - u^*(\bar{\omega}) = \phi(\bar{\lambda},\,\omega,\,u^*(\omega)) - \phi(\bar{\lambda},\,\bar{\omega},\,u^*(\bar{\omega})).$$

[10]Notice that $\phi_{21} = \phi_{31} = 0$ implies

$$\frac{\partial u'(\omega)}{\partial\lambda} = \frac{\partial}{\partial\lambda}\left[\frac{\phi_2(\lambda,\,\omega,\,u(\omega))}{1-\phi_3(\lambda,\,\omega,\,u(\omega))}\right] = 0,$$

i.e., $u'(\omega)$ is independent of λ, which is the case under the first-best. Now notice that equation (FOC) implies

$$\frac{d\phi(\bar{\lambda},\,\omega,\,u(\omega))}{d\omega} = \phi_2(\bar{\lambda},\,\omega,\,u(\omega)) + \phi_3(\bar{\lambda},\,\omega,\,u(\omega))u'(\omega) = \frac{\phi_2(\bar{\lambda},\,\omega,\,u(\omega))}{1-\phi_3(\bar{\lambda},\,\omega,\,u(\omega))}.$$

So the distributions of $u^*(\omega)$ and ω have the same shape, and both the wage functions $u(\omega)$ and $u^*(\omega)$ yield the same wage at $\bar{\omega}$. Take an wealth level $\omega_2 > \bar{\omega}$. Positive assortment implies $\lambda(\omega_2) > \bar{\lambda}$. Then

$$\begin{aligned} u(\omega_2) - u(\bar{\omega}) &= \int_{\bar{\omega}}^{\omega_2} \left. \frac{\phi_2(\lambda,\, \omega,\, u(\omega))}{1 - \phi_3(\lambda,\, \omega,\, u(\omega))} \right|_{\lambda = l(\omega)} d\omega \\ &> \int_{\bar{\omega}}^{\omega_2} \frac{\phi_2(\bar{\lambda},\, \omega,\, u(\omega))}{1 - \phi_3(\bar{\lambda},\, \omega,\, u(\omega))} d\omega \\ &= \phi(\bar{\lambda},\, \omega_2,\, u(\omega_2)) - \phi(\bar{\lambda},\, \bar{\omega},\, u(\bar{\omega})) \end{aligned}$$

The above together with $u(\bar{\omega}) = u^*(\bar{\omega})$ imply that

$$\begin{aligned} & u(\omega_2) > u^*(\omega_2) + [\phi(\bar{\lambda},\, \omega_2,\, u(\omega_2)) - \phi(\bar{\lambda},\, \omega_2,\, u^*(\omega_2))] \\ & \qquad = u^*(\omega_2) + \phi_3(\bar{\lambda},\, \omega,\, u(\omega))[u(\omega_2) - u^*(\omega_2)] \\ \iff & \quad [1 - \phi_3(\bar{\lambda},\, \omega,\, u(\omega))][u(\omega_2) - u^*(\omega_2)] > 0 \\ \iff & \quad u(\omega_2) > u^*(\omega_2). \end{aligned}$$

Similarly, for a wealth level $\omega_1 < \bar{\omega}$ it is easy to show that $u(\omega_1) > u^*(\omega_1)$. In a Walrasian equilibrium, low-wealth and high-wealth agents choose to work in lands with low- and high-productivity, respectively instead of lands of intermediate-quality (equal to $\bar{\lambda}$) because their wages are higher with the principals they are matched with. This means that the density of $u(\omega)$ can be obtained from $u^*(\omega)$ by shifting a positive mass from the left tail to the right, i.e., the distribution of $u(\omega)$ is positively skewed relative to the distribution of $u(\omega)$. ■

References

Ackerberg, D. and M. Botticini (2002), "Endogenous Matching and the Empirical Determinants of Contract Form." *Journal of Political Economy*, 110, 564–591.

Aghion, P. and P. Bolton (1997), "A Theory of Trickle-Down Growth and Development." *The Review of Economic Studies*, 64, 151–172.

Banerjee, A. V., P. Gertler, and M. Ghatak (2002), "Empowerment and Efficiency: Tenancy Reform in West Bengal." *Journal of Political Economy*, 110, 239–280.

Basu, K. (1992), "Limited Liability and the Existence of Share Tenancy." *Journal of Development Economics*, 38, 203–220.

Besley, T. and M. Ghatak (2005), "Competition and Incentives with Motivated Agents." *The American Economic Review*, 95, 616–636.

Since the wealth levels ω_1 and ω_2 are arbitrarily chosen, equation (G.1) is equivalent to

$$u'(\omega) = \frac{\phi_2(\bar{\lambda},\, \omega,\, u(\omega))}{1 - \phi_3(\bar{\lambda},\, \omega,\, u(\omega))},$$

i.e., $u'(\omega)$ is independent of λ, and hence the distributions of wage and initial wealth will have the same shape.

Braido, L. (2008), “Evidence on the Incentive Properties of Share Contracts.” *Journal of Law and Economics*, 51, 327–349.

Chakraborty, A. and A. Citanna (2005), “Occupational Choice, Incentives and Wealth Distribution.” *Journal of Economic Theory*, 122, 206–224.

Dam, K. (2010), “Optimal Assignment of CEOs to Firms: Implications for Market Power, Executive Compensation and Managerial Incentive.” Mimeo, Centro de Investigación y Docencia Económicas.

Edmans, A., X. Gabaix, and A. Landier (2009), “A Multiplicative Model of Optimal CEO Incentives in Market Equilibrium.” *The Review of Financial Studies*, 22, 4881–4917.

Eswaran, M. and A. Kotwal (1985), “A Theory of Contractual Structure in Agriculture.” *The American Economic Review*, 75, 352–367.

Ghatak, M. and A. Karaivanov (2010), “Contractual Structure in Agriculture with Endogenous Matching.” Mimeo, London School of Economics.

Ghatak, M. and P. Pandey (2000), “Contract Choice in Agriculture with Joint Moral Hazard in Effort and Risk.” *Journal of Development Economics*, 63, 303–326.

Grossman, S. and O. Hart (1983), “An Analysis of the Principal-Agent Problem.” *Econometrica*, 51, 7–45.

Kremer, M. (1993), “The O-Ring Theory of Economic Development.” *The Quarterly Journal of Economics*, 108, 551–575.

Laffont, J.-J. and M. Matoussi (1995), “Moral Hazard, Financial Constraints and Sharecropping in El Oulja.” *The Review of Economic Studies*, 62, 381–399.

Legros, P. and A. Newman (2007), “Beauty is a Beast, Frog is a Prince: Assortative Matching in a Nontransferable World.” *Econometrica*, 75, 1073–1102.

Mookherjee, D. and D. Ray (2002), “Contractual Structure and Wealth Accumulation.” *The American Economic Review*, 92, 818–849.

Newbery, D. (1977), “Risk Sharing, Sharecropping and Uncertain Labour Markets.” *The Review of Economic Studies*, 44, 585–594.

Rao, C. H. (1971), “Uncertainty, Entrepreneurship, and Sharecropping in India.” *Journal of Political Economy*, 79, 578–595.

Ray, T. (2005), “Sharecropping, Land Exploitation and Land Improving Investments.” *Japanese Economic Review*, 56, 127–143.

Ray, T. and N. Singh (2001), “Limited Liability, Contractual Choice, and the Tenancy Ladder.” *Journal of Development Economics*, 66, 289–303.

Rosen, S. (1974), “Hedonic Prices and Implicit Markets: Product Differentiation in Pure Competition.” *Journal of Political Economy*, 82, 34–55.

Sattinger, M. (1975), "Comparative Advantage and the Distributions of Earnings and Abilities." *Econometrica*, 43, 455–468.

Sattinger, M. (1979), "Differential Rents and the Distribution of Earnings." *Oxford Economic Papers*, 31, 60–71.

Sengupta, K. (1997), "Limited Liability, Moral Hazard and Share Tenancy." *Journal of Development Economics*, 52, 393–407.

Serfes, K. (2005), "Risk Sharing vs. Incentives: Contract Design under Two-Sided Heterogeneity." *Economics Letters*, 88, 343–349.

Shetty, S. (1988), "Limited Liability, Wealth Differences, and the Tenancy Ladder in Agrarian." *Journal of Development Economics*, 29, 1–22.

Stiglitz, J. (1974), "Incentives and Risk Sharing in Sharecropping." *The Review of Economic Studies*, 41, 219–255.

Teulings, C. (1995), "The Wage Distribution in a Model of the Assignment of Skills to Jobs." *Journal of Political Economy*, 103, 280–315.