Abstract

We provide a complete characterization of optimal strategies for both players in non-symmetric discrete General Lotto games, where one of the players has an advantage over the other. By this we complete the characterization given in Hart [2008], where the strategies for symmetric case were fully characterized and some of the optimal strategies for the non-symmetric case were obtained. Our results can be used to solve the remaining cases of Colonel Blotto games.

Discrete General Lotto games – the non-symmetric case

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1 Introduction

The *General Lotto games* are allocation games introduced in Hart [2008] as a generalization and a technical tool for studying *Colonel Blotto games*, a classic example of allocation games, where two players compete on different fronts allocating to them their limited resources. The Blotto games were introduced by Borel [1921] and most variations of the classic games remained unsolved. The allocation games of this type find application in areas such as political economics, all-pay auctions and tournaments. For example, Merolla et al. [2005] argue that this kind of games is suitable to describe the allocation problem facing candidates in U.S. presidential races. They discuss, in particular, how the 2000 US presidential election can been modelled as a Colonel Blotto game. They show in their analysis that Gore could have utilized a strategy that would have won the election, but that such a strategy was not identifiable ex ante.

General Lotto games, apart from being applicable to solve Colonel Blotto games, have wide applications as well. Before we discuss these applications let us briefly introduce the game. A more detailed definition will be given later.

Given $a, b > 0$, the General Lotto game $\Gamma(a, b)$ is defined as follows. There are two players, A and B, who simultaneously chose non-negative random variables X and Y,[1] respectively, with expectations $\mathbf{E}(X) = a$ and $\mathbf{E}(Y) = b$. The payoff in the game is given by

$$H(X, Y) := \mathbf{P}(X > Y) - \mathbf{P}(X < Y). \tag{1}$$

A general Lotto Game $\Gamma(a, b)$ is called *symmetric* if $a = b$ and it is called *non-symmetric* otherwise. Putting additional requirements that X and Y are integer valued leads to *discrete General Lotto games.*

In Myerson [1993] a model of political competition between two and more candidates is studied where each candidate decides how to distribute his campaign promises among the electorate. These promises could be seen, for example, as a promise about how the budget of a candidate will be distributed among the electorate. They are modelled by *offer distributions*, probability distributions over non-negative real numbers. These offers specify a promise of a fraction of the budget to each voter. An interpretation of a distribution F is that probability of a given interval is a fraction of voters for whom a value from this interval is promised. Budget constraints of each candidate is expressed

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[1] Throughout the paper we will identify random variables with their distributions, to simplify the presentation.

as constraints on the average offer per voter that a candidate can promise. The budgets constraints of all the candidates are assumed to be equal and this model results in a multi-player symmetric General Lotto game. It is further extended by Sahuguet and Persico [2006] to a situation where there are only two candidates but with unequal budgets constraints. This leads to a non-symmetric General Lotto game. A different model was studied by Dekel et al. [2008], who investigate *vote buying games* where two parties, alternately, make one of the two possible offers: up-front payments or campaign promises. Two games with these two different types of offers are studied with the assumption of a minimal unit of exchange. The model considered is essentially a sequential variant of General Lotto game.

Sahuguet and Persico [2006] connect the non-symmetric General Lotto games with complete information all-pay auctions, as studied by Baye et al. [1996]. In this kind of auctions equilibria in pure strategies do not exist. The mixed strategies are probability distributions over possible bids. As shown by Sahuguet and Persico [2006], there is a correspondence between the budget constraints in the model of political competition they study and the object valuations in all-pay auctions, as well as between the equilibrium mixed strategies in the auctions game and the political promises game. All pay auctions can be also used to link General Lotto games to tournaments, as shown by Groh et al. [2010]. They study elimination tournaments with heterogeneous contestants, modelling each match as an all pay auction. The task of the players is to decide how the effort should be distributed at each stage.

A symmetric continuous variant of General Lotto games was solved by [Bell and Cover, 1980, Section 2] while Sahuguet and Persico [2006] provided the solution for the non-symmetric case of these games. The latter proof uses a reduction to "all-pay-auctions", thus providing a link between the multi-object auctions and General Lotto games. A significantly easier proof based on first principles was given by Hart [2008].

A discrete variant of General Lotto games was solved by Hart [2008]. However, not for every value of a and b the complete characterization was obtained (although examples of optimal strategies are provided for each case). In this paper we fill in the missing places by providing complete characterization of the optimal strategies in discrete General Lotto games. Such a characterization is useful for two reasons. Firstly, using the optimal strategies for discrete General Lotto games one could try to solve the unsolved variants of the discrete Colonel Blotto games (see Hart [2008] for the cases solved so far). Secondly, as we discussed above, General Lotto games are of interest on their own due to their connection to political economics, multi-object auctions and tournaments. In these applications a continuous case was mostly used, however the discrete case is a natural variant with the minimal unit of exchange.

The paper is structured as follows. In Section 2 we describe the connection with Colonel Blotto game and formally define the discrete General Lotto game. In Section 3 we give the complete characterization of the optimal strategies for the game. In Section 4 we give some concisions and remarks regarding the application of the solutions found to Colonel Blotto games.

2 Discrete General Lotto Game

Before defining the discrete General Lotto Game we give a description of the connection between this game and the Colonel Blotto game discovered by Hart [2008]. The Colonel

Blotto game is a classic example of allocation games, where two players compete on different fronts allocating to them their limited resources (see Borel [1921], Tukey [1949], Shubik [1982]). The game $\mathcal{B}(A, B; K)$ is defined as follows. There are two players A and B having $A \geq 1$ and $B \geq 1$ tokens, respectively, to distribute simultaneously over K urns. Thus a pure strategy of player A is a K-partition, $x = \langle x_1, \ldots, x_K \rangle$, of A, so that $x_1 + \ldots + x_K = A$ and each x_i is a natural number. Similarly, a pure strategy of player B is a K-partition, $y = \langle y_1, \ldots, y_K \rangle$, of B, so that $y_1 + \ldots + y_K = B$ and each y_i is a natural number.

After the tokens are distributed, the payoff of each player is computed as follows. For each urn where a player has a strictly larger number of tokens placed he receives the score 1, while for each urn where a player has a strictly smaller number of tokens placed, he receives the score -1. The score on the tied urns is 0 for each player. The overall payoff is the average of payoffs obtained for all urns, that is, given the strategies x and y of A and B, respectively, it is

$$h_{\mathcal{B}}(x, y) = \frac{1}{K} \sum_{i=1}^{K} \operatorname{sign}(x_i - y_i).$$

The Colonel Blotto is a zero-sum game.

The Colonel Lotto game, denoted by $\mathcal{L}(A, B; K)$, is a symmetrized-across-urns variant of the Colonel Blotto games. That is the urns are indistinguishable and players simultaneously divide their tokens into K groups, which are then randomly paired. Thus, again, the strategies of the players are K-partitions and the payoff of each player is an average over all possible pairings, that is, given the strategies x and y of A and B, respectively, it is

$$h_{\mathcal{L}}(x, y) = \frac{1}{K^2} \sum_{i=1}^{K} \sum_{j=1}^{K} \operatorname{sign}(x_i - y_j).$$

To see the connection between the Colonel Blotto and Colonel Lotto games, given a pure strategy x of player A, let $\sigma(x)$ denote a mixed strategy that assigns equal probability, $\frac{1}{K!}$, to each permutation of x. Similarly, given a mixed strategy ξ of player A, let $\sigma(\xi)$ denote a mixed strategy obtained by replacing each pure strategy x in the support of ξ by $\sigma(x)$. The strategies $\sigma(x)$ and $\sigma(\xi)$ are called *symmetric across urns*. As was observed in Hart [2008], $h_{\mathcal{B}}(\sigma(\xi), y) = h_{\mathcal{L}}(\xi, y)$, for any pure strategy y of player B. Consequently, $h_{\mathcal{B}}(\sigma(\xi), \eta) = h_{\mathcal{L}}(\xi, \eta)$, for any mixed strategy η of player B. Analogously for the strategies of player B. Hence the following observation can be made

Observation 1 (Hart). *The Colonel Blotto game $\mathcal{B}(A, B; K)$ and the Colonel Lotto game $\mathcal{L}(A, B; K)$ have the same value. Moreover, the mapping σ maps the optimal strategies in the Colonel Lotto game onto the optimal strategies in the Colonel Blotto game that are symmetric across urns.*

Having linked the Colonel Blotto and Colonel Lotto games we are ready to define the General Lotto game and see the link between them. Notice that any K-partition $\langle z_1, \ldots, z_K \rangle$ of a natural number C can be seen as a discrete random variable Z with values in the set $\{z_1, \ldots, z_K\}$ and the distribution obtained by assigning to each $z_1, \ldots, z_K$ the probability $\frac{1}{K}$. The expected value of Z is then $\mathbf{E}(Z) = \frac{C}{K}$, which is the average number of tokens per urn. This construction links the pure strategies x and y or players A and B in Colonel Lotto game with discrete integer valued random variables X and Y.

The strategies of players A and B in Colonel Lotto game could be seen as non-negative, integer valued random variables bounded by A and B and having expectations A/K and B/K, respectively. The payoff $h_{\mathcal{L}}(x,y)$ can be then written as

$$h_{\mathcal{L}}(x,y) = H(X,Y) = \mathbf{P}(X > Y) - \mathbf{P}(X < Y).$$

General Lotto game is a generalization of Colonel Lotto game which allows for strategies of the players to be unbounded random variables. The General Lotto game $\Gamma(a,b)$ is a two player game where the pure strategies of players A and B are non-negative, integer valued random variables X and Y with expectations a and b, respectively. Following Hart [2008] we identify random variables with their distributions. Thus every random variable X is $\sum_{i=0}^{\infty} p_i \mathbf{1}_i$, where $p_i = \mathbf{P}(X = i)$ and $\mathbf{1}_i$ denotes Dirac's measure which puts probability 1 on i. Thus the set of strategies of player A is a set $\mathcal{X}$ of all non-negative, integer valued random variables $\mathcal{X}$ such that all $X \in \mathcal{X}$ have the same expectation a (but their distributions are different). Similarly for player B. The payoff of the game is

$$H(X,Y) = \mathbf{P}(X > Y) - \mathbf{P}(X < Y).$$

Notice that every strategy in the Colonel Lotto game $\mathcal{L}(A,B;K)$ is a strategy in the General Lotto game $\Gamma(A/K, B/K; K)$, although the opposite is not necessarily true. However, every optimal strategy in a General Lotto game which is a strategy in the corresponding Colonel Lotto game is an optimal strategy there. Hence one of the approaches to find optimal strategies for Colonel Lotto games (and, further, for Colonel Blotto games) one could find the optimal strategies in General Lotto games and see which of them are the strategies in the aforementioned game. This was partially done in Hart [2008], where, in particular, the symmetric case of $A = B$ was covered. However, most of non-symmetric cases were only partially solved.

In the next section we give the complete characterization of the optimal strategies in General Lotto games.

3 Solution of the Discrete General Lotto Game

All random variables considered from now on are non-negative and integer-valued. As we mentioned above, every random variable X is $\sum_{i=0}^{\infty} p_i \mathbf{1}_i$, where $p_i = \mathbf{P}(X = i)$ and $\mathbf{1}_i$ denotes Dirac's measure which puts probability 1 on i. Also $\mathbf{E}(X) = \sum_i i\mathbf{P}(X = i) = \sum_i \mathbf{P}(X \geq i)$. Expected payoff of player A from using strategy X against strategy Y of player B is:

$$\begin{aligned} H(X,Y) &= \sum_{i=0}^{\infty} p_i[\mathbf{P}(i > Y) - \mathbf{P}(i < Y)] = \sum_{i=0}^{\infty} p_i[1 - \mathbf{P}(Y \geq i)] - \mathbf{P}(Y \geq i+1)] \\ &= 1 - \sum_{i=0}^{\infty} p_i[\mathbf{P}(Y \geq i) + \mathbf{P}(Y \geq i+1)]. \end{aligned}$$

Notice that H satisfies the following properties:

$$H(X,Y) = -H(Y,X), \tag{2}$$

$$H(\alpha X_1 + \beta X_2, Y) = \alpha H(X_1, Y) + \beta H(X_2, Y). \tag{3}$$

The following two distributions where crucial for players strategies discovered in Hart [2008]:

$$U_{\mathrm{O}}^{m} := U(\{1, 3, \ldots, 2m-1\}) = \sum_{i=1}^{m} \left(\frac{1}{m}\right) \mathbf{1}_{2i-1}, \text{ and}$$
$$U_{\mathrm{E}}^{m} := U(\{0, 2, \ldots, 2m\}) = \sum_{i=1}^{m+1} \left(\frac{1}{m+1}\right) \mathbf{1}_{2i}.$$

Distributions U_{O}^{m} and U_{E}^{m} can be thought of as "uniform on odd numbers" and "uniform on even numbers", respectively. We will use

$$\mathcal{U}^{m} = \{U_{\mathrm{E}}^{m}, U_{\mathrm{O}}^{m}\}$$

to denote the set of these distributions. We will also $\vec{u}_{\mathrm{O}}^{m}$ and $\vec{u}_{\mathrm{E}}^{m}$ to denote stochastic vectors representing these distributions.

As was shown in Hart [2008], for every Y it holds that

$$H(U_{\mathrm{O}}^{m}, Y) = 1 - \left(\frac{1}{m}\right) \sum_{i=1}^{2m} \mathbf{P}(Y \geq i) \geq 1 - \frac{\mathbf{E}(Y)}{m}, \tag{4}$$

with equality if and only if $\sum_{j=2m+1}^{+\infty} \mathbf{P}(Y \geq j) = 0$ or, in other words, $Y \leq 2m$. For every Y it also holds that

$$H(U_{\mathrm{E}}^{m}, Y) = 1 - \left(\frac{1}{m+1}\right) \left(1 + \sum_{i=1}^{2m+1} \mathbf{P}(Y \geq i)\right) \geq 1 - \frac{\mathbf{E}(Y)+1}{m+1}, \tag{5}$$

with equality if and only if $\sum_{j=2m+2}^{+\infty} \mathbf{P}(Y \geq j) = 0$ or, in other words, $Y \leq 2m+1$.

We extend this repertoire with the following distributions: W_j^m (with $1 \leq j \leq m-1$), defined for $m \geq 2$, and V_j^m (with $1 \leq j \leq m$) defined for $m \geq 1$, represented by stochastic vectors:

$$\vec{w}_j^{m} := \frac{1}{2m}[1, \underbrace{0, 2, \ldots, 0, 2}_{2(j-1)}, 0, 1, \underbrace{2, 0, \ldots, 2, 0}_{2(m-j)}]^T,$$
$$\vec{v}_j^{m} := \frac{1}{2m+1}[\underbrace{0, 2, \ldots, 0, 2}_{2(j-1)}, 0, 1, 2\, \underbrace{0, 2, \ldots, 0, 2}_{2(m-j)}]^T.$$

Distribution W_j^m could be thought of as distribution U_{O}^{m} distorted at the first $2j+1$ positions with a sort of 2-moving average, so that $\mathbf{P}(W_j^m = i) = (\mathbf{P}(U_{\mathrm{O}}^{j} = i-1) + \mathbf{P}(U_{\mathrm{O}}^{j} = i+1))/2$, for $0 \leq i \leq 2j$ (where $\mathbf{P}(U_{\mathrm{O}}^{j} = -1) = 0$). Similarly, distribution V_j^m could be thought of as distribution U_{E}^{m} distorted at the first $2j$ positions with a sort of 2-moving average, so that $\mathbf{P}(V_j^m = i) = (\mathbf{P}(U_{\mathrm{E}}^{j-1} = i-1) + \mathbf{P}(U_{\mathrm{E}}^{j-1} = i+1))/2$, for $0 \leq i \leq 2j-1$ (where $\mathbf{P}(U_{\mathrm{E}}^{j-1} = -1) = 0$). It could be also thought of as distribution W_j^{m+1} 'shifted to the left' by one position.

We will also use

$$\mathcal{W}^{m} = \{W_1^m, \ldots, W_{m-1}^m\}$$

to denote the set of distributions W_j^m, as well as

$$\mathcal{V}^m = \{V_1^m, \dots, V_m^m\}$$

to denote the set of distributions V_j^m. These sets are defined for $m \geq 0$. In the case of $m < 2$ we assume that $\mathcal{W}^m = \varnothing$. Similarly, in the case of $m < 1$ we assume that $\mathcal{V}^m = \varnothing$.

Additionally, will consider the following distribution, defined for $m \geq 1$:

$$U_{\mathrm{O}\uparrow 1}^m := U(\{2, 4, \dots, 2m-2\}) = \sum_{i=1}^{m-1} \left(\frac{1}{m-1}\right) \mathbf{1}_{2i},$$

which could be thought of as uniform on even numbers from 2 to $2m-2$, or as the distribution U_{O}^{m-1} 'shifted to the right' by one position. We will also use $\vec{u}_{\mathrm{O}\uparrow 1}^m$ to denote stochastic vector associated with this distribution.

Let $p_i = \mathbf{P}(Y = i)$. For every Y it holds that

$$\begin{aligned} H(W_j^m, Y) = 1 - \left(\frac{1}{2m}\right) \Bigg(& \mathbf{P}(Y \geq 0) + \mathbf{P}(Y \geq 1) + 2 \sum_{i=1}^{j-1} [\mathbf{P}(Y \geq 2i) + \mathbf{P}(Y \geq 2i+1)] \\ & + \mathbf{P}(Y \geq 2j) + \mathbf{P}(Y \geq 2j+1) + 2 \sum_{i=j+1}^{m} [\mathbf{P}(Y \geq 2i-1) + \mathbf{P}(Y \geq 2i)] \Bigg) \\ = 1 - \left(\frac{1}{2m}\right) & \left(p_0 - p_{2j} + 2 \sum_{i=1}^{2m} \mathbf{P}(Y \geq i)\right) \geq 1 - \frac{\mathbf{E}(Y)}{m} + \frac{p_{2j} - p_0}{2m} \end{aligned} \tag{6}$$

with equality if and only if $\sum_{j=2m+1}^{+\infty} \mathbf{P}(Y \geq j) = 0$ or, in other words, $Y \leq 2m$.

$$\begin{aligned} H(V_j^m, Y) = 1 - \left(\frac{1}{2m+1}\right) \Bigg(& 2 \sum_{i=1}^{j-1} [\mathbf{P}(Y \geq 2i-1) + \mathbf{P}(Y \geq 2i)] \\ & + \mathbf{P}(Y \geq 2j-1) + \mathbf{P}(Y \geq 2j) + 2 \sum_{i=j}^{m} [\mathbf{P}(Y \geq 2i) + \mathbf{P}(Y \geq 2i+1)] \Bigg) \\ = 1 - \left(\frac{1}{2m+1}\right) & \left(-p_{2j-1} + 2 \sum_{i=1}^{2m+1} \mathbf{P}(Y \geq i)\right) \geq 1 - \frac{2\mathbf{E}(Y)}{2m+1} + \frac{p_{2j-1}}{2m+1} \end{aligned} \tag{7}$$

with equality if and only if $\sum_{j=2m+2}^{+\infty} \mathbf{P}(Y \geq j) = 0$ or, in other words, $Y \leq 2m+1$.

$$\begin{aligned} H(U_{\mathrm{O}\uparrow 1}^m, Y) &= 1 - \left(\frac{1}{m-1}\right) \left(\sum_{i=1}^{m-1} [\mathbf{P}(Y \geq 2i) + \mathbf{P}(Y \geq 2i+1)]\right) \\ &= 1 - \left(\frac{1}{m-1}\right) \left(-\mathbf{P}(Y \geq 1) + \sum_{i=1}^{2m-1} \mathbf{P}(Y \geq i)\right) \\ &= 1 - \left(\frac{1}{m-1}\right) \left(p_0 - 1 + \sum_{i=1}^{2m-1} \mathbf{P}(Y \geq i)\right) \geq 1 - \frac{\mathbf{E}(Y) - 1}{m-1} - \frac{p_0}{m-1} \end{aligned} \tag{8}$$

with equality if and only if $\sum_{j=2m}^{+\infty} \mathbf{P}(Y \geq j) = 0$ or, in other words, $Y \leq 2m - 1$.

Before starting the analysis, we introduce some additional notation that will be used. Given a distribution X, a set of distributions $\mathcal{Y}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, we will use

$$\lambda_1 X + \lambda_2 \mathcal{Y} = \{\lambda_1 X + \lambda_2 Y : Y \in \mathcal{Y}\}$$

to denote the set of distributions that can be obtained by linearly combining X and the distributions from $\mathcal{Y}$ with coefficients λ_1 and λ_2, respectively. Similarly, given two sets of distributions $\mathcal{X}$ and $\mathcal{Y}$ as well as $\lambda_1, \lambda_2 \in \mathbb{R}$, we will use

$$\lambda_1 \mathcal{X} + \lambda_2 \mathcal{Y} = \{\lambda_1 X + \lambda_2 Y : X \in \mathcal{X} \text{ and } Y \in \mathcal{Y}\}$$

to denote the set of distributions that can be obtained by linearly combining distributions from $\mathcal{X}$ and $\mathcal{Y}$ with coefficients λ_1 and λ_2, respectively. Given a set of distributions $\mathcal{X}$ we will use $\operatorname{conv}(\mathcal{X})$ to denote the set of all convex combinations of distributions from $\mathcal{X}$.

The following cases where left incomplete in Hart [2008]:

- a is an integer and $b < a$,
- a is not an integer and $b = \lfloor a \rfloor$,
- a is not an integer and $b < \lfloor a \rfloor$.

We start the analysis with the first case, where a is an integer and $b < a$. The following theorem characterizing this case was shown in Hart [2008].

Theorem 1 (Hart)**.** *Let $a > b > 0$, where a is an integer. Then the value of General Lotto game $\Gamma(a, b)$ is*

$$\operatorname{val}\Gamma(a, b) = \frac{a - b}{a} = 1 - \frac{b}{a}.$$

The optimal strategies are as follows:

(i). Strategy U_{O}^{a} is the unique optimal strategy of Player A.

(ii). The strategies $(1 - b/a)\mathbf{1}_0 + (b/a)Z$ with $Z \in \operatorname{conv}(\mathcal{U}^a)$ are optimal strategies of Player B.

(iii). Every optimal strategy Y of Player B *satisfies $Y \leq 2a$ and*

$$1 - \frac{a}{b} \leq \mathbf{P}(Y = 0) \leq 1 - \frac{b}{a + 1}.$$

What is missing in this case is the complete characterization of the optimal strategies for the disadvantaged player B. We characterize them in two theorems covering the case where $b \leq a - 1$ first and then the case where $a - 1 < b < a$.

Theorem 2. *Let $a - 1 \geq b > 0$, where a is an integer. The strategy Y is optimal for Player* B *if and only if*

$$Y = \left(1 - \frac{b}{a}\right)\mathbf{1}_0 + \left(\frac{b}{a}\right)Z, \text{ with } Z \in \operatorname{conv}\big(\mathcal{U}^a \cup \mathcal{W}^a \cup \{U_{\mathrm{O}\uparrow 1}^{a}\}\big).$$

Proof. Suppose that Y is an optimal strategy for Player B. By Theorem 1 we have $\mathbf{P}(Y = 0) \geq 1 - (a/b)$, hence Y can be written as

$$Y = \left(1 - \frac{b}{a}\right) \mathbf{1}_0 + \left(\frac{b}{a}\right) Z.$$

Since Y is optimal and, by Theorem 1, $\operatorname{val} \Gamma(a,b) = 1 - b/a$ so for any X with $\mathbf{E}(X) = a$ it must hold that

$$\begin{aligned} 1 - \frac{b}{a} \geq H(X,Y) &= \left(1 - \frac{b}{a}\right) H(X, \mathbf{1}_0) + \left(\frac{b}{a}\right) H(X,Z) \\ &= \left(1 - \frac{b}{a}\right) \mathbf{P}(X > 0) + \left(\frac{b}{a}\right) H(X,Z). \end{aligned} \tag{9}$$

Thus Z is optimal (i.e. such that Y is optimal) if and only if

$$H(Z,X) \geq -\left(\frac{a-b}{b}\right)(1 - \mathbf{P}(X > 0)) = -\left(\frac{a-b}{b}\right) p_0, \tag{10}$$

where $p_0 = \mathbf{P}(X = 0)$.

Consider distributions $T^a_{i,j} = \lambda \mathbf{1}_i + (1-\lambda)\mathbf{1}_j$ with $\mathbf{E}(T^a_{i,j}) = a$ (i.e. with $\lambda = (j-a)/(j-i)$). Take any $T^a_{i,j}$ with $0 < i \leq a \leq j$. For optimal Z, from (10), we have:

$$H(Z, T^a_{i,j}) = \lambda H(Z, \mathbf{1}_i) + (1-\lambda) H(Z, \mathbf{1}_j) \geq 0.$$

Let $w_i = H(Z, \mathbf{1}_i)$. Then we have

$$(j-a)w_i + (a-i)w_j \geq 0. \tag{11}$$

Since U^a_{O} is non-zero for all positive and odd $j \leq 2a-1$, so for any odd i and j such that $1 \leq i \leq a \leq j \leq 2a-1$ we have $U^a_{\mathrm{O}} = \tau T^a_{i,j} + (1-\tau)W$ for some $0 < \tau < 1$ and $W \geq 0$ with $\mathbf{E}(W) = a$. Since U^a_{O} is optimal and $\mathbf{P}(U^a_{\mathrm{O}} = 0) = 0$, so for optimal Z it must be that $H(Z, U^a_{\mathrm{O}}) = 0$. Thus $\tau H(Z, T^a_{i,j}) + (1-\tau) H(Z,W) = 0$ and since, by optimality of Z, $H(Z, T^a_{i,j}) \geq 0$ and $H(Z,W) \geq 0$, so $H(Z, T^a_{i,j}) = 0$. Hence for i and j odd and such that $1 \leq i \leq a \leq j \leq 2a-1$, (11) becomes equality.

Suppose that a is even. Taking $i = a-1$ from (11) we get $w_j \geq (a-j)w_{a-1}$ (with equality for positive and odd $j \leq 2a-1$). In particular, this yields $w_{a-1} = -w_{a+1}$. On the other hand, taking $j = a+1$ from (11) we get $w_i \geq (i-a)w_{a+1}$ and, further, $w_i \geq (a-i)w_{a-1}$. Hence for all $i > 0$ it holds that $w_i \geq (a-i)w_{a-1}$ (with equality for positive and odd $i \leq 2a-1$). For odd $1 \leq i \leq 2a-1$ this implies

$$w_i - w_{i+1} \leq w_{a-1}. \tag{12}$$

On the other hand, for even $2 \leq i \leq 2a-2$ this implies

$$w_i - w_{i+1} \geq w_{a-1}. \tag{13}$$

Let $q_i = \mathbf{P}(Z = i)$. Then $w_i - w_{i+1} = q_i + q_{i+1}$ and, from $(12-13)$ we get $q_i + q_{i+1} \leq w_{a-1}$ (for all odd $1 \leq i \leq 2a-1$) and $q_i + q_{i+1} \geq w_{a-1}$ (for all even $2 \leq i \leq 2a-2$). Hence for all odd $1 \leq i \leq 2a-3$ we have $q_i + q_{i+1} \leq q_{i+1} + q_{i+2}$ and for all even $2 \leq i \leq 2a-2$ we have $q_i + q_{i+1} \geq q_{i+1} + q_{i+2}$. Thus there exist $d_i \geq 0$ (with $1 \leq i \leq 2a-2$) such that

$$q_i - q_{i+2} + d_i = 0, \text{for odd } 1 \leq i \leq 2a-2 \tag{14}$$

$$-q_i + q_{i+2} + d_i = 0, \text{for even } 1 \leq i \leq 2a-2. \tag{15}$$

In the case of odd $1 \le i \le 2a-1$, (11) becomes equality and it yields $w_i = (a-i)w_{a-1}$. Thus $w_i - w_{i+2} = 2w_{a-1}$ (for odd $1 \le i \le 2a-3$) and so $w_i - w_{i+2} = w_{i+2} - w_{i+4}$ (for odd $1 \le i \le 2a-5$). Since $w_i - w_{i+2} = q_i + 2q_{i+1} + q_{i+2}$, so this implies

$$q_i + 2q_{i+1} - 2q_{i+3} - q_{i+4} = 0, \text{for odd } 1 \le i \le 2a-5. \tag{16}$$

Moreover, since $w_{2a-1} - w_{2a+1} \le 2w_{a-1}$ (as $w_{2a-1} = -(a-1)w_{a-1}$ and $w_{2a+1} \ge -(a+1)w_{a-1}$), so in the case of $i = 2a-3$ we have inequality $w_{2a-3} - w_{2a-1} \ge w_{2a-1} - w_{2a+1}$. Thus there exist $d_{2a-1} \ge 0$ such that

$$q_{2a-3} + 2q_{2a-2} - 2q_{2a} - d_{2a-1} = 0 \tag{17}$$

(recall that, by Theorem 1, $q_{2a+1} = 0$). Equations (14 – 17) can be obtained for even a as well, taking $i = a-2$, $j = a-2$ and noticing that $w_{a-2} = w_{a+2}$.

Observe also that since $\sum_{i=0}^{\infty} q_i = 1$ and $\sum_{i=0}^{\infty} iq_i = a$, so $\sum_{i=0}^{\infty}(i-a)q_i = 0$. Since $Z \le 2a$, so in this case

$$\sum_{i=0}^{2a}(i-a)q_i = 0. \tag{18}$$

Lemma 1. *The set of solutions of the system of Equations (14 – 18) with additional constraints:*

$$0 \le q_i \le 1, \text{ for all } 0 \le i \le 2a, \tag{19}$$

$$d_i \ge 0, \text{ for all } 1 \le i \le 2a-1, \tag{20}$$

$$q_0 + \ldots + q_{2a} = 1, \tag{21}$$

is $\operatorname{conv}\left(\left\{\begin{bmatrix} \vec{z}_{-1} \\ \vec{d}_{-1} \end{bmatrix}, \ldots, \begin{bmatrix} \vec{z}_a \\ \vec{d}_a \end{bmatrix}\right\}\right)$, *where*

$$\vec{z}_{-1} = \vec{u}^a_{\mathrm{O}}, \quad \vec{z}_0 = \vec{u}^a_{\mathrm{E}}, \quad \vec{z}_i = \vec{w}^a_i, \text{ for } 1 \le i \le a-2,$$
$$\vec{z}_{a-1} = \frac{2a}{a+1}\vec{w}^a_{a-1} - \frac{a-1}{a+1}\vec{u}^a_{\mathrm{O}\uparrow 1}, \quad \vec{z}_a = \vec{u}^a_{\mathrm{O}\uparrow 1},$$

and $\vec{d_i}$, $-1 \le i \le a$, *satisfy Constraints (20).*

(Proof of Lemma 1 is moved to the Appendix).

If Z is optimal, then it must satisfy Equations (14 – 18) with Constraints (19 – 21). Hence, by Lemma 1, it must be that

$$Z = \lambda_{\mathrm{O}} U^a_{\mathrm{O}} + \lambda_{\mathrm{E}} U^a_{\mathrm{E}} + \sum_{i=1}^{a-2} \lambda_i W^a_i + \lambda_{a-1}\left(\frac{2a}{a+1}W^a_{a-1} - \frac{a-1}{a+1}U^a_{\mathrm{O}\uparrow 1}\right) + \lambda_{\mathrm{O}\uparrow 1} U^a_{\mathrm{O}\uparrow 1}, \tag{22}$$

with $\lambda_{\mathrm{O}} + \lambda_{\mathrm{E}} + \sum_{i=1}^{a-1} \lambda_i + \lambda_{\mathrm{O}\uparrow 1} = 1$ and $\lambda_{\mathrm{O}}, \lambda_{\mathrm{E}}, \lambda_{\mathrm{O}\uparrow 1}, \lambda_i \ge 0$, for all $1 \le i \le a-1$. Consider any distribution $T^a_{i,2a}$ with $1 \le i < a$. By (3), (4 – 8) and the fact that $\mathbf{E}(T^a_{i,2a}) = a$ we have

$$H(Z, T^a_{i,2a}) = -\lambda_{a-1}\frac{a-1}{a+1}\frac{p_{2a}}{a-1} + \lambda_{\mathrm{O}\uparrow 1}\frac{p_{2a}}{a-1} = -\left(\frac{\lambda_{a-1}}{a+1} - \frac{\lambda_{\mathrm{O}\uparrow 1}}{a-1}\right)p_{2a}, \tag{23}$$

where $p_{2a} = \mathbf{P}(T^a_{i,2a} = 2a) > 0$. On the other hand, by (10), it must be that $H(Z, T^a_{i,2a}) \geq 0$. Thus it must be that $\lambda_{\mathrm{O}\uparrow 1} \geq \frac{a-1}{a+1}\lambda_{a-1}$. Hence any optimal Z can be represented as

$$Z = \lambda_{\mathrm{O}} U^a_{\mathrm{O}} + \lambda_{\mathrm{E}} U^a_{\mathrm{E}} + \sum_{i=1}^{a-1} \lambda_i W^a_i + \lambda'_{\mathrm{O}\uparrow 1} U^a_{\mathrm{O}\uparrow 1},$$

where $\lambda'_{\mathrm{O}\uparrow 1} = \lambda_{\mathrm{O}\uparrow 1} - \frac{a-1}{a+1}\lambda_{a-1} \geq 0$ and $\lambda_{\mathrm{O}} + \lambda_{\mathrm{E}} + \sum_{i=1}^{a-1} \lambda_i + \lambda'_{\mathrm{O}\uparrow 1} = 1$. Therefore $Z \in \mathrm{conv}\Big(\mathcal{U}^a \cup \mathcal{W}^a \cup \{U^a_{\mathrm{O}\uparrow 1}\}\Big)$.

On the other hand it can be easily checked that for any $Z \in \mathcal{U}^a \cup \mathcal{W}^a \cup \{U^a_{\mathrm{O}\uparrow 1}\}$, $H(Z, X) \geq -\left(\frac{a-b}{b}\right) p_0$, if $b \leq a-1$. Hence $H(Z, X) \geq -\left(\frac{a-b}{b}\right) p_0$, for any $Z \in \mathrm{conv}\Big(\mathcal{U}^a \cup \mathcal{W}^a \cup \{U^a_{\mathrm{O}\uparrow 1}\}\Big)$. Thus Z satisfies (10), which implies that Z is optimal (i.e. such that Y is optimal). □

In the case of $b < a$ with integer a and b close to a the structure of optimal strategies for B is like in the case of $b \leq a-1$, but not every Z from Theorem 2 leads to an optimal strategy. Theorem 3 below characterize completely all the Z that do do, thus providing complete characterization of optimal strategies for B in this case as well.

Theorem 3. *Let $a = m$ and $b = m - \beta$, where $m \geq 1$ is an integer and $0 < \beta < 1$. The strategy Y is optimal for Player* B *if and only if*

$$Y = \left(\frac{\beta}{m}\right) \mathbf{1}_0 + \left(1 - \frac{\beta}{m}\right) Z, \text{ with } Z \in \mathrm{conv}\Big(\mathcal{U}^m \cup \mathcal{Y}^{m,\beta}\Big), \text{ where}$$

- $\mathcal{Y}^{m,\beta} = \varnothing$, *if* $m = 1$,
- $\mathcal{Y}^{m,\beta} = (\beta\sigma\mathcal{W}^m + (1-\beta\sigma)\mathcal{U}^m) \cup \Big(\beta\delta U^m_{\mathrm{O}\uparrow 1} + (1-\beta\delta)\mathcal{U}^m\Big)$, *if* $m \geq 2$ *and* $0 < \beta \leq \frac{m}{2m+1}$,
- $\mathcal{Y}^{m,\beta} = \mathcal{W}^m \cup \Big(\beta\delta U^m_{\mathrm{O}\uparrow 1} + (1-\beta\delta)\mathcal{U}^m\Big) \cup \Big((1-(1-\beta)\sigma\varrho)\, U^m_{\mathrm{O}\uparrow 1} + (1-\beta)\sigma\varrho\mathcal{W}^m\Big)$, *if* $m \geq 2$ *and* $\frac{m}{2m+1} < \beta < 1$, *where*

$$\delta = \frac{m-1}{m-\beta}, \quad \sigma = \frac{2m}{m-\beta}, \quad \varrho = \frac{m}{m+1}.$$

Proof. It is easy to check that for any $Z \in \mathcal{U}^m \cup \mathcal{Y}^{m,\beta}$ and any X with $\mathbf{E}(X) = m$, $H(Z, X) \geq -\left(\frac{a-b}{b}\right) p_0$, in the two cases given above. Hence $H(Z, X) \geq -\left(\frac{a-b}{b}\right) p_0$, for any $\vec{z} \in \mathrm{conv}\big(\mathcal{U}^m \cup \mathcal{Y}^{m,\beta}\big)$. Thus Z satisfies Inequality (10), which, as we observed in proof of Theorem 2, means that Z is optimal (i.e. such that Y is optimal).

What remains to be shown is the left to right implication, i.e. that if Z is optimal, then $Z \in \mathrm{conv}\big(\mathcal{U}^m \cup \mathcal{Y}^{m,\beta}\big)$. Consider the case with $m = 1$ first. By Theorem 1, $Z \leq 2$ in this case and we need to find the values of q_0, q_1 and q_2, where $q_i = \mathbf{P}(Z = i)$. From $q_0 + q_1 + q_2 = 1$ and $q_1 + 2q_2 = 1$ (as $\mathbf{E}(Z) = 2$), we get $q_0 = q_2$. Hence any Z must be a convex combination of $\mathcal{U}^1$, which completes the proof of this case.

Suppose now that $m \geq 2$. As was already shown in proof of Theorem 2, if Z is optimal, then it must be that $Z \in \mathrm{conv}\Big(\mathcal{U}^m \cup \mathcal{W}^m \cup \{U^m_{\mathrm{O}\uparrow 1}\}\Big)$. Thus any optimal Z can be represented as

$$Z = \lambda_{\mathcal{U}} U^m + \lambda_{\mathcal{W}} W^m + \lambda_{\mathrm{O}\uparrow 1} U^m_{\mathrm{O}\uparrow 1}, \tag{24}$$

where $U^m \in \mathrm{conv}(\mathcal{U}^m)$, $W^m \in \mathrm{conv}(\mathcal{W}^m)$, $\lambda_{\mathcal{U}}+\lambda_{\mathcal{W}}+\lambda_{\mathrm{O}\uparrow 1} = 1$ and $0 \le \lambda_{\mathcal{U}}, \lambda_{\mathcal{W}}, \lambda_{\mathrm{O}\uparrow 1} \le 1$. Consider a strategy $T^m_{0,j}$ (as defined in proof of Theorem 2), with odd j such that $m+1 \le j \le 2m-1$. Then, by $(4-6)$ and (8):

$$H(Z, T^m_{0,j}) = -p_0\left(\frac{\lambda_{\mathcal{W}}}{2m}\right) - p_0\left(\frac{\lambda_{\mathrm{O}\uparrow 1}}{m-1}\right). \tag{25}$$

Since Z is optimal, so it must satisfy (10) for any X with $\mathbf{E}(X) = 1$. This, together with (25), implies

$$\lambda_{\mathcal{W}}\frac{\delta}{\sigma} + \lambda_{\mathrm{O}\uparrow 1} \le \beta\delta. \tag{26}$$

where $\delta = \frac{m-1}{m-\beta}$ and $\sigma = \frac{2m}{m-\beta}$.

Suppose that $0 < \beta \le \frac{m}{2m+1}$ (in which case $0 \le \beta\sigma \le 1$). Inequality (26) implies that $\lambda_{\mathcal{W}} \le \beta\sigma$ and $\lambda_{\mathrm{O}\uparrow 1} \le \beta\delta$. Hence $\lambda_{\mathcal{W}}$ and $\lambda_{\mathrm{O}\uparrow 1}$ can be represented as $\alpha_1\beta\sigma$ and $\alpha_2\beta\delta$, respectively, with $0 \le \alpha_1, \alpha_2 \le 1$. From this and from (26) we also get $\alpha_1 + \alpha_2 \le 1$. Now, equation (24) can be rewritten as:

$$Z = \alpha_3 U^m + \alpha_1((1-\beta\sigma)U^m + \beta\sigma W^m) + \alpha_2((1-\beta\delta)U^m + \beta\delta U^m_{\mathrm{O}\uparrow 1}),$$

where $\alpha_3 = \lambda_{\mathcal{U}} - \alpha_1(1-\beta\sigma) - \alpha_2(1-\beta\delta) = \lambda_{\mathcal{U}} + \lambda_{\mathcal{W}} + \lambda_{\mathrm{O}\uparrow 1} - (\alpha_1+\alpha_2) = 1-(\alpha_1+\alpha_2)$. This shows that any optimal Z can be represented as a convex combination of vectors in $\mathcal{U}^m \cup (\beta\sigma\mathcal{W}^m + (1-\beta\sigma)\,\mathcal{U}^m) \cup \left(\beta\delta U^m_{\mathrm{O}\uparrow 1} + (1-\beta\delta)\,\mathcal{U}^m\right)$.

Suppose that $\frac{m}{2m+1} < \beta < 1$ (in which case $0 < (1-\beta)\sigma\varrho < 1$). By (26), $\lambda_{\mathrm{O}\uparrow 1} \le \beta\delta - \lambda_{\mathcal{W}}\frac{\delta}{\sigma}$. Hence $\lambda_{\mathrm{O}\uparrow 1}$ can be represented as $\alpha\left(\beta\delta - \lambda_{\mathcal{W}}\frac{\delta}{\sigma}\right)$, where $0 \le \alpha \le 1$. We rewrite Equation (24) as

$$\begin{aligned} Z = \alpha_1 U^m + \alpha_2 W^m + \alpha_3\left((1-\beta\delta)\,U^m + \beta\delta U^m_{\mathrm{O}\uparrow 1}\right) + \\ \alpha_4\left((1-\beta)\,\sigma\varrho W^m + (1-(1-\beta)\,\sigma\varrho)\,U^m_{\mathrm{O}\uparrow 1}\right) \end{aligned}$$

$\alpha_1 = \lambda_{\mathcal{U}} - \alpha_3(1-\beta\delta)$, $\alpha_2 = \lambda_{\mathcal{W}} - \alpha_4(1-\beta)\sigma\varrho = \lambda_{\mathcal{W}}(1-\alpha)$, $\alpha_3 = \alpha\frac{1-\beta\delta-\lambda_{\mathcal{W}}(\sigma-\delta)/\sigma}{1-\beta\delta}$ and $\alpha_4 = \lambda_{\mathcal{W}}\alpha\frac{\delta(1-\beta)(\beta\sigma-1)}{(1-\beta\delta)(1-(1-\beta)\sigma\varrho)} = \lambda_{\mathcal{W}}\alpha\frac{(m-\beta)(m+1)}{2m^2(1-\beta)}$. It is easy to check that $\alpha_3\beta\delta + \alpha_4(1-(1-\beta)\sigma\varrho) = \lambda_{\mathrm{O}\uparrow 1}$ and, consequently, that $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 1$. It is also easy to see that $\alpha_2, \alpha_4 \ge 0$. By the fact that $\beta > \frac{m}{2m+1}$ we also have $\alpha_3 \ge 0$. For α_1 notice that adding $\frac{\sigma-\delta}{\sigma}\lambda_{\mathcal{W}}$ to both sides of (26) and using the fact that $\lambda_{\mathcal{W}} + \lambda_{\mathrm{O}\uparrow 1} = 1 - \lambda_{\mathcal{U}}$, from (26) we get

$$\lambda_{\mathcal{U}} \ge 1 - \beta\delta - \lambda_{\mathcal{W}}\frac{\sigma-\delta}{\sigma}. \tag{27}$$

From this it follows that $\alpha_1 \ge 0$. This shows that if $\frac{m}{2m+1} < \beta < 1$, then any optimal Z can be represented as a convex combination of vectors in $\mathcal{U}^m \cup \mathcal{W}^m \cup \left(\beta\delta U^m_{\mathrm{O}\uparrow 1} + (1-\beta\delta)\,\mathcal{U}^m\right) \cup \left((1-(1-\beta)\sigma\varrho)\,U^m_{\mathrm{O}\uparrow 1} + (1-\beta)\sigma\varrho\mathcal{W}^m\right)$. $\square$

As an example consider a game $\Gamma(4,1)$. Then the value of the game is $3/4$ and the strategy $Y^0 = (25/32)\mathbf{1}_0 + (1/16)\mathbf{1}_2 + (1/32)\mathbf{1}_4 + (1/16)\mathbf{1}_5 + (1/16)\mathbf{1}_7$, given as an example in Hart [2008] of one of the strategies not captured by Theorem 1, is $(3/4)\mathbf{1}_0 + (1/4)W^4_2$.

Now we move to the case of $b \le \lfloor a \rfloor$. The following theorem characterizing this case was shown in Hart [2008].

Theorem 4 (Hart)**.** *Let $a = m+\alpha$ and $b \le m$, where $m \ge 1$ is an integer and $0 < \alpha < 1$. Then the value of General Lotto game $\Gamma(a, b)$ is*

$$\operatorname{val}\Gamma(a,b) = (1-\alpha)\frac{\lfloor a\rfloor - b}{\lfloor a\rfloor} + \alpha\frac{\lceil a\rceil - b}{\lceil a\rceil} = 1 - \frac{(1-\alpha)b}{m} - \frac{\alpha b}{m+1}.$$

The optimal strategies are as follows:

(i). Strategy $Y^* = (1-b/m)\mathbf{1}_0 + (b/m)U^m_{\mathrm{E}}$ is the unique optimal strategy of Player B*.*

(ii). The strategy $X^* = (1-\alpha)U^m_{\mathrm{O}} + \alpha U^{m+1}_{\mathrm{O}}$ is an optimal strategy of Player A *and, when $b = m$, so are $(1-\alpha)V + \alpha U^{m+1}_{\mathrm{O}}$ for all $v \in \operatorname{conv}(\mathcal{U}^m)$.*

(iii). Every optimal strategy X of Player A *satisfies $Y \le 2m+1$; moreover, it also satisfies $X \ge 1$, when $b < m$, and*

$$\mathbf{P}(X=0) \le \frac{1-\alpha}{m+1},$$

when $b = m$.

What is missing is the complete characterization of optimal strategies from the advantaged player A. We give this characterization in the theorem below.

Theorem 5. *Let $a = m+\alpha$ and $b \le m$, where $m \ge 1$ is an integer and $0 < \alpha < 1$. The strategy X is optimal for Player* A *if and only if*

$$X \in \operatorname{conv}(\mathcal{U}^{m,\alpha} \cup \mathcal{X}^{m,\alpha}), \textit{where}$$

- $\mathcal{U}^{m,\alpha} = (1-\alpha)\mathcal{U}^m + \alpha U^{m+1}_{\mathrm{O}}$, *if $b = m$,*
- $\mathcal{U}^{m,\alpha} = \left\{(1-\alpha)U^m_{\mathrm{O}} + \alpha U^{m+1}_{\mathrm{O}}\right\}$, *if $b < m$*

and

- $\mathcal{X}^{m,\alpha} = \alpha\delta\mathcal{V}^m + (1-\alpha\delta)\,\mathcal{U}^m$, *if $0 < \alpha \le \frac{m+1}{2m+1}$ and $b = m$,*
- $\mathcal{X}^{m,\alpha} = \alpha\delta\mathcal{V}^m + (1-\alpha\delta)\,U^m_{\mathrm{O}}$, *if $0 < \alpha \le \frac{m+1}{2m+1}$ and $b < m$,*
- $\mathcal{X}^{m,\alpha} = (1-\alpha)\sigma\mathcal{V}^m + (1-(1-\alpha)\sigma)\,U^{m+1}_{\mathrm{O}}$, *if $\frac{m+1}{2m+1} < \alpha < 1$, where*

$$\delta = \frac{2m+1}{m+1}, \quad \sigma = \frac{2m+1}{m}.$$

Proof. Suppose that X is an optimal strategy for player A. Consider any strategy Y of player B of the form $(1-b/m)\,\mathbf{1}_0 + (b/m)Z$, where $\mathbf{E}(Z) = m$. Then

$$\begin{aligned} H(X,Y) &= \left(1-\frac{b}{m}\right)H(X,\mathbf{1}_0) + \left(\frac{b}{m}\right)H(X,Z) \\ &= \left(1-\frac{b}{m}\right)(1-p_0) + \left(\frac{b}{m}\right)H(X,Z), \end{aligned} \tag{28}$$

where $p_0 = \mathbf{P}(X=0)$. Since $\mathbf{E}(Y) = b$ so, by Theorem 4 and Equation (28),

$$H(X,Z) \ge \frac{\alpha}{m+1} + p_0\left(\frac{m-b}{b}\right), \tag{29}$$

for any Z with $\mathbf{E}(Z) = m$. Since, by Theorem 4, any optimal X satisfies $\mathbf{P}(X = 0) = 0$ if $b < m$, so (29) can be replaced with

$$H(X, Z) \geq \frac{\alpha}{m+1}, \tag{30}$$

Let $T^m_{i,j}$, with $0 < i \leq m \leq j$ be defined like in proof of Theorem 2. By Equation (30) for any optimal X we have

$$H(Z, T^m_{i,j}) = \lambda H(X, \mathbf{1}_i) + (1-\lambda) H(X, \mathbf{1}_j) \geq \frac{\alpha}{m+1}.$$

Like in proof of Theorem 2 we take $w_i = H(Z, \mathbf{1}_i)$ to obtain

$$(j-m)w_i + (m-i)w_j \geq \frac{\alpha(j-i)}{m+1}. \tag{31}$$

Since the strategy $(1 - b/m)\,\mathbf{1}_0 + (b/m)U^m_{\mathrm{E}}$ is optimal for player B, so for any optimal X we have equality in (30) for $Z = U^m_{\mathrm{E}}$, as well as for $Z = T^m_{i,j}$, with even $0 \leq i \leq m \leq j \leq 2m$ (c.f. proof of Theorem 2 for similar analysis and arguments used there). Hence for i and j even and such that $0 \leq i \leq m \leq j \leq 2m$, (31) becomes equality.

Suppose that m is odd. Taking $i = m-1$ from (31) we get

$$w_j \geq -(j-m)w_{m-1} + \frac{\alpha(j-m+1)}{m+1} \tag{32}$$

(with equality for even $m \leq j \leq 2m$). Similarly, taking $j = m+1$ we get

$$w_i \geq -(m-i)w_{m+1} + \frac{\alpha(m+1-i)}{m+1} \tag{33}$$

(with equality for even $0 \leq i \leq m$). Since $m-1$ and $m+1$ are even so, from (32) we get

$$w_{m+1} = -w_{m-1} + \frac{2\alpha}{m+1}. \tag{34}$$

From this and from (33) we find out that (32) holds for all $j \geq 0$, with equality for all even $0 \leq j \leq 2m$. For even $0 \leq j \leq 2m$ this implies

$$w_j - w_{j+1} \leq w_{m-1} - \frac{\alpha}{m+1}. \tag{35}$$

On the other hand, for odd $1 \leq j \leq 2m-1$ this implies

$$w_j - w_{j+1} \geq w_{m-1} - \frac{\alpha}{m+1}. \tag{36}$$

Let $p_j = \mathbf{P}(X = j)$. Then $w_j - w_{j+1} = p_j + p_{j+1}$ and, from $(35-36)$ we get $p_j + p_{j+1} \leq p_{j+1} + p_{j+2}$ (for all even $0 \leq j \leq 2m-2$). and $p_j + p_{j+1} \geq p_{j+1} + p_{j+2}$ (for all odd $1 \leq j \leq 2m-1$). Thus there exist $d_j \geq 0$ (with $0 \leq j \leq 2m-1$) such that

$$p_j - p_{j+2} + d_j = 0, \text{for even } 0 \leq j \leq 2m-2 \tag{37}$$

$$-p_j + p_{j+2} + d_j = 0, \text{for odd } 1 \leq j \leq 2m-1. \tag{38}$$

In the case of even $0 \leq j \leq 2m-2$, (32) becomes equality and it yields

$$w_j - w_{j+2} = 2w_{m-1} - \frac{2\alpha}{m+1},$$

for all even $0 \le j \le 2m-2$. Thus $w_j - w_{j+2} = w_{j+2} - w_{j+4}$ (for all even $1 \le j \le 2m-4$) and, since $w_i - w_{i+2} = p_i + 2p_{i+1} + p_{i+2}$, so this implies

$$p_j + 2p_{j+1} - 2p_{j+3} - p_{j+4} = 0, \text{for even } 0 \le j \le 2m-4. \tag{39}$$

Moreover, in the case of $j = 2m-2$ we have inequality $w_{2m-2} - w_{2m} \ge w_{2m} - w_{2m+2}$. Thus there exist $d_{2m} \ge 0$ such that

$$p_{2m-2} + 2p_{2m-1} - 2p_{2m+1} - d_{2m} = 0 \tag{40}$$

(recall that, by Theorem 4, $p_{2m+2} = 0$). Equations (37 – 40) can be obtained for odd m as well, taking $i = m-2$ and $j = m+2$.

Observe also that since $\sum_{i=0}^{\infty} p_i = 1$ and $\sum_{i=0}^{\infty} ip_i = m$, so $\sum_{i=0}^{\infty}(i-m)p_i = 0$. Since $Z \le 2m+1$, so in this case

$$\sum_{i=0}^{2m+1} (i-m)p_i = 0. \tag{41}$$

Lemma 2. *The set of solutions of the system of Equations (37 – 41) with additional constraints:*

$$0 \le p_i \le 1, \ \textit{for all}\ 0 \le i \le 2m+1, \tag{42}$$

$$d_i \ge 0, \ \textit{for all}\ 0 \le i \le 2m, \tag{43}$$

$$p_0 + \ldots + p_{2m+1} = 1 \tag{44}$$

is $\sum_{i=0}^{m+1} \lambda_i \begin{bmatrix} \vec{z}_i \\ \vec{d}_i \end{bmatrix}$ *with* $\sum_{i=0}^{m+1} \lambda_i = 1$,

$$\vec{z}_{m+1} = (1-\alpha)\vec{u}_{\mathrm{O}}^{m} + \alpha\vec{u}_{\mathrm{O}}^{m+1},$$

and

- *in the case of* $0 < \alpha \le \frac{m+1}{2m+1}$*:* $\lambda_i \ge 0$*, for all* $0 \le i \le m$ *and* $i = m+1$*,* $\lambda_0 + \lambda_m \ge 0$*, and*

  $$\vec{z}_0 = (1-\alpha)\vec{u}_{\mathrm{E}}^{m} + \alpha\delta\vec{v}_m^{m} + \alpha(1-\delta)\vec{u}_{\mathrm{O}}^{m}, \quad \vec{z}_i = \alpha\delta\vec{v}_i^{m} + (1-\alpha\delta)\vec{u}_{\mathrm{O}}^{m},$$

  where $1 \le i \le m$*,* $\delta = \frac{2m+1}{m+1}$ *and* $\vec{d}_j$*, with* $0 \le j \le m+1$*, satisfy Constraints (43);*

- *in the case of* $\frac{m+1}{2m+1} < \alpha < 1$*:* $\lambda_i \ge 0$*, for all* $0 \le i \le m$*, and*

  $$\vec{z}_0 = (1-\alpha)\vec{u}_{\mathrm{E}}^{m} + \alpha\vec{u}_{\mathrm{O}}^{m+1}, \quad \vec{z}_i = (1-\alpha)\sigma\vec{v}_i^{m} + (1-(1-\alpha)\sigma)\vec{u}_{\mathrm{O}}^{m+1},$$

  where $1 \le i \le m$*,* $\sigma = \frac{2m+1}{m}$ *and* $\vec{d}_j$*, with* $0 \le j \le m+1$*, satisfy Constraints (43).*

(Proof of Lemma 2 is moved to the Appendix).

Let $\vec{x}$ be a stochastic vector representing X. If X is optimal, then it must satisfy Equations (37 – 41) with Constraints (42 – 44). Hence, by Lemma 2, it must be that

$$\vec{x} = \sum_{i=0}^{m+1} \lambda_i \vec{z}_i, \tag{45}$$

with $\sum_{i=0}^{m+1} \lambda_i = 1$ and additional properties depending on the value of α.

Suppose first that $0 < \alpha \le \frac{m+1}{2m+1}$ and $b < m$. Then, by Theorem 4, it must be that $\lambda_0 = 0$ and, consequently, $\lambda_m \ge 0$. Hence any optimal $X \in \mathrm{conv}(\mathcal{U}^{m,\alpha} \cup \mathcal{X}^{m,\alpha})$ with $\mathcal{U}^{m,\alpha} = \{(1-\alpha)U_{\mathrm{O}}^m + \alpha U_{\mathrm{O}}^{m+1}\}$ and $\mathcal{X}^{m,\alpha} = \alpha\delta\mathcal{V}^m + (1-\alpha\delta)\, U_{\mathrm{O}}^m$.

Secondly, suppose that $0 < \alpha \le \frac{m+1}{2m+1}$ and $b = m$. Consider any distribution $T_{i,2m+1}^m$ with even $1 \le i \le m$. By (3), $(4-7)$ and the fact that $\mathbf{E}(T_{i,2m+1}^m) = m$ we have

$$H(X, T_{i,2m+1}^m) = \frac{\alpha}{m+1}\sum_{i=0}^{m+1}\lambda_i - q_{2m+1}\frac{\alpha}{m+1}\lambda_0 + q_{2m+1}\frac{1-\alpha\delta}{m}\sum_{i=1}^{m}\lambda_i + q_{2m+1}\frac{1-\alpha}{m}\lambda_{m+1}$$
$$= \frac{\alpha}{m+1} + q_{2m+1}\left(\frac{1-\alpha\delta}{m}\sum_{i=1}^{m}\lambda_i + \frac{1-\alpha}{m}\lambda_{m+1} - \frac{\alpha}{m+1}\lambda_0\right),$$

where $q_{2m+1} = \mathbf{P}(T_{i,2m+1}^m = 2m+1) > 0$. On the other hand, by (30), it must be that $H(X, T_{i,2m+1}^m) \ge \frac{\alpha}{m+1}$. Thus it must be that

$$\frac{\alpha}{m+1}\lambda_0 \le \frac{1-\alpha\delta}{m}\sum_{i=1}^{m}\lambda_i + \frac{1-\alpha}{m}\lambda_{m+1},$$

which can be reduced to

$$\lambda_0 \le \frac{1-\alpha\delta}{1-\alpha}\sum_{i=0}^{m}\lambda_i + \lambda_{m+1}.$$

Thus

$$\lambda_0 = \beta\frac{1-\alpha\delta}{1-\alpha}\sum_{i=0}^{m}\lambda_i + \lambda_{m+1},$$

where $0 \le \beta \le 1$. From this and from (45) we get $\vec{x} = \lambda_0'\vec{z}_0' + \sum_{i=1}^{m}(\lambda_i'\vec{z}_i + \lambda_i''\vec{z}_i') + \lambda_{m+1}' z_{m+1}$ where

$$\begin{aligned} \lambda_0' &= \beta\lambda_{m+1}, & \lambda_{m+1}' &= (1-\beta)\lambda_{m+1}, \\ \lambda_i' &= \beta\lambda_i, & \lambda_i'' &= (1-\beta)\lambda_i, \quad \text{for } 1 \le i \le m-1, \\ \lambda_m' &= \beta\left(\lambda_0 + \lambda_m\right), & \lambda_m'' &= (1-\beta)\left(\lambda_0 + \lambda_m\right) \end{aligned}$$

and

$$\begin{aligned} \vec{z}_0' &= \vec{z}_{m+1} + \vec{z}_0 - \vec{z}_m = \alpha\vec{u}_{\mathrm{O}}^{m+1} + (1-\alpha)\vec{u}_{\mathrm{E}}^m, \\ \vec{z}_i' &= \vec{z}_i + \frac{1-\alpha\delta}{1-\alpha}\left(\vec{z}_0 - \vec{z}_m\right) = \alpha\delta\vec{v}_i^m + (1-\alpha\delta)\vec{u}_{\mathrm{E}}^m, \text{ for } 1 \le i \le m. \end{aligned}$$

It is easy to see that $\sum_{i=0}^{m+1}\lambda_i' + \sum_{i=1}^{m}\lambda_i'' = 1$, $\lambda_i' \ge 0$, for all $0 \le i \le m+1$, and $\lambda_i'' \ge 0$, for all $1 \le i \le m$. Hence any optimal $X \in \mathrm{conv}(\mathcal{U}^{m,\alpha} \cup \mathcal{X}^{m,\alpha})$ with $\mathcal{U}^{m,\alpha} = (1-\alpha)\mathcal{U}^m + \alpha U_{\mathrm{O}}^{m+1}$ and $\mathcal{X}^{m,\alpha} = \alpha\delta\mathcal{V}^m + (1-\alpha\delta)\,\mathcal{U}^m$.

Lastly, suppose that $\frac{m+1}{2m+1} < \alpha < 1$. Consider any distribution $T_{i,2m+1}^m$ with even $1 \le i \le m$. By (3), $(4-7)$ and the fact that $\mathbf{E}(T_{i,2m+1}^m) = m$ we have

$$H(X, T_{i,2m+1}^m) = \frac{\alpha}{m+1}\sum_{i=0}^{m+1}\lambda_i + q_{2m+1}\frac{1-\alpha}{m}\lambda_{m+1} = \frac{\alpha}{m+1} + \lambda_{m+1}q_{2m+1}\frac{1-\alpha}{m},$$

where $q_{2m+1} = \mathbf{P}(T^m_{i,2m+1} = 2m+1) > 0$. On the other hand, by (30), it must be that $H(X, T^m_{i,2m+1}) \geq \frac{\alpha}{m+1}$. Thus it must be that $\lambda_{m+1} \geq 0$. Moreover, by Theorem 4, it must be that $\lambda_0 = 0$ in the case of $m = b$. Hence any optimal $X \in \mathrm{conv}(\mathcal{U}^{m,\alpha} \cup \mathcal{X}^{m,\alpha})$ with $\mathcal{X}^{m,\alpha} = (1-\alpha)\sigma\mathcal{V}^m + (1-(1-\alpha)\sigma)\, U^{m+1}_{\mathrm{O}}$ and $\mathcal{U}^{m,\alpha} = (1-\alpha)\mathcal{U}^m + \alpha U^{m+1}_{\mathrm{O}}$ (if $b = m$) and $\mathcal{U}^{m,\alpha} = \{(1-\alpha)U^m_{\mathrm{O}} + \alpha U^{m+1}_{\mathrm{O}}\}$ (if $b < m$).

To see that the strategies found above are optimal, by Theorem 4, it is enough to check that

$$H(X, Y) \geq 1 - \frac{(1-\alpha)b}{m} - \frac{\alpha b}{m+1}, \tag{46}$$

for any $X \in \mathrm{conv}(\mathcal{U}^{m,\alpha} \cup \mathcal{X}^{m,\alpha})$ and any Y with $\mathbf{E}(Y) = b$. Using (3) and $(4-7)$ it can be easily checked that (46) is satisfied for any $X \in \mathcal{U}^{m,\alpha} \cup \mathcal{X}^{m,\alpha}$, for any case listed in the theorem. Hence it is also satisfied for any $X \in \mathrm{conv}(\mathcal{U}^{m,\alpha} \cup \mathcal{X}^{m,\alpha})$. $\square$

As an example consider a game $\Gamma(3/2, 1)$. Then the strategy $X = (1/2)\mathbf{1}_1 + (1/2)\mathbf{1}_2$, given as an example in Hart [2008] of one of the strategies not captured by Theorem 4, is $(3/4)V^1_1 + (1/4)U^1_{\mathrm{O}}$. Consider also a game $\Gamma(5/2, 1/2)$. Then the strategy $X = (5/12)\mathbf{1}_1 + (1/4)\mathbf{1}_3 + (1/3)\mathbf{1}_4$, given in Hart [2008] as another example of optimal strategies not captured by Theorem 4, is $(5/6)V^2_1 + (1/6)U^2_{\mathrm{O}}$.

4 Conclusions

In this paper we have found the missing optimal strategies for the players in discrete General Lotto games introduced in Hart [2008]. This could allow for solving the missing cases of Colonel Blotto games, which we reserve for future research.

Appendix

In the analysis below we will use standard notation $\mathbf{1}_i$ to denote the i'th unit vector, $\mathbf{I}_n$ to denote the $n \times n$ unit matrix and $\mathbf{0}_{m,n}$ to denote the $m \times n$ zero matrix. We will drop subscripts denoting the dimension of these matrices if it is clear from the context. Given a sequence of elements $a_1 \ \ldots \ a_n$ we will use the notation $(a_1 \ \ldots \ a_n)^m$ to denote a sequence obtained by repeating the sequence m times. Hence, for example, $\left[\ 1\ (0\ 2)^2\ 0\ \right]^T$ denotes the vector $[1\ 0\ 2\ 0\ 2\ 0]$. If $m \leq 0$, then we will a convention that $(a_1 \ \ldots \ a_n)^m$ denotes the empty sequence. So, for example, $\left[\ 1\ (0\ 2)^0\ 0\ \right]^T$ denotes the vector $[1\ 0]$.

In two of the lemmas we prove below we compute the basis of a null space of matrices of the form $\left[\begin{array}{c|c} f & \mathbf{f} \\ \hline \vec{0} & \mathbf{B}_n \end{array}\right]$ (in the case of Lemma 1) or of the form $\left[\begin{array}{c} \mathbf{f} \\ \hline \mathbf{B}_n \end{array}\right]$ (in the case of Lemma 2), where $\mathbf{f}$ is a row vector and $\mathbf{B}_n$ is a $3(n-1) \times (4n-1)$ matrix of the form

$$\mathbf{B}_n = \left[\begin{array}{c} \mathbf{G}_n \\ \hline \mathbf{H}_n \end{array}\right], \tag{47}$$

where

$$\mathbf{G}_n = \left[\ \tilde{\mathbf{G}}_n \,\middle|\, \mathbf{I}_{2n-2} \,\middle|\, \vec{0}\ \right], \quad \tilde{\mathbf{G}}_n = \left[\begin{array}{c} \mathbf{g}_1 \\ -\mathbf{g}_2 \\ \vdots \\ \mathbf{g}_{2n-3} \\ -\mathbf{g}_{2n-2} \end{array}\right],$$

$$\mathbf{g}_i = \left[\ (0)^{i-1} \quad 1 \quad 0 \quad -1 \quad (0)^{2n-i-2}\ \right], \text{ for } 1 \leq i \leq 2n-2,$$

$$\mathbf{H}_n = \left[\begin{array}{c|c|c} \tilde{\mathbf{H}}_n & \mathbf{0} & \vec{0} \\ \hline \mathbf{h}_{n-1} & (0)^{2n-2} & -1 \end{array}\right], \; \tilde{\mathbf{H}}_n = \begin{bmatrix} \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_{n-2} \end{bmatrix},$$

$$\mathbf{h}_i = \begin{cases} \left[\begin{array}{ccccccc} (0)^{2(i-1)} & 1 & 2 & 0 & -2 & -1 & (0)^{2(n-i)-3} \end{array}\right], & \text{if } 1 \le i \le n-2, \\ \left[\begin{array}{ccccc} (0)^{2(i-1)} & 1 & 2 & 0 & -2 \end{array}\right], & \text{if } i = n-1. \end{cases}$$

The computation is by Gaussian elimination and before we give the proofs of the lemmas we show how $\mathbf{B}_n$ can be reduced by Gaussian elimination to a matrix $\mathbf{B}_n^{(2)}$, which will be used in those proofs. The process of elimination is as follows. First we add to each row i of $\mathbf{G}_n$ the sum of rows $j > i$ of $\mathbf{G}_n$ with the same parity as i and multiply even rows of the resulting matrix by -1. By doing this we obtain

$$\mathbf{G}_n^{(1)} = \left[\begin{array}{c|c|c|c} \mathbf{I}_{2n-2} & -\vec{g}_{-1} & -\vec{g}_0 & \tilde{\mathbf{G}}_n^{(1)} \end{array}\right], \text{ with}$$

$$\vec{g}_{-1} = \left[(1\ 0)^{n-1}\right]^T, \quad \vec{g}_0 = \left[(0\ 1)^{n-1}\right]^T, \quad \tilde{\mathbf{G}}_n^{(1)} = \begin{bmatrix} \mathbf{g}_1^{(1)} \\ -\mathbf{g}_2^{(1)} \\ \vdots \\ \mathbf{g}_{2n-3}^{(1)} \\ -\mathbf{g}_{2n-2}^{(1)} \end{bmatrix}, \text{ where}$$

$$\mathbf{g}_i^{(1)} = \begin{cases} \left[\begin{array}{ccc} (0)^{2j} & (1\ 0)^{n-j-1} & 0 \end{array}\right], & \text{if } i = 2j+1, \\ \left[\begin{array}{ccc} (0)^{2j} & (0\ 1)^{n-j-1} & 0 \end{array}\right], & \text{if } i = 2j+2. \end{cases}$$

Next, we eliminate the first $2n+1$ columns of matrix $\mathbf{H}_n$ using rows of $\mathbf{G}_n^{(1)}$, obtaining:

$$\mathbf{H}_n^{(1)} = \left[\begin{array}{c|c|c|c} \mathbf{0} & \hat{e}_{n-1} & \vec{0} & \tilde{\mathbf{H}}_n^{(1)} \end{array}\right], \text{ where}$$

$$\tilde{\mathbf{H}}_n^{(1)} = \begin{bmatrix} \mathbf{h}_1^{(1)} \\ \vdots \\ \mathbf{h}_{n-1}^{(1)} \end{bmatrix}, \text{ with}$$

$$\mathbf{h}_i^{(1)} = \left[\begin{array}{ccccc} (0)^{2(i-1)} & -1 & 2 & -1 & (0)^{2(n-i-1)} \end{array}\right], \text{ for } 1 \le i \le n-1.$$

We proceed further by adding to each row $2j-1$ of $\mathbf{G}_n^{(1)}$ the sum of rows $k \ge j$ of $\mathbf{H}_n^{(1)}$ with the same parity as j obtaining:

$$\mathbf{G}_n^{(2)} = \left[\begin{array}{c|c|c|c} \mathbf{I}_{2n-2} & -\vec{g}'_{-1} & -\vec{g}_0 & \tilde{\mathbf{G}}_n^{(2)} \end{array}\right], where$$

$$\vec{g}'_{-1} = \begin{cases} \left[(1\ 0\ 0\ 0)^{(n-1)/2}\right]^T, & \text{if } n \text{ is odd,} \\ \left[\begin{array}{ccc} (0\ 0\ 1\ 0)^{n/2-1} & 0 & 0 \end{array}\right]^T, & \text{if } n \text{ is even.} \end{cases} \quad \text{and}$$

$$\tilde{\mathbf{G}}_n^{(2)} = \begin{bmatrix} \mathbf{g}_1^{(2)} \\ -\mathbf{g}_2^{(2)} \\ \vdots \\ \mathbf{g}_{2n-3}^{(2)} \\ -\mathbf{g}_{2n-2}^{(2)} \end{bmatrix}, \text{ with}$$

$$\mathbf{g}_i^{(2)} = \begin{cases} \left[\begin{array}{ccccc} (0)^{2j} & (0\ 2\ 0\ 0)^{(n-j)/2-1} & 0 & 2 & -1 \end{array}\right], & \text{if } i = 2j+1 \text{ and } n-j \text{ is even,} \\ \left[\begin{array}{ccc} (0)^{2j} & (0\ 2\ 0\ 0)^{(n-j+1)/2-1} & 0 \end{array}\right], & \text{if } i = 2j+1 \text{ and } n-j \text{ is odd,} \\ \mathbf{g}_i^{(1)}, & \text{if } i \text{ is even.} \end{cases}$$

Next we add to each row i of $\mathbf{H}_n^{(1)}$ the sum of rows $j > i$ of $\mathbf{H}_n^{(1)}$ with the same parity as i and subtract from it the sum of rows $j > i$ of $\mathbf{H}_n^{(1)}$ with different parity to i. Multiplying the result by -1 we obtain:

$$\mathbf{H}_n^{(2)} = \left[\ \mathbf{0}\ \middle|\ \vec{h}_{-1}\ \middle|\ \vec{0}\ \middle|\ \tilde{\mathbf{H}}_n^{(2)}\ \right], \text{ where}$$

$$\vec{h}_{-1} = \begin{cases} \left[(1\ -1)^{(n-1)/2}\right]^T, & \text{if } n \text{ is odd,} \\ \left[\ (-1\ 1)^{n/2-1}\quad -1\ \right]^T, & \text{if } n \text{ is even,} \end{cases} \quad \text{and}$$

$$\tilde{\mathbf{H}}_n^{(2)} = \begin{bmatrix} \mathbf{h}_1^{(2)} \\ \vdots \\ \mathbf{h}_{n-1}^{(2)} \end{bmatrix}, \text{ with}$$

$$\mathbf{h}_i^{(2)} = \begin{cases} \left[\ (0)^{2(i-1)}\quad 1\quad -2\quad (0\ 2\ 0\ -2)^{(n-i-1)/2}\quad 1\ \right], & \text{if } n-i \text{ is odd,} \\ \left[\ (0)^{2(i-1)}\quad 1\quad -2\quad (0\ 2\ 0\ -2)^{(n-i)/2-1}\quad 0\quad 2\quad -1\ \right], & \text{if } n-i \text{ is even.} \end{cases}$$

Thus we obtain the matrix

$$\mathbf{B}_n^{(2)} = \left[\begin{array}{c} \mathbf{G}_n^{(2)} \\ \hline \mathbf{H}_n^{(2)} \end{array}\right]. \tag{48}$$

Now we are ready to give proofs of Lemmas 1 – 2.

Proof of Lemma 1. Matrix representation of the system of Equations (14 – 18) is

$$\mathbf{A}_a \cdot \begin{bmatrix} \vec{q} \\ \vec{d} \end{bmatrix} = \vec{0}, \qquad \text{where } \mathbf{A}_a = \left[\begin{array}{c|c} f_1 & \mathbf{f}_2 \\ \hline \vec{0} & \mathbf{B}_a \end{array}\right], \tag{49}$$

$\mathbf{B}_a$ is defined in Equation (47) and $\left[\ f_1\ \middle|\ \mathbf{f}_2\ \right] = \mathbf{f} = \left[\ -a\quad -(a-1)\quad \ldots\quad a-1\quad a\ \middle|\ (0)^{2a-1}\ \right]$.

Any solution of (49) is an element of the null space of $\mathbf{A}_a$, $\mathbf{Ker}(\mathbf{A}_a)$. To find its basis we proceed by the standard methods, applying Gaussian elimination to $\mathbf{A}_a$ first. Firstly, we reduce $\mathbf{B}_a$ to $\mathbf{B}_a^{(2)}$, as given in Equation (48). Next, we eliminate first elements in columns $2..2a-2$ of $\mathbf{f}$ with rows of $\mathbf{G}_a^{(2)}$. Dividing the result by $-a$ we get:

$$\mathbf{f}^{(1)} = \begin{cases} \left[\ 1\quad (0)^{2a-2}\quad 0\quad -1\quad (0\ -1\ 0\ 0)^{(a+1)/2-1}\quad 0\ \right], & \text{if } a \text{ is odd,} \\ \left[\ 1\quad (0)^{2a-2}\quad -\frac{1}{2}\quad -1\quad (0\ -1\ 0\ 0)^{a/2-1}\quad 0\quad -1\quad \frac{1}{2}\ \right], & \text{if } a \text{ is even.} \end{cases}$$

The resulting matrix $\mathbf{A}_a^{(1)}$, written column-wise, is:

$$\mathbf{A}_a^{(1)} = \left[\begin{array}{c|cccccccc} \mathbf{I}_{2a-1} & -\vec{g}_{-1} & -\vec{g}_0 & \vec{0} & -\vec{g}_1 & \ldots & \vec{0} & -\vec{g}_{a-1} & -\vec{g}_a \\ \hline \mathbf{0} & -\vec{h}_{-1} & \vec{0} & \hat{e}_1 & -\vec{h}_1 & \ldots & \hat{e}_{a-2} & -\vec{h}_{a-1} & -\vec{h}_a \end{array}\right],$$

where

$$\vec{g}_{-1} = \left[0\ (1\ 0\ 0\ 0)^{(a-1)/2}\right]^T, \qquad \vec{h}_{-1} = \left[(-1\ 1)^{(a-1)/2}\right]^T, \text{ if } a \text{ is odd,}$$

$$\vec{g}_{-1} = \left[\frac{1}{2}\ 0\ 0\ (1\ 0\ 0\ 0)^{a/2-1}\right]^T, \qquad \vec{h}_{-1} = \left[(1\ -1)^{a/2-1}\ 1\right]^T, \text{ if } a \text{ is even,}$$

$$\vec{g}_0 = \left[(1\ 0)^{a-1}\ 1\right]^T,$$

$$\vec{g}_{2j-1} = \left[1\ -2\ (1\ 0\ 1\ -2)^{j-1}\ 1\ (0)^{2a-4j}\right]^T,$$
$$\vec{h}_{2j-1} = \left[(2\ -2)^{j-1}\ 2\ (0)^{a-2j}\right]^T, \ 1 \le j \le \left\lceil\frac{a-1}{2}\right\rceil,$$

$$\vec{g}_{2j} = \left[0\ 0\ (1\ -2\ 1\ 0)^j\ (0)^{2a-4j-3}\right]^T,$$
$$\vec{h}_{2j} = \left[(-2\ 2)^j\ (0)^{a-2j-1}\right]^T, 1 \le j \le \left\lfloor \frac{a-1}{2} \right\rfloor,$$

$$\vec{g}_a = \left[0\ (0\ 0\ 1\ 0)^{(a-1)/2}\right]^T, \qquad \vec{h}_a = \left[(1\ -1)^{(a-1)/2}\right]^T, \text{ if } a \text{ is odd},$$
$$\vec{g}_a = \left[-\frac{1}{2}\ 1\ 0\ (0\ 0\ 1\ 0)^{a/2-1}\right]^T, \qquad \vec{h}_a = \left[(-1\ 1)^{a/2-1}\ -1\right]^T, \text{ if } a \text{ is even},$$

Notice that there are $a+2$ columns of $\mathbf{A}_a^{(1)}$ that are associated with free variables. These are the columns with indexes $2a$, $2(a+i)+1$ (with $0 \le i \le a-1$) and $4a-1$, i.e. the columns where in the upper part of the matrix there are vectors $-\vec{g}_i$ with $-1 \le i \le a$.

To obtain the basis for the null space of $\mathbf{A}_a$ we multiply the free variable columns by -1 and then fill in the rows by adding $\hat{e}_i^T$ at positions $i = 2a$, $2(a+j)+1$ (with $0 \le j \le a-1$) and $4a-1$. The columns in thus obtained matrix form a basis of the null space, $\mathbf{Ker}(\mathbf{A}_a) = \operatorname{span}\{\vec{x}_{-1}, \vec{x}_0, \vec{x}_1, \ldots, \vec{x}_a\}$, where

$$\vec{x}_{-1} = \begin{cases} \left[\ 0\quad 1\quad (0\ 0\ 0\ 1)^{(a-1)/2}\quad 0 \,\middle|\, (-1\ 01\ 0)^{(a-1)/2}\quad 0\ \right]^T, & \text{if } a \text{ is odd}, \\ \left[\ \frac{1}{2}\quad (0\ 0\ 1\ 0)^{a/2} \,\middle|\, (1\ 0\ -1\ 0)^{a/2-1}\quad 1\quad 0\quad 0\ \right]^T, & \text{if } a \text{ is even}, \end{cases}$$
$$\vec{x}_0 = \left[\ (1\ 0)^a\quad 1 \,\middle|\, (0)^{2a-1}\ \right]^T,$$
$$\vec{x}_{2j-1} = \left[\ 1\quad -2\quad (1\ 0\ 1\ -2)^{j-1}\quad 1\quad (0)^{2a-4j+2} \,\middle|\, (2\ 0\ -2\ 0)^{j-1}\quad 2\quad 1\quad (0)^{2a-4j+1}\ \right]^T,$$
$$1 \le j \le \left\lceil \frac{a-1}{2} \right\rceil,$$
$$\vec{x}_{2j} = \left[\ 0\quad 0\quad (1\ -2\ 1\ 0)^j\quad (0)^{2a-4j-1} \,\middle|\, (-2\ 0\ 2\ 0)^{j-1}\quad -2\quad 0\quad 2\quad 1\quad (0)^{2a-4j-1}\ \right]^T,$$
$$1 \le j \le \left\lfloor \frac{a-1}{2} \right\rfloor,$$
$$\vec{x}_a = \begin{cases} \left[\ 0\quad 0\quad (0\ 1\ 0\ 0)^{(a-1)/2}\quad 0 \,\middle|\, (1\ 0\ -1\ 0)^{(a-1)/2}\quad 1\ \right]^T, & \text{if } a \text{ is odd}, \\ \left[\ -\frac{1}{2}\quad (1\ 0\ 0\ 0)^{a/2} \,\middle|\, (-1\ 0\ 1\ 0)^{a/2-1}\quad -1\quad 0\quad 1\ \right]^T, & \text{if } a \text{ is even}, \end{cases}$$

First we change the basis to $\{\vec{x}'_{-1}, \ldots, \vec{x}'_a\}$, where

$$\vec{x}'_{-1} = \frac{1}{a}(\vec{x}_{-1} + \vec{x}_a) = \begin{bmatrix} \vec{u}^a_{\mathrm{O}} \\ \vec{d}_{-1} \end{bmatrix}, \text{ where } \vec{d}_{-1} = \frac{1}{m}\left[\ (0)^{2a-2}\quad 1\ \right]^T,$$
$$\vec{x}'_0 = \frac{1}{a+1}\vec{x}_0 = \begin{bmatrix} \vec{u}^a_{\mathrm{E}} \\ \vec{d}_0 \end{bmatrix}, \text{ where } \vec{d}_0 = \vec{0},$$
$$\vec{x}'_1 = \frac{1}{2a}\left(2\vec{x}'_{-1} + \vec{x}_1\right) = \begin{bmatrix} \vec{w}^a_1 \\ \vec{d}_1 \end{bmatrix}, \text{ where } \vec{d}_1 = \frac{1}{2m}\left[\ 2\quad 1\quad (0)^{2a-4}\quad 2\ \right]^T,$$
$$\vec{x}'_i = \frac{1}{2a}\left(2\vec{x}'_{-1} + \vec{x}_{i-1} + \vec{x}_i\right) = \begin{bmatrix} \vec{w}^a_i \\ \vec{d}_i \end{bmatrix}, \text{ where } \vec{d}_i = \frac{1}{2m}\left[\ (0)^{2i-3}\quad 1\quad 2\quad 1\quad (0)^{2(a-i-1)}\quad 2\ \right]^T$$
and $2 \le i \le a-2$,
$$\vec{x}'_{a-1} = \frac{2}{a+1}\vec{x}_{-1} = \begin{bmatrix} \frac{2a}{a+1}\vec{w}^a_{a-1} - \frac{a-1}{a+1}\vec{u}^a_{\mathrm{O}\uparrow 1} \\ \vec{d}_{a-1} \end{bmatrix}, \text{ where } \vec{d}_{a-1} = \frac{1}{a+1}\left[\ 2\quad 0\quad 0\ \right]^T, \text{ if } a = 2,$$
$$\vec{x}'_{a-1} = \frac{1}{a+1}\left(2\vec{x}_{-1} + \vec{x}_{a-2}\right) = \begin{bmatrix} \frac{2a}{a+1}\vec{w}^a_{a-1} - \frac{a-1}{a+1}\vec{u}^a_{\mathrm{O}\uparrow 1} \\ \vec{d}_{a-1} \end{bmatrix},$$
$$\text{where } \vec{d}_{a-1} = \frac{1}{a+1}\left[\ (0)^{2a-5}\quad 1\quad 2\quad 0\quad 0\ \right]^T, \text{if } a \ge 3,$$
$$\vec{x}'_a = \frac{1}{a-1}\left(\vec{x}_{a-1} + 2\vec{x}_a\right) = \begin{bmatrix} \vec{u}^a_{\mathrm{O}\uparrow 1} \\ \vec{d}_a \end{bmatrix}, \text{ where } \vec{d}_a = \frac{1}{m-1}\left[\ (0)^{2a-3}\quad 1\quad 2\ \right]^T.$$

Any solution $\vec{x} = [q_0, \ldots, q_{2a}, d_1, \ldots, d_{2a-1}]$ of (49) is a linear combination of the vectors above, that is

$$\vec{x} = \sum_{i=-1}^{a} \lambda_i \vec{x}'_i.$$

Since $q_1 = \lambda_{-1}\frac{1}{a}$, $q_{2a} = \lambda_0\frac{1}{a+1}$, $d_{2a-2} = \lambda_a\frac{1}{a-1}$, $d_{2a-3} = \lambda_{a-1}\frac{1}{a+1}$ and $d_{2i-1} = \lambda_i\frac{1}{a}$, for $1 \le i \le a-2$ so, from Constraints (19 – 20), we get that $\lambda_i \ge 0$, for all $-1 \le i \le a$. Additionally, from Constraint (21) and the fact that $\sum_{j=0}^{2a} x'_{ij} = 1$, for all $-1 \le i \le a$, we have:

$$\sum_{i=-1}^{a} \lambda_i = \sum_{i=-1}^{a} \lambda_i \sum_{j=0}^{2a} x'_{ij} = \sum_{i=0}^{2a} q_i = 1.$$

Hence the set solutions of the system of Equation (14 – 18) with Constraints (19 – 20) is $\mathrm{conv}\big(\{\vec{x}'_{-1}, \ldots, \vec{x}'_a\}\big)$. □

Proof of Lemma 2. Matrix representation of the system of Equations (37 – 41) is

$$\mathbf{A}_m \cdot \begin{bmatrix} \vec{p} \\ \vec{d} \end{bmatrix} = \vec{0}, \qquad \text{where } \mathbf{A}_m = \left[\begin{array}{c} \mathbf{f} \\ \hline \mathbf{B}_{m+1} \end{array}\right], \tag{50}$$

$\mathbf{B}_{m+1}$ is defined in Equation (47) and

$$\mathbf{f} = \left[\begin{array}{ccccc|c} -m-\alpha & -(m-1)-\alpha & \ldots & m-\alpha & m+1-\alpha & (0)^{2m+1} \end{array}\right].$$

Like in proof of Lemma 1 to find solutions of (50) we find a basis of the null space of $\mathbf{A}_m$ using Gaussian elimination. $\mathbf{B}_{m+1}$ can be reduced to $\mathbf{B}^{(2)}_{m+1}$, as given in Equation (48). Next, we eliminate first $2m$ elements of $\mathbf{f}$ with rows of $\mathbf{G}^{(2)}_{m+1}$. Dividing the result by $(m+1)(1-\alpha)$ we get:

$$\mathbf{f}^{(1)} = \begin{cases} \left[\begin{array}{ccccc} (0)^{2m} & -u & 1 & (0\ t\ 0\ 0)^{m/2} & -w \end{array}\right], & \text{if } m \text{ is even,} \\ \left[\begin{array}{ccccccc} (0)^{2m} & -u & 1 & (0\ t\ 0\ 0)^{(m-1)/2} & 0 & t & -w \end{array}\right], & \text{if } m \text{ is odd, where} \end{cases}$$

$$u = \begin{cases} \frac{m+2}{2(m+1)}\frac{\alpha}{1-\alpha}, & \text{if } m \text{ is even,} \\ -\frac{1}{2}, & \text{if } m \text{ is odd,} \end{cases}$$

$$t = \frac{m+1+\alpha}{(m+1)(1-\alpha)}$$

$$w = \begin{cases} \frac{m}{2(m+1)}\frac{\alpha}{1-\alpha}, & \text{if } m \text{ is even,} \\ \frac{1}{2}\frac{1+\alpha}{1-\alpha}, & \text{if } m \text{ is odd.} \end{cases}$$

Next, we move the first row below block $\mathbf{G}^{(2)}_{m+1}$ obtaining $\mathbf{A}^{(1)}_m = \left[\begin{array}{c} \mathbf{G}^{(2)}_{m+1} \\ \hline \mathbf{f}^{(1)} \\ \hline \mathbf{H}^{(2)}_{m+1} \end{array}\right]$. Adding row $\mathbf{f}^{(1)}$ to rows of $\mathbf{G}^{(2)}_{m+1}$ we get matrix $\mathbf{A}^{(2)}_m$, which, written column-wise, is:

$$\mathbf{A}^{(2)}_m = \left[\begin{array}{c|cccccccccc} \mathbf{I}_{2m} & -\vec{g}_0 & \vec{0} & \vec{0} & -\vec{g}_1 & \vec{0} & -\vec{g}_2 & \ldots & \vec{0} & -\vec{g}_m & -\vec{g}_{m+1} \\ \hline \mathbf{0} & -\vec{h}_0 & \hat{e}_1 & \hat{e}_2 & -\vec{h}_1 & \vec{0} & -\vec{h}_2 & \ldots & \hat{e}_{m+2} & -\vec{h}_m & -\vec{h}_{m+1} \end{array}\right],$$

where

$$\vec{g}_0 = \left[0\ u\ (1\ u\ 0\ u)^{(m-1)/2}\right]^T, \qquad \vec{h}_{-1} = \left[(1\ \ -1)^{(m-1)/2\ 1}\right]^T, \text{ if } m \text{ is odd,}$$

$$\vec{g}_0 = \left[(1\ u\ 0\ u)^{m/2}\right]^T, \qquad \vec{h}_{-1} = \left[(-1\ 1)^{m/2}\right]^T, \text{ if } m \text{ is even,}$$

$$\vec{g}_{2j-1} = \left[(-2\ 1\!-t\ 0\ 1\!-t)^{j-1}\ \ -2\ 1\!-t\ (0\ \ -t)^{m-2j+1}\right]^T,$$
$$\vec{h}_{2j-1} = \left[(2\ \ -2)^j\ 2\ (0)^{m-2j-1}\right]^T, 1 \le j \le \left\lceil \frac{m}{2} \right\rceil,$$

$$\vec{g}_{2j} = \left[(0\ 1\ \ -2\ 1)^j\ (0)^{2m-4j}\right]^T,$$
$$\vec{h}_{2j} = \left[(-2\ 2)^j\ (0)^{m-2j}\right]^T, 1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor,$$

$$\vec{g}_{m+1} = \left[1\ w\ (0\ w\ 1\ w)^{(m-1)/2}\right]^T, \qquad \vec{h}_{m+2} = \left[(-1\ 1)^{(m-1)/2}\ \ -1\right]^T, \text{ if } m \text{ is odd},$$
$$\vec{g}_{m+1} = \left[(0\ w\ 1\ w)^{m/2}\right]^T, \qquad \vec{h}_{m+2} = \left[(1\ \ -1)^{m/2}\right]^T, \text{ if } m \text{ is even}.$$

Notice that there are $m+2$ columns of $\mathbf{A}_m^{(2)}$ that are associated with free variables. There are the columns with indexes $2m+1$, $2(m+i+1)$ (with $1 \le i \le m$) and $4m+3$, i.e. the columns where in the upper part of the matrix there are vectors $-\vec{g}_i$ with $0 \le i \le m+1$.

To obtain the basis for the null space of $\mathbf{A}_m$ we multiply the free variable columns by -1 and then fill in the rows by adding $\hat{e}_i^T$ at positions $i = 2m+1$, $2(m+j+1)$ (with $1 \le j \le m$) and $4m+3$. The columns in thus obtained matrix form a basis of the null space, $\mathbf{Ker}(\mathbf{A}_m) = \operatorname{span}\{\vec{x}_0, \vec{x}_1, \ldots, \vec{x}_{m+1}\}$, where

$$\vec{x}_0 = \begin{cases} \left[\ (0\ u\ 1\ u)^{(m+1)/2} \,\middle|\, (1\ 0 -1\ 0)^{(m-1)/2}\ \ 1\ \ 0\ \ 0\ \right]^T, & \text{if } m \text{ is odd}, \\ \left[\ 1\ \ u\ \ (0\ u\ 1\ u)^{m/2} \,\middle|\, (-1\ 0\ 1\ 0)^{m/2}\ \ 0\ \right]^T, & \text{if } m \text{ is even}, \end{cases}$$

$$\vec{x}_{2j-1} = \left[\ (-2\ 1\!-t\ 0\ 1\!-t)^{j-1}\ \ -2\ \ 1\!-t\ \ (0\ \ -t)^{m-2j+2} \,\middle|\, (2\ 0\ \ -2\ 0)^{j-1}\ \ 2\ \ 1\ \ (0)^{2m-4j+3}\ \right]^T,$$
$$1 \le j \le \left\lceil \frac{m}{2} \right\rceil,$$

$$\vec{x}_{2j} = \left[\ (0\ 1\ \ -2\ 1)^j\ \ (0)^{2m-4j+2} \,\middle|\, (-2\ 0\ 2\ 0)^{j-1}\ \ -2\ \ 0\ \ 2\ \ 1\ \ (0)^{2m-4j+1}\ \right]^T,$$
$$1 \le j \le \left\lfloor \frac{m}{2} \right\rfloor,$$

$$\vec{x}_{m+1} = \begin{cases} \left[\ (1\ w\ 0\ w)^{(m+1)/2} \,\middle|\, (-1\ 0\ 1\ 0)^{(m-1)/2}\ \ -1\ \ 0\ \ 1\ \right]^T, & \text{if } m \text{ is odd}, \\ \left[\ 0\ \ w\ \ (1\ w\ 0\ w)^{m/2} \,\middle|\, (1\ 0\ \ -1\ 0)^{m/2}\ \ 1\ \right]^T, & \text{if } m \text{ is even}, \end{cases}$$

First we change the basis to $\{\vec{x}'_0, \ldots, \vec{x}'_{m+1}\}$, where

$$\vec{x}'_{m+1} = \frac{1-\alpha}{m}\left(\vec{x}_m + 2\vec{x}_{m+1}\right) = \begin{bmatrix} (1-\alpha)\vec{u}_{\mathrm{O}}^m + \alpha \vec{u}_{\mathrm{O}}^{m+1} \\ \vec{d}_{m+1} \end{bmatrix}, \text{ with } \vec{d}_{m+1} = \frac{1-\alpha}{m}\left[\ (0)^{2m-1}\ \ 1\ \ 2\ \right]^T,$$

and, in the case of $\frac{m+1}{2m+1} < \alpha < 1$,

$$\vec{x}'_0 = \frac{1-\alpha}{m+1}\left(\vec{x}_0 + \vec{x}_{m+1}\right) = \begin{bmatrix} (1-\alpha)\vec{u}_{\mathrm{E}}^m + \alpha \vec{u}_{\mathrm{O}}^{m+1} \\ \vec{d}_0 \end{bmatrix}, \text{ with } \vec{d}_0 = \frac{1-\alpha}{m+1}\left[\ (0)^{2m}\ \ 1\ \right]^T,$$

$$\vec{x}'_1 = \frac{1-\alpha}{m}\left(2\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \vec{x}_1\right) = \begin{bmatrix} (1-\alpha)\sigma \vec{v}_1^m + (1-(1-\alpha)\sigma)\vec{u}_{\mathrm{O}}^{m+1} \\ \vec{d}_1 \end{bmatrix},$$
$$\text{with } \sigma = \frac{2m+1}{m} \text{ and } \vec{d}_1 = \frac{1-\alpha}{m}\left[\ 2\ \ 1\ \ (0)^{2m-2}\ \ 2\ \right]^T,$$

$$\vec{x}'_i = \frac{1-\alpha}{m}\left(2\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \vec{x}_{i-1} + \vec{x}_i\right) = \begin{bmatrix} (1-\alpha)\sigma \vec{v}_i^m + (1-(1-\alpha)\sigma)\vec{u}_{\mathrm{O}}^{m+1} \\ \vec{d}_i \end{bmatrix}, \text{ for } 2 \le i \le m,$$
$$\text{with } \vec{d}_i = \frac{1-\alpha}{m}\left[\ (0)^{2i-3}\ \ 1\ \ 2\ \ 1\ \ (0)^{2(m-i)}\ \ 2\ \right]^T, \text{ for } 2 \le i \le m,$$

while in the case of $0 < \alpha \leq \frac{m+1}{2m+1}$,

$$\vec{x}'_1 = \frac{\alpha}{m+1}\left(2\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \gamma\left(\vec{x}_m + 2\vec{x}_{m+1}\right) + \vec{x}_1\right) = \begin{bmatrix} \alpha\delta\vec{v}_1^m + (1-\alpha\delta)\vec{u}_{\mathrm{O}}^m \\ \vec{d}_1 \end{bmatrix},$$

$$\text{with } \delta = \frac{2m+1}{m+1},\ \gamma = \frac{m+1-\alpha(2m+1)}{m\alpha}$$

$$\text{and } \vec{d}_1 = \frac{\alpha}{m+1}\begin{bmatrix} 2 & 1 & (0)^{2m-3} & \gamma & 2(1+\gamma) \end{bmatrix}^T,$$

$$\vec{x}'_i = \frac{\alpha}{m+1}\left(2\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \gamma\left(\vec{x}_m + 2\vec{x}_{m+1}\right) + \vec{x}_{i-1} + \vec{x}_i\right) = \begin{bmatrix} \alpha\delta\vec{v}_i^m + (1-\alpha\delta)\vec{u}_{\mathrm{O}}^m \\ \vec{d}_i \end{bmatrix},$$

$$\text{with } \vec{d}_i = \frac{\alpha}{m+1}\begin{bmatrix} (0)^{2i-3} & 1 & 2 & 1 & (0)^{2(m-i)-1} & \gamma & 2(1+\gamma) \end{bmatrix}^T, \text{for } 2 \leq i \leq m-1,$$

$$\vec{x}'_m = \frac{\alpha}{m+1}\left(2\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \gamma\left(\vec{x}_m + 2\vec{x}_{m+1}\right) + \vec{x}_{m-1} + \vec{x}_m\right) = \begin{bmatrix} \alpha\delta\vec{v}_i^m + (1-\alpha\delta)\vec{u}_{\mathrm{O}}^m \\ \vec{d}_i \end{bmatrix},$$

$$\text{with } \vec{d}_m = \frac{\alpha}{m+1}\begin{bmatrix} (0)^{2m-3} & 1 & 2 & 1+\gamma & 2(1+\gamma) \end{bmatrix}^T,$$

$$\vec{x}'_0 = \frac{1-\alpha}{m+1}\left(\vec{x}_0 + \vec{x}_{m+1}\right) + \vec{x}'_m - \vec{x}'_{m+1} = \begin{bmatrix} (1-\alpha)\vec{u}_{\mathrm{E}}^m + \alpha\delta\vec{v}_m^m + \alpha(1-\delta)\vec{u}_{\mathrm{O}}^m \\ \vec{d}_0 \end{bmatrix},$$

$$\text{with } \vec{d}_0 = \frac{\alpha}{m+1}\begin{bmatrix} (0)^{2m-3} & 1 & 2 & 0 & \frac{1-\alpha}{\alpha} \end{bmatrix}^T.$$

Any solution $\vec{x} = [p_0, \ldots, p_{2m+1}, d_0, \ldots, d_{2m}]$ of (50) is a linear combination of the vectors above, that is

$$\vec{x} = \sum_{i=0}^{m+1} \lambda_i \vec{x}'_i.$$

From Constraint (44) and the fact that $\sum_{j=0}^{2m+1} x'_{ij} = 1$, for all $0 \leq i \leq m+1$, we have:

$$\sum_{i=0}^{m+1} \lambda_i = \sum_{i=0}^{m+1} \lambda_i \sum_{j=0}^{2m+1} x'_{ij} = \sum_{i=0}^{2m+1} p_i = 1.$$

Suppose that $\frac{m+1}{2m+1} < \alpha < 1$. Then $p_0 = \lambda_0 \frac{1-\alpha}{m+1}$ and $d_{2i-1} = \lambda_i \frac{2(1-\alpha)}{m}$, for $1 \leq i \leq m$. Hence, by Constraints $(42-43)$, we get that $\lambda_i \geq 0$, for all $0 \leq i \leq m$.

Now, suppose that $0 < \alpha \leq \frac{m+1}{2m+1}$. Then $p_0 = \lambda_0 \frac{1-\alpha}{m+1}$, $d_{2i-1} = \lambda_i \frac{2\alpha}{m+1}$, for $1 \leq i \leq m-1$, $p_{2m+1} = \lambda_{m+1} \frac{\alpha}{m+1}$ and $d_{2m-1} = (\lambda_m + \lambda_0) \frac{2\alpha}{m+1}$. Thus, by Constraints (43), we get that $\lambda_i \geq 0$, for all $0 \leq i \leq m-1$ and $i = m+1$, as well as $\lambda_0 + \lambda_m \geq 0$. □

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