WEAK GROUP STRATEGY-PROOF AND QUEUE-EFFICIENT MECHANISMS FOR THE QUEUEING PROBLEM WITH MULTIPLE MACHINES

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ABSTRACT. We first provide the complete characterization of mechanisms that satisfy weak group strategy-proofness and queue-efficiency in the multiple machine queueing problem with two agents. For any such mechanism, there can be at most one point of discontinuity in the transfer map. We then state a necessary condition for mechanisms to satisfy queue-efficiency, weak group strategy-proofness and continuity, with more than two agents. Finally, we provide a class of mechanisms that satisfy queue-efficiency, weak group strategy-proofness and continuity.
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1. INTRODUCTION

We address the queueing problem with possibly multiple, possibly non-identical machines from a group incentive point of view when agents have quasilinear preferences over positions in queue and monetary transfers. A queueing problem involves a set of agents wanting to consume a service provided by one or many machines, and a set of machines which can only serve the agents sequentially (one by one). Such a problem with n agents and m machines has the following features: (i) each agent has exactly one job to complete using any one of these machines, (ii) each machine can process only one job at a time, (iii) the jobs are identical across agents so that for a given machine, they take the same time to get processed, (iv) the machines are non-identical with respect to the time taken to complete the job.

This model captures a multitude of real life situations; a typical example would be the problem of provision of the quickest possible service to n customers waiting at m cashier windows. Similar situations arise in a printing press, truckload transportation, people waiting on ATM machines, amateur astronomers waiting to use public telescopes and whole host of other possibilities. Maniquet [14] discusses many other interesting applications of this problem in the single machine context. Apart from the aforementioned practical relevance, queueing models are also important from a theoretical point of view. Mitra and Sen [18] show that for any multiple heterogenous

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good allocation problem; if there exists a mechanism satisfying efficiency and strategy-proofness, then the underlying structure of the problem must be like that of a queueing problem. Such wide applicability of the queueing model has led to an extensive literature[1].

The planner wants to ensure queue-efficiency, that is, minimize the aggregate waiting cost of provision of the service to the agents. This requires the agents to reveal their waiting costs to the planner. In doing so, they have the incentive to misreport so as to ensure a favorable outcome (distinct from the socially optimal one). Thus, the planner runs into a problem of information extraction; and so needs to apply a cost revelation mechanism. Such a mechanism should ideally rule out the possibility of agents colluding amongst themselves to misreport, and therefore, be group strategy-proof.

Such an information extraction problem has been analyzed by Vickrey [22], Clarke [4] and Groves [6] leading to the formulation of VCG mechanisms, which are sufficient for queue-efficiency and (individual) strategy-proofness. For smoothly connected domains Holmström [10] established the uniqueness of VCG mechanisms in this regard.

Our goal is to identify the class of mechanisms that satisfy queue-efficiency, group strategy-proofness and continuity in the multiple machine queueing problem. Any such mechanism must be strategy-proof. Hence, we try to identify the VCG mechanisms that satisfy group strategy-proofness and continuity. As discussed in Postlewaite and Wettstein [20], continuity of mechanisms ensures that they are robust to small misspecifications of the characteristics of agents.

As discussed in Mitra and Mutuswami [17], there can be two variants of group strategy-proofness, weak and strong. The former requires that no group of agents be able to misreport so that all of them are strictly better off. The latter requires that no group of agents be able to misreport so that at least one member is strictly better off and no other member is strictly worse off. It follows from the definitions that the latter implies the former, not conversely.

Mitra and Mutuswami [17] show that there does not exist any mechanism that satisfies queue-efficiency and strong group strategy-proofness, in a single machine queueing context. They further argue that the notion of strong group strategy-proofness presumes the ability of agents to arrange credible side payments. This is not always reasonable as reporting honestly is a weakly dominant strategy for VCG mechanisms. Hence, we focus on weak group strategy-proof mechanisms[2].

Mitra and Mutuswami [17] investigate mechanisms immune to coalitional misreports with single machine and identify a necessary condition for mechanisms to satisfy queue-efficiency, pairwise strategy-proofness and weakly linearity. They remark that for two or four agents, no mechanism satisfies the aforementioned properties and budget balance. They also completely characterize the class of mechanisms that satisfy

[1] Dolan [5], Suijs [21], Mitra [15], Moulin [19], Hashimoto and Saitoh [8], Kayi and Ramaekers [13], Maniquet [14], Chun [2], Katta and Sethuraman [12], Kar, Mitra and Mutuswami [11], Chun and Heo [3], Mitra and Mutuswami [17].

[2] This notion of group strategy-proofness has also been used in different contexts by Barbera, Berga and Moreno [1] and Hatsumi and Serizawa [9].

queue-efficiency, pairwise strategy-proofness, weakly linearity and the fairness property of equal treatment of equals. They call this class k-pivotal mechanisms and show that they satisfy weak group strategy-proofness.

This paper extends the single machine queueing problem in Mitra and Mutuswami [17], to multiple machines. Unlike the single machine setting, with multiple machines more than one agents may have to wait the same time to get service.

We completely characterize the mechanisms that satisfy queue-efficiency and weak group strategy-proofness for two agents. When there are two agents, queue-efficiency and weak group strategy-proofness imply lower semi-continuity of the transfer map. By considering all possible deviations three possible cases emerge; (a) flat straight line, (b) positively sloped straight line, and (c) initially positively sloped but later flat straight line (a kink occurs in the map). We see that discontinuity in map can only occur for case (c), that too at the kink point with the only possibility of a sudden fall in value at that point. Thus, there can be at most one point of discontinuity in the transfer map, that too with both side limits being equal (at that point). Further, such a point of discontinuity, if present, can only occur for one agent.

For n agents, we provide a necessary condition for mechanisms to satisfy queue-efficiency, weak group strategy-proofness and continuity, when no two agent has to wait the same time to get service. This result is shown to be a generalization of the necessity result Theorem 3.7 of Mitra and Mutuswami [17]. We also provide a class of mechanisms that satisfy queue-efficiency, weak group strategy-proofness and continuity. This class contains the k-pivotal mechanisms of Mitra and Mutuswami [17], as a special subclass.

Section 2 states the model. Section 3.1 states the two agent results while the section 3.2 states the $n > 2$ results. Section 4 discusses possible extensions. Section 5 states the conclusion and section 6 is the appendix.

2. The Model

Let $N = \{1, \ldots, n\}$, $n \geq 2$ be the set of agents with identical jobs[3] and $M = \{1, \ldots, m\}$ be the set of machines. Each machine j is identified with a speed of $s_j \in (0, 1]$ which is the time taken by the machine to process one job. W.l.o.g. we assume $s_1 \leq s_2 \leq \ldots \leq s_m$. Each agent i is identified with $\theta_i \in \mathbb{R}_+$ which denotes the disutility incurred by i per unit of waiting time. Let $\theta = (\theta_i)_{i \in N}$ denote the profile of waiting costs and θ_{-i} denote the cost vector $(\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$. The cost of waiting on

[3] If agents have non-identical jobs that are private information, then an agent's utility from the service depends *directly* on the announcements of processing times of other agents. Then, strategy-proofness is not possible and is replaced by "*implementation in ex-post equilibrium*", which is Bayesian incentive compatibility under all priors (Hain and Mitra [7]). However, in most cases, processing times are public information (the planner may easily check them once machines process jobs).

machine j, to agent i, in position k is given by $ks_j\theta_i$.[4] Agents have quasilinear preferences over positions and money. So an agent i waiting on machine j in kth position with money $t_i \in \mathbb{R}$ gets a utility $-ks_j\theta_i + t_i$.

The planner wants to ensure queue-efficiency, which means that the n jobs need to be scheduled in such a way that the aggregate waiting cost is minimized. To attain queue-efficiency, the planner needs to pick the smallest n numbers from the set of all possible waiting times $\{\{ks_j\}_{k\in N}\}_{j\in M}$, arrange them in a non-decreasing order and then assign these waiting times to agents in such a way that queue-efficiency is achieved. Let $(z(1), \dots, z(n))$ denote the smallest n waiting times arranged in such an order.

Any ranking of n agents can be represented by an injection $\hat{\sigma} : N \to \mathbb{N}$, where agent i is ranked $\hat{\sigma}_i$. Let Σ be the set of all such injections. For any profile of waiting costs θ, the planner picks an efficient ranking of agents $\sigma(\theta) = (\sigma_i(\theta))_{i\in N}$ such that $\sigma(\theta) \in \text{argmin}_{\hat{\sigma}\in\Sigma} \sum_{i=1}^{n} \hat{\sigma}_i\theta_i$; and then assigns to each agent i a waiting time $z(\sigma_i(\theta))$. This efficient ranking is unique if and only if no two agents have the same waiting cost per unit time. To ensure that efficient ranking be a single valued selection, a *tie-breaking* rule is required. A *strict order* $\succ$ is defined on N, for this purpose. This relation is used to break ties in the following manner; if any two agents i and j have same waiting cost per unit time, then $\sigma_i(\theta) < \sigma_j(\theta)$ iff $i \succ j$. Also define for any profile of waiting costs θ, $P'_i(\theta) := \{k \in N | \sigma_k(\theta) > \sigma_i(\theta)\}$ and $P_i(\theta) := \{k \in N | \sigma_k(\theta) < \sigma_i(\theta)\}$. Therefore, $P'_i(\theta)$ and $P_i(\theta)$ denote the set of agents ranked after agent i and before agent i, respectively, in the efficient ranking. Also note that $\forall\, \theta \in \mathbb{R}^n_+$, $\forall\, i \neq j \in N$, $\sigma_i(\theta_{-j})$ denotes the efficient rank of agent i in the profile of costs θ_{-j}.

If waiting costs are private information, agents will have incentive to misreport. Under incomplete information, planner has to design a mechanism to extract information. A mechanism associates to any profile of waiting costs $\theta \in \mathbb{R}^n_+$, a tuple $(\hat{\sigma}(\theta), \tau(\theta)) \subset \mathbb{N} \times \mathbb{R}^n$ where $\hat{\sigma}(\theta) \in \Sigma$ and $\tau(\theta) = (\tau_i(\theta))_{i\in N}$. Under this mechanism, any agent i gets rank $\hat{\sigma}_i(\theta)$ and a transfer $\tau_i(\theta)$. The utility to agent i for any reported profile of costs θ is $u(\hat{\sigma}_i(\theta), \tau_i(\theta); \theta'_i) = -z(\hat{\sigma}_i(\theta))\theta'_i + \tau_i(\theta)$, where θ'_i is the true waiting cost of agent i.[5] We assume that $\tau_i(0, 0, \dots, 0) = 0$ *irrespective* of the tie-breaking rule chosen, $\forall\, i \in N$. This means that transfers are independent of agent specific constants no matter what the tie-breaking rule. It reinforces the fairness perception that agents should incur zero disutility, if all of them have zero waiting costs. We also assume that $\forall\, n \in \mathbb{N}$, there exists $s = 1, \dots, n-1$ such that $z(s) \neq z(s+1)$. Otherwise, the planner does not require any information about waiting costs to attain queue-efficiency.

[4] An agent incurs a cost of waiting until a machine ends processing its job; unlike in Mitra and Mutuswami [17] where an agent incurs a cost of waiting *until* a machine starts processing it. This allows for situations where an agent prefers the kth (some $k > 1$) position on the queue of a faster machine then the first position on the queue of a slower machine.

[5] For notational simplicity, we make an implicit efficiency assumption that an agent is allocated any one of the n smallest waiting times $(z(1), \dots, z(n))$. As we shall restrict our attention to mechanisms verifying such a property, it has no incidence.

Definition 1. A mechanism $(\hat{\sigma}, \tau)$ is *queue-efficient* (Q-EFF) if $\forall\, \theta \in \mathbb{R}^n_+$,

$$\hat{\sigma}(\theta) \in \operatorname{argmin}_{\tilde{\sigma}\in\Sigma} \sum_{i=1}^{n} z(\tilde{\sigma}_i)\theta_i$$

In other words, a mechanism $(\hat{\sigma}, \tau)$ is Q-EFF if $\hat{\sigma}(\theta) = \sigma(\theta), \forall\, \theta \in \mathbb{R}^n_+$.

Definition 2. A mechanism $(\hat{\sigma}, \tau)$ is *strategy-proof* if $\forall\ i \in N$, $\forall\ \theta_i, \theta'_i \in \mathbb{R}_+$ and $\forall$ $\theta_{-i} \in \mathbb{R}^{n-1}_+$,

$$u(\hat{\sigma}_i(\theta_i, \theta_{-i}), \tau_i(\theta_i, \theta_{-i}); \theta_i) \geq u(\hat{\sigma}_i(\theta'_i, \theta_{-i}), \tau_i(\theta'_i, \theta_{-i}); \theta_i)$$

A strategy-proof mechanism guarantees that revealing the true waiting cost is a weakly dominant strategy for every agent. If a mechanism achieves queue-efficiency and strategy-proofness, then we say that the queue-efficient decision is implementable in dominant strategies. However, there remains the possibility of agents forming coalitions and misreporting together. Ideally a mechanism should also be immune to such coalitional misreporting. Hence, we define a stronger incentive compatibility criterion. First, we introduce the following notation. For any $\theta, \theta' \in \mathbb{R}^n_+$; θ' is an S-variant of θ if $\forall\ i \notin S$, $\theta_i = \theta'_i$, for any non-empty $S \subseteq N$. The profile of waiting costs θ' is said to be an order preserving S-variant of θ if $\forall\, i \in S$, $\hat{\sigma}_i(\theta) = \hat{\sigma}_i(\theta')$.

Definition 3. A mechanism $(\hat{\sigma}, \tau)$ is *weak group strategy-proof* (WGS) if $\forall\, \theta \in \mathbb{R}^n_+$, $\forall\, S \subseteq N$, there exists $i \in S$ such that

$$u(\hat{\sigma}_i(\theta), \tau_i(\theta); \theta_i) \geq u(\hat{\sigma}_i(\theta'), \tau_i(\theta'); \theta_i)$$

where θ' is an S-variant of θ.

Thus, WGS property ensures that any coalition misreporting together would have at least one member who would not be strictly better off. For any singleton coalition this condition reduces to strategy-proofness. For any coalition with no more that 2 members, this condition reduces to *pairwise strategy-proofness* (e.g. Mitra and Mutuswami [17]) defined below.

Definition 4. A mechanism $(\hat{\sigma}, \tau)$ is *pairwise strategy-proof* if $\forall\, \theta \in \mathbb{R}^n_+$ and $\forall\, S \subseteq N$ such that $|S| \leq 2$, there exists $i \in S$ such that

$$u(\hat{\sigma}_i(\theta), \tau_i(\theta); \theta_i) \geq u(\hat{\sigma}_i(\theta'), \tau_i(\theta'); \theta_i)$$

where θ' is an S-variant of θ.

The following definition specifies a fairness property for mechanisms. It requires that any two agents with same waiting costs must get same utility.

Definition 5. A mechanism $(\hat{\sigma}, \tau)$ satisfies *equal treatment of equals* if $\forall\, \theta \in \mathbb{R}^n_+$ and $\forall\, i \neq j \in N$,

$$\theta_i = \theta_j \implies u(\hat{\sigma}_i(\theta), \tau_i(\theta); \theta_i) = u(\hat{\sigma}_j(\theta), \tau_j(\theta); \theta_j)$$

Implementing queue-efficiency should not entail wastage of resources. The following definition captures this aspect by requiring that sum of transfers, for any profile of costs, never exceeds zero.

Definition 6. A mechanism $(\hat{\sigma}, \tau)$ is *feasible* if $\forall\, \theta \in \mathbb{R}^n_+$,

$$\sum_{i \in N} \tau_i(\theta) \leq 0$$

Result 1. A Q-EFF mechanism (σ, τ) is strategy-proof *if and only if* $\forall\, \theta \in \mathbb{R}^n_+$ and $\forall\, i \in N$,

$$\tau_i(\theta) = -\sum_{j \neq i} z(\sigma_j(\theta))\theta_j + h_i(\theta_{-i})$$

Proof: Since the domain of cost profiles $\mathbb{R}^n_+$ is convex, the result follows from Theorem 2 of Holmström [10]. □

Any Q-EFF mechanism with transfers given by Result 1 is known as a Vickrey-Clarke-Groves (VCG) mechanism.

Result 2. A Q-EFF mechanism (σ, τ) is strategy-proof *if and only if* $\forall\, \theta \in \mathbb{R}^n_+$ and $\forall\, i \in N$,

$$\tau_i(\theta) = -\sum_{j \in P'_i(\theta)} \left[z(\sigma_j(\theta)) - z(\sigma_j(\theta_{-i})) \right] \theta_j + g_i(\theta_{-i})$$

Proof: The result follows from Result 1 by substituting $h_i(\theta_{-i}) = \sum_{j \neq i} z(\sigma_j(\theta_{-i}))\theta_j + g_i(\theta_{-i})$. □

This paper attempts to specify the class of mechanisms that satisfy Q-EFF, WGS and continuity. From definition it follows that WGS mechanisms are necessarily strategy-proof. Thus, we need to search the class of transfers given by Result 2 for a WGS and continuous transfer map. Effectively, the additional restrictions of WGS and continuity impose a structure on $g_i(\theta_{-i})$ function. The latter imposes what follows.

Definition 7. A Q-EFF and WGS mechanism (σ, τ) is *upper semi continuous* (USC) if $\forall\, i \in N$ and $\forall\, \alpha \in \mathbb{R}_+$, the set $\{x \in \mathbb{R}^{n-1}_+ : g_i(x) \geq \alpha\}$ is closed in $\mathbb{R}_+$.

Definition 8. A Q-EFF and WGS mechanism (σ, τ) is *lower semi continuous* (LSC) if $\forall\, i \in N$ and $\forall\, \alpha \in \mathbb{R}_+$, the set $\{x \in \mathbb{R}^{n-1}_+ : g_i(x) \leq \alpha\}$ is closed in $\mathbb{R}_+$.

Definition 9. A Q-EFF and WGS mechanism (σ, τ) is *continuous* if it is USC as well as LSC.

3. Results

3.1. Prelude to Main Results.

We discuss the 2 agent case first, because it is the building block of n agent result. Suppose $N = \{1, 2\}$. To keep the incentive problem non-trivial, we assume $z(1) \neq z(2)$.

Theorem 1. In a 2 agent multiple machine queueing problem, a Q-EFF mechanism (σ, τ) is WGS *if and only if* there exists $\eta \in [0, \infty]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

- $\forall\, \theta \in \mathbb{R}^2_+$ and $\forall\, i \neq j \in \{1, 2\}$,

$$g_i(\theta_j) = \begin{cases} (z(2) - z(1)) \min\{\theta_j, \eta\} & \text{if } \theta_j \neq \eta \\ \alpha_i (z(2) - z(1))\eta & \text{if } \theta_j = \eta \end{cases}$$

- $\max\{\alpha_1, \alpha_2\} = 1$

Sketch of the Proof: Consider a Q-EFF and WGS mechanism (σ, τ). Let $g(.)$ be the maps associated with $\tau(.)$. For each $\theta, \theta' \in \mathbb{R}^2_+$ such that θ' is an order-preserving $\{1,2\}$-variant of θ, the following holds. W.l.o.g. assume that $\sigma_1(\theta) = \sigma_1(\theta') = 1$, $\sigma_2(\theta) = \sigma_2(\theta') = 2$ and $1 \succ 2$. Therefore, $\theta_1 \geq \theta_2$ and $\theta'_1 \geq \theta'_2$. Consider the $\{1,2\}$-deviation from true profile θ to misreport θ'. WGS requires either

$$g_1(\theta_2) - g_1(\theta'_2) \geq [z(2) - z(1)][\theta_2 - \theta'_2] \tag{1}$$

or

$$g_2(\theta_1) - g_2(\theta'_1) \geq 0. \tag{2}$$

Consider the $\{1,2\}$-deviation from true profile θ' to misreport θ. WGS requires either

$$g_1(\theta_2) - g_1(\theta'_2) \leq [z(2) - z(1)][\theta_2 - \theta'_2] \tag{3}$$

or

$$g_2(\theta_1) - g_2(\theta'_1) \leq 0. \tag{4}$$

If any of these equations hold with equality, WGS is ensured by the mechanism irrespective of whether $\{1,2\}$ deviates from θ to θ' or from θ' to θ. Such a situation will be referred to as *WGS holding with equality*. If from each of the two pairs of equations, one holds with strict inequality, WGS requires that the two equations thus chosen must have their inequalities in the opposite direction. Such a situation will be referred to as *WGS holding with strict inequality*. E.g. if (1) & (4) hold with inequality, we can rewrite the equations as follows.

$$g_1(\theta_2) - g_1(\theta'_2) > [z(2) - z(1)][\theta_2 - \theta'_2] \tag{5}$$

$$g_2(\theta_1) - g_2(\theta'_1) < 0 \tag{6}$$

The first step of the proof (Proposition 1 in section 6.1.1 of Appendix) is to prove that if there is a pair of profiles θ and θ' such that WGS holds with strict inequality, w.l.o.g. θ and θ' such that (5) and (6) hold, then $\theta_2 \leq \theta_1 < \theta'_2 \leq \theta'_1$ and the $g(.)$ maps are such that they have a kink in their graphs at some point $\eta \in [\theta_1, \theta'_2]$. Only at this point there can be a discontinuity for at most one agent, that too with both side limits being equal. The second step of the proof (Proposition 2 in section 6.1.2 of Appendix) is to prove that if there is no pair $\theta, \theta' \in \mathbb{R}^2_+$ such that WGS holds with strict inequality, then the $g(.)$ maps are such that; *either* $\forall\, i \neq j \in \{1,2\}$ and $\forall\, \theta_j \in \mathbb{R}_+$, $g_i(\theta_j) = 0$ *or* $\forall\, i \neq j \in \{1,2\}$ and $\forall\, \theta_j \in \mathbb{R}_+$, $g_i(\theta_j) = [z(2) - z(1)]\theta_j$. Necessity follows from Propositions 1 and 2.

Since all the logical arguments involved in proving these propositions are reversible, sufficiency for order preserving deviations follows. We prove the sufficiency for order interchanging deviations in the last step of the proof (Section 6.1.2 of Appendix). □

Figure 1 shows the graphical representation of Theorem 1 when $\eta \in (0, +\infty)$ and $\alpha_2 < 1$. The η value decides the position of the *kink* point while α value decides whether there is any discontinuity at the kink point or not. Also Theorem 1 implies that there cannot be discontinuity in both $g(.)$ maps. We interpret the values of α and η in the following way;

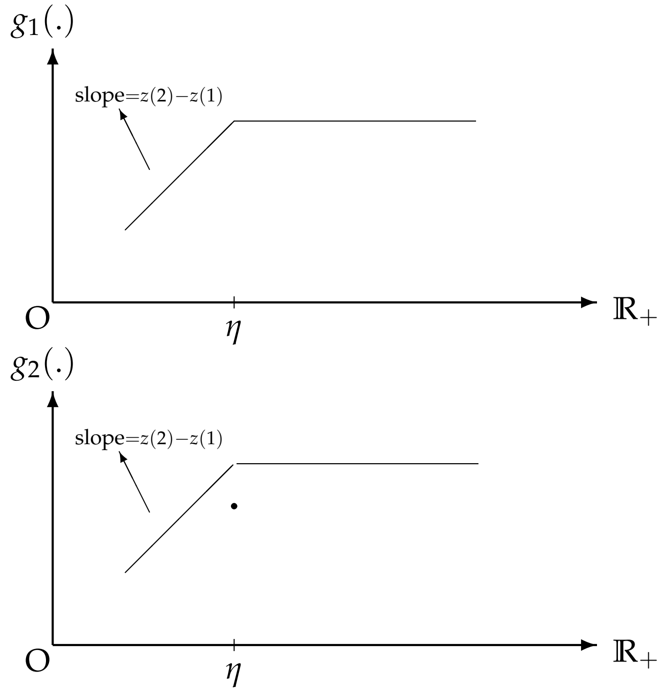


FIGURE 1

(1) If $\alpha_t = 1, \forall\, t = 1, 2$, then there is no discontinuity in either of the $g(.)$, no matter what the value of η.
(2) If $\eta = 0$ then the $g(.)$ maps are horizontal straight lines along the x-axis, irrespective of the values of α.
(3) If $\eta = \infty$ then the values of α become immaterial since then $g(.)$ maps will simply be upward sloping straight lines with the slope $z(2) - z(1)$.

Remark 1. In the single machine setting, we may normalize the speed of the machine to be equal to 1, that is, $z(2) - z(1) = 1$ (as e.g. Mitra and Mutuswami [17]). Then, the expression in Theorem 1 reduces to the following. There exists $\eta \in [0, \infty]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

- $\forall\, \theta \in \mathbb{R}^2_+$ and $\forall\, i \neq j \in \{1, 2\}$,

$$g_i(\theta_j) = \begin{cases} \min\{\theta_j, \eta\} & \text{if } \theta_j \neq \eta \\ \alpha_i \eta & \text{if } \theta_j = \eta \end{cases}$$

- $\max\{\alpha_1, \alpha_2\} = 1$.

Remark 2. From Theorem 1 it follows that when the number of agents is 2; any mechanism will satisfy Q-EFF and WGS only if the number of discontinuities in the mechanism does not exceed one. Moreover, any discontinuity, if present, can occur for only one of the two agents. Such discontinuity must occur at the kink point of the transfer map and must have equal both side limits.

Corollary 1. In a 2 agent multiple machine queueing problem, a Q-EFF and WGS mechanism is *Lower Semi-Continuous*.

From the Corollary 1, it is obvious that in a two agent case, imposition of the USC property, leads to *Continuity* of the mechanism. Hence, the following result;

Result 3. In a 2 agent multiple machine queueing problem, a Q-EFF mechanism (σ, τ) is WGS and USC *if and only if* there exists $\eta \in [0, \infty]$ such that $\forall\, \theta \in \mathbb{R}^2_+$ and $\forall\, i \neq j \in \{1, 2\}$,

$$g_i(\theta_j) = (z(2) - z(1)) \min\{\eta, \theta_j\}.$$

The discontinuity in the two agent result is 'mild', in the sense that the both side limits at the only possible point of discontinuity (the kink point), are equal. Also, if we impose the fairness requirement of equal treatment of equals on Theorem 1, the possibility of discontinuity gets eliminated (as $\alpha_1 = \alpha_2$) and we get the continuous mechanism specified by Result 3. Hence, the axiom of continuity has technical as well as fairness justifications in a 2 agent multiple machine queueing problem.

3.2. **Main Results.** Suppose $N = \{1, 2, \ldots, n\}$. Let $\forall\, i = 1, 2, \ldots, n-1, \Delta z(i) \stackrel{def}{=} z(i+1) - z(i)$.

We use Theorem 1 to obtain the n agent $g(.)$ maps associated with queue-efficient, weak group strategy-proof and *continuous* mechanisms. However, proof of Theorem 1 requires that each agent have a different waiting time (that is, $z(2) \neq z(1)$). Therefore, any n agent $g(.)$ map obtained by aggregating the 2 agent $g(.)$ maps given by Theorem 1, must implicitly assume that for all $i = 1, \ldots, n-1$, $\Delta z(i) \neq 0$. For the rest of the paper, we assume the same.

Theorem 2. In a multiple machine queueing problem, a Q-EFF mechanism (σ, τ) satisfies WGS and continuity only if there exist non-negative $\big((\eta_{ij}(s))_{j\neq i}\big)_{s=1,\ldots,n-1}$ such that $\forall\, i \in N$ and $\forall\, \theta_{-i} \in \mathbb{R}^{n-1}_+$,

$$g_i(\theta_{-i}) = \sum_{j\neq i} \Delta z(\sigma_j(\theta_{-i})) \min\{\theta_j, \eta_{ij}(\sigma(\theta_{-i}))\}$$

Proof: Pick a $\theta \in \mathbb{R}^n_+$ such that $\sigma_1(\theta_{-2}) = \sigma_2(\theta_{-1})$, that is, agents 1 and 2 are adjacently ranked in the efficient ranking for profile θ. While analyzing the impact of change in 2's report on the $g_1(.)$ function; we assume that the announcements of other agents, that is, the vector θ_{-1-2} is constant. The impact of θ_{-1-2} can be deemed to enter the $g_1(.)$ function through a constant intercept term $F_{12}(\theta_{-1-2})$. Also upfront, we cannot rule out the possibility of this $F_{12}(.)$ depending on $\sigma_2(\theta_{-1})$, that is, the rank of agent 2 when 1 is not around. We can then invoke Result 3 to write that

$$\begin{aligned} g_1(\theta_{-1}) &= \Delta z(\sigma_2(\theta_{-1})) \min\{\theta_2, \eta_{12}(\sigma_2(\theta_{-1}); \theta_{-1-2})\} \\ &+ F_{12}(\sigma_2(\theta_{-1}); \theta_{-1-2}) \end{aligned} \tag{7}$$

Consider a profile $\theta' = (\theta'_1, \theta_{-1})$ such that $\sigma_1(\theta'_{-3}) = \sigma_3(\theta'_{-1})$, that is, agents 1 and 3 are adjacently ranked in the efficient ranking. Again invoking Result 3, now for agents $\{1, 3\}$, we can write

$$\begin{aligned} g_1(\theta_{-1}) &= \Delta z(\sigma_3(\theta_{-1})) \min\{\theta_3, \eta_{13}(\sigma_3(\theta_{-1}); \theta_{-1-3})\} \\ &+ F_{13}(\sigma_3(\theta_{-1}); \theta_{-1-3}) \end{aligned} \tag{8}$$

The left hand side of both the above equations are the same. This means that the $F_{13}(.)$ must contain $\Delta z(\sigma_2(\theta_{-1}))\min\{\theta_2, \eta_{12}(.)\}$. This in turn implies that (i) the $\eta_{12}(.)$ function may contain $\sigma_3(\theta_{-1})$ as its argument, (ii) the $\eta_{12}(.)$ does not depend on θ_3, (iii) the $F_{13}(.)$ term may contain $\sigma_2(\theta_{-1})$ as its argument and (iv) the $F_{13}(.)$ does not depend θ_2 as argument.
Arguing similarly for the terms $F_{12}(.)$ and $\Delta z(\sigma_3(\theta_{-1}))\min\{\theta_3, \eta_{13}(.)\}$ in the right hand side of (7) and (8), respectively; we can write that

$$\begin{aligned} g_1(\theta_{-1}) &= \Delta z(\sigma_2(\theta_{-1}))\min\{\theta_2, \eta_{12}(\sigma_2(\theta_{-1}), \sigma_3(\theta_{-1}); \theta_{-1-2-3})\} \\ &+ \Delta z(\sigma_3(\theta_{-1}))\min\{\theta_3, \eta_{13}(\sigma_2(\theta_{-1}), \sigma_3(\theta_{-1}); \theta_{-1-2-3})\} \\ &+ F_{123}(\sigma_2(\theta_{-1}), \sigma_3(\theta_{-1}); \theta_{-1-2-3}) \end{aligned}$$

Continuing this recursion for all $j \neq 1$, we get that

$$g_1(\theta_{-1}) = \sum_{j\neq 1} \Delta z(\sigma_j(\theta_{-1}))\min\{\theta_j, \eta_{1j}(\sigma(\theta_{-1}))\} + C(\sigma(\theta_{-1}))$$

Note that $\tau_1(0, 0, \ldots, 0) = 0$ *irrespective* of what tie-breaking rule we use. Hence, it must be that $C(\sigma(\theta_{-1})) = 0$ and $\eta_{1j}(\sigma(\theta_{-1})) \in [0, \infty], \forall\, j \in N - \{1\}$ and $\forall\, \theta_{-1} \in \mathbb{R}^{n-1}_{+}$. Arguing similarly, we can establish the result for all $i \in N$. □

Remark 3. In the single machine setting, we may normalize the speed of the machine to be equal to 1, that is, $\forall\, i = 1, 2, \ldots, n-1, \Delta z(i) = 1$ (as e.g. Mitra and Mutuswami [17]). Therefore, the expression in Theorem 2 reduces to the following. There exist non-negative $\big((\eta_{ij}(s))_{j\neq i}\big)_{s=1,\ldots,n-1}$ such that $\forall\, i \in N$ and $\forall\, \theta_{-i} \in \mathbb{R}^{n-1}_{+}$,

$$g_i(\theta_{-i}) = \sum_{j\neq i} \min\{\theta_j, \eta_{ij}(\sigma(\theta_{-i}))\}.$$

Remark 4. In proving Theorem 2, we check only for 2 agent coalitional deviations. Therefore, in statement of Theorem 2, we can substitute the WGS axiom with the pair-wise strategy-proofness axiom of Mitra and Mutuswami [17]. This implies that Theorem 2 is a generalization of Theorem 3.7 in Mitra and Mutuswami [17]. This becomes clear when (i) we assume *weak linearity* instead of *continuity* in Theorem 2 and (ii) consider the single machine setting. Indeed, weak linearity implies that for all $i \neq j \in N$ and for all possible ranking $\bar{s}$ among the agents other than i (or j), $\eta_{ij}(\bar{s}) \in \{0, \infty\}$. For all such $\bar{s}$, it follows from Theorem 1 that $\eta_{ij}(\bar{s}) = \eta_{ji}(\bar{s})$. Further, for all $\theta \in \mathbb{R}^n_+$ with $\theta_1 > \theta_2 > \ldots > \theta_n$ and $\forall\, i \neq k \in N, \theta_i > \theta_k \implies \eta_{ik}(\sigma(\theta_{-i})) \geq \eta_{ki}(\sigma(\theta_{-k}))$.[6] Therefore, in the single machine setting (that is, when $\forall\, i = 1, \ldots, n-1, \Delta z(i) = 1$), the necessity result Theorem 2 implies Theorem 3.7 in Mitra and Mutuswami [17]; not conversely.

The complete characterization of mechanisms that satisfy queue-efficiency, weak group strategy-proofness and continuity, would require proving that the transfers in Theorem 2 along with queue-efficiency, satisfy weak group strategy-proofness. This turns out to be difficult since the η terms depend on identity of agents, as well as the

[6]Suppose $\exists\, \theta$ such that $\theta_1 > \ldots > \theta_i > \theta_j > \theta_k > \ldots > \theta_n$ and $\eta_{ik}(\sigma(\theta_{-i})) = 0 < \eta_{ki}(\sigma(\theta_{-k})) = \infty$. Then, WGS is violated in an order preserving $\{i, k\}$ deviation from θ to $(\theta'_i, \theta'_k, \theta_{-i-k})$ where $\theta'_i > \theta_i$ and $\theta'_k < \theta_k$.

$\sigma(\theta_{-i})$ vector. Instead, in the following theorem, we generate a class of continuous mechanisms that satisfy Q-EFF and WGS, by allowing the η terms to depend only on the rank $\sigma_j(\theta_{-i})$ (instead of the vector of ranks $\sigma(\theta_{-i})$).

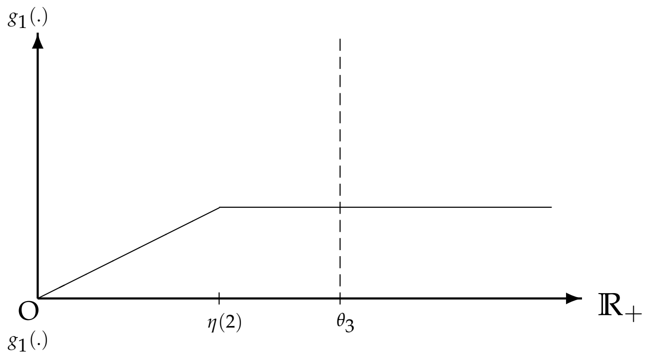
Theorem 3. In a multiple machine queueing problem, a Q-EFF mechanism (σ, τ) satisfies WGS and continuity if there exists $(\eta(s))_{s=1,\ldots,n-1}$ such that $\forall\, i \in N$ and $\forall\, \theta_{-i} \in \mathbb{R}_+^{n-1}$,

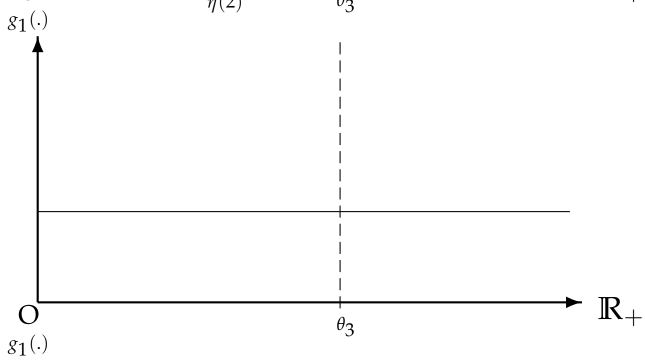
$$g_i(\theta_{-i}) = \sum_{j \neq i} \Delta z(\sigma_j(\theta_{-i})) \min\{\theta_j, \eta(\sigma_j(\theta_{-i}))\} \tag{9}$$

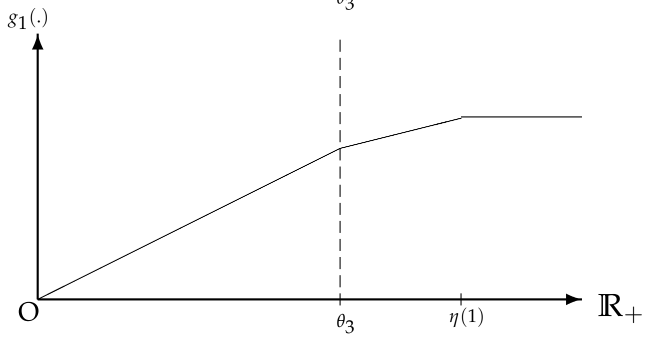
$$\forall\, s = 1, 2, \ldots, n-2, \quad \eta(s+1) \geq \eta(s) \tag{10}$$

Proof: See Appendix. □

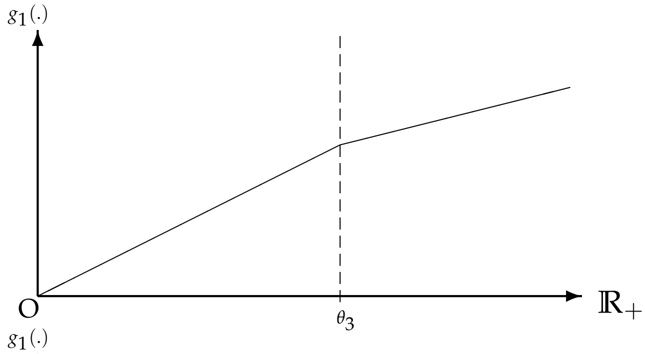
The following set of graphs capture the implication of Theorem 3 when $n = 3$. (9) and (10) imply that 3 agent $g(.)$ maps must look like one of the following five figures[7].

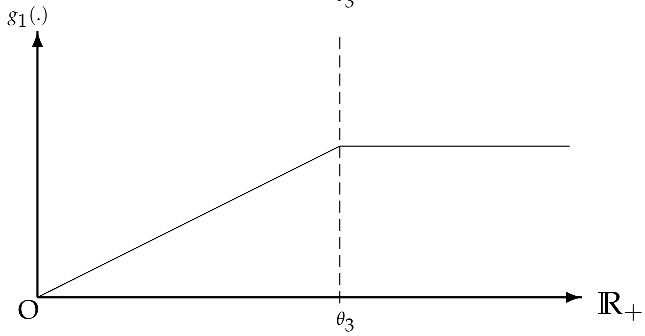






[7]The second map is drawn with a positive intercept to emphasize the flat curvature.





The following is a queue-efficient, weak group strategy-proof and continuous mechanism, in a 3 agent single machine queueing setting, which *does not belong* to the class of mechanisms specified by (9) and (10) in Theorem 3.

Example 1. Let $N = \{1,2,3\}$ and $\Delta z(1) = \Delta z(2) = 1$. Therefore, the $\sigma(\theta_{-i})$ term in Theorem 2 can be either $(1,2)$ or $(2,1)$. Let $\eta_{12}(1,2) = 0$ and $\eta_{ij}(1,2) = \eta_{ij}(2,1) = \infty$ for all $ij \neq 12$. Using the expression in Remark 3, we can write that $\forall\, \theta \in \mathbb{R}^3_+$,

- if $\theta_1 > \theta_2 > \theta_3$ then $\tau_1(\theta) = -\theta_2, \tau_2(\theta) = 0, \tau_3(\theta) = \theta_1 + \theta_2$
- if $\theta_1 > \theta_3 > \theta_2$ then $\tau_1(\theta) = 0, \tau_2(\theta) = \theta_3, \tau_3(\theta) = \theta_1$
- if $\theta_2 > \theta_1 > \theta_3$ then $\tau_1(\theta) = 0, \tau_2(\theta) = -\theta_1, \tau_3(\theta) = \theta_1 + \theta_2$
- if $\theta_2 > \theta_3 > \theta_1$ then $\tau_1(\theta) = \theta_3, \tau_2(\theta) = 0, \tau_3(\theta) = \theta_2$
- if $\theta_3 > \theta_1 > \theta_2$ then $\tau_1(\theta) = \theta_3, \tau_2(\theta) = \theta_1 + \theta_3, \tau_3(\theta) = 0$
- if $\theta_3 > \theta_2 > \theta_1$ then $\tau_1(\theta) = \theta_2 + \theta_3, \tau_2(\theta) = \theta_3, \tau_3(\theta) = 0$

It can easily be checked that these continuous transfers along with queue-efficiency satisfy weak group strategy-proofness.

Remark 5. In the single machine setting, we may normalize the speed of the machine to be equal to 1, that is, $\forall\, i = 1, 2, \ldots, n-1, \Delta z(i) = 1$ (as e.g. Mitra and Mutuswami [17]). Therefore, the expression in Theorem 3 reduces to the following. There exist non-negative $(\eta(s))_{s=1,\ldots,n-1}$ such that $\forall\, i \in N$ and $\forall\, \theta_{-i} \in \mathbb{R}^{n-1}_+$,

- $g_i(\theta_{-i}) = \sum_{j \neq i} \min\{\theta_j, \eta(\sigma_j(\theta_{-i}))\}$
- $\forall\, s = 1, \ldots, n-2, \eta(s+1) \geq \eta(s)$.

We can easily show that k-pivotal mechanisms introduced by Mitra and Mutuswami [17] is a special subclass of the class of mechanisms given by Theorem 3. Indeed, $\forall\ k =$

$1, \ldots, n$, (9) reduces to the k-pivotal mechanism when

$$\eta(s) = \begin{cases} \infty & \text{if } s \geq k \\ 0 & \text{if } s < k \end{cases}$$

From this arrangement of η values, we may formulate a generalization of k-pivotal mechanism for the multiple machine setting. For this purpose, we need the following notation: $\forall\, s = 1, \ldots, n$, let $\theta(s) := \{\theta_j | \sigma_j(\theta) = s\}$. Then, $\forall\, k = 1, \ldots, n$, $\forall\, \theta \in \mathbb{R}^n_+$ and $\forall\, i \in N$,

$$\tau_i^k(\theta) = \begin{cases} \sum_{s=\sigma_i(\theta)+1,\ldots,k} \Delta z(s-1)\theta(s) & \text{if } \sigma_i(\theta) < k \\ 0 & \text{if } \sigma_i(\theta) = k \\ \sum_{s=k,\ldots,\sigma_i(\theta)-1} \Delta z(s)\theta(s) & \text{if } \sigma_i(\theta) > k \end{cases}$$

Remark 6. An interesting subclass of the mechanisms described by Theorem 3 are the feasible mechanisms. For all such mechanisms, $\forall\, \theta \in \mathbb{R}^n_+$,

$$\sum_{i \in N} \tau_i(\theta) = -\sum_{s=2,3,\ldots,n} (s-1)\Delta z(s-1)\theta(s) + \sum_{s=1,2\ldots,n} [(n-s)\Delta z(s) \min\{\theta(s), \eta(s)\} + (s-1)\Delta z(s-1) \min\{\theta(s), \eta(s-1)\}]$$

It can easily be seen that feasible mechanisms in the class provided by Theorem 3 must have $\eta(1) = 0$. This means that; in the panel of graphs before, the third and fourth possibilities are ruled out.

Remark 7. The class of mechanisms given by (9) and (10) in Theorem 3 continue to satisfy queue-efficiency, weak group strategy-proofness and continuity, *even* when there exists $s = 1, \ldots, n-1$ such that $z(s) = z(s+1)$ (that is, there are agents who have to wait the same time to get service).

Remark 8. The class of mechanisms specified by (9) and (10) in Theorem 3 is quite large. These mechanisms also satisfy the fairness property of equal treatment of equals. We hope that these are the only mechanisms that satisfy queue-efficiency, weak group strategy-proofness, equal treatment of equals and continuity, but cannot say for sure.

4. Discussion

The queueing problem assumes that (i) machine speeds are constant across jobs, and (ii) per unit time waiting cost of each agent is constant over time. Relaxing assumption (i) would mean that each machine j is associated with a sequence $\left\{s_j^t\right\}_{t=1}^{\infty}$ where $s_j^t > 0$, $\forall\, t$. Any agent placed on kth position in the queue for machine j would have to wait $\sum_{t=1}^{k} s_j^t$ to get his job completed. Since these speeds are known to the planner; the planner can choose the n smallest numbers out of the set $\left\{\left\{\sum_{t=1}^{k} s_j^t\right\}_{k=1}^{n}\right\}_{j=1}^{m}$ and arrange them in a non-decreasing order to get the $z \equiv (z(1), \ldots, z(n))$ vector. The rest of the analysis would remain same as above.

Relaxing the assumption (ii) is more difficult. If we measure time as a discrete variable and assume the waiting cost to vary with time t; we get that the cost to agent i

upon being assigned the rank k is

$$\sum_{t=1}^{[z(k)]} \theta_i(t) + \{z(k) - [z(k)]\}\, \theta_i([z(k)] + 1)$$

where $[x]$ is the integer nearest to x but smaller than x, $\forall\, x > 0$. This leads to a different definition of queue-efficiency and may well lead to different results. Identifying the necessary and sufficient conditions for mechanisms to satisfy Q-EFF and WGS in this setting, would be an interesting but very difficult problem. The other alternative is to consider time as a continuous variable. Mitra [16] proves that implementing queue-efficiency in dominant strategies with balanced budget, requires the cost function to be linear in time.

As in Theorems 2 and 3, kinked mechanisms may also be obtained in analysis of group strategy-proofness in related fields like the indivisible good (single and multiple) allocation problem and the public good provision problem.

5. Conclusion

We analyze queueing problem with multiple machines and identical jobs. The crucial aspect here is how the $g_i(\theta_{-i})$ function (which can be any arbitrary function if we require only strategy-proofness) behaves when we require weak group strategy-proofness. We assume continuity, which is weaker than *weak linearity* (unlike in Mitra and Mutuswami [17]) and this results in transfer maps having kinks. Continuity is not demanding in this structure from technical perspective because the 2 agent queue-efficient and weak group strategy-proof mechanism is *lower semi-continuous*. The same holds true from a fairness perspective, too, because any such 2 agent mechanism satisfying equal treatment of equals is continuous.

Our results show that if we restrict the $g_i(.)$ function to be continuous then it must be that (i) it is piecewise linear and (ii) if θ_j $(j \neq i)$ changes without changing the queue order then it cannot have a flat stretch followed by an increasing stretch. This feature prevails even in the single machine case. We also provide a class of mechanisms satisfying queue-efficiency, weak group strategy-proofness and continuity. The k-pivotal mechanisms introduced by Mitra and Mutuswami [17], are a special subclass of this class.

6. Appendix

6.1. **Proof of Theorem 1:** $\forall\, S \subseteq N$, $\forall\, \theta, \theta' \in \mathbb{R}^n_+$ such that θ' is an S-variant of θ, and $\forall\, i \in S$, let $l_i(\theta, \theta') := u(\hat{\sigma}_i(\theta), \tau_i(\theta); \theta_i) - u(\hat{\sigma}_i(\theta'), \tau_i(\theta'); \theta_i)$ capture the change in utility to member i of the misreporting coalition S as they deviate from (truth) profile θ to (misreport) profile θ'. In what follows, we prove the required necessity and sufficiency.

6.1.1. **Proof of Only If in Theorem 1:** Consider a Q-EFF and WGS mechanism (σ, τ). If a pair of order preserving $\{1,2\}$-variants exists such that WGS amongst them holds with strict inequality then from Proposition 1 (below) we can obtain the $g(.)$ maps. The expression in theorem captures this case, when η takes a finite positive value. If for all pairs of order preserving $\{1,2\}$-variants, WGS holds with equality, then from Proposition 2 (below), we obtain the corresponding $g(.)$ maps. The expression in theorem captures these two subcases, when $\eta = 0$ or when (with a slight abuse of notation) $\eta = \infty$. Since we are proving necessity in this subsection, we need not consider the order interchanging deviations.

Proposition 1. If there is a pair $\theta, \theta' \in \mathbb{R}^2_+$ such that WGS holds with strict inequality, w.l.o.g. θ and θ' are such that (5) and (6) hold, then there are $\eta \in [\theta_1, \theta'_2]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

- $\forall\, i \neq j \in \{1,2\}$,

$$g_i(\theta_j) = \begin{cases} (z(2) - z(1)) \min\{\theta_j, \eta\} & \text{if } \theta_j \neq \eta \\ \alpha_i (z(2) - z(1))\eta & \text{if } \theta_j = \eta \end{cases}$$

- $\max\{\alpha_1, \alpha_2\} = 1$.

Proof: Let $\theta, \theta' \in \mathbb{R}^2_+$ such that WGS holds with strict inequality. W.l.o.g. assume θ and θ' are such that (5) and (6). Claims 1 to 10 (below) prove the proposition.

Claim 1. $\theta_2 \leq \theta_1 < \theta'_2 \leq \theta'_1$.

Proof: Let us eliminate the other possibilities, namely the following five cases;
Case 1: $\theta'_1 \geq \theta_2, \theta_1 \geq \theta'_2$
Design a $\{1,2\}$ deviation from $\alpha \equiv (\theta_1, \theta'_2)$ to $\beta \equiv (\theta'_1, \theta_2)$. By (5) and (6), $l_t(\alpha, \beta) < 0$, $\forall\, t = 1, 2$ and thus WGS is violated.
Case 2: $\theta'_2 = \theta'_1 < \theta_2 < \theta_1$
Design a $\{1,2\}$ deviation from $\beta \equiv (\theta_2, \theta'_2)$ to $\alpha \equiv (\theta'_1, \theta_2)$. By (5), $-z(2)\theta_2 + g_1(\theta_2) > -z(1)\theta_2 - [z(2) - z(1)]\theta'_2 + g_1(\theta'_2) \implies l_1(\beta, \alpha) < 0$. Thus, $WGS \implies l_2(\beta, \alpha) \geq 0$ which means $-z(1)\theta'_2 - [z(2) - z(1)]\theta'_1 + g_2(\theta'_1) \leq -z(2)\theta'_2 + g_2(\theta_2) \implies$ (**a**) $g_2(\theta'_1) \leq g_2(\theta_2)$. In a $\{1,2\}$ deviation from $\tilde{\beta} \equiv (\theta_1, \theta'_2)$ to $\tilde{\alpha} \equiv (\theta_2, \theta_2)$; from (5), $l_1(\tilde{\beta}, \tilde{\alpha}) < 0$ and so $WGS \implies l_2(\tilde{\beta}, \tilde{\alpha}) \geq 0 \implies$ (**b**) $g_2(\theta_1) \geq g_2(\theta_2)$. Combining conditions (**a**) and (**b**) we get that $g_2(\theta_1) \geq g_2(\theta'_1)$ which contradicts (6).
Case 3: $\theta'_2 = \theta'_1 < \theta_2 = \theta_1$
In a $\{1,2\}$ deviation from $\alpha \equiv (\theta_1, \theta'_2)$ to $\beta \equiv (\theta'_1, \theta_2)$; (5) implies that $-z(2)\theta_2 + g_1(\theta_2) > -z(1)\theta_2 - [z(2) - z(1)]\theta'_2 + g_1(\theta'_2) \Rightarrow l_1(\alpha, \beta) < 0$. Similarly (6) implies that $g_2(\theta'_1) - [z(2) - z(1)]\theta'_1 - z(1)\theta'_2 > g_2(\theta_1) - z(2)\theta'_2 \Rightarrow l_2(\alpha, \beta) < 0$ which violates WGS.
Case 4: $\theta'_2 < \theta'_1 < \theta_2 = \theta_1$
Design a $\{1,2\}$ deviation from $\beta \equiv (\theta_1, \theta'_1)$ to $\alpha \equiv (\theta'_1, \theta_2)$. (6) implies that $g_2(\theta'_1) - z(1)\theta'_1 - [z(2) - z(1)]\theta'_1 > g_2(\theta_1) - z(2)\theta'_1 \Rightarrow l_2(\beta, \alpha) < 0$. So $WGS \implies l_1(\beta, \alpha) \geq 0 \implies$ (**c**) $g_1(\theta_2) - g_1(\theta'_1) \leq [z(2) - z(1)][\theta_2 - \theta'_1]$. For an $\{1,2\}$ deviation from $\tilde{\beta} \equiv (\theta_1, \theta'_2)$ to $\tilde{\alpha} \equiv (\theta'_1, \theta'_1)$; (6) implies that $l_2(\tilde{\beta}, \tilde{\alpha}) < 0$, and so $WGS \implies l_1(\tilde{\beta}, \tilde{\alpha}) \geq 0$ which implies that (**d**) $g_1(\theta'_1) - g_1(\theta'_2) \leq [z(2) - z(1)][\theta'_1 - \theta'_2]$. Then, (5) minus (**d**) we get that $g_1(\theta_2) - g_1(\theta'_1) > [z(2) - z(1)][\theta_2 - \theta'_1]$ which contradicts (**c**).
Case 5: $\theta'_2 < \theta'_1 < \theta_2 < \theta_1$
Consider four profiles (θ_2, θ'_2), (θ_1, θ_2), (θ_1, θ'_2), and (θ_2, θ_2). Given (5), if $g_2(\theta_2) < g_2(\theta_1)$ then a $\{1,2\}$ coalition deviation from the first profile to the second makes both agents

strictly better off; and if $g_2(\theta_2) > g_2(\theta_1)$ then a $\{1,2\}$ coalition deviation from the third profile to the fourth leads to both agents being strictly better off. Therefore, $WGS \implies$ (**e**) $g_2(\theta_2) = g_2(\theta_1)$. For a pair of profiles $\alpha \equiv (\theta_2, \theta_1')$ and $\beta \equiv (\theta_1', \theta_2)$; from (6) and (**e**) it follows that $g_2(\theta_1') > g_2(\theta_2) \implies g_2(\theta_1') - [z(2) - z(1)]\theta_1' - z(1)\theta_1' > g_2(\theta_2) - z(2)\theta_1' \implies l_2(\alpha, \beta) < 0$. So $WGS \implies l_1(\alpha,\beta) \geq 0 \implies$ (**f**) $g_1(\theta_2) - g_1(\theta_1') \leq [z(2) - z(1)][\theta_2 - \theta_1']$.
Consider four profiles (θ_1, θ_2'), (θ_1', θ_1'), (θ_1, θ_1'), and (θ_1', θ_2'). If $g_1(\theta_1') - g_1(\theta_2') > [z(2) - z(1)](\theta_1' - \theta_2')$, then in a $\{1,2\}$ deviation from the first profile to the second; from (6) it follows that both agents are strictly better off. Again if, $g_1(\theta_1') - g_1(\theta_2') < [z(2) - z(1)](\theta_1' - \theta_2')$, then in a deviation from the third profile to the fourth; from (6) it follows that both agents are strictly better off. Thus, $WGS \implies g_1(\theta_1') - g_1(\theta_2') = [z(2) - z(1)](\theta_1' - \theta_2')$. Using (**f**), we can say that $g_1(\theta_2) - g_1(\theta_2') \leq [z(2) - z(1)][\theta_2 - \theta_2']$, which then, contradicts (5). □

By Claim 1 we know that $\theta_2 \leq \theta_1 < \theta_2' \leq \theta_1'$. W.l.o.g., assume $\theta_2 < \theta_1 < \theta_2' < \theta_1'$ and continue the proof[8].

Claim 2.

A: $\forall\, x, y \leq \theta_1$, $g_1(x) - g_1(y) = [z(2) - z(1)][x - y]$
B: $\forall\, x, y < \theta_1$, $g_2(x) - g_2(y) = [z(2) - z(1)][x - y]$
C: $\forall\, x, y \geq \theta_2'$, $g_2(x) - g_2(y) = 0$
D: $\forall\, x, y > \theta_2'$, $g_1(x) - g_1(y) = 0$
E: $\forall\, x \in (\theta_1, \theta_2')$, $g_2(x) \leq g_2(\theta_1')$
F: *If* $\exists\, x \in (\theta_1, \theta_2')$ *such that* $g_2(x) < g_2(\theta_1')$, *then*

$$g_1(\theta_1) - g_1(y) = [z(2) - z(1)][\theta_1 - y], \forall\, y \in (\theta_1, x]$$

Proof:
A: For any $x, y \leq \theta_1, x \neq y$; consider the profiles $(\theta_1, x), (\theta_1', y)\ (\theta_1, y), (\theta_1', x)$. If $g_1(x) - g_1(y) > [z(2) - z(1)][x - y]$ then consider a $\{1,2\}$ deviation from the third profile to the fourth; and if $g_1(x) - g_1(y) < [z(2) - z(1)][x - y]$ then consider a deviation from the first profile to the second. In both cases, by (6), WGS is violated.
B: Pick any x, y, ρ, x' such that $x < y < \rho < x' \leq \theta_1$. If $g_2(x) - g_2(y) > [z(2) - z(1)][x - y]$ then in a $\{1,2\}$ deviation from $\beta \equiv (y, \rho)$ to $\alpha \equiv (x, x')$; $l_2(\beta, \alpha) < 0$. From case **A**, $g_1(x') - g_1(\rho) > 0 \implies l_1(\beta, \alpha) < 0$ which violates WGS. If $g_2(x) - g_2(y) < [z(2) - z(1)][x - y]$, then in a $\{1,2\}$ deviation from (x, ρ) to (y, x'), as before, WGS is violated.
C: Pick any $x, y \geq \theta_2'$. If $g_2(x) > g_2(y)$ then in a $\{1,2\}$ deviation from $\beta \equiv (y, \theta_2')$ to $\alpha \equiv (x, \theta_2)$, given (5); $l_t(\beta, \alpha) < 0, \forall\, t = 1, 2$. If $g_2(x) < g_2(y)$ then in a deviation from (x, θ_2') to (y, θ_2), using (5), $l_t(\beta, \alpha) < 0, \forall\, t = 1, 2$. In both cases WGS is violated.
D: Pick any x, y, x', ρ such that $x > y > x' > \rho \geq \theta_2'$. From the case **C**, we get that $g_2(\rho) - g_2(x') = 0 > [z(2) - z(1)][\rho - x']$. If $g_1(x) > g_1(y)$, then $l_t((x', y), (\rho, x)) < 0, \forall\, t = 1, 2$; and if $g_1(x) < g_1(y)$, then $l_t((x', x), (\rho, y)) < 0, \forall\, t$. In both cases WGS is violated.

[8] This rules out the possibility that $\theta_1 = 0$. We will discuss the implications of that possibility in Remark 9.

E: Say $\exists\, x \in (\theta_1, \theta_2')$ such that $g_2(x) > g_2(\theta_1')$. Then, in a deviation from profile $\alpha \equiv (\theta_1', \theta_2')$ to $\beta \equiv (x, \theta_2)$, $l_2(\alpha,\beta) < 0$; while by (5), $l_1(\alpha,\beta) < 0$. Thus, WGS is violated.
F: For any $y \in (\theta_1, x]$, if $g_1(\theta_1) - g_1(y) > [z(2) - z(1)][\theta_1 - y]$ then $l_1(\alpha,\beta) < 0$ where $\alpha \equiv (x, y)$ to $\beta \equiv (\theta_1', \theta_1)$ while $g_2(x) < g_2(\theta_1') \implies l_2(\alpha,\beta) < 0$. If $g_1(\theta_1) - g_1(y) < [z(2) - z(1)][\theta_1 - y]$, then similarly, it can be shown that $l_t((x,\theta_1),(\theta_1', y)) < 0, \forall\, t = 1, 2$. □

The implications of all the subcases of Claim 2 is depicted in the following set of Figures 2 - 7;

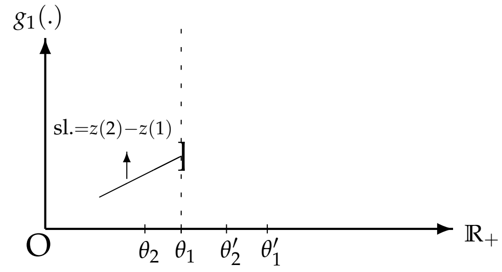


FIGURE 2. Claim 2A

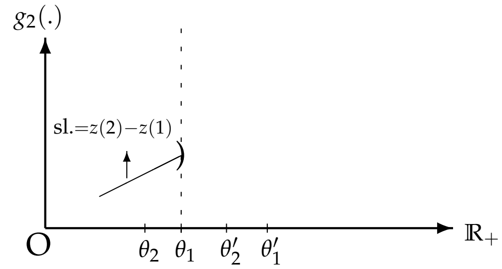


FIGURE 3. Claim 2B

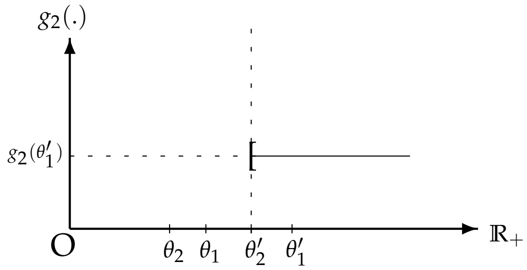


FIGURE 4. Claim 2C

Claim 3. Either (i) $\forall\, y \geq \theta_2'$, $g_1(y) = g_1(\theta_2')$, or (ii) there is $K \in \mathbb{R}$ such that $\forall\, y > \theta_2'$, $g_1(y) = K > g_1(\theta_2')$.

Proof: If $\exists\, y > \theta_2'$ such that $g_1(y) < g_1(\theta_2')$ then $l_1(\alpha,\beta) < 0$ where $\alpha \equiv (x', y)$ and $\beta \equiv (\theta_1, \theta_2')$ with $x' \in (\theta_2', y)$. Then, $WGS \implies l_2(\alpha,\beta) \geq 0 \implies g_2(\theta_1) \leq g_2(x') - [z(2) - z(1)][x' - \theta_1]$. By Claim 2C; $g_2(x')$ is constant, $\forall\, x' \geq \theta_2'$. Since there is no upper bound on y and so, on x'; there can be no lower bound on the value $g_2(\theta_1)$.

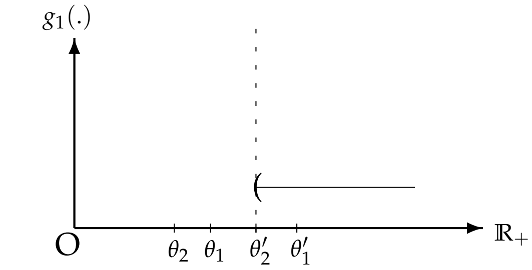


FIGURE 5. Claim 2D

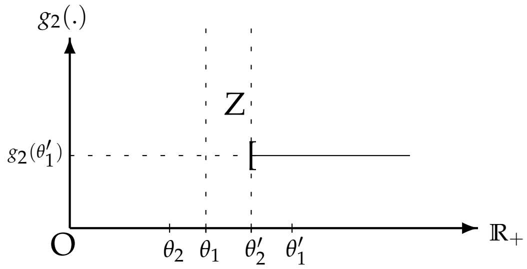


FIGURE 6. Claim 2E $\implies$ interior of zone Z is vacant

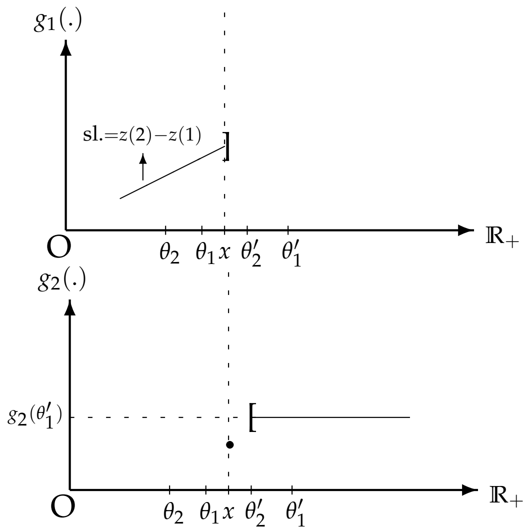


FIGURE 7. Claim 2F

Therefore, $g_2(\theta_1) < 0$ and so, there can be no upper bound on $|g_2(\theta_1)|$. Since $\forall\, i = 1, 2$, $g_i : \mathbb{R}_+ \mapsto \mathbb{R}$, $|g_2(\theta_1)| \in \mathbb{R}$. However, from the *archimedean property* of $\mathbb{R}$, it follows that there exists $n \in \mathbb{N}$ such that $|g_2(\theta_1)| < n$ and so, contradiction. Therefore, $\forall\, y > \theta_2'$, $g_1(y) \geq g_1(\theta_2')$. Using Claim 2D the result follows. □

The graphical implication of Claim 3 is given by Figure 8. In the following Claim 4 we analyze the implication of Claim 3i.

Claim 4. If $\forall\, y \geq \theta_2'$, $g_1(y) = g_1(\theta_2')$, then $g_1(x) \leq g_1(\theta_2'), \forall\, x \in [\theta_1, \theta_2']$

Proof: If $\exists\, x \in [\theta_1, \theta_2')$ such that $g_1(x) > g_1(\theta_2')$ then consider a $\{1, 2\}$ deviation from profile $\alpha \equiv (\delta, y)$ to $\beta \equiv (\theta_2, x)$ where $\theta_2' \leq \delta < y$. Therefore, $g_1(y) = g_1(\theta_2') < g_1(x)$,

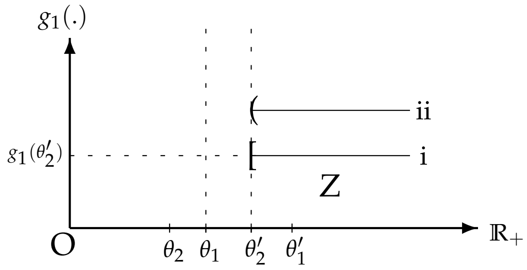


FIGURE 8. Claim 3 $\implies$ interior of zone Z is vacant $\implies$ either (i) or (ii) must hold

which implies that $l_1(\alpha,\beta) < 0$. Then, $WGS \implies l_2(\alpha,\beta) \geq 0 \implies g_2(\theta_2) \leq g_2(\delta) - [z(2)-z(1)][\delta - \theta_2]$. Since there is no upper bound on y, and so, on δ; using Claim 2C and arguing as in Claim 3, we arrive at a contradiction. □

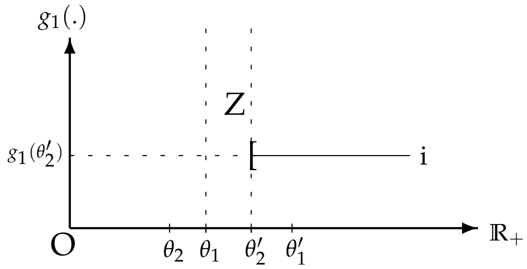


FIGURE 9. Claim 4 $\implies$ [Claim 3i $\implies$ interior of zone Z is vacant]

The graphical implication of Claim 4 is given by Figure 9. Claim 4 states that if Claim 3i holds, then $g_1(\theta_1) \leq g_1(\theta_2')$. Therefore, Claims 3 and 4 imply three possibilities;

(1) For all $y \geq \theta_2'$, $g_1(y) = g_1(\theta_2')$ and $g_1(\theta_1) = g_1(\theta_2')$.
(2) For all $y \geq \theta_2'$, $g_1(y) = g_1(\theta_2')$ and $g_1(\theta_1) < g_1(\theta_2')$.
(3) There is $K \in \mathbb{R}$ such that $\forall\, y > \theta_2'$, $g_1(y) = K > g_1(\theta_2')$.

Claims 5 and 6 state the implications of possibility (1), Claims 7 and 8 state the implications of possibility (2) and Claims 9 and 10 state the implications of possibility (3). All three possibilities imply the map in Figure 1. However, the kink point η varies. Possibility (1) implies that $\eta = \theta_1$ (shown in Figure 11) while possibility (2) implies that for some $\omega \in (\theta_1, \theta_2')$, $\eta = \omega$ (shown in Figure 13). Finally, possibility (3) implies that $\eta = \theta_2'$ (shown in Figure 15).

Claim 5. If $\forall\, y \geq \theta_2'$, $g_1(y) = g_1(\theta_2')$ and $g_1(\theta_1) = g_1(\theta_2')$, then $\forall\, x \in (\theta_1, \theta_2')$, $\forall\, t \in \{1,2\}$,

$$g_t(x) = g_t(\theta_2')$$

Proof: If $\exists\, x \in (\theta_1, \theta_2')$ such that $g_2(x) < g_2(\theta_2')$, then $l_2(\alpha,\beta) < 0$ when $\alpha \equiv (x, x')$ and $\beta \equiv (\theta_2', \theta_1)$ with $x' \in (\theta_1, x)$. Since $x' \in (\theta_1, \theta_2')$, by Claim 4, $g_1(x') \leq g_1(\theta_2') = g_1(\theta_1) \implies g_1(\theta_1) - g_1(x') > [z(2)-z(1)][\theta_1 - x'] \implies l_1(\alpha,\beta) < 0$. Therefore, $WGS \implies g_2(x) \geq g_2(\theta_2')$. Then, from Claim 2C and 2E, it follows that (**a**) $g_2(x) = g_2(\theta_2'), \forall\, x \in (\theta_1, \theta_2')$.
If $\exists\, x \in (\theta_1, \theta_2')$ such that $g_1(x) < g_1(\theta_2')$ then $l_1(\alpha', \beta') < 0$ where $\alpha' \equiv (\delta, x)$ and $\beta' \equiv (\epsilon, \theta_2')$ with $\theta_1 < \epsilon < \delta < x$. Since $\delta, \epsilon \in (\theta_1, \theta_2')$, by condition (**a**), we get

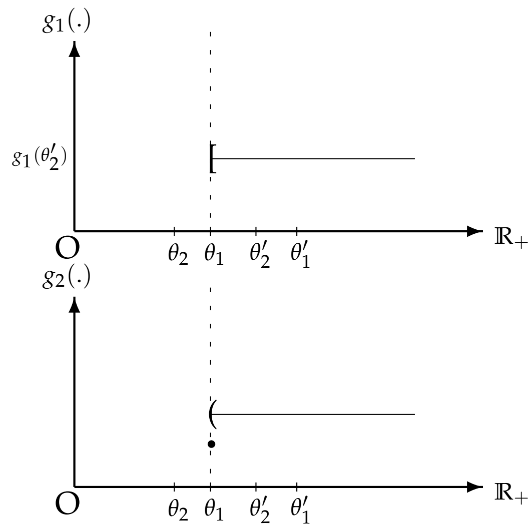


FIGURE 10. Claim 5

$g_2(\epsilon) - g_2(\delta) = 0 > [z(2) - z(1)][\epsilon - \delta] \implies l_2(\alpha', \beta') < 0$. Thus, $WGS \implies g_1(x) \geq g_1(\theta_2')$ which coupled with Claim 4 implies that $g_1(x) = g_1(\theta_2')$. □

The graphical implication of Claim 5 is given by Figure 10.

Claim 6. If $\forall\, y \geq \theta_2'$, $g_1(y) = g_1(\theta_2')$ and $g_1(\theta_1) = g_1(\theta_2')$, then $\forall\, t \in \{1, 2\}$,

$$\lim_{x \to \theta_1 -} g_t(x) = g_t(\theta_2')$$

Proof: If $\exists\, \nu \in [\theta_2, \theta_1)$ such that $g_2(\nu) > g_2(\theta_2')$, then $l_2(\alpha, \beta) < 0$ where $\alpha \equiv (\theta_2', \theta_2')$ and $\beta \equiv (\nu, \theta_2)$. From (5) it follows that $l_1(\alpha, \beta) < 0$.[9] Therefore, $WGS \implies g_2(\nu) \leq g_2(\theta_2')$. By Claim 2B; $\forall\, \nu_2 < \nu_1 < \theta_1$, $g_2(\nu_1) > g_2(\nu_2)$. Thus, $g_2(\nu) \leq g_2(\theta_2')$, $\forall\, \nu < \theta_1$ which in turn implies that $\lim_{x \to \theta_1 -} g_2(x) \overset{def}{=} T \leq g_2(\theta_2')$. Given Claim 2B, if $T < g_2(\theta_2')$ then $\exists\, \epsilon > 0$ such that $g_2(\theta_2') - g_2(x) > \epsilon$, $\forall\, x < \theta_1$. Then, $\exists\, \delta < \theta_1$ and $\rho \in (\theta_1, \theta_2')$ such that $\rho - \delta < \frac{\epsilon}{(z(2)-z(1))}$. Therefore, by Claim 5 $g_2(\rho) = g_2(\theta_2') \implies g_2(\rho) - g_2(\delta) > \epsilon > [z(2) - z(1)][\rho - \delta]$. This implies that $l_2(\alpha', \beta') < 0$ when $\alpha' \equiv (\delta, \xi)$ and $\beta' \equiv (\rho, \theta_2')$ with $\xi \in (\delta, \theta_1)$. By Claim 2A, $g_1(\xi) < g_1(\theta_1) = g_1(\theta_2')$, since $\xi < \theta_1$. Therefore, $l_1(\alpha', \beta') < 0$ and so $WGS \implies T = g_2(\theta_2')$.

For $t = 1$; from Claim 2A and the condition $g_1(\theta_1) = g_1(\theta_2')$ we get that $g_1(\theta_2') - g_1(x) = [z(2) - z(1)][\theta_2' - x]$. Therefore, as x tends to θ_2', the result is established. □

Claims 2, 5 and 6 imply that if Claim 3i holds true and $g_1(\theta_1) = g_1(\theta_2')$ then $g(.)$ map for each agent will look like in Figure 11 where the kink point $\eta = \theta_1$. Note that there is only one point of discontinuity and that too for a single agent, here agent 2.

Let us now consider the other possible implication of Claim 3i, that is, $g_1(\theta_1) < g_1(\theta_2')$. Given Claim 2A; (5) implies that $g_1(\theta_1) - g_1(\theta_2') > [z(2) - z(1)][\theta_1 - \theta_2']$. Since $g_1(\theta_1) < g_1(\theta_2')$; we can extend the straight line with slope $z(2) - z(1)$ passing through the

[9]In case $\theta_2 = \theta_1$, we choose any $\nu, \nu' < \theta_1$ such that $\nu < \nu'$. Claim 2A and (5) imply that $l_1((\theta_2', \theta_2'), (\nu, \nu')) < 0$. The rest of the proof remains same.

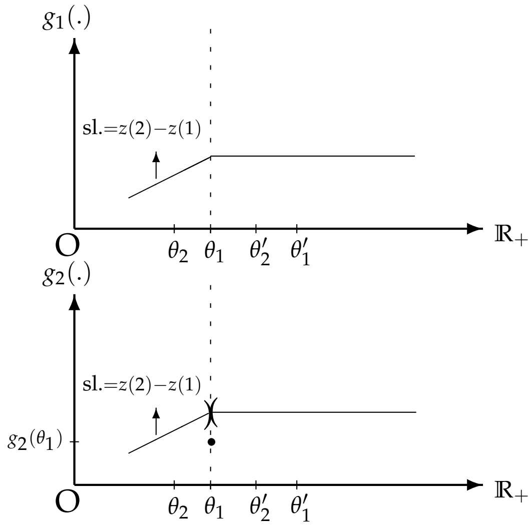


FIGURE 11. Claims 2, 5 and 6 put together.

$(\theta_2, g_1(\theta_2))$ point in the $g_1(.)$ map; on the right side of θ_1, to find a number $\omega \in (\theta_1, \theta_2')$ such that[10]

$$g_1(\theta_2) + [z(2) - z(1)][\omega - \theta_2] = g_1(\theta_2') \tag{11}$$

By Claim 4 and (11) above;

$$g_1(\theta_2) - g_1(x) > [z(2) - z(1)][\theta_2 - x], \forall\, x > \omega \tag{12}$$

Claim 7. If $\forall\, y \geq \theta_2'$, $g_1(y) = g_1(\theta_2')$ and $g_1(\theta_1) < g_1(\theta_2')$, then

- **A:** $\forall\, t \in \{1,2\}, g_t(x) = g_t(\theta_2'), \forall\, x \in (\omega, \theta_2']$.
- **B:** $\forall\, t \in \{1,2\}, \forall\, x < \omega, g_t(\theta_2) - g_t(x) = [z(2) - z(1)][\theta_2 - x]$.
- **C:** For some $t \in \{1,2\}$, $g_t(\omega) = g_t(\theta_2')$.

Proof:

A: For $t = 2$; pick any x, x' such that $\omega < x' < x < \theta_2'$. Then, by (12), $l_1(\alpha, \beta) < 0$ when $\alpha \equiv (x, x')$ and $\beta \equiv (\theta_2', \theta_2)$. Therefore, by Claim 2E , $WGS \implies$ (**a**) $g_2(x) = g_2(\theta_2'), \forall\, x \in (\omega, \theta_2')$.

For $t = 1$; if $\exists\, x \in (\omega, \theta_2')$ such that $g_1(x) < g_1(\theta_2')$ then consider a deviation from $\alpha' \equiv (\rho, x)$ and $\beta' \equiv (\delta, \theta_2')$ where $\omega < \delta < \rho < x$. By condition (**a**); $g_2(\delta) - g_2(\rho) = 0 > [z(2) - z(1)][\delta - \rho] \implies l_2(\alpha', \beta') < 0$. Therefore, by Claim 4, $WGS \implies g_1(x) = g_1(\theta_2'), \forall\, x \in (\omega, \theta_2')$.

B: For $t = 2$, from Claim 2B it follows that the statement is satisfied for any $x < \theta_1$. If $g_2(\theta_2) - g_2(\theta_1) > [z(2) - z(1)][\theta_2 - \theta_1]$ then $l_2(\alpha, \beta) < 0$ when $\alpha \equiv (\theta_1, \psi)$ and $\beta \equiv (\theta_2, \theta_2')$, where $\psi \in (\theta_1, \omega)$. Hence, by Claim 4, $WGS \implies g_1(\psi) = g_1(\theta_2'), \forall\, \psi \in (\theta_1, \omega)$. This coupled with (11), implies that $g_1(\psi) - g_1(\theta_2) > [z(2) - z(1)][\psi - \theta_2] \implies l_1(\beta', \alpha') < 0$ where $\alpha' \equiv (\theta_1', \psi)$ and $\beta' \equiv (\theta_1, \theta_2)$. Then, from (6), it follows that $l_2(\beta', \alpha') < 0$ and so WGS is violated. Thus, $WGS \implies g_2(\theta_2) - g_2(\theta_1) \leq [z(2) - z(1)][\theta_2 - \theta_1]$. If this equation holds with strict inequality then $l_2(\alpha'', \beta'') < 0$ when

[10]We are not assuming continuity; but simply extending the line continuously to locate the value ω.

$\alpha'' \equiv (\theta_2, \theta_1)$ and $\beta'' \equiv (\theta_1, \theta_2')$ while $g_1(\theta_1) < g_1(\theta_2') \implies l_1(\alpha'', \beta'') < 0$. Therefore, $WGS \implies$ (**b**) $g_2(\theta_2) - g_2(\theta_1) = [z(2) - z(1)][\theta_2 - \theta_1]$.
We now show that (**b**) holds even if θ_1 is replaced by any real number lying in the open interval (θ_1, ω). If $\exists\ \psi \in (\theta_1, \omega)$ such that $g_1(\psi) = g_1(\theta_2')$ then (11) implies that $g_1(\psi) - g_1(\theta_1) > [z(2) - z(1)](\psi - \theta_1)$ and so, from (6); $l_t((\theta_1, \theta_1), (\theta_1', \psi)) < 0, \forall\, t = 1, 2$. Hence, by Claim 4 , $WGS \implies$ (**c**) $g_1(\psi) < g_1(\theta_2')$. But from (**c**) it follows that $l_1((\theta_2, \psi), (x, \theta_2')) < 0$ where $x \in (\theta_1, \psi)$; and so $WGS \implies$ (**d**) $g_2(\theta_2) - g_2(x) \geq [z(2) - z(1)][\theta_2 - x]$. If (**d**) holds with strict inequality then (**c**) $\implies l_t((x, \psi), (\theta_2, \theta_2')) < 0, \forall\, t = 1, 2$ which violates WGS. Therefore, (**d**) must hold with equality. Using (**b**) and Claim 2B, then, we complete the proof for $t = 2$.
For $t = 1$, pick any ψ, ϵ such that $\theta_1 < \psi < \epsilon < \omega$. As proved in the paragraph above, we can say that $g_2(\epsilon) - g_2(\psi) = [z(2) - z(1)][\epsilon - \psi] > 0$ which implies that $l_2((\psi, x), (\epsilon, \theta_1)) < 0$ where $x \in (\theta_1, \psi)$. Then, $WGS \implies$ (**e**) $g_1(x) - g_1(\theta_1) \geq [z(2) - z(1)][x - \theta_1]$. If (**e**) holds with strict inequality then $l_1((x, \theta_1), (\epsilon, x)) < 0$. Again, from the statement proved in the previous paragraph $\epsilon > x \implies g_2(\epsilon) > g_2(x) \implies l_2((x, \theta_1), (\epsilon, x)) < 0$ and so WGS is violated. Therefore, (**e**) must hold with equality. Claim 2A, then, completes the proof for $t = 1$.

C: If $g_1(\omega) < g_1(\theta_2')$ then from (11) it follows that $g_1(\omega) < g_1(\theta_2') \implies g_1(\theta_2) - g_1(\omega) > [z(2) - z(1)][\theta_2 - \omega] \implies l_1((\omega, \omega), (\theta_2', \theta_2)) < 0$ and so by Claim 2E, $WGS \implies g_2(\omega) = g_2(\theta_2')$. Therefore, given Claim 4, we can say that there always exists $t' \in \{1, 2\}$ such that $g_{t'}(\omega) = g_{t'}(\theta_2')$. □
The graphical implications of Claim 7 are given by Figure 12.

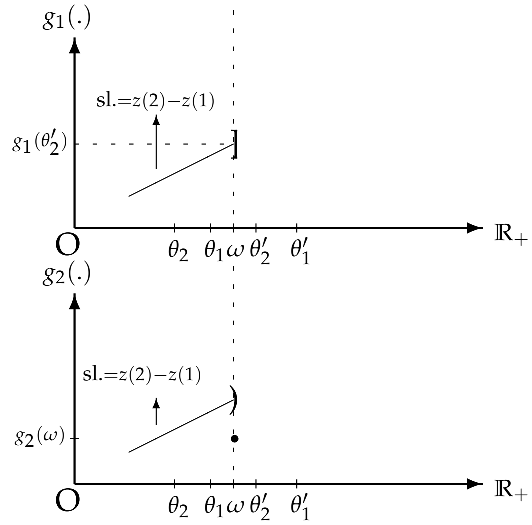


FIGURE 12. Claim 7 with $g_1(\omega) = g_1(\theta_2')$

Claim 8. If $\forall\, y \geq \theta_2', g_1(y) = g_1(\theta_2')$ and $g_1(\theta_1) < g_1(\theta_2')$, then $\forall\, t = 1, 2$,

$$\lim_{x \to \omega -} g_t(x) = g_t(\theta_2')$$

Proof: From Claim 7C, w.l.o.g. assume that $g_1(\omega) = g_1(\theta_2')$. Then, from (11) and Claim 7B we get that $g_1(\theta_2') - g_1(x) = [z(2) - z(1)][\omega - x], \forall\, x \leq \omega$. Therefore, as x tends to ω, the result is established for $t = 1$.
For $t = 2$; Claim 2E and Claim 7B imply that $g_2(x) \leq g_2(\theta_2'), \forall\, x < \theta_2'$. Thus, $\lim_{x \to \omega-} g_2(x) \stackrel{def}{=} T' \leq g_2(\theta_2')$. As in Claim 6, the possibility of $T' < g_2(\theta_2')$ can be ruled out. □
Claims 2, 7 and 8 imply that if Claim 3i holds true with $g_1(\theta_1) < g_1(\theta_2')$ then discontinuity in the $g(.)$ maps, if present, shall occur only at a single point (at the kink point) and for at most one agent out of the two. The $g(.)$ map for each agent will look like in Figure 13 where the kink point $\eta = \omega \in (\theta_1, \theta_2')$.

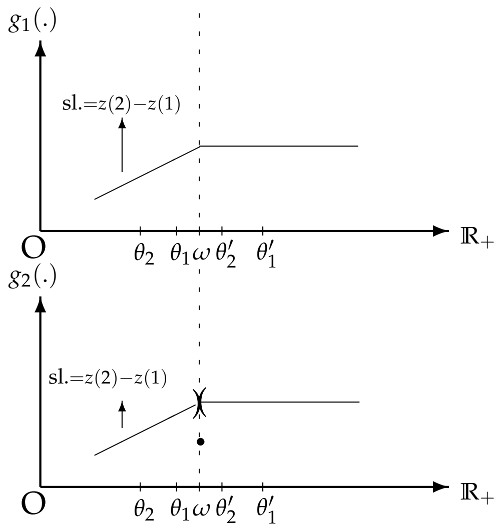


FIGURE 13. Claims 2, 7 and 8 put together, with $g_1(\omega) = g_1(\theta_2')$

Now that the consequences of two possible implications of Claim 3i have been analyzed, let us move to the implications of Claim 3ii.

Claim 9. If there is $K \in \mathbb{R}$ such that $\forall\, y > \theta_2', g_1(y) = K > g_1(\theta_2')$, then

A: $\forall\, x, x' < \theta_2', \forall\, t \in \{1, 2\}, g_t(x) - g_t(x') = [z(2) - z(1)][x - x']$.
B: $\forall\, x \leq \theta_2', g_2(\theta_2') - g_2(x) = [z(2) - z(1)][\theta_2' - x]$.

Proof:
A: For $t = 2$, pick any $x, x' < \theta_2'$. If $g_2(x) - g_2(x') > [z(2) - z(1)][x - x']$, then consider the deviation from $l_2((x', \theta_2'), (x, y)) < 0$, where $y > \theta_2'$. Also $g_1(y) = K > g_1(\theta_2') \implies l_1((x', \theta_2'), (x, y)) < 0$. Hence, $WGS \implies g_2(x) - g_2(x') \leq [z(2) - z(1)][x - x']$. If this holds with strict inequality then again $l_t((x, \theta_2'), (x', y)) < 0, \forall\, t = 1, 2$. Thus, $WGS \implies$ (**a**) $g_2(x) - g_2(x') = [z(2) - z(1)][x - x']$.
For $t = 1$, pick any ν, ϵ such that $\epsilon < \nu < \theta_2'$ and any $x, x' < \epsilon$. By (**a**), $g_2(\nu) - g_2(\epsilon) = [z(2) - z(1)][\nu - \epsilon] > 0$. By checking the deviation from (ϵ, x') to (ν, x) and then the deviation from (ϵ, x) to (ν, x'), we see that WGS is violated unless $g_1(x') - g_1(x) = [z(2) - z(1)][x' - x]$.

B: Pick any ε, y such that $\theta_2' < \varepsilon < y$. By assumption, $g_1(y) = K > g_1(\theta_2') \implies l_1((\theta_2, \theta_2'), (\varepsilon, y)) < 0$. Then, $WGS \implies$ (**a**) $g_2(\theta_2) - g_2(\varepsilon) \geq [z(2) - z(1)][\theta_2 - \varepsilon]$, $\forall\, \varepsilon > \theta_2'$. Also by Claim 2C, $g_2(\theta_2') = g_2(\varepsilon)$, which coupled with (**a**) implies that $g_2(\theta_2) - g_2(\theta_2') \geq [z(2) - z(1)][\theta_2 - \varepsilon]$. Since ε was chosen arbitrarily, this equation must hold for all $\varepsilon > \theta_2'$. This implies that (**b**) $g_2(\theta_2) - g_2(\theta_2') \geq [z(2) - z(1)][\theta_2 - \theta_2']$ (otherwise, $g_2(\theta_2) - g_2(\theta_2') = [z(2) - z(1)][\theta_2 - \theta_2'] - \nu$ for some $\nu > 0$, and so we can find some $\varepsilon' \in \left(\theta_2', \theta_2' + \frac{\nu}{z(2)-z(1)}\right)$ such that **(a)** is violated).

If (**b**) holds with strict inequality, then there exists $\zeta < \theta_2'$ such that $g_2(\zeta) > g_2(\theta_2')$. We can say this because, if $\forall\, \zeta < \theta_2'$, $g_2(\zeta) \leq g_2(\theta_2')$, then by invoking case **A** for $t = 2$, we get that $g_2(\theta_2) - g_2(\zeta) = (\theta_2 - \zeta) \implies g_2(\theta_2) - g_2(\theta_2') \leq (\theta_2 - \zeta), \forall\, \zeta < \theta_2'$. Then, as ζ tends to θ_2', in limit this violates *condition* (**b**) *holding with strict inequality*. Therefore, $l_2((\theta_2', \theta_2'), (\zeta, x')) < 0$ where $x' < \zeta$. From case **A** (for $t = 1$) and (5), $l_1((\theta_2', \theta_2'), (\zeta, x')) < 0$. Thus, WGS requires that condition (**b**) hold with equality. This along with case **A** (for $t = 2$) completes the proof. □

The graphical exposition of this Claim 9 is given by Figure 14.

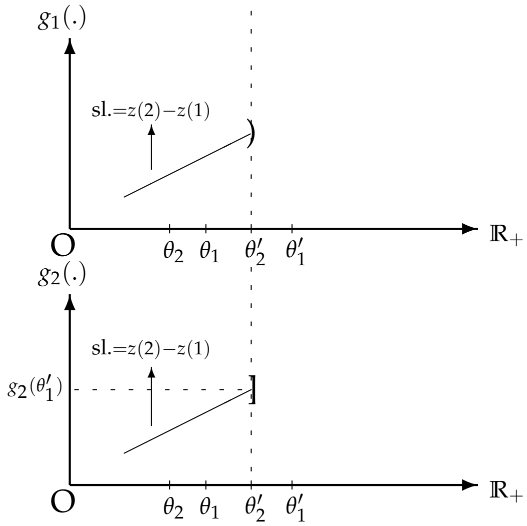


FIGURE 14. Claim 9

Claim 10. If $\forall\, y > \theta_2', g_1(y) = K > g_1(\theta_2')$, then $\forall\, t = 1, 2$,

$$\lim_{x \to \theta_2' -} g_2(x) = g_2(\theta_2') \text{ and } \lim_{x \to \theta_2' -} g_1(x) = K$$

Proof: Given Claim 9B, as $\{x\} \to \theta_2'$, the result is established for $t = 2$.
For $t = 1$; if $\exists\, x \in (\theta_1, \theta_2')$ such that $g_1(x) > K$ then as in Claim 4, WGS is violated. Again from Claim 9A, $\forall\, \zeta < \theta_2'$, $g_1(.)$ is an increasing in ζ. Therefore, $g_1(x) \leq K$, $\forall\, x < \theta_2' \implies \lim_{x \to \theta_2' -} g_1(x) \stackrel{def}{=} T'' \leq K$. If $T'' < K$ then as in Claim 6, we can design a violation of WGS. □

Claims 2, 9 and 10 imply that if $g_1(y) = K > g_1(\theta'_2), \forall\, y > \theta'_2$, then there is only one point of discontinuity (at the kink point $\eta = \theta'_2$) and that too for a single agent, here agent 1 (as shown in Figure 15).

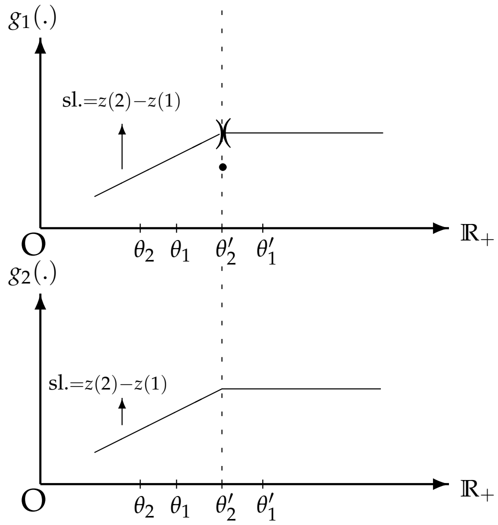


FIGURE 15. Claims 2, 9 and 10 put together

Proposition 2. If there is no pair $\theta, \theta' \in \mathbb{R}^2_+$ such that WGS holds with strict inequality, then *either* $\forall\, i \neq j \in \{1,2\}$ and $\forall\, \theta_j \in \mathbb{R}_+$, $g_i(\theta_j) = 0$ *or* $\forall\, i \neq j \in \{1,2\}$ and $\forall\, \theta_j \in \mathbb{R}_+$, $g_i(\theta_j) = [z(2) - z(1)]\theta_j$.

Proof: Assume that there is no pair $\theta, \theta' \in \mathbb{R}^2_+$ such that WGS holds with strict inequality. Let $\theta, \theta', \theta'' \in \mathbb{R}^2_+$ be such that θ', θ'' are order preserving $\{1,2\}$-profiles of θ. W.l.o.g. assume that 1 precedes 2 in the efficient ranking for all three profiles. By assumption, WGS *holds with equality* for both pairs θ, θ' and θ, θ''. W.l.o.g. suppose that for the pair θ, θ', (**ai**) $g_1(\theta_2) - g_1(\theta'_2) = [z(2) - z(1)](\theta_2 - \theta'_2)$ and (**aii**) $g_2(\theta_1) - g_2(\theta'_1) > 0$; while for the pair θ, θ'', (**bi**) $g_1(\theta_2) - g_1(\theta''_2) > [z(2) - z(1)](\theta_2 - \theta''_2)$ and (**bii**) $g_2(\theta_1) - g_2(\theta''_1) = 0$. (**ai**) & (**bi**) $\implies$ $g_1(\theta'_2) - g_1(\theta''_2) > [z(2) - z(1)](\theta_2 - \theta''_2) \implies l_1(\theta'', \theta') < 0$ while (**aii**) & (**bii**) $\implies$ $g_2(\theta''_1) > g_2(\theta'_1) \implies l_2(\theta'', \theta') > 0$. Therefore, WGS *holds with strict inequality* for the pair θ', θ'' and hence, contradiction. In the same way, for any other combination of $>$ and $<$ amongst the inequalities (**aii**) and (**bi**); either WGS is violated or WGS *holds with strict inequality*. Therefore, if (**ai**) holds with equality then (**bi**) holds with equality; and if (**aii**) holds with equality then (**bii**) holds with equality. Since $\theta, \theta', \theta''$ are chosen arbitrarily, this means that either (**c**) $\forall\, x, y \in \mathbb{R}_+, g_1(x) - g_1(y) = [z(2) - z(1)][x - y]$ or (**d**) $\forall\, x, y \in \mathbb{R}_+, g_2(x) = g_2(y)$.[11]

If (**c**) holds, then it can be shown, as in Claim 2B that $\forall\, x', y' \in \mathbb{R}_+$ $g_2(x') - g_2(y') = [z(2) - z(1)][x' - y']$. If (**d**) holds then it can be shown, as in Claim 2D that $\forall\, x', y' \in$

[11]It may be that WGS *holds with equality* in such a way that $g_1(\theta_2) - g_1(x) = [z(2) - z(1)][\theta_2 - x]$ & $g_2(\theta_1) - g_2(y) = 0$ where $x \in \{\theta'_2, \theta''_2\}$ and $y \in \{\theta'_1, \theta''_1\}$ respectively. Then, arguing as above we could say that $\forall\, m, n > 0$, $g_1(m) - g_1(n) = [z(2) - z(1)][m - n]$ and $g_2(m) - g_2(n) = 0$. Then, it is easy to check that $l_i((m, n), (m - \epsilon, n + \epsilon)) < 0, \forall\, i = 1, 2$ when $0 < \epsilon < m < n$.

$\mathbb{R}_{++}$ $g_1(x') = g_1(y')$. There remains a possibility that $\lim_{x \to 0+} g_1(x) \overset{def}{=} \bar{T}_1 \neq g_1(0) = 0$. Pick any ζ, ζ' such that $0 < \zeta' < \zeta$. If $\bar{T}_1 > 0$ then $g_1(\zeta) > 0 \implies l_1((0,0),(\zeta',\zeta)) < 0$ while from (**d**) we know that $g_2(\zeta') = g_2(0) \implies l_2((0,0),(\zeta',\zeta)) > 0$. If $\bar{T}_1 < 0$ then $g_1(\zeta) < g_1(0) \implies l_1((\zeta',\zeta),(0,0)) < 0$ while $g_2(\zeta) = g_2(0) \implies l_2((\zeta',\zeta),(0,0)) > 0$. Thus, in both cases WGS holds with strict inequality and hence, contradiction. Therefore, $\bar{T}_1 = 0$ which implies that $g_t(x) = 0, \forall\, x \geq 0, \forall\, t = 1,2$. □

6.1.2. **Proof of If in Theorem 1:** Consider a mechanism (σ, τ) such that the $g(.)$ maps associated with $\tau(.)$ are as in Theorem 1. Assume there exists a $\{1,2\}$-deviation such that WGS is violated. All the logical arguments involved in proving Propositions 1 and 2 are reversible. Thus, this deviation is not order preserving. Since there are only two agents, there is only one type of order interchanging deviation. Pick any such deviation, say, from $\beta \equiv (\zeta_1, \zeta_2)$ to $\alpha \equiv (\rho_1, \rho_2)$ where $\rho_1, \rho_2, \zeta_1, \zeta_2$ are any four arbitrary non-negative numbers such that (w.l.o.g.) $\rho_1 \geq \rho_2$ and $\zeta_1 < \zeta_2$. Therefore, 1 precedes 2 in the efficient ranking for α while 2 precedes 1 in the efficient ranking for β. Then, **(i)** $l_1(\beta,\alpha) = [z(2) - z(1)][\rho_2 - \zeta_1] - g_1(\rho_2) + g_1(\zeta_2)$ and **(ii)** $l_2(\beta,\alpha) = [z(2) - z(1)][\zeta_2 - \zeta_1] + g_2(\zeta_1) - g_2(\rho_1)$. It will be shown that for any possible values of the arbitrarily chosen four numbers; there exists one agent $t^* \in \{1,2\}$ such that t^* is not strictly better off in a deviation from β to α.
There are two possible cases, namely;
Case A: $\rho_2 < \eta$
If $\zeta_2 < \eta$ then $g_1(\rho_2) - g_1(\zeta_2) = [z(2) - z(1)][\rho_2 - \zeta_2]$ which means that **(i)** $\implies l_1(\beta,\alpha) = [z(2) - z(1)][\zeta_2 - \zeta_1] > 0$. Hence, $t^* = 1$.
If $\zeta_2 \geq \eta$ then $g_1(\rho_2) - g_1(\zeta_2) = [z(2) - z(1)][\rho_2 - \eta] \implies l_1(\beta,\alpha) = [z(2) - z(1)](\eta - \zeta_1)$. If $\zeta_1 \leq \eta$ then $t^* = 1$. If $\zeta_1 > \eta$, then $l_1(\beta,\alpha) < 0$; but $g_2(\zeta_1) = [z(2) - z(1)]\eta$ and the fact that $g_2(\rho_1) \leq [z(2) - z(1)]\eta$ imply that $g_2(\zeta_1) - g_2(\rho_1) \geq 0$. From **(ii)**, it then follows that $l_2(\beta,\alpha) \geq [z(2) - z(1)][\zeta_2 - \zeta_1] > 0$ and so $t^* = 2$.
Case B: $\rho_2 \geq \eta$
If $\zeta_2 < \eta$ then $g_1(\rho_2) - g_1(\zeta_2) = [z(2) - z(1)][\eta - \zeta_2] \implies l_1(\beta,\alpha) > 0$ and so $t^* = 1$.
If $\zeta_2 \geq \eta$ then $g_1(\rho_2) - g_1(\zeta_2) = 0 \implies l_1(\beta,\alpha) = [z(2) - z(1)][\rho_2 - \zeta_1]$. If $\zeta_1 \leq \rho_2$ then $t^* = 1$. If $\zeta_1 > \rho_2$ then $l_1(\beta,\alpha) < 0$; but $\zeta_1 > \rho_2 \geq \eta$ and so, as in case $g_2(\zeta_1) - g_2(\rho_1) \geq 0 \implies l_2(\beta,\alpha) \geq [z(2) - z(1)][\zeta_2 - \zeta_1] > 0$ and so $t^* = 2$.
Thus, given the $g(.)$ maps, no order interchanging $\{1,2\}$ deviation violates WGS. □

Remark 9. It is possible that in (5); $\theta_1 = 0$.[12] In that case all the Claims other than 2A, 2B and 6 go through. Since $g_1(0) = 0$; in case Claim 3i holds with $g_1(\theta_1) = g_1(\theta'_2)$, the $g_1(.)$ map is a horizontal straight line along the x-axis. As in Claim 2D, it can be proved that the $g_2(.)$ map, too, is a horizontal straight line. There remains a possibility of jump discontinuity at the origin in $g_2(.)$ map. It can further be shown that such a jump, if present, can only occur in an upward direction[13]. To capture this possibility we would need to assume that $\tau_t(0,0,\ldots,0) = C_t > 0$ in the expression for the $g_t(.)$

[12] Recall that 1 precedes 2 in both profiles θ and θ'. Therefore, $\theta_1 = 0 \implies \theta_2 = 0$ so that $\sigma_1(0,0) < \sigma_2(0,0)$.

[13] If the jump is in downward direction then $l_t((x,y),(0,0)) < 0$ where $0 < y < x$.

map for all t. The relevant map containing the implications of $\theta_1 = 0$ would then be given by $\eta = 0$ and $\alpha_2 < 1$. Hence, our assumption of transfers independent of agent specific constants, that is $\tau_i(0,0,\dots,0) = 0, \forall\, i \in N$, rules out the case where $\theta_1 = 0$.[14].

6.2. **Proof of Theorem 3:** First we need to prove the following lemma, which says that given (9) and (10) there will always be an agent whose transfer turns out be independent of the announcements of all other agents. Recall that $\forall\, \theta \in \mathbb{R}^n_+$ and $\forall\, r = 1,2,\dots,n$, $\theta(r)$ denotes the waiting cost of the agent ranked r in the efficient ranking for profile θ.

Lemma 1. If (9) and (10) hold then $\forall\, \theta \in \mathbb{R}^n_+$, $\exists\, m(\theta) \in N$ such that $\tau_{m(\theta)}(\theta) = Constant$.

Proof of Lemma: Define $s' = \min\{s = 1,2,\dots,n-1 : \theta(s) \leq \eta(s)\}$, $\forall\, \theta \in \mathbb{R}^n_+$. Then, choose the agent $m(\theta)$ so that

$$\sigma_{m(\theta)}(\theta) = \begin{cases} n & \text{when } \{s'\} = \emptyset \\ s' & \text{when } \{s'\} \neq \emptyset \end{cases}$$

We will show that $\tau_{m(\theta)}(\theta) = Constant$ in each of the following two cases;
Case 1: $\{s'\} = \emptyset$
In this case $\sigma_{m(\theta)}(\theta) = n$ which implies that (i) $\tau_{m(\theta)}(\theta) = g_{m(\theta)}(\theta_{-m(\theta)})$ and (ii) $\theta_{-m(\theta)}(s) = \theta(s) > \eta(s)$ for all $s = 1,2,\dots,n-1$. Therefore, using (i) & (ii) we can write that

$$\begin{aligned} \tau_{m(\theta)}(\theta) &= \sum_{j \neq m(\theta)} \Delta z(\sigma_j(\theta_{-m(\theta)})) \min\left\{\theta_j, \eta(\sigma_j(\theta_{-m(\theta)}))\right\} \\ &= \sum_{k=1,2,\dots,n-1} \Delta z(k)\ \min\left\{\theta_{-m(\theta)}(k), \eta(k)\right\} \\ &= \sum_{k=1,2,\dots,n-1} \Delta z(k).\eta(k) = Constant \end{aligned}$$

Case 2: $\{s'\} \neq \emptyset$
In this case $\sigma_{m(\theta)}(\theta) = s' < n$; which implies that (**a**) $\theta_{-m(\theta)}(k) = \theta(k) > \eta(k)$, $\forall\, k = 1,2,\dots,s'-1$. From (10) we get that $\eta(s') \leq \eta(k)$, $\forall\, k = s'+1, s'+2,\dots,n-1$ and so we can say that (**b**) $\theta_{-m(\theta)}(k) = \theta(k+1) \leq \theta(s') \leq \eta(s') \leq \eta(k)$, $\forall\ k = s', s'+1,\dots,n-1$. Note that $\forall\, j \in P'_{m(\theta)}(\theta)$, $z(\sigma_j(\theta)) - z(\sigma_j(\theta_{-i})) = z(\sigma_j(\theta_{-i})+1) - z(\sigma_j(\theta_{-i})) = \Delta z(\sigma_j(\theta_{-i}))$. Using (**a**) and (**b**), we can write that

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[14]If $\theta_1 = 0$ then $\sigma_1(\theta) = 1 \implies \theta_2 = 0$ Putting $g_t(0) = 0$ for all t in (5) and (6) we get that (i) $g_1(\theta'_2) < \theta'_2$ and (ii) $g_2(\theta'_1) < 0$. Then, WGS gets violated in a deviation from θ' to $\theta \equiv (0,0)$ and so, contradiction.

$$\begin{aligned}\tau_{m(\theta)}(\theta) &= -\sum_{j\in P'_{m(\theta)}(\theta)} \Delta z(\sigma_j(\theta_{-m(\theta)}))\theta_j + \sum_{j\neq m(\theta)} \Delta z(\sigma_j(\theta_{-m(\theta)}))\min\left\{\theta_j, \eta(\sigma_j(\theta_{-m(\theta)}))\right\}\\ &= -\sum_{k=s',s'+1,\ldots,n-1} \Delta z(k)\theta_{-m(\theta)}(k) + \sum_{k=1,2,\ldots,n-1} \Delta z(k)\min\left\{\theta_{-m(\theta)}(k), \eta(k)\right\}\\ &= \sum_{k=1,2,\ldots,s'-1} \Delta z(k)\eta(k) = Constant\end{aligned}$$

□

Pick any non-empty $S \subseteq N$ and $\theta, \theta' \in \mathbb{R}^n_+$ such that θ is an S-variant of θ'. Suppose coalition S deviates from θ' to θ and this deviation violates WGS. For notational simplicity we suppress the argument θ in the term $m(\theta)$ and write just m.

Claim 11. $m \notin S$

Proof of Claim: Say $m \in S$. Lemma 1 implies that $\tau_m(\theta)$ is independent of the announcements of its coalition partners. Therefore, for agent m, this coalitional deviation is only as good as a unilateral deviation. But then strategy-proofness contradicts WGS being violated. □

Identify the agent $a \overset{def}{=} argmax\{\sigma_j(\theta) | j \in S \cap P_m(\theta)\}$ and the rank $r_a \overset{def}{=} \sigma_a(\theta)$. Similarly, agent $b \overset{def}{=} argmax\{\sigma_j(\theta) | j \in S \cap P'_m(\theta)\}$ and the rank $r_b \overset{def}{=} \sigma_b(\theta)$. Therefore, a is the *last ranked member of S preceding agent m* and b is the *first ranked member of S succeeding agent m*; in the efficient ranking for profile of costs θ. Also note that if not both, either of the agents a and b, must exist.
From the definition of m, there are two possible cases;
Case 1: $\sigma_m(\theta) = n$
Here, only a is well defined. As before we can write

$$\tau_a(\theta) = -\sum_{k=r_a,r_a+1,\ldots,n-1} \Delta z(k)\theta_{-a}(k) + \sum_{k=1,2,\ldots,n-1} \Delta z(k)\min\{\theta_{-a}(k), \eta(k)\}$$

Note that $\sigma_m(\theta) = n \implies$ either $\{s'\} = \emptyset$ or $s' = n \implies \theta(k) > \eta(k), \forall\, k = 1, 2, \ldots, n-1$. Therefore, since $a' < n$; (**i**) $\theta_{-a}(k) = \theta(k) > \eta(k)$, $\forall\, k = 1, 2, \ldots, r_a - 1$ and from (10); (**ii**) $\theta_{-a}(k) = \theta(k+1) > \eta(k+1) \geq \eta(k)$, $\forall\, k = r_a, r_a+1, \ldots, n-2$. Also (**iii**) $\theta_{-a}(n-1) = \theta_m$. Using **(i)**, **(ii)** and **(iii)**, we can write that;

$$\begin{aligned}\tau_a(\theta) &= \sum_{k=1,2,\ldots,r_a-1} \Delta z(k)\eta(k) + \sum_{k=r_a,r_a+1,\ldots,n-2} \Delta z(k)\left[\eta(k) - \theta(k+1)\right]\\ &+ \Delta z(n-1)\left[\min\{\theta_m, \eta(n-1)\} - \theta_m\right]\end{aligned}$$

By the definition of a; the numbers $\{\theta(k+1)\}_{k=r_a}^{n-2}$ are waiting costs of agents who are not members of S. Given $m \notin S$, this means that $\tau_a(\theta)$ does not depend on the misreports of members of $S - \{a\}$. Therefore, arguing as in Claim 11, we can arrive at a contradiction.
Case 2: $\sigma_m(\theta) = s' < n$

Once again, if a exists;

$$\begin{aligned}\tau_a(\theta) &= -\sum_{k=r_a,r_a+1,\ldots,n-1} \Delta z(k)\theta_{-a}(k) + \sum_{k=1,2,\ldots,n-1} \Delta z(k) \min\{\theta_{-a}(k), \eta(k)\} \\ &= \sum_{k=a,r_a+1,\ldots,s'-2} \Delta z(k) \left[\min\{\theta_{-a}(k), \eta(k)\} - \theta_{-a}(k)\right] \\ &+ \sum_{k=s',s'+1,\ldots,n-1} \Delta z(k) \left[\min\{\theta_{-a}(k), \eta(k)\} - \theta_{-a}(k)\right] \\ &+ \Delta z(s'-1) \left[\min\left\{\theta_{-a}(s'-1), \eta(s'-1)\right\} - \theta_{-a}(s'-1)\right] \\ &+ \sum_{k=1,2,\ldots,r_a-1} \Delta z(k) \min\{\theta_{-a}(k), \eta(k)\}\end{aligned}$$

By definition; (**a**) $s' < n \implies \theta(k) > \eta(k), \forall\, k = 1, 2, \ldots, s'-1$. By construction, $r_a < s'$ and so (**i**) $\theta_{-a}(s'-1) = \theta(s') = \theta_m$. Then, using (**a**) we can say that (**ii**) $\theta_{-a}(k) = \theta(k) > \eta(k)$, $\forall\, k = 1, 2, \ldots, r_a - 1$. From the construction of the rank s' and (10), it follows that (**iii**) $\theta_{-a}(k) = \theta(k+1) > \eta(k+1) \geq \eta(k)$, $\forall\, k = r_a, r_a+1, \ldots, s'-2$ and (**iv**) $\theta_{-a}(k) = \theta(k+1) \leq \theta(s') \leq \eta(s') \leq \eta(k)$, $\forall\, k = s', s'+1, \ldots, n-1$. Using conditions **(i)-(iv)**, we can write that

$$\begin{aligned}\tau_a(\theta) &= \sum_{1,2,\ldots,r_a-1} \Delta z(k)\eta(k) + \sum_{r_a,r_a+1,\ldots,s'-2} \Delta z(k) \left[\eta(k) - \theta(k+1)\right] \\ &+ \Delta z(s'-1) \left[\min\left\{\theta_m, \eta(s'-1)\right\} - \theta_m\right]\end{aligned}$$

Arguing as in Case 1; we can see that $\tau_a(\theta)$ in independent of reports of members of $S - \{a\}$. Therefore, as in Claim 11, we reach a contradiction.

Similarly, if b exists;

$$\begin{aligned}\tau_b(\theta) &= -\sum_{k=r_b,r_b+1,\ldots,n-1} \Delta z(k)\theta_{-b}(k) + \sum_{k=1,2,\ldots,n-1} \Delta z(k) \min\{\theta_{-b}(k), \eta(k)\} \\ &= \sum_{k=1,2,\ldots,r_b-1} \Delta z(k) \min\{\theta_{-b}(k), \eta(k)\} \\ &+ \sum_{k=r_b,r_b+1,\ldots,n-1} \Delta z(k) \left[\min\{\theta_{-b}(k), \eta(k)\} - \theta_{-b}(k)\right]\end{aligned}$$

By definition $r_b > s'$. Then, using (10) and the condition (**a**) we get that **(v)** $\theta_{-b}(s') = \theta(s') = \theta_m \leq \eta(s')$; **(vi)** $\theta_{-b}(k) = \theta(k) > \eta(k)$, $\forall\, k = 1, 2, \ldots, s'-1$; **(vii)** $\theta_{-b}(k) = \theta(k) \leq \theta(s') \leq \eta(s') \leq \eta(k)$, $\forall\, k = s'+1, s'+2, \ldots, r_b - 1$ and **(viii)** $\theta_{-b}(k) = \theta(k+1) \leq \theta(s') \leq \eta(s') \leq \eta(k)$, $\forall\, k = r_b, r_b+1, \ldots, n-1$. Conditions **(v)-(viii)** then imply that

$$\tau_b(\theta) = -\sum_{k=1,2,\ldots,s'-1} \Delta z(k)\eta(k) + \Delta z(s)\theta_m + \sum_{k=s'+1,s'+2,\ldots,r_b-1} \Delta z(k)\theta(k)$$

By definition of b, the numbers $\{\theta(k)\}_{k=s'+1}^{r_b-1}$ are waiting costs of members of $N - S$. Since $m \notin S$, it can therefore be said that $\tau_b(\theta)$ is independent of misreports of members of $S - \{b\}$. Thus, as in Claim 11, we reach a contradiction. □

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