On Monotonicity and Monotone Differences in Mechanism Design

(PRELIMINARY)

Levent Ulku
ITAM
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Abstract: I characterize incentives in a single-dimensional private information environment without using monotonicity or monotone differences conditions. I apply the characterization in an example where a seller faces a buyer whose values exhibit habit formation, i.e., current consumption causes disutility in the future. The seller-optimal mechanism of this example can be constructed even when the monotone differences condition fails. For certain parameter values the optimal mechanism is nonmonotone.

1 Introduction

Monotonicity and monotone differences are two key conditions in mechanism design: in single dimensional private infromation environments (or if private information belongs to a finite set) monotone allocation functions are implementable if values satisfy a monotone differences property. In this note I show that neither monotonicity, nor the monotone differences property is necessary for implementability in general. I give a characterization result for implementation without referring to these two conditions. I apply the characterization in a Myersonesque buyer-seller example where buyer's value exhibits intertemporal allocation externality in the form of habit persistence: agent's consumption today generates disutility in the future. The existence of a habit parameter renders the agent's value nonlinear, in fact, nonmonotone in type. Monotone differences property can not be satisfied for a large set of parameter values. I identify the seller-optimal mechanism and show that it may be nonmonotone for certain habit parameter values.

I consider a simple single-agent setting for brevity. All proofs as well as an outline of an extension to multiagent models with interdependent values are collected in the appendices.

2 Environment

Consider a standard mechanism design environment with a single agent.[1] Let A be a set of allocations. An *outcome* is a pair (a, x) where $a \in A$ is allocated to the agent in return for payment $x \in \Re$. The agent's resulting utility is $v(a, t) - x$ where $t \in T = [0, 1]$ is his type. I will assume that for all a, $v(a, \cdot)$ is continuously differentiable and denote its derivative by $v_2(a, \cdot)$. Importantly, v need not be monotone in t. Throughout the text (s, t) refers to a pair of types, not necessarily distinct.

An *allocation function* is a function $q : [0, 1] \rightarrow A$ mapping types into allocations. A *mechanism* is a pair (q, x) where $x : [0, 1] \rightarrow \Re$ determines monetary transfers. $q(\cdot)$ is *implementable* if there exists $x(\cdot)$ such that the agent can not gain by misreporting his type to the mechanism (q, x), i.e., $v(q(t), t) - x(t) \geq v(q(s), t) - x(s)$ for every (s, t). If this is the case, the mechanism (q, x) is said to be *incentive compatible*, or payments x implement q.[2]

3 Implementability

The literature contains several related results which establish sufficient conditions for implementation of allocation functions. A typical such result indicates:

Proposition 1 *$q(\cdot)$ is implementable if there exists a binary relation $\preceq$ on A such that*

1. *$s \leq t$ implies $q(s) \preceq q(t)$, and*
2. *$a' \preceq a$ implies $v_2(a', t) \leq v_2(a, t)$ for every t.*

Note that a binary relation on A satisfying the conditions of Proposition 1 can not be empty. Furthermore its restriction to $\{q(t) : t \in [0, 1]\}$ is complete and transitive. The first condition orders A in such a way that $q(\cdot)$ is monotone, and the second condition requires $v(\cdot, \cdot)$ to satisfy monotone differences with respect to this order on A and the usual less-than-or-equal-to order on $[0, 1]$.

[1] As in Chung and Ely (2002) the single-agent environment can be interpreted as a reduction of a multiagent environment where the focus is on the incentives faced by any one agent, given the private information of all others. The results in the next section extend therefore to multiagent environments with interdependent values, with the notion of incentive compatibility taken as ex post Nash incentive compatibility. These straightforward extensions are outlined in Apendix 2.

[2] The focus here is on incentive compatibility. Given incentive compatibility, individual rationality can be attained for most mechanisms at no additional cost. See Lemma 1 in the next section.

As a trivial example, consider a constant allocation function $q(t) = a$ for all t. Let the binary relation $\preceq$ on A consist only of $\{(a, a)\}$. Now the two conditions of Proposition 1 are satisfied with respect to $\preceq$. Thus constant allocation functions are trivially implementable. Now suppose that $q(s) \neq q(t)$ for some $s < t$ and that no other type gets a third allocation. Then Proposition 1 indicates that q is implementable if $v_2(q(s), y) \leq v_2(q(t), y)$ for all y. This condition may or may not be satisfied.

Proposition 1 is related to Proposition 3 in Bergemann and Välimäki (2002), Theorem 5.1 in Jehiel and Moldovanu (2001) and Lemma 2 in Crémer and McLean (1985). All these papers study the implementation of efficient (i.e., ex post welfare maximizing) allocation functions in multiagent problems with interdependent values. However a close reading of their arguments reveals that their implementability results apply to *any* allocation function, including inefficient ones. Appendix 2 presents the multiagent extension of Proposition 1 and discusses its relation to aforementioned results.[3]

This sufficient condition is not necessary for implementability however, and the next section presents an example to this effect. This observation raises the question of whether there exists a sufficient *and* necessary condition for implementability.

Proposition 2 *The following are equivalent:*

1. *The allocation function $q(\cdot)$ satisfies*

$$\int_s^t v_2(q(y), y)dy \leq v(q(t), t) - v(q(t), s) \text{ for every } (s, t). \tag{1}$$

2. *The mechanism $(q, \hat{x}_q)$ is incentive compatible where*

$$\hat{x}_q(t) = v(q(t), t) - \int_0^t v_2(q(y), y)dy \text{ for every } t. \tag{2}$$

3. *The allocation function $q(\cdot)$ is implementable.*

The implementability condition (1) may look unappealing, but it is a suitable generalization of well known monotonicity conditions. For example if $A = [0, 1]$ and $v(q, t) = qt$, then (1) is equivalent to the

[3] In my single-agent environment, values are trivially private and efficient implementation is never an issue. If $v(q(t), t) \geq v(a, t)$ for all a and t, then payments $x(t) = 0$ for all t implement q. This is precisely a Vickrey-Clarke-Groves mechanism in a single-agent environment.

condition that $q(\cdot)$ is nondecreasing. Furthermore, (1) gives rise to the usual revenue equivalence result: statements 2 and 3 are equivalent. Suppose that some payment function x implements q and define $\hat{x}_q$ as in (2). Then $\hat{x}_q(s) - x(s) = \hat{x}_q(t) - x(t)$ for every (s, t). Hence $(q, \hat{x}_q)$ is incentive compatible as well. On the other hand if $(q, \hat{x}_q)$ is not incentive compatible, then q is not implementable. For if q is implemented by some x, then x and $\hat{x}_q$ can only differ by a constant and $(q, \hat{x}_q)$ would have to be incentive compatible. Thus, for implementation purposes, restricting attention to implementability by $\hat{x}_q$ is without loss of generality. In what follows, I will use the implementability of an allocation q and the incentive compatibility of the mechanism $(q, \hat{x}_q)$ interchangeably.

In order to economize on space, I introduce the following expressions. For any allocation function q, and any (s, t),

$$\begin{aligned} L_q(s,t) &:= \int_s^t v_2(q(y), y)dy, \text{ and} \\ R_q(s,t) &:= v(q(t), t) - v(q(t), s) \end{aligned}$$

are the left- and right-hand sides of (1). Note that $L_q(s, t) = -L_q(t, s)$ and (1) is equivalent to the statement

$$L_q(t,s) \geq -R_q(s,t) \text{ for all } (s,t).$$

Now, to make sense of (1), suppose that for some allocation function q, the mechanism $(q, \hat{x}_q)$ is **not** incentive compatible. Then there is a pair (s, t) such that type s is better off reporting type t, i.e.,

$$\begin{gathered} v(q(s), s) - v(q(t), s) < \hat{x}_q(s) - \hat{x}_q(t) \\ \Leftrightarrow \\ v(q(s), s) - v(q(t), s) < v(q(s), s) - L_q(0, s) - v(q(t), t) + L_q(0, t) \\ \Leftrightarrow \\ R_q(s,t) < L_q(s,t). \end{gathered}$$

This is precisely the converse of the implementability condition (1).

Individual rationality An important consideration in mechanism design is participation, i.e., the issue of whether the truth-telling equilibrium of a mechanism gives the agent a payoff at least as large as his outside option. In standard models, the outside option is taken to be independent of the agent's type, and it is normalized to zero. Suppose this is the case. Then a mechanism $(q, \hat{x}_q)$ is *individually rational* if for every t, $v(q(t), t) - \hat{x}_q(t) \geq 0$. The left hand side of this inequality is precisely $L_q(0, t)$ and is commonly referred to as the agent's *information* *rent.* In general $(q, \hat{x}_q)$ need not be individually rational even if it is incentive compatible.

Lemma 1 *Suppose that the mechanism $(q, \hat{x}_q)$ is incentive compatible. Then it is also individually rational if $R_q(t, 0) \leq 0$ for all t.*

It is common in models of mechanism design to assume that values are increasing in types, an assumption I have not made in this paper. Under this assumption the sufficient condition in Lemma 1 trivially follows: for all t, $0 \leq v(q(0), t) - v(q(0), 0) = -R_q(t, 0)$. In problems where v is not monotone in t, Lemma 1 indicates that an incentive compatible mechanism $(q, \hat{x}_q)$ is also individually rational if all types of the agent receive the same payoff from the allocation of the smallest type. The optimal mechanisms in the environment of the next section will satisfy this rather mild condition.

4 Application: Optimal mechanism design

In this section, I will find the optimal mechanism for a seller in a simple parametrized environment which introduces habit formation in an otherwise standard framework. The optimal mechanism is, of course, incentive compatible and in order to solve for it, I will first need to identify implementable allocation functions. The environment is such that buyer's value fails monotone differences for a large set of parameter values. Hence in order to characterize incentives for these parameters, Proposition 1 is not of much help: there is no way to order allocations in such a way that its condition 2 is satisfied. (To be precise, Proposition 1 only identifies constant allocation functions as implementable.) Moreover I will show that the optimal mechanism is nonmonotone for a subset of these parameter values. This means that Proposition 1 would not identify the optimal allocation function as implementable. However, Proposition 2 applies in characterizing incentives for all parameter values.

Let $A = \{0, 1\}$ be the set of allocations and

$$v(a, t) = at - \beta at^2$$

give the agent's values. The number β parametrizes the environment. The type space is $[0, 1]$. The interpretation is as follows. The agent is purchasing an indivisible object from a seller. The allocation $a = 1$ corresponds to sale and $a = 0$ indicates no sale. The payment x takes place at time zero. Consumption takes place at time 1 but it generates utilities in two periods: a at time 1 and $-\beta a$ at time 2. The agent's type t is his discount rate and his value is the sum of discounted utilities. Thus if $\beta \neq 0$, the problem exhibits intertemporal allocation externalities in the form of habit formation: consumption at time 1 generates a disutility of $-\beta$ at time 2. I will refer to β as the habit parameter and maintain the following assumption.

Assumption: $\beta \in (0, 1)$.

Note that v is not monotone in t.

The problem of optimal mechanism design is that of finding an incentive compatible and individually rational mechanism (q, x) such that the expectation of $x(\cdot)$ is at a maximum. The formulation and solution of this problem require a statistical distribution assumption on t.

Assumption: t is uniform on $[0, 1]$.

Suppose that the seller has no costs and the buyer's outside option is zero regardless of his type so that the optimal mechanism design problem is:

$$\max_{(q,x)} \int_0^1 x(y)dy$$
$$\text{s.t.} \begin{cases} q(s)(s-\beta s^2) - x(s) \geq q(t)(s-\beta s^2) - x(t) \text{ for all } (s,t), \\ q(s)(s-\beta s^2) - x(s) \geq 0 \text{ for all } s. \end{cases}$$

The next proposition identifies the optimal mechanism for every β.

Proposition 3 *The seller's optimal mechanism* $(q^*, \hat{x}_{q^*})$ *is such that*

$$q^*(t) = 1 \text{ if and only if } \underline{\tau}(\beta) \leq t \leq \overline{\tau}(\beta)$$

where the cutoff types are given by

$$[\underline{\tau}(\beta), \overline{\tau}(\beta)] = \begin{cases} \left[\frac{1+\beta-\sqrt{\beta^2-\beta+1}}{3\beta}, 1\right] & \text{if} \quad 0 < \beta \leq \frac{5+\sqrt{5}}{10} \\ \left[\frac{1-\beta}{\beta}, 1\right] & \text{if} \quad \frac{5+\sqrt{5}}{10} < \beta \leq \frac{3+\sqrt{3}}{6} \\ \left[\frac{3-\sqrt{3}}{6\beta}, \frac{3+\sqrt{3}}{6\beta}\right] & \text{if} \quad \frac{3+\sqrt{3}}{6} < \beta < 1 \end{cases}$$

and the payments $\hat{x}_{q^*}$ *are defined as in (2).*

The following picture is meant to clarify the optimal mechanisms in Proposition 3.[4] For every habit parameter β, measured on the horizontal

[4] The boundaries $\underline{\tau}(\beta)$ and $\overline{\tau}(\beta)$ of the shaded region are plotted as constant slope functions even though they clearly are not, with the exception $\overline{\tau}(\beta) = 1$ if $0 < \beta < \frac{3+\sqrt{3}}{6}$. However the slopes do not change a lot in the domain $(0, 1)$. Thus, even though the figure lacks mathematical precision, it gives the right idea about how optimal mechanisms change as a function of β.

axis, the corresponding vertical slice of the shaded area gives the set of types, measured on the vertical axis, that are optimally allocated the object. This slice is given for β_0 in the figure. If some pair (β, t) is outside the shaded region, then the optimal mechanism does not allocate the object to type t when the habit parameter is β. Note that if $\beta > \frac{3+\sqrt{3}}{6}$, then the optimal mechanism is nonmonotone: low *and* high types are screened out.

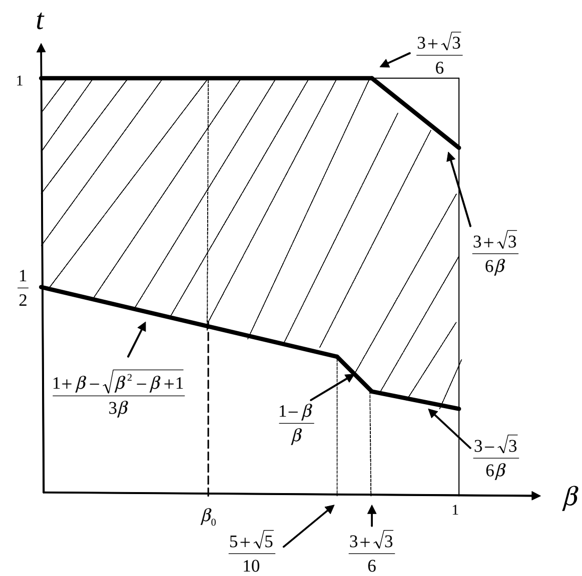

FIGURE 1: OPTIMAL MECHANISMS IN PROPOSITION 3

I follow with several observations regarding the optimal mechanisms.

1. For all β, $0 < \underline{\tau}(\beta) < \overline{\tau}(\beta) \leq 1$, and for $\beta > \frac{3+\sqrt{3}}{6}$, $\overline{\tau}(\beta) < 1$. In particular, $q^*(0) = 0$. Hence $v(q^*(0), t) = 0$ for all t and, by Lemma 1, $(q^*, \hat{x}_{q^*})$ is individually rational whenever it is incentive compatible.

2. Using L'Hospital's rule, $\lim_{\beta \to 0} \underline{\tau}(\beta) = \frac{1}{2}$, the lowest type that optimally receives the object in the standard model with no habit formation.

3. Let us define the agent's *virtual* valuation by

$$\begin{aligned} u(a,t) &= v(a,t) - (1-t)v_2(a,t) \\ &= a\left[(t-\beta t^2) - (1-t)(1-2\beta t)\right] \\ &= a(-3\beta t^2 + 2(1+\beta)t - 1) \end{aligned}$$

The methods of Myerson (1981) can be used to show that if (1) $q^*(t)$ solves $\max_a u(a,t)$ for every t and (2) $q^*(\cdot)$ is implementable, then $(q^*, \hat{x}_{q^*})$ is an optimal mechanism. Suppose that $\beta \in (0, \frac{1}{2}]$

so that values satisfy monotone differences when A is ordered by $0 \preceq 1$: $v_2(0,t) = 0 \leq 1 - 2\beta = v_2(1,t)$ for all t. Now Proposition 1 indicates that all monotone allocation functions are implementable. Consequently if the allocation function defined by themaximizarion of virtual utility is monotone, then it is part of the optimal mechanism. One can easily check that this allocation function is given by

$$\bar{q}(t) = \begin{cases} 1 & \text{if} \quad \frac{1+\beta-\sqrt{\beta^2-\beta+1}}{3\beta} \leq t \leq 1 \\ 0 & \text{if} \quad 0 \leq t < \frac{1+\beta-\sqrt{\beta^2-\beta+1}}{3\beta} \end{cases} \tag{3}$$

and is monotone. Hence the optimal mechanism can be found using Proposition 1.

4. If $\frac{1}{2} < \beta$, then the monotone differences property fails. Yet the proof of the proposition indicates that if $\beta \in (\frac{1}{2}, \frac{5+\sqrt{5}}{10}]$, the allocation function $\bar{q}$ in (3) satisfies the implementability condition (1). In other words, standard methods yield the optimal mechanism using Proposition 1 even in the absence of monotone differences.

5. If $\frac{5+\sqrt{5}}{10} < \beta$, then the allocation function $\bar{q}$ in (3) fails the implementability condition (1). To see this note that

$$L_{\bar{q}}(1,0) = \underline{\tau}(\beta) - \beta\underline{\tau}^2(\beta) - 1 + \beta > 0 = R_{\bar{q}}(1,0),$$

in other words, type $s = 1$ has a strict incentive to report $t = 0$ to the mechanism $(\bar{q}, \hat{x}_{\bar{q}})$. Yet, the optimal mechanism for these habit parameters can be solved for using the observation that there is a one-to-one relationship between implementable allocation functions and allocation functions supported by take-it-or-leave-it offers at some price. Solving for the optimal take-it-or-leave-it price, then, gives rise to an equivalent direct revelation mechanism which is optimal. If β is high enough, namely if $\frac{3+\sqrt{3}}{6} < \beta$, then the optimal mechanism is nonmonotone, low *and* high values of t are screened out by the seller. This is intuitive: high types discount future habit cost at a higher rate. If this cost is large enough, then the seller is better off excluding high types in favor of types with higher discounted values.

6. The payments in the optimal mechanism are

$$\begin{aligned}\hat{x}_{q^*}(t) &= v(1, \underline{\tau}(\beta)) \\ &= \underline{\tau}(\beta) - \beta\underline{\tau}^2(\beta) \\ &= \begin{cases} \frac{1+2\beta-2\beta^2-(1-2\beta)\sqrt{\beta^2-\beta+1}}{9\beta} & \text{if} \quad 0 < \beta \leq \frac{5+\sqrt{5}}{10} \\ 1-\beta & \text{if} \quad \frac{5+\sqrt{5}}{10} < \beta \leq \frac{3+\sqrt{3}}{6} \\ \frac{1}{6\beta} & \text{if} \quad \frac{3+\sqrt{3}}{6} < \beta < 1 \end{cases}\end{aligned}$$

and they correspond to optimal prices as the proof of Proposition 3 in Appendix 1 shows.

5 Conclusion

In this paper, I give a characterization of incentive compatibility that is especially useful when values are not monotone in private information and put it to use in a buyer-seller example featuring habit formation. The characterization builds on the usual envelope condition and is independent of monotonicity and monotone differences conditions commonly assumed in the literature. The characterization result can be used to establish individual rationality of most mechanisms, this is the content of Lemma 1. It is also useful, as the parametrized example of Section 4 shows, to show that standard methods of mechanism design may apply when values fail monotone differences and the resulting mechanisms can be nonmonotone.

The model I analyze is one with a single agent. This restriction is mostly for brevity. Results of Section 3 extend to multiagent environments and these extensions are in Appendix 2.

A multiagent version of the environment in Section 4 may also be of interest. Proposition 3 trivially extends to such a multiagent environment only in part. To solve for the optimal mechanism when $\beta > \frac{5+\sqrt{5}}{10}$, I make use of an equivalence between an optimal mechanism design problem and an optimal pricing problem, which fails to hold in multiagent environments.

References

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Appendix 1: Proofs

Proof of Proposition 1 Using Proposition 2 -whose proof is to follow- I need only show that if there is an order $\preceq$ on A such that (i) $s \leq t$ implies $q(s) \preceq q(t)$, and (ii) $a' \preceq a$ implies $v_2(a', t) \leq v_2(a, t)$ for every t, then the implementability condition (1) follows. Suppose such $\preceq$ exists and take $s < t$. Then

$$\int_s^t v_2(q(y), y)dy \leq \int_s^t v_2(q(t), y)dy = v(q(t), t) - v(q(t), s).$$

Similarly if $t < s$. Thus q satisfies (1) and the proof is complete. ■

Proof of Proposition 2 ($1 \Rightarrow 2$) Suppose that q satisfies (1). Then the mechanism $(q, \hat{x}_q)$ is incentive compatible as for every (s, t)

$$\begin{aligned}\hat{x}_q(t) - \hat{x}_q(s) &= v(q(t), t) - v(q(s), s) - \int_s^t v_2(q(y), y)dy \\ &\geq v(q(t), s) - v(q(s), s).\end{aligned}$$

($2 \Rightarrow 3$) This trivially follows from definitions.

($3 \Rightarrow 1$) To see that (1) is necessary for implementation, suppose that for some payment function x, (q, x) is incentive compatible. By the envelope theorem (Milgrom and Segal 2002)

$$v(q(t), t) - x(t) = v(q(s), s) - x(s) + \int_s^t v_2(q(y), y)dy$$

for every (s, t). Rearranging and using incentive compatibility,

$$\begin{aligned}v(q(t), t) &= v(q(s), s) - x(s) + x(t) + \int_s^t v_2(q(y), y)dy \\ &\geq v(q(t), s) + \int_s^t v_2(q(y), y)dy\end{aligned}$$

and (1) follows. ■

Proof of Lemma 1 Suppose that $(q, \hat{x}_q)$ is incentive compatible and that $v(q(0), t) \geq v(q(0), 0)$ for all t. Then for all t

$$v(q(t), t) - \hat{x}_q(t) = L_q(0, t) \geq -R_q(t, 0) \geq 0$$

where the equality follows from the definition of payments $\hat{x}_q$ in (2), the first inequality follows from the implementation condition (1) and the second from hypothesis. ■

Proof of Proposition 3 First note that $q^*(0) = 0$ for all β. Thus, by Lemma 1, $(q^*, \hat{x}_{q^*})$ is feasible in the optimal mechanism design problem if it is incentive compatibile.

I will divide the proof into three parts depending on the value of β.

Part 1 Suppose that $\beta \in (0, 1/2]$. As noted in the main text, monotone differences property is satisfied when $0 \preceq 1$, as $v_2(0, t) = 0 \leq 1 - 2\beta t = v_2(1, t)$ for all t. Now the allocation function (3) defined in the main text is nondecreasing and therefore implementable by Proposition 1. This shows feasibility. To see that $(q^*, \hat{x}_{q^*})$ is optimal, note that the allocation functions in Proposition 3 and in equation (3) coincide for this parameter range. Thus q^* maximizes virtual utility. Optimality now follows from the definition (2) of payments $\hat{x}_{q^*}$.

Part 2 Suppose that $\beta \in (1/2, (5 + \sqrt{5})/10]$. Note that q^* maximizes virtual utility for this parameter range as well. Hence q^* is part of an optimal mechanism if it is implementable. I will show that q^* satisfies (1) and is therefore implementable.

To begin, note that

$$0 < \underline{\tau}(\beta) < 1/2\beta < 1 < 1/\beta.$$

In order to apply the characterization result, first take (s, t) such that $s < \underline{\tau}(\beta) \leq t$. It follows that

$$\begin{aligned} L_{q^*}(s, t) &= \int_{\underline{\tau}(\beta)}^{t} (1 - 2\beta y) dy \\ &= t - \underline{\tau}(\beta) - \beta t^2 + \beta[\underline{\tau}(\beta)]^2 \\ &< t - s - \beta t^2 + \beta s^2 \\ &= R_{q^*}(s, t) \end{aligned}$$

where the inequality follows because $s < \underline{\tau}(\beta) < -1/2\beta$.

Now take (s,t) such that $t < \tau(\beta) \le s$ so that

$$\begin{aligned} L_{q^*}(s,t) &= \int_s^{\underline{\tau}(\beta)} (1-2\beta y)dy \\ &= \underline{\tau}(\beta) - s - \beta[\underline{\tau}(\beta)]^2 + \beta s^2 \\ &\le 0 \\ &= R_{q^*}(s,t). \end{aligned}$$

The weak inequality follows from the following observations: let $f(y) = y - \beta y^2$ on $[0,1]$. I need to show $f(\underline{\tau}(\beta)) \le f(s)$. Note that f is concave and maximized at $1/2\beta$. Hence if $\underline{\tau}(\beta) \le s \le 1/2\beta$, the inequality follows trivially. Otherwise, note that

$$\min_{\frac{1}{2\beta} < y \le 1} f(y) = f(1) = 1 - \beta.$$

Thus, it suffices to show that $f(\underline{\tau}(\beta)) \le f(1)$. But this follows as $\beta \le (5+\sqrt{5})/10$. Thus q^* satisfies (1) and is therefore implementable.

Part 3 If $\beta > (5+\sqrt{5})/10$, then the allocation function in equation (3) is no longer implementable. I will derive the optimal mechanism by solving an optimal pricing problem. I need some preparations.

Let $p \ge 0$ be a price for the object. If the seller makes a take-it-or-leave-it offer at p, then any type whose value weakly exceeds p will purchase the object. (See the figures in Appendix 4.) Hence each p gives rise to an allocation function q^p defined by

$$q^p(t) = \begin{cases} 1 & \text{if} \quad t - \beta t^2 \ge p, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, if $p > \frac{1}{4\beta} = \max_t t - \beta t^2$, then $q^p(t) = 0$ for all p. Such prices earn the seller a revenue of zero. I will concentrate on prices $p \in [0, \frac{1}{4\beta}]$. In this case q^p becomes

$$q^p(t) = \begin{cases} 1 & \text{if} \quad \frac{1-\sqrt{1-4\beta p}}{2\beta} \le t \le \frac{1+\sqrt{1-4\beta p}}{2\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

Calculating the payments $\hat{x}_{q^p}$ as in (2), I get

$$\hat{x}_{q^p}(t) = \begin{cases} p & \text{if} \quad \frac{1-\sqrt{1-4\beta p}}{2\beta} \le t \le \frac{1+\sqrt{1-4\beta p}}{2\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the indirect mechanism of a take-it-or-leave-it price at p is identifiable with the direct revelation mechanism $(q^p, \hat{x}_{q^p})$. On the other hand, if a mechanism $(q, \hat{x}_q)$ is implementable, then it satisfies the following monotonicity property:

$$\text{if } q(s) = 1 \text{ and } t - \beta t^2 > s - \beta s^2 \text{ then } q(t) = 1. \qquad (*)$$

For otherwise for some (s, t) such that $q(s) = 1$, $q(t) = 0$ and $t - \beta t^2 > s - \beta s^2$, we would have

$$L_q(s, t) \leq R_q(s, t) = 0$$

and

$$0 \leq -L_q(s, t) = L_q(t, s) \leq R_q(t, s) = s - \beta s^2 - t + \beta t^2 < 0,$$

a contradiction to implementability of q. The monotonicity property (*) in turn implies that the allocation function q is supported by a take-it-or-leave-it price at $p = t_0 + \beta t_0^2$ where $t_0 = \inf\{t : q(t) = 1\}$. But $t_0 + \beta t_0^2$ is precisely $\hat{x}_q(t)$ for any t such that $q(t) = 1$. In other words, any incentive compatible mechanism $(q, \hat{x}_q)$ is virtually a take-it-or-leave-it offer.

In order to solve for the optimal mechanism, then, I need only solve for the optimal price p for the seller and then convert it to the equivalent direct mechanism. In order to formulate the optimal pricing problem, first fix any habit parameter $\beta \in (\frac{\sqrt{5}+5}{10}, 1)$. Using the uniformity of t, for every price $p \geq 0$, the expected revenue becomes

$$\begin{aligned} R(p|\beta) &= p \Pr\{t - \beta t^2 \geq p\} \\ &= \begin{cases} p\left(1 - \frac{1-\sqrt{1-4\beta p}}{2\beta}\right) & \text{if} \quad 0 \leq p \leq 1 - \beta, \\ \frac{p\sqrt{1-4\beta p}}{\beta} & \text{if} \quad 1 - \beta < p \leq \frac{1}{4\beta}, \\ 0 & \text{if} \quad \frac{1}{4\beta} < p. \end{cases} \end{aligned}$$

Note that $0 < 1 - \beta < \frac{1}{4\beta}$ so the expected revenue function is well defined. Also note that $R(\cdot|\beta)$ is continous. To show this I need only check continuity at the break points $p = 1 - \beta$ and $p = \frac{1}{4\beta}$. To check continuity at $p = 1 - \beta$ fix any β. Continuity from the left is obvious. Note

$$\begin{aligned} \lim_{p \downarrow 1-\beta} R(p|\beta) &= \lim_{p \downarrow 1-\beta} \frac{p\sqrt{1-4\beta p}}{\beta} \\ &= \frac{(1-\beta)\sqrt{1-4\beta(1-\beta)}}{\beta} \\ &= \frac{(1-\beta)(2\beta-1)}{\beta} \\ &= R(1-\beta|\beta). \end{aligned}$$

$R(p|\beta)$ is also continuous at $p = \frac{1}{4\beta}$ as $R(\frac{1}{4\beta}|\beta) = 0 = R(p|\beta)$ for all $p > \frac{1}{4\beta}$.

Now I will solve

$$\max_p R(p|\beta)$$

piece-by-piece, first on $[0, 1-\beta]$, and then on $(1-\beta, \frac{1}{4\beta}]$.

I claim that

$$\{1-\beta\} = \arg \max_{0 \le p \le 1-\beta} R(p|\beta).$$

To see this, first note that $R(0|\beta) = 0$. Moreover for every $p \in [0, 1-\beta]$, the first and second derivatives of expected revenue with respect to price are

$$R'(p|\beta) = 1 + \frac{1}{2\beta}\left(\sqrt{1-4p\beta} - 1\right) - \frac{p}{\sqrt{1-4p\beta}}$$
$$R''(p|\beta) = -2p\frac{\beta}{(1-4p\beta)^{\frac{3}{2}}} - \frac{2}{\sqrt{1-4p\beta}} = -2 < 0$$

Thus $R(\cdot|\beta)$ is strictly concave on $[0, 1-\beta]$. The first derivative evaluated at $p = 0$ is $R'(0|\beta) = 1 > 0$, and therefore the expected revenue is increasing around 0. Finally the first derivative evaluated at $p = 1-\beta$ is

$$R'(1-\beta|\beta) = \frac{\beta - 1}{\beta} - \frac{1-\beta}{2\beta - 1} + 1 > 0.$$

(In fact $R'(1-\beta|\beta) = 0$ if $\beta = \frac{\sqrt{5}+5}{10}$.) Thus $R(\cdot|\beta)$ is strictly increasing on $[0, 1-\beta]$ and is maximized at $p = 1-\beta$.

Next, I note that

$$\beta \in \left(\frac{\sqrt{3}+3}{6}, 1\right) \Rightarrow \arg \max_{1-\beta < p \le \frac{1}{4\beta}} R(p|\beta) = \left\{\frac{1}{6\beta}\right\}, \text{ and}$$
$$\beta \in \left(\frac{\sqrt{5}+5}{10}, \frac{\sqrt{3}+3}{6}\right] \Rightarrow R(1-\beta|\beta) = \sup_{1-\beta < p \le \frac{1}{4\beta}} R(p|\beta).$$

Consequently the expected revenue maximizing price is

$$p^*(\beta) = \begin{cases} \frac{1}{6\beta} & \text{if } \frac{\sqrt{3}+3}{6} < \beta < 1, \text{ and} \\ -\beta + 1 & \text{if } \frac{\sqrt{5}+5}{10} < \beta \le \frac{\sqrt{3}+3}{6}. \end{cases}$$

The optimal price and the allocation function it supports is presented in the next two figures.

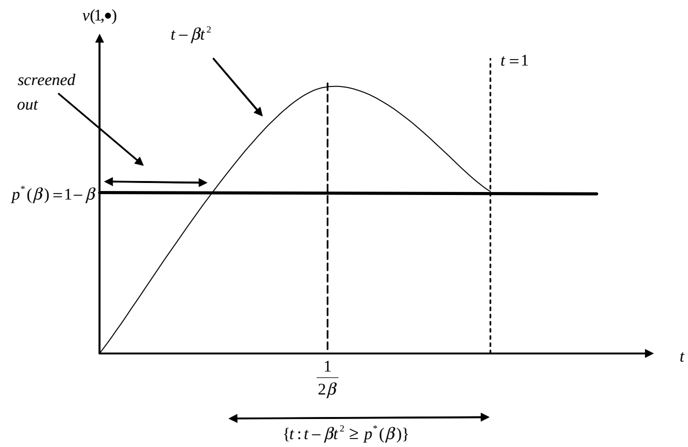

FIGURE 2: OPTIMAL PRICE when $\frac{\sqrt{5}+5}{10}<\beta\leq\frac{\sqrt{3}+3}{6}$

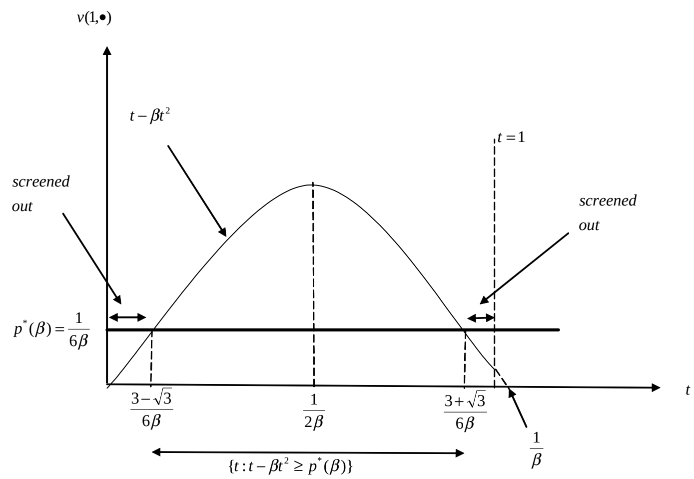

FIGURE 3: OPTIMAL PRICE when $\frac{\sqrt{3}+3}{6}<\beta<1$

All that remains to show is that the allocation function q^* in Proposition 3 is such that for every $\beta \geq \frac{5+\sqrt{5}}{10}$ by $q^*(t) = 1$ if and only if $t - \beta t^2 \geq p^*(\beta)$. I skip this straightforward step and the proof is complete. ■

Appendix 2: Extension to multiple agents with interdependent values

Consider the following standard multiagent model with interdependent values. Let A be a set of allocations and N be a set of agents, both finite. Endow each $i \in N$ with the type space $T_i = [\underline{t}_i, \bar{t}_i]$ and valuation function $v_i : A \times T \rightarrow \Re$ where $T = \times_{j \in N} T_j$. Assume that each v_i is continuously differentiable in t_i and denote the derivative by $v_i'(a, \cdot, t_{-i})$. An allocation function is a map $q : T \to A$. A mechanism is a list $(q, (x_i)_{i \in N})$ where $x_i : T \to \Re$ for every i.

A mechanism $(q, (x_i)_{i \in N})$ is *ex post Nash incentive compatible* if for every i, for every pair (s_i, t_i) of i's types, and for every type profile t_{-i} of the rest of the agents,

$$v_i(q(s_i, t_{-i}), t_i, t_{-i}) - x_i(s_i, t_{-i}) \leq v_i(q(t_i, t_{-i}), t_i, t_{-i}) - x_i(t_i, t_{-i}).$$

An allocation function q is *ex post Nash implementable* if for some $(x_i)_{i \in N}$, the mechanism $(q, (x_i)_{i \in N})$ is ex post Nash incentive compatible.

Proposition 4 (Extension of Proposition 1 to a multiagent environment) *An allocation function $q(\cdot)$ is ex post Nash implementable if for every i and t_{-i}, there exists a binary relation $\preceq_{t_{-i}}$ on A such that*

1. *$s_i \leq t_i$ implies $q(s_i, t_{-i}) \preceq_{t_{-i}} q(t_i, t_{-i})$ and*
2. *$a' \preceq_{t_{-i}} a$ implies $v_i'(a', t_i, t_{-i}) \leq v_i'(a, t_i, t_{-i})$ for every t_i.*

The proof uses Proposition 5 below and is identical to the proof of Proposition 1 except for the necessary changes in notation.

If we make the additional assumption that q is efficient, then Proposition 4 is an exact analog of Theorem 5.1 in Jehiel and Moldovanu (2001) and Proposition 3 in Bergemann and Välimäki (2002).

In the linear model of Jehiel and Moldovanu, $v_i(a, t) = \sum_{j \in N} \alpha_{ij}(a) t_j$ and monotone differences property takes the following form: for every i and t_{-i}, $a' \preceq_{t_{-i}} a \Rightarrow \alpha_{ii}(a') \leq \alpha_{ii}(a)$. Jehiel and Moldovanu start with an arbitrary unordered allocation set A and order it for every i and t_{-i} such that monotone differences property is satisfied. They are interested in implementing the efficient allocation rule $q^E(\cdot)$, which is obtained by choosing

$$q^E(t) \in \arg\max_{a \in A} \sum_{i \in N} \sum_{j \in N} \alpha_{ij}(a) t_j$$

for every t. The efficient allocation, of course, need not satisfy the monotonicity property. Jehiel and Moldovanu further impose the condition

$$\text{for all } i \text{ and } t_{-i},\ a' \preceq_{t_{-i}} a \Rightarrow \sum_{j \in N} \alpha_{ji}(a') < \sum_{j \in N} \alpha_{ji}(a).$$

If this condition is satisfied then for every i and t_{-i}, the map

$$(a, t_i) \mapsto \sum_{i \in N} \sum_{j \in N} \alpha_{ij}(a) t_j$$

satisfies the single crossing property. Consequently q^E is monotone and implementation follows.

In Bergemann and Välimäki (2002) values are nonlinear in types and therefore the monotone differences property is more involved. Their Proposition 3 is precisely Proposition 4 above for efficient allocation functions. Crémer and McLean (1985) give an analogous result using a mechanism tailored to their setting with finite type spaces.

The multiagent version of Proposition 2 is as follows.

Proposition 5 (Extension of Proposition 2 to a multiagent environment) *The following statements are equivalent:*

1. *The allocation function $q(\cdot)$ satisfies*

$$\int_{s_i}^{t_i} v_i(q(y, t_{-i}), y, t_{-i})dy \leq v_i(q(t_i, t_{-i}), t_i, t_{-i}) - v_i(q(t_i, t_{-i}), s_i, t_{-i})$$

for every (i, t_{-i}, s_i, t_i).

2. *The mechanism $(q, (\hat{x}_{iq})_{i \in N})$ is ex post Nash incentive compatible where*

$$\hat{x}_{iq}(t_i, t_{-i}) = v_i(q(t_i, t_{-i}), t_i, t_{-i}) - \int_{\underline{t}_i}^{t_i} v_i(q(y, t_{-i}), y, t_{-i})dy$$

for every (i, t_i, t_{-i}).

3. *The allocation function $q(\cdot)$ is ex post Nash implementable.*

The proof is identical to the proof of Proposition 2 except for the necessary changes in notation.