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(Weak) Correlation and Sunspots in Duopoly

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Abstract

For duopoly models, we consider the notion of *weak* correlation using *simple* symmetric devices that the players choose to commit to in equilibrium. In a linear duopoly game, we prove that *Nash-centric* devices involving a sunspot structure are simple symmetric weak correlated equilibria. Any small perturbation from such a structure fails to be an equilibrium.

Keywords: Duopoly, Weak correlation, Simple device, Sunspots.

JEL Classification Numbers: C72.

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1 INTRODUCTION

We know from the pioneering works by Azariadis (1981) and Cass and Shell (1983) that extrinsic uncertainty matters in competitive economies. Does it matter in strategic markets as well? The answer we get from the literature is unfortunately partial and incomplete, hence, inconclusive and confusing, to some extent.

First of all, we should note that two notions of extrinsic uncertainty, formulated as sunspot equilibrium in competitive markets (Azariadis 1981, Cass and Shell 1983) and correlated equilibrium in non-cooperative games, as introduced by Aumann (1974, 1987), are very similar in nature and indeed closely connected, as formally presented by Polemarchakis and Ray (2006) in a strategic market game played by overlapping generations.

On one hand, we know from a recent literature (Peck 1994, Forges and Peck 1995, Dávila 1999 among others) that indeed sunspot equilibria exist in strategic market games *a la* Shapley and Shubik (1977). On the other hand, in oligopoly markets, Liu (1996) and Yi (1997) proved that the only correlated equilibrium for such games is the Cournot-Nash equilibrium of the market, a result that Neyman (1997) and Ui (2008) subsequently generalised for a large class of potential games, of which oligopoly models are special cases. Although correlation may not achieve anything more than the Nash outcome, as one rightly reckons, a weaker notion of correlation may be able to improve upon the Nash equilibrium in oligopoly models. Indeed, such a concept was introduced by Moulin and Vial (1978) and was used by Gerard-Varet and Moulin (1978) in duopoly games to achieve an improvement over the Cournot-Nash outcome.

There are, however, a few gaps worth mentioning in the above results. First, from the analysis by Gerard-Varet and Moulin (1978), we do learn, in specific duopoly games, under what conditions Nash equilibrium can be locally improved upon. However, the specific structure of the correlation device that does the trick has not been characterised. Also, we do not know whether such a correlation device necessarily has to be local (close) to the Nash equilibrium or not. Second, from the analysis of Polemarchakis and Ray (2006), we do not learn the connection, if there is any, between (weak) correlated equilibrium and sunspot equilibrium in oligopoly models, in particular, duopoly games in the literature.

The purpose of this paper is precisely to bridge the gaps in the above literature and thus is two fold. We would like to find a general (non-local) structure of a weak correlated equilibrium and if possible, link it to a sunspot structure for strategic markets. To achieve these two results, we analyse arguably the most fundamental and surely the simplest of models in strategic markets, that of a duopoly game with linear demand and constant marginal cost, called here the *linear duopoly game*.

In this paper, we consider a specific form of correlation devices, that we call a Simple Symmetric Correlation Device (k-SSCD) to analyse weak correlation. The device is called so because the discrete probability distribution is symmetric and the support of it is finite. Such a device was introduced by Ganguly and Ray (2005) to analyse mediation in cheap talk games. We apply the notion of weak correlation introduced by Moulin and Vial (1978) and fully characterise an equilibrium concept that we call k-Simple Symmetric Weak Correlated Equilibrium (k-SSWCE) for the linear duopoly game (Theorem 1). Clearly, the deterministic device that chooses only the Cournot-Nash point is a k-SSWCE; however, unlike Aumann's correlated equilibrium, this is not the only device which is an equilibrium according to our notion. We identify a particular sunspot structure, that we call a Nash-centric device, which is a symmetric anti-diagonal probability distribution over quantities around the Nash equilibrium. We prove that any Nash-centric device is a k-SSWCE. This result (Theorem 2) holds for any such Nash-centric devices, regardless of its dimension, probabilities and the distances between the quantity levels, as long as this sunspot structure is maintained. Moreover, we prove that any small perturbation from this structure is not an equilibrium; in this sense, any Nash-centric device is a *locally* unique equilibrium.

Our results identify a specific random device that the players are willing to commit to, in equilibrium. The device involves a number of equi-distant quantity levels around the Nash equilibrium with any (anti-diagonal) probability values. Moreover, this Nash-centric device is a public randomisation over non-Nash points, which is a feature of a sunspot structure. Thus, weak correlation, like sunspot, matters in the strategic market model of the linear duopoly game. Finally, note that the expected payoff from the Nash-centric device is equal to that of the Nash payoff for the linear duopoly game. Our weak correlated equilibrium, thus, offers an explanation of how to achieve the Nash equilibrium payoff, in expectation, which often is difficult for the players to get by themselves, as many experimental papers indicate (Bone, Drouvelis and Ray 2011).

2 SET-UP

2.1 Correlation and Weak Correlation

Fix any finite normal form game, $G = [N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}]$, with set of players, $N = \{1, ..., n\}$, finite pure strategy sets, $S_1, ..., S_n$ with $S = \prod_{i \in N} S_i$, and payoff functions, $u_1, ..., u_n$, $u_i : S \to \Re$, for all i.

Definition 1 *A (direct or canonical) correlation device μ is a probability distribution over S.*

A normal form game, G, can be extended by using a direct correlation device. For correlation *a la* Aumann (1974, 1987), the device first selects a strategy profile $s(= (s_1, ..., s_n))$ according to μ, and then sends the private recommendation s_i to each player i. The extended game G_μ is the game where the correlation device μ selects and sends recommendations to the players, and then the players play the original game G. A (pure)[1] strategy for player i in the game G_μ is a map $\sigma_i : S_i \to S_i$ and the corresponding (ex-ante, expected) payoff is given by, $u_i^*(\sigma_1, ..., \sigma_n) = \sum_{s \in S} \mu(s) u_i(\sigma_1(s_1), ..., \sigma_n(s_n))$. The *obedient* strategy profile is the identity map $\sigma_i^*(s_i) = s_i$, for all i, with payoff to player i given by $u_i^*(\sigma^*) = \sum_{s \in S} \mu(s) u_i(s)$. The device is called a correlated equilibrium (Aumann 1974, 1987) if all the players follow the recommended strategies, i.e., the obedient strategy profile constitutes a Nash equilibrium of the extended game G_μ. Formally, with the notation $s_{-i} \in S_{-i} = \prod_{j \neq i} S_j$,

Definition 2 *μ is a (direct or canonical) correlated equilibrium of the game G if $\sum_{s \in S} \mu(s) u_i(s) \geq \sum_{s \in S} \mu(s) u_i(t_i, s_{-i})$, for all i, for all $t_i \in S_i$.*

One may use a direct correlation device, μ, in a different way. For a weaker notion of correlation *a la* Moulin and Vial (1978), a game G is extended to a game G_μ' in which the strategies of a player is either to commit to the device or to play any strategy in G. If all the players commit to the device, the players do not then play the game G, however, get the outcome chosen by the device according to the probability distribution. Thus, the expected payoff for any player i, when the device is accepted by all the players, is simply $\sum_{s \in S} \mu(s) u_i(s)$. Note that this is same as the payoff of the obedient strategy profile under the correlated equilibrium *a la* Aumann as above. If one of the players unilaterally deviates, while the others commit to the device, the deviant faces the *marginal* probability distribution μ' over $s_{-i} \in S_{-i}$ which is given by $\mu'(s_{-i}) = \sum_{s_i \in S_i} \mu(s_i, s_{-i})$. The weak correlated equilibrium[2] notion suggests that the players will commit to the device if the expected payoff from the device is higher than that from playing any other strategy. Formally,

Definition 3 *μ is a (direct or canonical) weak correlated equilibrium of the game G if $\sum_{s \in S} \mu(s) u_i(s) \geq \sum_{s_{-i} \in S_{-i}} \mu'(s_{-i}) u_i(t_i, s_{-i})$, for all i, for all $t_i \in S_i$.*

From the system of inequalities in the above definitions, it is clear that indeed weak correlated equilibrium is weaker than correlated equilibrium as a concept.[3] Also it is obvious that any Nash

[1] One can also think of behavioral strategies in any extended game. We, however, in this paper, restrict ourselves to pure strategies only.

[2] Although this notion is due to Moulin and Vial (1978), they have not named the equilibrium concept. They have called the device used for correlation a correlation scheme. Young (2004, Chapter 3, page 30) called this equilibrium the coarse correlated equilibrium, while Forgó (2010) called it the weak correlated equilibrium.

[3] It is also easy to prove that the set of correlated and weak correlated equilibria coincide for the case of 2×2 games. However, as Moulin and Vial (1978) demonstrated, there are games involving 2 players and 3 strategies each for which the set of weak correlated equilibria is strictly larger than the sets of correlated and Nash equilibria.

equilibrium and any convex combination of several Nash equilibria of any given game G, *corresponds* to a weak correlated and a correlated equilibrium.[4]

2.2 Linear Duopoly

In this paper, we use the simplest form of oligopoly models, that of a duopoly market with linear demand function and constant marginal cost. Consider two quantity-setting firms, each of whose strategy is to choose a quantity level $q \in Q = \{q : q \geq 0\}$. The profit functions for the firms are given by $\pi_1(q^1, q^2) = aq^1 - b(q^1)^2 - bq^1q^2 - cq^1$ and $\pi_2(q^1, q^2) = aq^2 - b(q^2)^2 - bq^1q^2 - cq^2$, where q^1 and q^2 are quantity choices of firms 1 and 2, respectively. As it is well-known, the Nash equilibrium outcome of this game is $q^1 = q^2 = \frac{a-c}{3b}$ and the Nash equilibrium payoff to each firm is $\frac{(a-c)^2}{9b}$. For the rest of the paper this two-person game will be called the *linear duopoly game*.

Liu (1996) analysed an oligopoly model with n firms, each with a constant marginal cost, c^i for firm i $(i = 1, ..., n)$ operating in a market with linear demand and proved that the only correlated equilibrium of this game is the unique Cournot-Nash equilibrium. Our game clearly is a special case of Liu's model with $n = 2$, and $c^1 = c^2 = c$. Neyman (1997) and Ui (2008) generalised Liu's result for potential games. Our game is a potential game with a smooth and concave potential function, f, given by, $f(q^1, q^2) = (a - c)(q^1 + q^2) - b[(q^1)^2 + q^1q^2 + (q^2)^2]$. Therefore, the linear duopoly game has a unique correlated equilibrium *a la* Aumann that coincides with the Nash equilibrium of the game. One may also wish to apply the result by Gerard-Varet and Moulin (1978) for local improvement upon the Cournot-Nash by weak correlation, in our game. Their theorem, unfortunately, does not apply to the linear duopoly game; however, as pointed out in their paper (Gerard-Varet and Moulin, 1978, page 133), it can be directly proved that Nash equilibrium of this game can not be improved upon by weak correlation (according to their notion of improvement).

2.3 Simple Devices

We now consider a specific form of correlation devices for any game with a continuum of strategy sets. Although the strategy sets in games such as the linear duopoly game are continuum, a device may involve only finitely many strategies, i.e., the support of the probability distribution in the direct correlation device may be finite. The structure of such a *simple* device was introduced in Ganguly and Ray (2005) who analysed simple mediation in cheap-talk games. Here, we consider such simple devices to analyse weak correlation in the linear duopoly game. Moreover, we impose symmetry in

[4] Formally, let $NE(G)$, $CONV(G)$, $WCE(G)$ and $CE(G)$ denote, respectively, the sets of all Nash equilibria, convex combination of Nash equilibria, weak correlated equilibria and correlated equilibria for any game G. Clearly, $NE(G) \subseteq CONV(G) \subseteq CE(G) \subseteq WCE(G)$.

the probability distribution and we restrict the device to use the same quantities for both the players. Formally, the specific form of the device we consider in this paper is defined below.

Definition 4 *A k-Simple Symmetric Correlation Device (k-SSCD), $[P; Q_c]$, is a symmetric probability distribution P over $Q_c \times Q_c$, where, $Q_c = \{q_1 < q_2 < .... < q_k; q_i \in Q\}$, and $P = \{(p_{ij})_{i=1,2....k, j=1,2...k}\}$ with each $p_{ij} \in [0, 1]$, $p_{ij} = p_{ji}$ and $\sum_{ij} p_{ij} = 1$.*

A k-SSCD, $[P; Q_c] = [\{P = (p_{ij})_{i=1,2....k, j=1,2...k}\}; \{q_i\}_{i=1,2....k}]$, can be represented in a tabular form as shown below.

strategies	q_1	q_2	...	...	...	q_k
q_1	p_{11}	p_{12}	...	...	...	p_{1k}
q_2	p_{21}	p_{22}	...	...	...	...
...	...	...	...	...	...	...
...	...	...	...	...	...	...
...	...	...	...	...	...	...
q_k	p_{k1}	...	...	...	...	p_{kk}

The interpretation of a k-SSCD is that the players are given a choice to commit to the device. If both players commit, the device will then pick the strategies q_i and q_j for the two players respectively, with probability p_{ij}; the players do not play the game, however, get the payoffs, respectively, $\pi_1(q_i, q_j)$ and $\pi_2(q_i, q_j)$ that correspond to the chosen strategy profile (q_i, q_j). Thus, if both players commit to the device, the expected payoffs for the two players are the same (by symmetry) and is given by $E(\pi_C) = \sum_{ij} p_{ij}\pi_1(q_i, q_j) = \sum_{ij} p_{ij}\pi_2(q_i, q_j) = \sum_{ij} p_{ij}[(a - c)q_i - bq_i^2 - bq_iq_j]$.

One of the players may unilaterally deviate to choose any strategy in the game, while the other does commit to the device. Note that, although the device, $[P; Q_c]$, involves only finitely many strategies in Q_c, the deviation for a player is however not restricted; any strategy $q \in Q$ (even outside the domain, Q_c, of the device) can be played by a player if he doesn't commit to the device. The deviant faces the *marginal* probability distribution $\acute{p}$ over $q_j \in Q_c$ which is given by $\acute{p}(q_j) = \sum_{q_i \in Q_c} p(q_i, q_j)$. Let $E(\pi \mid q)$ denote the expected payoff of any deviating player (by symmetry) from playing q. Clearly, $E(\pi \mid q) = \sum_{q_j \in Q_c} \acute{p}(q_j)\pi(q, q_j) = \sum_{q_j \in Q_c} \acute{p}(q_j)[(a - c)q - bq^2 - bqq_j]$. As before, the equilibrium condition requires that the device be accepted by both players. Formally,

Definition 5 *A k-SSCD, $[P; Q_c] = [\{P = (p_{ij})_{i=1,2....k, j=1,2...k}\}; \{q_i\}_{i=1,2....k}]$, is called a k-Simple Symmetric Weak Correlated Equilibrium (k-SSWCE) if both the players commit to the device, i.e., given that the other player is committing to the device, a player does not deviate to play any other* *strategy* $q \in Q$*, that is,* $E(\pi_C) = \sum_{ij} p_{ij}[(a-c)q_i - bq_i^2 - bq_iq_j] \geq E(\pi \mid q) = \sum_{q_j \in Q_c} p(q_j)[(a-c)q - bq^2 - bqq_j]$*, for all* $q \in Q_c$*.*

We now consider different types of k-SSCD, $[\{P = (p_{ij})_{i=1,2....k},_{j=1,2...k}\}; \{q_i\}_{i=1,2....k}]$, for our analysis in this paper.

A k-SSCD *with equi-distant quantities* is a device for which $q_i = q_1 + (i-1)\partial$, $1 \leq i \leq k$, and thus can be denoted by $[\{P = (p_{ij})_{i=1,2....k},_{j=1,2...k}\}; q_1; \partial]$.

A *public* or *sunspot* k-SSCD is a device for which the probability distribution P is such that whenever $p_{ij} > 0$, the *conditional probability* of q_i, given q_j, is 1, and vice versa (in other words, each row and each column in the probability distribution matrix has one and only one positive element).

An *anti-diagonal* k-SSCD is a device for which the probability distribution P is an *anti-diagonal distribution* in which only the anti-diagonal elements of the probability distribution matrix are strictly positive, i.e., $p_{ij} > 0$ only when $i + j = k + 1$ and $p_{ij} = 0$, when $i + j \neq k + 1$. Clearly, an anti-diagonal k-SSCD is a special case of a public or sunspot k-SSCD and can be characterised by its positive anti-diagonal elements only.

A *Nash-centric* k-SSCD for any odd k $(k = 2m+1)$ is a device with equi-distant quantities and anti-diagonal probability distribution for which the quantities are Nash-centric, i.e., $q_{m+1} = \frac{(a-c)}{3b}$ and $q_i = q_{m+1} - (m+1-i)\,\partial$ for $1 \leq i \leq k = 2m+1$. The anti-diagonal probability distribution associated with this device is characterised by its positive anti-diagonal elements only. Let p_i, for $1 \leq i \leq m$, be the probability of both strategy profiles $(q_i,\, q_{k+1-i})$ and $(q_{k+1-i},\, q_i)$ and $(1 - 2\sum\limits_{i=1}^{m} p_i) > 0$ be the probability attached to the Nash equilibrium strategy profile. Formally, $p_{i(k+1-i)} = p_{(k+1-i)i} = p_i$ for $1 \leq i \leq m$, $p_{(m+1)(m+1)} = 1 - 2\sum\limits_{i=1}^{m} p_i$ and $p_{ij} = 0$, otherwise. A Nash-centric device thus can be denoted by $[k = 2m+1; (p_i)_{1 \leq i \leq m}; \partial]$ and can be described in a tabular form as below.

strategies	q_1	q_2	...	...	q_{m+1}	...	...	....	q_k
$q_1 = q_{m+1} - m\partial$	0	0	0	0	0	0	0	0	$p_{1k} = p_1$
$q_2 = q_{m+1} - (m-1)\,\partial$	0	0	0	0	0	0	0	$p_{2(k-1)} = p_2$	0
...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...
$q_{m+1} = \frac{(a-c)}{3b}$	0	0	0	0	$1 - 2\sum\limits_{i=1}^{m} p_i$	0	0	0	0
...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...
$q_{k-1} = q_{m+1} + (m-1)\,\partial$	0	p_2	0	0	0	0	0	0	0
$q_k = q_{m+1} + m\partial$	p_1	0	0	0	0	0	0	0	0

3 RESULTS

We first characterise the condition for a k-SSCD to be a k-SSWCE for the linear duopoly game. Following Definition 5, a k-SSCD is in equilibrium when the expected payoff from the device is higher than that from any other unilateral deviation by a player. Hence, a necessary and sufficient condition for equilibrium is that the expected payoff from the device is higher than the maximum of the payoffs from any unilateral deviation. A k-SSWCE is thus characterised in the theorem below.

Theorem 1 *A k-SSCD, $[P; Q_c] = [\{P = (p_{ij})_{i=1,2....k, j=1,2...k}\}; \{q_i\}_{i=1,2....k}]$, is a k-SSWCE for the linear duopoly game if and only if $A \geq 0$, where A is given by,*

$$A = \tfrac{3}{2}(a-c)[\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij})] - \sum_{i=1}^{k} q_i^2[b\sum_{j=1}^{k} p_{ij} + bp_{ii} + \tfrac{b}{4}(\sum_{j=1}^{k} p_{ij})^2]$$
$$- \sum_{i=1}^{k-1}\sum_{j=2}^{k} q_i q_j[2bp_{ij} + \tfrac{b}{2}(\sum_{s=1}^{k} p_{is})(\sum_{s=1}^{k} p_{js})] - \tfrac{(a-c)^2}{4b}.$$

Proof. Note that for any k-SSCD, $[\{P = (p_{ij})_{i=1,2....k, j=1,2...k}\}; \{q_i\}_{i=1,2....k}]$, in the linear duopoly game, $E(\pi_C) = \sum_{ij} p_{ij}[(a-c)q_i - bq_i^2 - bq_iq_j]$

$$= (a-c)[\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij})] - b[\sum_{i=1}^{k} q_i^2(\sum_{j=1}^{k} p_{ij})] - b[\sum_{i=1}^{k} q_i(\textstyle\sum_{j=1}^{k} p_{ij}q_j)].$$

Also, if a player deviates to play any alternative strategy $q \in Q$, assuming the other player is following the k-SSCD, the (expected) payoff of the deviant from playing q is,

$$E(\pi \mid q) = \textstyle\sum_{q_j \in Q_c} p\'(q_j)[(a-c)q - bq^2 - bqq_j] = (a-c)q - bq^2 - bq[\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij})].$$

From Definition 5, in the previous section, note that for the k-SSCD to be a k-SSWCE, we must have, $E(\pi_C) \geq E(\pi \mid q)$ for all $q \in Q$, which holds true if and only if $E(\pi_C) \geq Max_{q\in Q}E(\pi \mid q)$.

The rest of the proof to show that the condition $E(\pi_C) \geq Max_{q\in Q}E(\pi \mid q)$ is equivalent to $A \geq 0$, is straightforward and thus has been relegated to the Appendix. ■

Clearly, the Nash equilibrium of the linear duopoly game can be viewed as a device with probability 1 on the Nash equilibrium quantity, $\frac{(a-c)}{3b}$, and is trivially a k-SSWCE. This fact is observed in the above characterisation. Note that at the Nash equilibrium point, the value of A stated in Theorem 1 is indeed 0, satisfying the condition weakly.

We can use the characterisation in Theorem 1, of a k-SSWCE, to find equilibrium conditions for any k-SSCD, for example, a k-SSCD with equi-distant quantities, $[\{P = (p_{ij})_{i=1,2....k, j=1,2...k}\}; q_1; \partial]$. The following corollary characterises probability distributions for such a k-SSWCE for the linear duopoly game.

Corollary 1 *For the linear duopoly game, any probability distribution, $\{P = (p_{ij})_{i=1,2....k, j=1,2...k}\}$, can be supported as a k-SSWCE with equi-distant quantities, with some q_1 and ∂, if and only if*

$$2[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k} p_{ij}]^2 - \sum_{i=2}^{k}(i-1)^2 p_{ii} - \sum_{i=2}^{k}(i-1)^2\sum_{j=1}^{k} p_{ij} - 2\textstyle\sum_{2\leq i<j\leq k}(i-1)(j-1)p_{ij} \geq 0.$$

The proof follows directly from Theorem 1 and hence has been postponed to the Appendix.

Corollary 1 is important to analyse any k-SSWCE with equi-distant quantities, in particular the Nash-centric devices described in the previous section. In what follows, we deal only with Nash-centric devices. All the results in the rest of the paper relate to the characterisations presented in Theorem 1 and Corollary 1.

3.1 Nash-centric Devices

As we have already noted, the unique Nash and correlated equilibrium *a la* Aumann of the linear duopoly game is a k-SSWCE. The question now is whether Nash equilibrium is the only k-SSWCE for this game or not. The answer is in the negative. We here find another possible structure of k-SSWCE. We focus on a specific k-SSCD, the Nash-centric device, as described earlier.

We first note that the expected payoff from a Nash-centric device, $E(\pi_{NC})$, is actually equal to that of the Nash equilibrium of the linear duopoly game. To see this, note that

$E(\pi_{NC}) = \sum_{i=1}^{m} p_i[(a-c)q_i - bq_i^2 - bq_i q_{2m+2-i}] + (1-2\sum_{i=1}^{m} p_i)\left[(a-c)q_{m+1} - bq_{m+1}^2 - bq_{m+1}q_{m+1}\right] + \sum_{i=1}^{m} p_i[(a-c)q_{2m+2-i} - bq_{2m+2-i}^2 - bq_{2m+2-i}q_i]$.

Now, using Nash-centric quantities, the above can be simplified to

$E(\pi_{NC}) = (a-c)q_{m+1} - bq_{m+1}^2 - bq_{m+1}q_{m+1} = \frac{(a-c)^2}{9b}$, which is indeed the Nash equilibrium payoff of the linear duopoly game.

We now prove that any such device is also a k-SSWCE for the linear duopoly game.

Theorem 2 *Any Nash-centric device is a k-SSWCE for the linear duopoly game.*

The direct proof of this theorem is very similar to that of Theorem 1 and hence has been postponed to the Appendix.

Note that Theorem 2 holds for any Nash-centric device, i.e., for any parametric values of $k(= 2m+1$ with $m \geq 1)$, $\partial > 0$ and any anti-diagonal probability distribution given by the probabilities p_i, for $1 \leq i \leq m$, maintaining the Nash-centric structure.

Any Nash-centric device is a special type of a public or sunspot k-SSCD, as defined earlier, and now by Theorem 2 above is also a k-SSWCE. We should point out that Nash-centric devices assign positive probabilities over non-Nash quantities as well. Such a *sunspot structure* is clearly not a correlated equilibrium *a la* Aumann, as it is well-known that a *public* device can only be a correlated equilibrium *a la* Aumann if and only if it is a convexification of pure Nash equilibria.

The question now naturally arises is whether Nash-centric devices are the only k-SSWCE or not. We tackle this uniqueness issue rather indirectly. We first prove that Nash-centric is the unique equilibrium structure among any k-SSCDs with equi-distant quantities and anti-diagonal probability distributions.

Proposition 1 *Nash-centric devices are the only k-SSWCE with equi-distant quantities and an anti-diagonal probability distribution.*

The proof of this proposition follows immediately from Corollary 1 and hence has been postponed to the Appendix.

In the following sub-section, we prove that this particular structure is not robust as an equilibrium by showing that any small perturbation from the Nash-centric device leads to a violation of the equilibrium condition. In this sense, Nash-centric devices are locally unique k-SSWCE.

3.2 Perturbations

To address the uniqueness issue, we consider different small perturbations of the Nash centric devices and show that none of these perturbed devices are in equilibrium. From these results, we conclude that the Nash centric devices are not robust to any small change, and hence, are locally unique.

To justify our claim, we consider small changes, respectively, in the probability distribution keeping the quantity levels unchanged and in the quantity levels keeping the probability distribution fixed. We also consider perturbed devices by adding one more quantity level. Proofs of all the propositions in this section follow directly form the equilibrium characterisations presented in Theorem 1 and Corollary 1.

3.2.1 Probability

We first consider a small change in the anti-diagonal probability distribution associated with a Nash-centric device, without changing the quantities, which still remain Nash-centric. We divide the perturbation in probabilities into two cases: first, we change one of the zero off-diagonal elements and then we change one of the zero diagonal elements.

First, for simplicity and without loss of generality, we change the first off-diagonal element and make it positive. Let us make $p_{12} = p_{21} > 0$, and as a consequence, the probability attached to the Nash equilibrium strategy profile, $p_{(m+1)(m+1)}$, becomes $(1-2\sum_{i=1}^{m} p_i - 2p_{12})$, with the rest of the distribution remaining intact. Let us call this new device an *off-diagonal-probability-perturbed Nash-centric device* and prove the following desired result.

Proposition 2 *Any off-diagonal-probability-perturbed Nash-centric device is not a k-SSWCE for the linear duopoly game.*

Now, for simplicity and without loss of generality, let us change the first diagonal element and make it positive, i.e., let us make $p_{11} > 0$, and as a consequence let us make the probability attached to the Nash equilibrium strategy profile $p_{(m+1)(m+1)} = 1 - 2\sum_{i=1}^{m} p_i - p_{11}$, keeping the rest of the distribution intact. Let us call this new device a *diagonal-probability-perturbed Nash-centric device* and prove the following desired result.

Proposition 3 *Any diagonal-probability-perturbed Nash-centric device is not a k-SSWCE for the linear duopoly game.*

Note that both the propositions above refer to perturbed devices for which the quantities are equi-distant. Hence, we can use the characterisation in Corollary 1 to prove these two propositions. The proofs are in the Appendix.

3.2.2 Quantity

We now consider a small perturbation in the quantity levels, keeping the anti-diagonal probability distribution fixed. For simplicity and without loss of generality, let us make the first quantity $q_1 = q_{m+1} - m\partial + \varepsilon$, and keep all other quantities to be Nash-centric and $\partial-$distance away from each other. Let us call this new device a *quantity-perturbed device.* Using the equilibrium condition in Theorem 1, we now prove that any such device is not a k-SSWCE for the linear duopoly game.

Proposition 4 *Any quantity-perturbed Nash-centric device is not a k-SSWCE for the linear duopoly game.*

Note that the above proposition refers to a device that does not involve all equi-distant quantities. Thus, unlike the previous propositions in this sub-section, here we can not use the characterisation in Corollary 1. The proof, however, directly follows from the characterisation in Theorem 1, and has been relegated to the Appendix.

3.2.3 Composition

Finally, we turn to another way of perturbing a Nash-centric device. We consider a new device composed of one additional quantity level (other than the Nash equilibrium quantity) along with the original Nash-centric device. Starting from a Nash-centric $k-$SSWCE, we construct a public $(k+1)-$SSCD by adding another quantity $\acute{q} > 0$ ($\neq \frac{(a-c)}{3b}$) for both the players, with probability ε for the strategy profile $(\acute{q}, \acute{q})$ coupled with the original Nash-centric device with probability $(1-\varepsilon)$.

Formally, given a Nash-centric device, $[k = 2m+1; (p_i)_{1\leq i\leq m}; \partial]$, we construct a $(k+1)-$SSCD $[\{P = (p_{ij})_{i=1,2....k+1, j=1,2...k+1}\}; \{q_i\}_{i=1,2....k+1}]$ where, the quantities are $q_1 = \acute{q}$, $q_{m+2} = \frac{(a-c)}{3b}$ and

$q_i = q_{m+2} - (m+2-i)\,\partial$ for $2 \leq i \leq k+1 (= 2m+2)$ and the probabilities are $p_{11} = \varepsilon$, $p_{1j} = p_{j1} = 0$ for $j = 2, ..., k+1$, $p_{i(k+3-i)} = p_{(k+3-i)i} = (1-\varepsilon)p_{i-1}$, for $2 \leq i \leq m+1$, $p_{(m+2)(m+2)} = (1-\varepsilon)(1 - 2\sum_{i=1}^{m} p_i)$, and $p_{ij} = 0$, otherwise. Note that $\sum_{i=2}^{k+1}\sum_{j=2}^{k+1} p_{ij} = (1-\varepsilon)$. We call this device a *composite device* and prove the following desired result.

Proposition 5 *Any composite device is not a $(k+1)-SSWCE$ for the linear duopoly game.*

The proof of the above proposition follows from Theorem 1 and thus has been postponed to the Appendix.

4 REMARKS

In this paper, we have analysed the notion of weak correlation in the simplest of the oligopoly models, that of a duopoly with linear market demand and constant marginal costs. We have defined and characterised an equilibrium notion that we call $k-$SSWCE for this linear duopoly game. We have identified a particular sunspot structure, that we call a Nash-centric device, which always serves as an equilibrium for such a game; moreover, any small perturbation from this device is not an equilibrium. Hence, in this sense Nash-centric devices are locally unique. However, the expected payoff from such a device is equal to the Nash equilibrium payoff. A few remarks are in order.

4.1 $k = 2$

We note that there does not exist any $2-$SSWCE for the linear duopoly game. To prove this claim one can use the equilibrium characterisation in Corollary 1, because a $2-$SSCD can trivially be viewed as a device with equi-distant quantities. The condition in Corollary 1 for $k = 2$ becomes $p_{12} - 2MN \geq 0$, where $M = p_{11} + p_{12}$ and $N = 1 - M$. The LHS of this condition can easily be rearranged to $-(p_{22}M + p_{11}N)$, which clearly is always < 0. Hence such an equilibrium does not exist.

4.2 Quadratic Models

We should point out that although any Nash-centric device is a $k-$SSWCE for the linear duopoly game, such devices may however fail to be so in non-linear models. We show this fact using the duopoly model with quadratic costs as studied by Yi (1997). Consider a duopoly game with payoff functions $\pi_1 = aq^1 - (q^1)^2(1+c) - \gamma q^1 q^2$ and $\pi_2 = aq^2 - (q^2)^2(1+c) - \gamma q^1 q^2$, where q^1 and q^2 are quantity choices of firms 1 and 2, respectively with $c > 0$ and $0 < \gamma \leq 1$. Call this game a *duopoly with quadratic costs*. The Nash equilibrium of this game is given by $q^1 = q^2 = \frac{a}{2(1+c)+\gamma}$, with payoffs

$\pi_1 = \pi_2 = \frac{a^2(1+c)}{[2(1+c)+\gamma]^2}$. We now formally prove that Nash-centric devices may not be equilibria for such a game.

Proposition 6 *Nash-centric devices may fail to be k-SSWCE in a duopoly with quadratic costs.*

The proof of the above proposition indicates that even for $k = 3$, any Nash-centric device may not be an equilibrium for a duopoly with quadratic costs. The proof is similar to that of Theorem 2 and hence has been relegated to the Appendix.

4.2.1 Other Devices

Finally, we note that $k-$SSWCE, even $2-$SSWCE, other than the Nash-centric devices, may exist for other duopoly models, such as the environmental games (Barrett 1994).[5] Barrett's environmental game can be viewed as a duopoly with quadratic demand and cost. Here, we consider a specific example of such a game and identify a 2-SSWCE for this game.

Consider a duopoly with payoff functions $\pi_1 = 4(q^1 + q^2) - [(q^1)^2 + (q^2)^2 + 2q^1q^2] - \frac{3(q^1)^2}{4}$ and $\pi_2 = 4(q^1 + q^2) - [(q^1)^2 + (q^2)^2 + 2q^1q^2] - \frac{3(q^2)^2}{4}$, where q^1 and q^2 are quantity choices of firms 1 and 2, respectively. The Nash equilibrium of this game is given by $q^1 = q^2 = \frac{8}{11} \approxeq 0.727$, with payoffs $\pi_1 = \pi_2 = \frac{400}{121} \approxeq 3.30578$.

We now show that the following 2-SSCD is a 2-SSWCE for this game.

strategies	$q_1 = \frac{1}{2}$	$q_2 = 1$
$q_1 = \frac{1}{2}$	$\frac{1}{50}$	$\frac{12}{25}$
$q_2 = 1$	$\frac{12}{25}$	$\frac{1}{50}$

Note that the (expected) payoff, $E(\pi_C)$, of a player from following the above device is given by $E(\pi_C) = 4q_1 + 4q_2 - \frac{283}{200}q_1^2 - \frac{283}{200}q_2^2 - \frac{48}{25}q_1q_2 = \frac{2617}{800} \approxeq 3.271$. Now, the (expected) payoff from playing any alternative strategy $q \in Q$, assuming the other player is following the device, denoted by $E(\pi \mid q)$, is $E(\pi \mid q) = \frac{5}{2}q - \frac{7}{4}q^2 + \frac{19}{8}$, which is maximised at $q^* = \frac{5}{7} \approxeq 0.7142$ with $MaxE(\pi \mid q) = \frac{183}{56} \approxeq 3.267$. Thus, $E(\pi_C) - MaxE(\pi \mid q) = 0.00339 > 0$. Hence, the above device is a 2-SSWCE for this game.

[5] In a parallel paper (Ray and Sen Gupta 2011), we extensively analyse (weak) correlation in an emission abatement game for two nations, as introduced in the seminal work in environmental games by Barrett (1994).

5 APPENDIX

We collect the proofs of all our results in this section.

Proof. of Theorem 1 (contd.).

Let $E(\pi^*)$ denote $Max_{q\in Q}E(\pi \mid q)$. Using the first order condition, $\frac{\partial E(\pi|q)}{\partial q} = 0$ (and confirming the second order condition), $E(\pi \mid q)$ is maximised at $q^* = \frac{1}{2b}[(a-c) - b\{\sum\limits_{i=1}^{k} q_i(\sum\limits_{j=1}^{k} p_{ij})\}]$.

Using above q^*,

$$E(\pi^*) = \frac{(a-c)}{2b}[(a-c) - b\{\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij})\}] - \frac{1}{4b}[(a-c) - b\{\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij})\}]^2$$
$$-\frac{1}{2}[(a-c) - b\{\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij})\}][\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij})]$$
$$= \frac{(a-c)^2}{4b} - \frac{(a-c)}{2}\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij}) + \frac{b}{4}\sum_{i=1}^{k} q_i^2(\sum_{j=1}^{k} p_{ij})^2 \ + \frac{b}{2}\sum_{i=1}^{k-1}\sum_{j=2}^{k} q_i q_j[(\sum_{s=1}^{k} p_{is})(\sum_{s=1}^{k} p_{js})].$$

For the result to hold, we must have, $E(\pi_C) \geq E(\pi^*)$, that is,

$$(a-c)\sum_{i=1}^{k} q_i \sum_{j=1}^{k} p_{ij} - b\sum_{i=1}^{k} q_i^2 \sum_{j=1}^{k} p_{ij} - b(\sum_{i=1}^{k} q_i \textstyle\sum_{j=1}^{k} p_{ij}q_j)$$
$$\geq \frac{(a-c)^2}{4b} - \frac{(a-c)}{2}\sum_{i=1}^{k} q_i(\sum_{j=1}^{k} p_{ij}) + \frac{b}{4}\sum_{i=1}^{k} q_i^2(\sum_{j=1}^{k} p_{ij})^2 \ + \frac{b}{2}\sum_{i=1}^{k-1}\sum_{j=2}^{k} q_i q_j[(\sum_{s=1}^{k} p_{is})(\sum_{s=1}^{k} p_{js})],$$

which gives us the condition $A \geq 0$, stated in the theorem. ■

Proof. of Corollary 1.

Given a distribution, $\{P = (p_{ij})_{i=1,2....k, j=1,2...k}\}$, consider a k-SSCD with equi-distant quantities $[\{P = (p_{ij})_{i=1,2....k, j=1,2...k}\}; q_1; \partial]$. We will use the equilibrium condition $A \geq 0$, in Theorem 1 for such a k-SSCD. Substituting the values of $q_i = q_1 + (i-1)\partial$, $1 \leq i \leq k$, and simplifying, the expression A in Theorem 1 becomes,

$$\frac{3}{2}(a-c)q_1 + \frac{3}{2}(a-c)\partial[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k} p_{ij}] - \frac{9}{4}bq_1^2 - \frac{9}{2}bq_1\partial[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k} p_{ij}]$$
$$-b\partial^2[\sum_{i=2}^{k}(i-1)^2\{p_{ii} + \sum_{j=1}^{k} p_{ij} + \frac{1}{4}(\sum_{j=1}^{k} p_{ij})^2\} + 2\textstyle\sum_{2\leq i<j\leq k}(i-1)(j-1)p_{ij}]$$
$$-\frac{1}{2}b\partial^2[\sum_{i=2}^{k-1}(i-1)\sum_{j=1}^{k} p_{ij}][\sum_{i=3}^{k}(i-1)\sum_{j=1}^{k} p_{ij}] - \frac{(a-c)^2}{4b}.$$

To be an equilibrium, the above expression needs to be ≥ 0, for some values of q_1 and ∂. For any $\partial > 0$, we can view this expression as a function of q_1. Thus, a necessary and sufficient condition for the existence of such an equilibrium is that the maximum value of this function be ≥ 0. Using the first order condition (and confirming the second order condition), this is maximised at

$$\hat{q_1} = \frac{1}{3b}[(a-c) - 3b\delta\{\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k} p_{ij}\}].$$

Using $\hat{q_1}$, the maximum value of this function is,

$\frac{3}{2}(a-c)\frac{1}{3b}[(a-c)-3b\delta\{\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}\}]+\frac{3}{2}(a-c)\delta[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]$
$-\frac{9b}{4}\frac{1}{(3b)^2}[(a-c)-3b\delta\{\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}\}]^2$
$-\frac{9b}{2}\delta[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]\frac{1}{3b}[(a-c)-3b\delta\{\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}\}]$
$-b\delta^2[\sum_{i=2}^{k}(i-1)^2\{p_{ii}+\sum_{j=1}^{k}p_{ij}+\frac{1}{4}(\sum_{j=1}^{k}p_{ij})^2\}+2\sum_{2\le i<j\le k}(i-1)(j-1)p_{ij}]$
$-\frac{b\partial^2}{2}[\sum_{i=2}^{k-1}(i-1)\sum_{j=1}^{k}p_{ij}][\sum_{i=3}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]-\frac{(a-c)^2}{4b}$
$=\frac{(a-c)^2}{2b}-\frac{3}{2}(a-c)\delta[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]+\frac{3}{2}(a-c)\delta[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]$
$-\frac{9b}{4}\cdot\frac{1}{9b^2}[(a-c)^2+9b^2\delta^2\{\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}\}^2-6(a-c)b\delta\{\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}\}]$
$-\frac{3}{2}(a-c)\delta[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]+\frac{9}{2}b\delta^2[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]^2$
$-b\delta^2[\sum_{i=2}^{k}(i-1)^2\{p_{ii}+\sum_{j=1}^{k}p_{ij}+\frac{1}{4}(\sum_{j=1}^{k}p_{ij})^2\}+2\{\sum_{2\le i<j\le k}(i-1)(j-1)p_{ij}\}]$
$-\frac{b\delta^2}{2}[(\sum_{i=2}^{k-1}(i-1)\sum_{j=1}^{k}p_{ij})(\sum_{i=3}^{k}(i-1)\sum_{j=1}^{k}p_{ij})]-\frac{(a-c)^2}{4b}$
$=\frac{-9b\delta^2}{4}[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]^2+\frac{9}{2}b\delta^2[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]^2$
$-\frac{b\delta^2}{4}[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]^2-b\delta^2[\sum_{i=2}^{k}(i-1)^2\{p_{ii}+\sum_{j=1}^{k}p_{ij}\}+2\{\sum_{2\le i<j\le k}(i-1)(j-1)p_{ij}\}]$
$=2b\delta^2[\sum_{i=2}^{k}(i-1)\sum_{j=1}^{k}p_{ij}]^2-b\delta^2[\sum_{i=2}^{k}(i-1)^2\{p_{ii}+\sum_{j=1}^{k}p_{ij}\}+2\{\sum_{2\le i<j\le k}(i-1)(j-1)p_{ij}\}].$

The equilibrium condition in Theorem 1 requires the above to be ≥ 0. As both b and ∂ are >0, this leads to the statement in the corollary. ■

Proof. of Theorem 2.

As we have already noted, the expected payoff from following any Nash-centric device is $E(\pi_{NC})=\frac{(a-c)^2}{9b}$, which is equal to the Nash equilibrium payoff of the linear duopoly game. Let $E(\pi\mid q)$ denote the (expected) payoff of the deviant from playing q. Note that $E(\pi\mid q)$ is given by,

$$\sum_{i=1}^{m}p_i[(a-c)q-bq^2-bqq_i]+[1-2\sum_{i=1}^{m}p_i]\left[(a-c)q-bq^2-bqq_{m+1}\right]+\sum_{i=1}^{m}p_i[(a-c)q-bq^2-bqq_{2m+2-i}]$$
$$=(a-c)q-bq^2-bqq_{m+1}.$$

As in the proof of Theorem 1, for the Nash-centric device to be a k-SSWCE, we must have, $E(\pi_{NC})\geq E(\pi\mid q)$ for all $q\in Q$, which holds true if and only if $E(\pi_{NC})\geq Max_{q\in Q}E(\pi\mid q)=E(\pi^*)$. Using the first order condition, $\frac{\partial E(\pi|q)}{\partial q}=0$ (and confirming the second order condition), $E(\pi\mid q)$ is maximised at $q^*=\frac{(a-c)-bq_{m+1}}{2b}=\frac{(a-c)-b\frac{(a-c)}{3b}}{2b}=\frac{(a-c)}{3b}$. Using q^*, $E(\pi^*)=(a-c)q^*-bq^{*2}-bq^*q_{m+1}=\frac{(a-c)^2}{9b}=E(\pi_{NC})$. Hence, $E(\pi_{NC})=E(\pi^*)$, and thus any Nash-centric device is a k-SSWCE. ■

Proof. of Proposition 1.

From Corollary 1, the equilibrium condition for a device with equi-distant quantities and an anti-diagonal probability distribution to be a k-SSWCE becomes,

$2[\sum_{i=1}^{m}(i-1)p_i + m(1-2\sum_{i=1}^{m}p_i) + \sum_{i=1}^{m}(2m+1-i)p_i]^2$

$-[\sum_{i=1}^{m}(i-1)^2p_i + m^2(1-2\sum_{i=1}^{m}p_i) + \sum_{i=1}^{m}(2m+1-i)^2p_i] - [m^2(1-2\sum_{i=1}^{m}p_i) + 2\sum_{i=2}^{m}(i-1)(2m+1-i)p_i] \geq 0.$

With further simplification, the LHS of the above turns out to be $\left[2(m)^2 - (m^2+m^2)\right]$, which indeed is 0, thereby weakly satisfying the condition. Thus an anti-diagonal probability distribution can be supported as a k-SSWCE with some equi-distant quantities. Now to prove the uniqueness of Nash-centric quantities, recall from the proof of Corollary 1, the expression in the condition is maximised at $\hat{q_1}$ which for any anti-diagonal probability distribution is,

$\hat{q_1} = \frac{1}{3b}[(a-c) - 3b\delta\{\sum_{i=2}^{2m+1}(i-1)\sum_{j=1}^{2m+1}p_{ij}\}]$

$= \frac{(a-c)}{3b} - \partial[\sum_{i=1}^{m}(i-1)p_i + m(1-2\sum_{i=1}^{m}p_i) + \sum_{i=1}^{m}(2m+1-i)p_i] = \frac{(a-c)}{3b} - m\partial$, which is indeed the Nash-centric quantity. Hence, among all devices with equi-distant quantities and anti-diagonal probability distributions, only Nash centric devices weakly satisfy the equilibrium condition. ■

Proof. of Proposition 2.

From Corollary 1, the equilibrium condition for the off-diagonal-probability-perturbed Nash centric device to be a k-SSWCE becomes,

$2[(p_{21}+p_2) + \sum_{i=3}^{m}(i-1)p_i + m(1-2p_{12}-2\sum_{i=1}^{m}p_i) + \sum_{i=1}^{m}(2m+1-i)p_i]^2$

$-[(p_{21}+p_2) + \sum_{i=3}^{m}(i-1)^2p_i + m^2(1-2p_{12}-2\sum_{i=1}^{m}p_i) + \sum_{i=1}^{m}(2m+1-i)^2p_i]$

$-[m^2(1-2p_{12}-2\sum_{i=1}^{m}p_i) + \{2\sum_{i=2}^{m}(i-1)(2m+1-i)p_i\}] \geq 0.$

Simplifying, the LHS of the above turns out to be,

$2\left(p_{21}+m-2mp_{21}\right)^2 - \left(p_{21}+m^2-2m^2p_{21}+m^2-2m^2p_{21}\right)$

$= \left[p_{21}\left(4m-4m^2-1\right) - 2p_{21}^2\left(4m-4m^2-1\right)\right] = \left(1+4m^2-4m\right)\left(2p_{21}^2-p_{21}\right),$

which is always < 0, because $\left(1+4m^2-4m\right) > 0$ for $m \geq 1$ and $(2p_{21}^2 - p_{21}) \geq 0$ for $p_{21} < \frac{1}{2}$. Hence, the equilibrium condition is violated. ■

Proof. of Proposition 3.

From Corollary 1, the equilibrium condition for the diagonal-probability-perturbed Nash centric device to be a k-SSWCE becomes,

$2[\sum_{i=2}^{m}(i-1)p_i + m(1-p_{11}-2\sum_{i=1}^{m}p_i) + \sum_{i=1}^{m}(2m+1-i)p_i]^2$

$-[\sum_{i=2}^{m}(i-1)^2p_i + m^2(1-p_{11}-2\sum_{i=1}^{m}p_i) + \sum_{i=1}^{m}(2m+1-i)^2p_i]$

$-[m^2(1-p_{11}-2\sum_{i=1}^{m}p_i)+\{2\sum_{i=2}^{m}(i-1)(2m+1-i)p_i\}]\geq 0.$

Simplifying, the LHS of the above turns out to be,

$2(m-mp_{11})^2-(m^2-m^2p_{11}+m^2-m^2p_{11})$

$=2m^2\left(1-p_{11}\right)^2-2m^2\left(1-p_{11}\right)=-2m^2p_{11}\left(1-p_{11}\right),$

which is always < 0. Hence, the equilibrium condition is violated. ■

Proof. of Proposition 4.

Following Theorem 1, for the quantity-perturbed Nash centric device the expression A turns out to be,

$$\begin{aligned}
&\tfrac{3}{2}(a-c)\sum_{i=1}^{m}p_iq_i+\tfrac{3}{2}(a-c)(1-2\sum_{i=1}^{m}p_i)q_{m+1}+\tfrac{3}{2}(a-c)\sum_{i=1}^{m}p_iq_{2m+2-i}\\
&-b\sum_{i=1}^{m}(q_i)^2[p_i+\tfrac{(p_i)^2}{4}]-b(q_{m+1})^2[2(1-2\sum_{i=1}^{m}p_i)+\tfrac{1}{4}(1-2\sum_{i=1}^{m}p_i)^2]\\
&-b\sum_{i=1}^{m}(q_{2m+2-i})^2[p_i+\tfrac{(p_i)^2}{4}]-b\sum_{i=1}^{m-1}\sum_{j=2}^{m}q_iq_j[\tfrac{p_ip_j}{2}]\\
&-\tfrac{b}{2}\sum_{i=1}^{m}q_iq_{m+1}p_i(1-2\sum_{i=1}^{m}p_i)-\tfrac{b}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}q_iq_{2m+2-j}(p_ip_j+4p_{i(2m+2-j)})\\
&-\tfrac{b}{2}q_{m+1}\sum_{i=1}^{m}q_{2m+2-i}p_i(1-2\sum_{i=1}^{m}p_i)-\tfrac{b}{2}\sum_{i=m+2}^{2m}\sum_{j=m+3}^{2m+1}q_iq_j(p_{2m+2-i}p_{2m+2-j})-\tfrac{(a-c)^2}{4b}.
\end{aligned}$$

Substituting the values of q_1 and other q_i for $i\neq 1$, and simplifying, the above expression becomes,

$$\begin{aligned}
&\tfrac{3}{2}(a-c)p_1\varepsilon+\tfrac{3}{2}(a-c)q_{m+1}-b\varepsilon^2[p_1+\tfrac{(p_1)^2}{4}]\\
&-2bq_{m+1}^2-bq_{m+1}^2[\sum_{i=1}^{m}p_i+\tfrac{1}{2}(1-2\sum_{i=1}^{m}p_i)]^2-\tfrac{(a-c)^2}{4b}-\tfrac{9}{2}b\varepsilon p_1q_{m+1}\\
&=\tfrac{3}{2}(a-c)q_{m+1}-\tfrac{9}{4}bq_{m+1}^2+\tfrac{3}{2}(a-c)p_1\varepsilon-b\varepsilon^2[p_1+\tfrac{(p_1)^2}{4}]-\tfrac{9}{2}b\varepsilon p_1q_{m+1}-\tfrac{(a-c)^2}{4b}.
\end{aligned}$$

From Theorem 1, for the quantity-perturbed Nash centric device to be a k-SSWCE, we need the following expression to be ≥ 0. Now substituting $q_{m+1}=\frac{(a-c)}{3b}$, the expression becomes,

$$\begin{aligned}
&\tfrac{(a-c)^2}{2b}-\tfrac{(a-c)^2}{4b}+\tfrac{3}{2}(a-c)p_1\varepsilon-b\varepsilon^2[p_1+\tfrac{(p_1)^2}{4}]-\tfrac{3}{2}(a-c)p_1\varepsilon-\tfrac{(a-c)^2}{4b}\\
&=-b\varepsilon^2[p_1+\tfrac{(p_1)^2}{4}],
\end{aligned}$$

which is always < 0. Hence, the equilibrium condition is violated. ■

Proof. of Proposition 5.

Following Theorem 1, for the composite device the expression A turns out to be,

$$\begin{aligned}
&\tfrac{3\varepsilon}{2}(a-c)\,q'+\tfrac{3}{2}(a-c)(1-\varepsilon)q_{m+1}(2\sum_{i=1}^{m}p_i+1-2\sum_{i=1}^{m}p_i)-bq'^2(2\varepsilon+\tfrac{\varepsilon^2}{4})\\
&-bq_{m+1}^2[4(1-\varepsilon)\sum_{i=1}^{m}p_i+2(1-\varepsilon)(1-2\sum_{i=1}^{m}p_i)+\tfrac{1}{4}(1-\varepsilon)^2(2\sum_{i=1}^{m}{p_i}^2)+\tfrac{1}{4}(1-\varepsilon)^2(1-2\sum_{i=1}^{m}p_i)^2]\\
&-\tfrac{b\varepsilon}{2}(1-\varepsilon)q'q_{m+1}-\tfrac{b}{2}q_{m+1}^2(1-\varepsilon)^2[\sum_{i=1}^{m}p_i^2+4\sum_{i=1}^{m-1}\sum_{j=2}^{m}p_ip_j+2\sum_{i=1}^{m}p_i(1-2\sum_{i=1}^{m}p_i)]-\tfrac{(a-c)^2}{4b}.
\end{aligned}$$

With further simplification, the above expression reduces to,

$$\begin{aligned}
&\tfrac{3\varepsilon}{2}(a-c)\,q'+\tfrac{3}{2}(a-c)(1-\varepsilon)q_{m+1}-bq'^2[2\varepsilon+\tfrac{\varepsilon^2}{4}]-2b(1-\varepsilon)q_{m+1}^2-\tfrac{b}{2}\varepsilon(1-\varepsilon)q'q_{m+1}-\tfrac{b}{4}(1-\varepsilon)^2q_{m+1}^2-\tfrac{(a-c)^2}{4b}\\
&=\tfrac{3}{2}(a-c)[\varepsilon q'+(1-\varepsilon)q_{m+1}]-2b[\varepsilon q'^2+(1-\varepsilon)q_{m+1}^2]-\tfrac{b}{4}[\varepsilon q'+(1-\varepsilon)q_{m+1}]^2-\tfrac{(a-c)^2}{4b}.
\end{aligned}$$

Substituting $q_{m+1}=\frac{(a-c)}{3b}$, and rearranging, we get,

$A = \frac{\acute{q}\varepsilon(a-c)(8+\varepsilon)}{6} - \frac{\acute{q}^2 b\varepsilon(8+\varepsilon)}{4} - \frac{\varepsilon(a-c)^2(8+\varepsilon)}{36b}$, which can be viewed a quadratic function in $\acute{q}$.

From Theorem 1, for the composite device to be a k-SSWCE, we need the above to be ≥ 0. Now note that at $\acute{q} = 0$, the value of the above function is < 0. Also note that the above function is maximized at $\acute{q} = \frac{(a-c)}{3b}$(the Nash equilibrium quantity) and the maximised value of the function is 0. Therefore, for any $\acute{q} > 0$, other than the Nash Equilibrium quantity, the value of A is < 0. Hence, the equilibrium condition is violated. ■

Proof. of Proposition 6.

We will prove this result for $k = 3$. Consider a 3− SSCD which is Nash-centric with anti-diagonal probabilities with $p_{13} = p_{31} = p < \frac{1}{2}$ and $p_{22} = 1 - 2p > 0$, and all other $p_{ij} = 0$. Following the proof of Theorem 2, the expected payoff, $E(\pi_{NC})$, from accepting the device is given by,

$$p\left[aq_1 - q_1^2(1+c) - \gamma q_1 q_3\right] + (1-2p)\left[aq_2 - q_2^2(1+c) - \gamma q_2 q_2\right] + p\left[aq_3 - q_3^2(1+c) - \gamma q_3 q_1\right]$$

$$= apq_1 + a(1-2p)q_2 + apq_3 - p(1+c)q_1^2 - (1-2p)(1+c)q_2^2 - p(1+c)q_3^2 - 2\gamma p q_1 q_3 - \gamma(1-2p)q_2^2.$$

Let $E(\pi \mid q)$ denote the (expected) payoff of the deviant from playing q.

$$E(\pi \mid q) = p\left[aq - q^2(1+c) - \gamma q q_1\right] + (1-2p)\left[aq - q^2(1+c) - \gamma q q_2\right] + p\left[aq - q^2(1+c) - \gamma q q_3\right]$$

$$= aq - q^2(1+c) - \gamma q\left[pq_1 + (1-2p)q_2 + pq_3\right].$$

As in the proof of Theorem 1, for the Nash-centric device to be a k-SSWCE, we must have, $E(\pi_{NC}) \geq E(\pi \mid q)$ for all $q \in Q$, which holds true if and only if $E(\pi_{NC}) \geq Max_{q \in Q} E(\pi \mid q) = E(\pi^*)$. Using the first order condition, $\frac{\partial E(\pi|q)}{\partial q} = 0$ (and confirming the second order condition), $E(\pi \mid q)$ is maximised at $q^* = \frac{1}{2(1+c)}[a - \gamma\{pq_1 + (1-2p)q_2 + pq_3\}]$. Substituting this value of q^*,

$$E(\pi^*) = \frac{a^2}{4(1+c)} + \frac{\gamma^2 p^2 q_1^2}{4(1+c)} + \frac{\gamma^2(1-2p)^2 q_2^2}{4(1+c)} + \frac{\gamma^2 p^2 q_3^2}{4(1+c)} + \frac{2p(1-2p)q_1 q_2}{4(1+c)}$$
$$+\frac{2p(1-2p)q_3 q_2}{4(1+c)} + \frac{2p^2 q_1 q_3}{4(1+c)} - \frac{a\gamma p q_1}{2(1+c)} - \frac{a\gamma(1-2p)q_2}{2(1+c)} - \frac{a\gamma p q_3}{2(1+c)}.$$

Therefore, the equilibrium condition becomes,

$$q_1[ap + \frac{a\gamma p}{2(1+c)}] + q_2[a(1-2p) + \frac{a\gamma(1-2p)}{2(1+c)}] + q_3[ap + \frac{a\gamma p}{2(1+c)}]$$
$$-q_1^2[p(1+c) + \frac{\gamma^2 p^2}{4(1+c)}] - q_2^2[(1-2p)(1+c) + \gamma(1-2p) + \frac{\gamma^2(1-2p)^2}{4(1+c)}]$$
$$-q_3^2[p(1+c) + \frac{\gamma^2 p^2}{4(1+c)}] - q_1 q_3[2p\gamma + \frac{2p^2}{4(1+c)}] - q_1 q_2[\frac{2p(1-2p)}{4(1+c)}] - q_3 q_2[\frac{2p(1-2p)}{4(1+c)}] - \frac{a^2}{4(1+c)} \geq 0.$$

Substituting the Nash-centric quantities, $q_2 = \frac{a}{2(1+c)+\gamma}$, $q_1 = q_2 - \partial$ and $q_3 = q_2 + \partial$ and simplifying further, the LHS of the above inequality becomes,

$$q_2[\frac{2(1+c)a + a\gamma}{2(1+c)}] - \partial^2[2p(1+c-\gamma) + \frac{p^2(\gamma^2-1)}{2(1+c)}] - q_2^2[1+c+\gamma + \frac{\gamma^2}{4(1+c)} + \frac{(6p^2-4p)(\gamma^2-1)}{4(1+c)}] - \frac{a^2}{4(1+c)}$$
$$= -2\partial^2 p(1-\gamma+c) + \frac{(1-\gamma^2)p(\partial^2 p + 3a^2 p - 2a^2)}{2(1+c)[2(1+c)+\gamma]^2}.$$

To complete the proof now, let us suitably choose $\partial = a$. Thereby the above expression becomes $-2a^2 p(1-\gamma+c) + \frac{(1-\gamma^2)2a^2 p(2p-1)}{2(1+c)[2(1+c)+\gamma]^2} = -2a^2 p(1-\gamma+c) - \frac{(1-\gamma^2)2a^2 p(1-2p)}{2(1+c)[2(1+c)+\gamma]^2}$, which clearly is always < 0. Hence, the equilibrium condition is not satisfied. ■

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