On the existence, efficiency and bubbles of a Ramsey equilibrium with endogenous labor supply and borrowing constraints*

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Abstract

In this paper, we study the existence of an intertemporal equilibrium in a Ramsey model with heterogenous discounting, elastic labor supply and borrowing constraints. Applying a fixed-point argument by Gale and Mas-Colell (1975), we prove the existence of an equilibrium in a truncated bounded economy. This equilibrium is also an equilibrium of any unbounded economy with the same fundamentals. Then, we prove the existence of an equilibrium in an infinite-horizon economy as a limit of a sequence of truncated economies. On the one hand, our paper generalizes Becker et al. (1991) because of the elastic labor supply and, on the other hand, Bosi and Seegmuller (2010) because of a proof of global existence. Our methodology can be also applied to other Ramsey models with different market imperfections. The issue of bubbles existence and efficiency is raised at the end of the paper.

Keywords: bubbles, efficiency, Ramsey model, heterogeneous agents, endogenous labor supply, borrowing constraint.

JEL classification: C62, D31, D91, G10.

1 Introduction

Ramsey (1928) remains the most influential paper in growth literature and an inexhaustible source of inspiration for theorists. One of the puzzling aspects of the model is the so-called Ramsey conjecture: "*... equilibrium would be attained by a division into two classes, the thrifty enjoying bliss and the improvident at the subsistence level*" (Ramsey (1928), p. 559). This sentence ends the paper and means that, in the long run, the most patient agents would hold all the

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capital, while the others would live at their subsistence level. The Ramsey conjecture was proved by Robert Becker more than half a century later.

Becker (1980) pioneers a series of works during three decades on the properties of a Ramsey equilibrium under heterogenous discounting.[1] He shows the existence of a long-run equilibrium where the most patient agent holds the capital of the economy, while the impatient ones consume their labor income. The existence of the steady state rests on the introduction of borrowing constraints that prevent agents to borrow against their future labor income.

The complete markets case is considered by other authors. Le Van and Vailakis (2003) prove that, when individuals are allowed to borrow against future income, the impatient agents borrow from the patient one and spend the rest of their life to work to refund the debt. In addition, their consumption asymptotically vanishes and there is no longer room for a steady state. The extension with elastic labor supply, which is pertinent for a comparison with our paper, is provided by Le Van et al. (2007).

Borrowing constraints are credit market imperfections that change the equilibrium properties in terms of: (1) optimality, (2) stationarity and (3) monotonicity.

(1) Optimality. Credit market incompleteness entails the failure of the first welfare theorem. As a matter of fact, it is no longer possible to prove the existence of a competitive equilibrium by studying the set of Pareto-efficient allocations as done by Le Van and Vailakis (2003) and Le Van et al. (2007), among others, in absence of market imperfections.

(2) Stationarity. Under borrowing constraints, there exists a stationary state where impatient agents consume. The steady state vanishes when these constraints are retired: in the complete markets counterpart, Le Van and Vailakis (2003) and Le Van et al. (2007) show that the convergence of the optimal capital sequence to a particular stock still holds, but this stock is not itself a steady state.

(3) Monotonicity. In presence of borrowing constraints, persistent cycles arise (Becker (1980), Becker and Foias (1987, 1994), Sorger (1994)). To understand the role of these constraints, it is worthy to compare with similar models where markets are complete: Le Van and Vailakis (2003) and Le Van et al. (2007) also find that, under discounting heterogeneity, the monotonicity property of the representative agent counterpart does not carry over and that a twisted turnpike property holds (see Mitra (1979) and Becker (2005)). The very difference with the class of models à la Becker is that the optimal capital sequence always converges in the long run and, thus, there is no room for persistent cycles.

What is the reason of persistence? Becker and Foias (1987, 1994) show that cycles of period two may occur when capital income monotonicity fails, that is capital income is decreasing in the capital stock.

Thereby, the Ramsey conjecture holds under perfect competition, but also under the kind of imperfection represented by financial constraints. However,

[1]For a survey on this literature, the reader is referred to Becker (2006).

the introduction of other forms of imperfections makes this conjecture fragile. Prominent examples are given by distortionary taxation and market power. Sarte (1997) and Sorger (2002) study a progressive capital income taxation, while Sorger (2002, 2005, 2008) and Becker and Foias (2007) focus on the strategic interaction in the capital market. They prove the possibility of a long-run non-degenerated distribution of capital where impatient agents hold capital.

Our paper addresses the difficult question of the existence of an intertemporal equilibrium under borrowing constraints. The usual proof of existence à la Negishi no longer applies because markets are imperfect.

Becker et al. (1991) have shown the existence of an intertemporal equilibrium under borrowing constraints with inelastic labor supply. The argument of the proof rests on the introduction of a tâtonnement map giving an equilibrium as a fixed point of the map.

Bosi and Seegmuller (2010) provide a local proof of existence of an intertemporal equilibrium with elastic labor supply. Their argument rests on the existence of a local fixed point for the policy function based on the local stability properties of the steady state.

The novelty of our paper is threefold.

(1) We generalize Becker et al. (1991) by considering an elastic labor supply.

(2) We go beyond Bosi and Seegmuller (2010) by providing a proof of global existence.

(3) We study the occurrence of bubbles under agents' heterogeneity and market imperfections.

We show the existence of an intertemporal equilibrium in presence of market imperfections by applying a method inspired by Florenzano (1999), a model with incomplete markets. This method is based on a Gale and Mas-Colell (1975) fixed-point argument and can be applied in other contexts.

The entire paper is devoted to the proof of existence and is articulated in three steps.

(1) We first consider a time-truncated economy. Since the feasible allocations sets of our economy are uniformly bounded, we prove that there exists an equilibrium in a time-truncated bounded economy by using a theorem by Gale and Mas-Colell (1975). Actually, this equilibrium turns out to be an equilibrium for the time-truncated economy.

(2) Second, we take the limit of a sequence of truncated unbounded economies and we prove the existence of an intertemporal equilibrium in the limit economy.

(3) Third, a definition of bubble is introduced and a condition for equilibrium efficiency is provided.

Most of the proofs are given in Appendices 1 to 4.

2 Firms

We consider a representative firm with no market power. The technology is represented by a constant returns to scale production function: $F(K_t, L_t)$, where

K_t and L_t are the aggregate capital and the aggregate labor. Profit maximization: $\max_{K_t,L_t}[p_tF(K_t,L_t)-r_tK_t-w_tL_t]$, gives $\partial F/\partial K_t = r_t/p_t$ and $\partial F/\partial L_t = w_t/p_t$. We introduce the set of nonnegative real numbers: $\mathbb{R}_+ \equiv \{x \in \mathbb{R} : x \geq 0\}$. Profit maximization is correctly defined under the following assumption.[2]

Assumption 1 *$F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is C^1, constant returns to scale, strictly increasing and concave. We assume that inputs are essential: $F(0,L) = F(K,0) = 0$. In addition, $F(K,L) \to +\infty$ when $L > 0$ and $K \to +\infty$ or when $K > 0$ and $L \to +\infty$.*

Let us introduce also boundary conditions on capital productivity when the labor supply is maximal and equal to m in order to simplify the proof of equilibrium existence.

Assumption 2 *$(\partial F/\partial K_t)(0,m) > \delta$ and $(\partial F/\partial K_t)(+\infty,m) < \delta$, where $\delta \in (0,1)$ denotes the rate of capital depreciation.*

3 Households

We consider an economy without population growth where m households work and consume. Each household i is endowed with k_{i0} units of capital at period 0 and 1 unit of leisure-time per period. Leisure demand of agent i at time t is denoted by λ_{it} and the individual labor supply is given by $l_{it} = 1 - \lambda_{it}$. Individual wealth and consumption demand at time t are denoted by k_{it} and c_{it}.

Initial capital endowments are supposed to be positive.

Assumption 3 *$k_{i0} > 0$ for $i = 1, \ldots, m$.*

It is known that, in economies with heterogenous discounting and no borrowing constraints, impatient agents borrow, consume more and work less in the short run, and that they consume less and work more in the long run to refund the debt to patient agents. In our model, agents are prevented from borrowing: $k_{it} \geq 0$ for $t = 1, 2, \ldots$ and $i = 1, \ldots, m$.

Each household maximizes a utility separable over time: $\sum_{t=0}^{T} \beta_i^t u_i(c_{it}, \lambda_{it})$, where $\beta_i \in (0,1)$ is the discount factor of agent i.

Assumption 4 *$u_i : \mathbb{R}_+^2 \to \mathbb{R}$ is C^1, strictly increasing and concave.*

[2] The shortcut of maximization of an aggregate profit rests on the following argument. Consider a large number q of firms that share the same technology and have no market power. Each firm j maximizes the profit $p_tF(k_{jt},l_{jt}) - r_tk_{jt} - w_tl_{jt}$ in every period: $t = 0, 1, \ldots$ This gives $\partial F/\partial k_{jt} = r_t/p_t$ and $\partial F/\partial l_{jt} = w_t/p_t$ which in turn implies that the ratio k_{jt}/l_{jt} is the same across the firms. Let $(K_t, L_t) \equiv \left(\sum_{j=1}^{q} k_{jt}, \sum_{j=1}^{q} l_{jt}\right)$ be the aggregate solution. We define an aggregate production function: $F(K_t, L_t)$. Since productivities $\partial F/\partial k_{jt}$ and $\partial F/\partial l_{jt}$ are homogeneous of degree zero, the aggregate solution is also solution of the aggregate program: $\max_{K_t,L_t}[p_tF(K_t,L_t) - r_tK_t - w_tL_t]$.

4 Definition of equilibrium

We define an infinite-horizon sequences of prices and quantities:

$$(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)_{i=1}^m, \mathbf{K}, \mathbf{L})$$

where

$$\begin{aligned} (\mathbf{p}, \mathbf{r}, \mathbf{w}) &\equiv ((p_t)_{t=0}^\infty, (r_t)_{t=0}^\infty, (w_t)_{t=0}^\infty) \in \mathbb{R}^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \\ (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) &\equiv ((c_{it})_{t=0}^\infty, (k_{it})_{t=1}^\infty, (\lambda_{it})_{t=0}^\infty) \in \mathbb{R}_+^\infty \times \mathbb{R}^\infty \times \mathbb{R}_+^\infty \\ (\mathbf{K}, \mathbf{L}) &\equiv ((K_t)_{t=0}^\infty, (L_t)_{t=0}^\infty) \in \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \end{aligned}$$

with $i = 1, \ldots, m$.

Definition 1 *A Walrasian equilibrium* $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ *satisfies the following conditions.*

(1) Price positivity: $\bar{p}_t, \bar{r}_t, \bar{w}_t > 0$ *for* $t = 0, 1, \ldots$

(2) Market clearing:

$$\begin{aligned} \textit{goods} &: \sum_{i=1}^m \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] = F\left(\bar{K}_t, \bar{L}_t\right) \\ \textit{capital} &: \bar{K}_t = \sum_{i=1}^m \bar{k}_{it} \\ \textit{labor} &: \bar{L}_t = \sum_{i=1}^m \bar{l}_{it} \end{aligned}$$

for $t = 0, 1, \ldots$, *where* $l_{it} = 1 - \lambda_{it}$ *denotes the individual labor supply.*

(3) Optimal production plans: $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t$ *is the value of the program:* $\max\left[\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t\right]$, *for* $t = 0, 1, \ldots$ *under the constraints* $K_t \geq 0$ *and* $L_t \geq 0$.

(4) Optimal consumption plans: $\sum_{t=0}^\infty \beta_i^t u_i\left(\bar{c}_{it}, \bar{\lambda}_{it}\right)$ *is the value of the program:* $\max \sum_{t=0}^\infty \beta_i^t u_i(c_{it}, \lambda_{it})$, *under the following constraints:*

$$\begin{aligned} \textit{budget constraint} &: \bar{p}_t\left[c_{it} + k_{it+1} - (1-\delta) k_{it}\right] \leq \bar{r}_t k_{it} + \bar{w}_t (1 - \lambda_{it}) \\ \textit{borrowing constraint} &: k_{it+1} \geq 0 \\ \textit{leisure endowment} &: 0 \leq \lambda_{it} \leq 1 \\ \textit{capital endowment} &: k_{i0} \geq 0 \textit{ given} \end{aligned}$$

for $t = 0, 1, \ldots$

The following claims are essential in our paper.

Claim 1 *Labor supply is bounded.*

Proof. At the individual level, because $l_{it} = 1 - \lambda_{it} \in [0, 1]$. At the aggregate level, because $0 \leq \sum_{i=1}^m l_{it} \leq m$. ■

Claim 2 *Under Assumptions 1 and 2, individual and aggregate capital supplies are bounded.*

Proof. At the individual level, because of the borrowing constraint, we have $0 \leq k_{it} \leq \sum_{h=1}^{m} k_{ht}$.

To prove that the individual capital supply is bounded, we prove that the aggregate capital supply is bounded. We want to show that $0 \leq \sum_{h=1}^{m} k_{ht} \leq \max \{x, \sum_{i=1}^{m} k_{i0}\} \equiv A$, where x is the unique solution of

$$x = (1-\delta)\, x + F\,(x, m) \tag{1}$$

Since F is C^1, increasing and concave, $F\,(0, L) = 0$ and

$$1 - \delta + (\partial F/\partial K_t)\,(0, m) > 1 > 1 - \delta + (\partial F/\partial K_t)\,(+\infty, m)$$

(Assumptions 1 and 2), the solution of (1) is unique. Moreover, $x \leq y$ implies

$$(1-\delta)\, y + F\,(y, m) \leq y \tag{2}$$

We notice that

$$\begin{aligned}\sum_{i=1}^{m} k_{it+1} &\leq \sum_{i=1}^{m} (c_{it} + k_{it+1}) \leq (1-\delta) \sum_{i=1}^{m} k_{it} + F\left(\sum_{i=1}^{m} k_{it}, \sum_{i=1}^{m} l_{it}\right) \\ &\leq (1-\delta) \sum_{i=1}^{m} k_{it} + F\left(\sum_{i=1}^{m} k_{it}, m\right)\end{aligned}$$

because F is increasing, the capital employed cannot exceed its aggregate supply $\sum_{i=1}^{m} k_{it}$ and $\sum_{i=1}^{m} l_{it} \leq m$. Let $x_t \equiv \sum_{i=1}^{m} k_{it}$. Then, $x_{t+1} \leq (1-\delta)\, x_t + F\,(x_t, m)$.

We observe that $x_0 \leq \max\{x, x_0\} \equiv A$. Therefore, $x_1 \leq (1-\delta)\, x_0 + F\,(x_0, m) \leq (1-\delta)\, A + F\,(A, m) \leq A$ because $x \leq A$ and, from (2), $(1-\delta)A + F(A, m) \leq A$. Iterating the argument, we find $x_t \leq A$ for $t = 0, 1, \ldots$ ■

Claim 3 *Under Assumptions 1 and 2, consumption is bounded.*

Proof. At the individual level, we have $0 \leq c_{it} \leq \sum_{h=1}^{m} c_{ht}$.

To prove that the individual consumption is bounded, we prove that the aggregate consumption is bounded.

$$\begin{aligned}\sum_{i=1}^{m} c_{it} &\leq \sum_{i=1}^{m} (c_{it} + k_{it+1}) \leq (1-\delta) \sum_{i=1}^{m} k_{it} + F\left(\sum_{i=1}^{m} k_{it}, m\right) \\ &\leq (1-\delta)\, A + F\,(A, m) \leq A\end{aligned}$$

■

5 On the existence of equilibrium in a finite-horizon economy

We consider an economy which goes on for $T+1$ periods: $t = 0, \ldots, T$.

Focus first on a bounded economy, that is choose sufficiently large bounds for quantities:

$$\begin{aligned}
X_i &\equiv \{(c_{i0}, \ldots, c_{iT}) : 0 \leq c_{it} \leq B_c\} = [0, B_c]^{T+1} \text{ with } A < B_c \\
Y_i &\equiv \{(k_{i1}, \ldots, k_{iT}) : 0 \leq k_{it} \leq B_k\} = [0, B_k]^{T} \text{ with } A < B_k \\
Z_i &\equiv \{(\lambda_{i0}, \ldots, \lambda_{iT}) : 0 \leq \lambda_{it} \leq 1\} = [0, 1]^{T+1} \\
Y &\equiv \{(K_0, \ldots, K_T) : 0 \leq K_t \leq B_K\} = [0, B_K]^{T+1} \text{ with } A < B_K \\
Z &\equiv \{(L_0, \ldots, L_T) : 0 \leq L_t \leq B_L\} = [0, B_L]^{T+1} \text{ with } m < B_L
\end{aligned}$$

We notice that k_{i0} is given and that the borrowing constraints (inequalities $k_{it} \geq 0$) capture the imperfection in the credit market.[3]

Let $\mathcal{E}^T$ denote this economy with technology and preferences as in Assumptions 1 to 4 and with X_i, Y_i and Z_i as the ith consumer-worker's bounded sets of consumption demand, capital supply and leisure demand respectively ($i = 1, \ldots, m$), and Y and Z as the firm's bounded sets of capital and labor demands respectively.

Proposition 1 *Under the Assumptions 1, 2, 3 and 4, there exists an equilibrium*

$$\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_h, \bar{\mathbf{k}}_h, \bar{\boldsymbol{\lambda}}_h\right)_{h=1}^{m}, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$$

for the finite-horizon bounded economy $\mathcal{E}^T$.

Proof. The proof is quite long and articulated in many claims (see Appendix 1). ■

Focus now on an unbounded economy.

Theorem 4 *Any equilibrium of $\mathcal{E}^T$ is an equilibrium for the finite-horizon unbounded economy.*

Proof. Let $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_h, \bar{\mathbf{k}}_h, \bar{\boldsymbol{\lambda}}_h\right)_{h=1}^{m}, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ with $\bar{p}_t, \bar{r}_t, \bar{w}_t > 0$, $t = 0, \ldots, T$, be an equilibrium of $\mathcal{E}^T$.

Let $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)$ verify $\sum_{t=0}^{T} \beta_i^t u_i (c_{it}, \lambda_{it}) > \sum_{t=0}^{T} \beta_i^t u_i \left(\bar{c}_{it}, \bar{\lambda}_{it}\right)$. We want to prove that this allocation violates at least one budget constraint, that is that there exists t such that

$$\bar{p}_t \left[c_{it} + k_{it+1} - (1 - \delta) k_{it}\right] > \bar{r}_t k_{it} + \bar{w}_t (1 - \lambda_{it}) \tag{3}$$

Focus on a strictly convex combination of $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)$ and $\left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)$:

$$\begin{aligned}
c_{it}(\gamma) &\equiv \gamma c_{it} + (1 - \gamma) \bar{c}_{it} \\
k_{it}(\gamma) &\equiv \gamma k_{it} + (1 - \gamma) \bar{k}_{it} \\
\lambda_{it}(\gamma) &\equiv \gamma \lambda_{it} + (1 - \gamma) \bar{\lambda}_{it}
\end{aligned} \tag{4}$$

[3] A possible generalization of credit constraints is $h_i \leq k_{it}$ with $h_i < 0$ given.

with $0 < \gamma < 1$. Notice that we assume that the bounds satisfy $B_c, B_k, B_K > A$ and $B_L > m$ in order ensure that we enter the bounded economy when the parameter γ is sufficiently close to 0.

Entering the bounded economy means $(\mathbf{c}_i(\gamma), \mathbf{k}_i(\gamma), \boldsymbol{\lambda}_i(\gamma)) \in X_i \times Y_i \times Z_i$. In this case, because of the concavity of the utility function, we find

$$\begin{aligned}\sum_{t=0}^{T} \beta_i^t u_i(c_{it}(\gamma), \lambda_{it}(\gamma)) &\geq \gamma \sum_{t=0}^{T} \beta_i^t u_i(c_{it}, \lambda_{it}) + (1-\gamma) \sum_{t=0}^{T} \beta_i^t u_i(\bar{c}_{it}, \bar{\lambda}_{it}) \\ &> \sum_{t=0}^{T} \beta_i^t u_i(\bar{c}_{it}, \bar{\lambda}_{it})\end{aligned}$$

Since $(\mathbf{c}_i(\gamma), \mathbf{k}_i(\gamma), \boldsymbol{\lambda}_i(\gamma)) \in X_i \times Y_i \times Z_i$ and $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_h, \bar{\mathbf{k}}_h, \bar{\boldsymbol{\lambda}}_h\right)_{h=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ is an equilibrium for this economy, there exists $t \in \{0, \ldots, T\}$ such that

$$\bar{p}_t [c_{it}(\gamma) + k_{it+1}(\gamma) - (1-\delta) k_{it}(\gamma)] > \bar{r}_t k_{it}(\gamma) + \bar{w}_t (1 - \lambda_{it}(\gamma))$$

Replacing (4), we obtain

$$\begin{aligned}&\bar{p}_t \left(\gamma c_{it} + (1-\gamma) \bar{c}_{it} + \gamma k_{it+1} + (1-\gamma) \bar{k}_{it+1} - (1-\delta) \left[\gamma k_{it} + (1-\gamma) \bar{k}_{it}\right]\right) \\ > \;& \bar{r}_t \left[\gamma k_{it} + (1-\gamma) \bar{k}_{it}\right] + \bar{w}_t \left(1 - \left[\gamma \lambda_{it} + (1-\gamma) \bar{\lambda}_{it}\right]\right)\end{aligned}$$

that is

$$\begin{aligned}&\gamma \bar{p}_t [c_{it} + k_{it+1} - (1-\delta) k_{it}] + (1-\gamma) \bar{p}_t \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta) \bar{k}_{it}\right] \\ > \;& \gamma [\bar{r}_t k_{it} + \bar{w}_t (1 - \lambda_{it})] + (1-\gamma) \left[\bar{r}_t \bar{k}_{it} + \bar{w}_t \left(1 - \bar{\lambda}_{it}\right)\right]\end{aligned}$$

Since $\bar{p}_t \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta) \bar{k}_{it}\right] = \bar{r}_t \bar{k}_{it} + \bar{w}_t \left(1 - \bar{\lambda}_{it}\right)$, we obtain (3). Thus $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_h, \bar{\mathbf{k}}_h, \bar{\boldsymbol{\lambda}}_h\right)_{h=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ is also an equilibrium for this unbounded economy. ■

6 On the existence of equilibrium in an infinite-horizon economy

In the section, we introduce a separable utility and, for simplicity, we denote by u_i the utility of consumption and by v_i that of leisure. If w_i is the utility defined on both these arguments, we have $w_i(c_{it}, \lambda_{it}) \equiv u_i(c_{it}) + v_i(\lambda_{it})$.

Assumption 5 *The utility function is separable:* $w_i(c_{it}, \lambda_{it}) \equiv u_i(c_{it}) + v_i(\lambda_{it})$, *with* $u_i, v_i : \mathbb{R}_+ \to \mathbb{R}$ *and* $u_i, v_i \in C^1$. *In addition, we assume that* $u_i(0) = v_i(0) = 0$, $u_i'(0) = v_i'(0) = +\infty$, $u_i'(c_{it}), v_i'(\lambda_{it}) > 0$ *for* $c_{it}, \lambda_{it} > 0$, *and that functions* u, v *are concave.*

Theorem 5 *Under the Assumptions 1, 2, 3 and 5, there exists an equilibrium in the infinite-horizon economy with endogenous labor supply and borrowing constraints.*

Proof. We consider a sequence of time-truncated economies and the associated equilibria. We prove that there exists a sequence of equilibria which converges, when the horizon T goes to infinity, to an equilibrium of the infinite-horizon economy. The proof is detailed in Appendix 2. ■

7 Bubbles

Let $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ denote an equilibrium.

Claim 6 *For any individual i, the equilibrium sequence of multipliers $\bar{\boldsymbol{\mu}}_i \equiv (\bar{\mu}_{it})_{t=0}^{\infty}$ exists. FOC of point (7) in Claim 14 are satisfied in the limit economy.*

Proof. The proof is given in Appendix 4.

Let

$$q_{t+1} \equiv \max_i \frac{\mu_{it+1} p_{t+1}}{\mu_{it} p_t}$$

Since $\bar{K}_t > 0$ for any t, there exists i such that $\bar{k}_{it+1} > 0$ and $\bar{\mu}_{it}\bar{p}_t = \bar{\mu}_{it+1}\left[\bar{p}_{t+1}(1-\delta) + \bar{r}_{t+1}\right]$. We observe that

$$\frac{\bar{\mu}_{it+1}\bar{p}_{t+1}}{\bar{\mu}_{it}\bar{p}_t} \leq \frac{1}{1-\delta+\bar{\rho}_{t+1}} \text{ and } \frac{\bar{\mu}_{it+1}\bar{p}_{t+1}}{\bar{\mu}_{it}\bar{p}_t} = \frac{1}{1-\delta+\bar{\rho}_{t+1}} \text{ if } \bar{k}_{it+1} > 0$$

Thus

$$\bar{q}_{t+1} = \max_i \frac{\beta_i u_i'\left(\bar{c}_{it+1}\right)}{u_i'\left(\bar{c}_{it}\right)} = \frac{1}{1-\delta+\bar{\rho}_{t+1}}$$

Then

$$1 = \bar{q}_t\left(1-\delta+\bar{\rho}_t\right)$$

Let

$$Q_0 \equiv 1 \tag{5}$$

$$Q_t \equiv \prod_{s=1}^{t} q_s \text{ for } t > 0 \tag{6}$$

Clearly,

$$\bar{Q}_0 = 1$$

$$\bar{Q}_t = \prod_{s=1}^{t} \frac{1}{1-\delta+\bar{\rho}_s} \text{ for } t > 0$$

$\bar{Q}_t$ is the present value of a unit of capital of period t. For any t, we obtain:

$$\bar{Q}_t = \bar{Q}_{t+1}\left(1-\delta+\bar{\rho}_{t+1}\right) \tag{7}$$

and, by induction,

$$\begin{aligned}
\bar{Q}_2 &= \bar{Q}_3 (1-\delta) + \bar{Q}_3 \bar{\rho}_3 \\
\bar{Q}_1 &= \bar{Q}_2 (1-\delta) + \bar{Q}_2 \bar{\rho}_2 = \bar{Q}_3 (1-\delta)^2 + \bar{Q}_3 \bar{\rho}_3 (1-\delta) + \bar{Q}_2 \bar{\rho}_2 \\
\bar{Q}_0 &= \bar{Q}_1 (1-\delta) + \bar{Q}_1 \bar{\rho}_1 = \bar{Q}_3 (1-\delta)^3 + \bar{Q}_3 \bar{\rho}_3 (1-\delta)^2 + \bar{Q}_2 \bar{\rho}_2 (1-\delta) + \bar{Q}_1 \bar{\rho}_1
\end{aligned}$$

that is

$$1 = \bar{Q}_0 = \bar{Q}_T (1-\delta)^T + \sum_{t=1}^{T} \bar{Q}_t \bar{\rho}_t (1-\delta)^{t-1}$$

Definition 2 *We define the fundamental value of capital as*

$$\bar{v}_0 \equiv \sum_{t=1}^{+\infty} \bar{Q}_t \bar{\rho}_t (1-\delta)^{t-1}$$

The economy experiences a bubble if $\lim_{T\to+\infty} \bar{Q}_T (1-\delta)^T > 0$. *Otherwise* ($\lim_{T\to+\infty} \bar{Q}_T (1-\delta)^T = 0$)*, we say that there is no bubble.*

If $i = 1$ denotes the most patient agent with $\beta_i < \beta_1$ for $i = 2, \ldots, m$, at the stationary equilibrium, $\bar{Q}_t$ coincides with the discount factor of the patient agent, the only one with $\bar{k}_{it} > 0$ in the long run:

$$\bar{Q}_t \equiv \prod_{s=1}^{t} \bar{q}_s = \prod_{s=1}^{t} \frac{\beta_1 u_1' (\bar{c}_{1t+1})}{u_1' (\bar{c}_{1t})} = \prod_{s=1}^{t} \beta_1 = \beta_1^t$$

8 Efficiency

As above, $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ denotes an equilibrium. Following Malinvaud (1953) and Becker and Mitra (2011), we introduce the following definition.

Definition 3 *An equilibrium is efficient if there exists no sequence of total consumption, capital and labor* (C_t, K_t, L_t) *which satisfies, for* $t = 0, 1, \ldots$

$$C_t + K_{t+1} - (1-\delta) K_t = F(K_t, L_t) \tag{8}$$

(feasibility) with

$$C_t \geq \bar{C}_t \text{ and } m - L_t \geq m - \bar{L}_t$$

for $t = 0, 1, \ldots$ *with at least one strict inequality for consumption or for leisure.* K_0*, the aggregate capital endowment, is given.*

Proposition 2 *Assume that* $\lim_{t\to\infty} \bar{Q}_t = 0$*, where* $\bar{Q}_t$ *is given by (6) and (5). Then, this equilibrium is efficient.*

Proof. The proof is given in Appendix 4. ■

9 Conclusion

In this paper, we have shown the existence of an intertemporal equilibrium with market imperfections (borrowing constraints). Applying the fixed-point theorem of Gale-Mas-Colell, we have proved the existence of an equilibrium in a finite-horizon bounded economy. This equilibrium turns out to be also an equilibrium of any unbounded economy with the same fundamentals. Eventually, we have shown the existence of an equilibrium in an infinite-horizon economy as a limit of a sequence of truncated economies by applying a uniform convergence argument.

The paper generalizes in one respect Becker et al. (1991) by considering an elastic labor supply, and, in another respect, Bosi and Seegmuller (2010) by providing a proof of global existence. Our methodology, inspired by Florenzano (1999), is quite general and can be applied to other Ramsey models with different market imperfections.

At the end of the paper, we have addressed the question of occurrence of bubbles and their efficiency.

10 Appendix 1: finite horizon

Let us prove Proposition 1.

We define a bounded price set:

$$P \equiv \{(\mathbf{p}, \mathbf{r}, \mathbf{w}) : -1 \leq p_t \leq 1, 0 \leq r_t \leq 1, 0 \leq w_t \leq 1, t = 0, \ldots, T\}$$

At this stage, we put no restriction on the sign of p_t. We will prove later the positivity of the good price through an equilibrium argument.

Focus now on the budget constraints:

$$p_t \left[c_{it} + k_{it+1} - (1 - \delta) k_{it}\right] \leq r_t k_{it} + w_t (1 - \lambda_{it})$$

for $t = 0, \ldots, T - 1$ and $p_T \left[c_{iT} - (1 - \delta) k_{iT}\right] \leq r_T k_{iT} + w_T (1 - \lambda_{iT})$.

In the spirit of Bergstrom (1976), we introduce modified budget sets:

$$\begin{aligned} & B_i (\mathbf{p}, \mathbf{r}, \mathbf{w}) \\ \equiv & \left\{ \begin{array}{c} (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in X_i \times Y_i \times Z_i : \\ p_t \left[c_{it} + k_{it+1} - (1 - \delta) k_{it}\right] < r_t k_{it} + w_t (1 - \lambda_{it}) + \gamma (p_t, r_t, w_t) \\ t = 0, \ldots, T - 1 \\ p_T \left[c_{iT} - (1 - \delta) k_{iT}\right] < r_T k_{iT} + w_T (1 - \lambda_{iT}) + \gamma (p_T, r_T, w_T) \end{array} \right\} \\ & C_i (\mathbf{p}, \mathbf{r}, \mathbf{w}) \\ \equiv & \left\{ \begin{array}{c} (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in X_i \times Y_i \times Z_i : \\ p_t \left[c_{it} + k_{it+1} - (1 - \delta) k_{it}\right] \leq r_t k_{it} + w_t (1 - \lambda_{it}) + \gamma (p_t, r_t, w_t) \\ t = 0, \ldots, T - 1 \\ p_T \left[c_{iT} - (1 - \delta) k_{iT}\right] \leq r_T k_{iT} + w_T (1 - \lambda_{iT}) + \gamma (p_T, r_T, w_T) \end{array} \right\} \end{aligned}$$

where $\gamma (p_t, r_t, w_t) \equiv 1 - \min \{1, |p_t| + r_t + w_t\}$.

Let $\bar{B}_i (\mathbf{p}, \mathbf{r}, \mathbf{w})$ denote the closure of $B_i (\mathbf{p}, \mathbf{r}, \mathbf{w})$.

Claim 7 *For every* $(\mathbf{p}, \mathbf{r}, \mathbf{w}) \in P$*, we have* $B_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \neq \varnothing$ *and* $C_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) = \bar{B}_i(\mathbf{p}, \mathbf{r}, \mathbf{w})$*.*

Proof. Without loss of generality, focus on the modified budget constraints of the first two periods:

$$p_0 [c_{i0} + k_{i1} - (1-\delta) k_{i0}] < r_0 k_{i0} + w_0 (1 - \lambda_{i0}) + \gamma (p_0, r_0, w_0) \quad (9)$$
$$p_1 [c_{i1} + k_{i2} - (1-\delta) k_{i1}] < r_1 k_{i1} + w_1 (1 - \lambda_{i1}) + \gamma (p_1, r_1, w_1) \quad (10)$$

We know that $-1 \leq p_t \leq 1$, $0 \leq r_t \leq 1$, $0 \leq w_t \leq 1$.

(1) Assume that $|p_0| + r_0 + w_0 < 1$. Then $\gamma(p_0, r_0, w_0) > 0$.

Assume B_c to be large enough to set $c_{i0} = (1-\delta) k_{i0}$ and choose $\lambda_{i0} = 1$ (we stay in $X_i \times Z_i$). Then the inequality (9) becomes $p_0 k_{i1} < r_0 k_{i0} + \gamma(p_0, r_0, w_0)$ and it is satisfied if $k_{i1} > 0$ is sufficiently close to zero.

Focus now on the second period and two subcases.

(1.1) Assume that $|p_1| + r_1 + w_1 < 1$. Then $\gamma(p_1, r_1, w_1) > 0$.

If $p_1 < 0$, choose c_{i1} sufficiently large (assume the upper bound B_c to be large enough) and the inequality (10) is satisfied.

If $p_1 \geq 0$, set $c_{i1} = k_{i2} = 0$ and the inequality (10) becomes $-p_1(1-\delta) k_{i1} < r_1 k_{i1} + w_1(1-\lambda_{i1}) + \gamma(p_1, r_1, w_1)$ and it is satisfied. Notice that, in this case, inequality (10) is satisfied also if $k_{i2} > 0$ but sufficiently close to zero.

(1.2) Assume that $|p_1| + r_1 + w_1 \geq 1$. Then $\gamma(p_1, r_1, w_1) = 0$.

If $p_1 < 0$, choose c_{i1} sufficiently large (assume the upper bound B_c to be large enough) and the inequality (10) is satisfied.

If $p_1 = 0$, choose $\lambda_{i1} = 0$. The inequality (10) becomes $0 < r_1 k_{i1} + w_1$ and, since either $r_1 > 0$ or $w_1 > 0$, it is satisfied because $k_{i1} > 0$ (see point (1)).

If $p_1 > 0$, set $c_{i1} = k_{i2} = 0$: the inequality (10) becomes $-p_1(1-\delta) k_{i1} < r_1 k_{i1} + w_1(1-\lambda_{i1})$ and is satisfied because $k_{i1} > 0$ (see point (1)) and $\delta < 1$. Notice that, in this case, inequality (10) is satisfied also if $k_{i2} > 0$ but sufficiently close to zero.

(2) Assume that $|p_0| + r_0 + w_0 \geq 1$. Then $\gamma(p_0, r_0, w_0) = 0$.

If $p_0 < 0$, assume B_c to be large enough to set $c_{i0} = (1-\delta) k_{i0}$ and choose $k_{i1} > 0$. Inequality (9) becomes $p_0 k_{i1} < r_0 k_{i0} + w_0(1-\lambda_{i0})$ and it is satisfied.

If $p_0 = 0$, we have either $r_0 > 0$ or $w_0 > 0$. Set $\lambda_{i0} = 0 < k_{i1}$. Inequality (9) becomes $0 < r_0 k_{i0} + w_0$. We can not exclude the case $r_0 = 0$ or $w_0 = 0$, but Assumption 3 ensures that inequality (9) is verified.

If $p_0 > 0$, set $c_{i0} = 0$ and $0 < k_{i1} < (1-\delta) k_{i0}$. Inequality (9) becomes $p_0 [k_{i1} - (1-\delta) k_{i0}] < r_0 k_{i0} + w_0(1-\lambda_{i0})$ and it is satisfied.

Focus on the second period and two subcases.

(2.1) Assume that $|p_1| + r_1 + w_1 < 1$. Then $\gamma(p_1, r_1, w_1) > 0$.

The same arguments of point (1.1) apply.

(2.2) Assume that $|p_1| + r_1 + w_1 \geq 1$. Then $\gamma(p_1, r_1, w_1) = 0$.

The same arguments of point (1.2) apply (just replace "see point (1)" with "see point (2)").

Thus, we have proved that, for whatever price system $(\mathbf{p}, \mathbf{r}, \mathbf{w}) \in P$, there exists $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in B_i(\mathbf{p}, \mathbf{r}, \mathbf{w})$. In addition, $B_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \neq \varnothing$ implies $C_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) = \bar{B}_i(\mathbf{p}, \mathbf{r}, \mathbf{w})$ for every $(\mathbf{p}, \mathbf{r}, \mathbf{w}) \in P$. ■

Claim 8 *B_i is a lower semi-continuous correspondence on P.*

Proof. We observe that B_i has an open graph. ■

Claim 9 *C_i is upper semi-continuous on P with closed convex values.*

Proof. We remark that the inequalities in the definition of C_i are affine and that $X_i^T \times Y_i^T \times Z_i^T$ is a compact convex set. Thus C_i has a closed graph with convex values. ■

In the spirit of Gale and Mas-Colell (1975, 1979), we introduce the reaction correspondences $\varphi_i \left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right)$, $i = 0, \ldots, m+1$ defined on $P \times \left[\times_{h=1}^m (X_h \times Y_h \times Z_h)\right] \times Y \times Z$, where $i = 0$ denotes an "additional" agent, $i = 1, \ldots, m$ the consumers, and $i = m+1$ the firm. These correspondences are defined as follows.

Agent $i = 0$ (the "additional" agent):

$$\varphi_0 \left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right) \equiv \left\{ \begin{array}{c} (\tilde{\mathbf{p}}, \tilde{\mathbf{r}}, \tilde{\mathbf{w}}) \in P : \\ \sum_{t=0}^T (\tilde{p}_t - p_t) \left(\sum_{i=1}^m [c_{it} + k_{it+1} - (1-\delta) k_{it}] - F(K_t, L_t)\right) \\ + \sum_{t=0}^T (\tilde{r}_t - r_t)\left(K_t - \sum_{i=1}^m k_{it}\right) \\ + \sum_{t=0}^T (\tilde{w}_t - w_t)\left(L_t - m + \sum_{i=1}^m \lambda_{it}\right) > 0 \end{array} \right\} \tag{11}$$

Agents $i = 1, \ldots, m$ (consumers-workers):

$$\varphi_i \left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right) \equiv \left\{ \begin{array}{c} B_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \text{ if } (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \notin C_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \\ B_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \cap [P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i) \times Y_i] \text{ if } (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in C_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \end{array} \right\}$$

where P_i is the ith agent's set of strictly preferred allocations: $P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i) \equiv \left\{ \left(\tilde{\mathbf{c}}_i, \tilde{\boldsymbol{\lambda}}_i\right) : \sum_{t=0}^T \beta_i^t u_i\left(\tilde{c}_{it}, \tilde{\lambda}_{it}\right) > \sum_{t=0}^T \beta_i^t u_i(c_{it}, \lambda_{it}) \right\}$.

Agent $i = m+1$ (the firm):

$$\varphi_{m+1} \left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right) \equiv \left\{ \begin{array}{c} \left(\tilde{\mathbf{K}}, \tilde{\mathbf{L}}\right) \in Y \times Z : \\ \sum_{t=0}^T \left[p_t F\left(\tilde{K}_t, \tilde{L}_t\right) - r_t \tilde{K}_t - w_t \tilde{L}_t\right] \\ > \sum_{t=0}^T [p_t F(K_t, L_t) - r_t K_t - w_t L_t] \end{array} \right\} \tag{12}$$

We observe that $\varphi_i : \Phi \to 2^{\Phi_i}$ where

$$\begin{aligned} \Phi &\equiv \Phi_0 \times \ldots \times \Phi_{m+1} \\ \Phi_0 &\equiv P \\ \Phi_i &\equiv X_i \times Y_i \times Z_i,\ i = 1, \ldots, m \\ \Phi_{m+1} &\equiv Y \times Z \end{aligned}$$

and 2^{Φ_i} denotes the set of subsets of Φ_i.

Claim 10 *φ_i is a lower semi-continuous convex-valued correspondence for $i = 0, \ldots, m+1$.*

Proof.

(1) Focus first on openness.

φ_0 has an open graph.

Consider φ_i with $i = 1, \ldots, m$. B_i is lower semi-continuous and has an open graph (Claim 8) in $X_i \times Y_i \times Z_i$. $P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i)$ has also an open graph in $X_i \times Z_i$, so $B_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \cap [P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i) \times Y_i]$ has an open graph in $X_i \times Y_i \times Z_i$.

φ_{m+1} has an open graph.

(2) Focus now on convexity.

The affinity of the function w.r.t. $(\tilde{\mathbf{p}}, \tilde{\mathbf{r}}, \tilde{\mathbf{w}})$ in the LHS of the inequality defining φ_0 implies the convexity of φ_0.

The affinity of the modified budget constraint implies the convexity of B_i for every $(\mathbf{p}, \mathbf{r}, \mathbf{w}) \in P$. The concavity of u_i implies the convexity of $P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i)$ for every $(\mathbf{c}_i, \boldsymbol{\lambda}_i) \in X_i \times Z_i$. Then $B_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \cap [P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i) \times Y_i]$ is convex and φ_i is convex-valued for $i = 1, \ldots, m$.

Concavity of F implies also the convexity of φ_{m+1}. ■

Let us simplify the notation

$$\begin{aligned}
\mathbf{v} &\equiv \left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right) \\
\mathbf{v}_0 &\equiv (\mathbf{p}, \mathbf{r}, \mathbf{w}) \\
\mathbf{v}_i &\equiv (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \text{ for } i = 1, \ldots, m \\
\mathbf{v}_{m+1} &\equiv (\mathbf{K}, \mathbf{L})
\end{aligned}$$

Lemma 1 *(a fixed-point argument) There exists $\mathbf{v} \in \Phi$ such that either $\varphi_i(\mathbf{v}) = \varnothing$ or $\mathbf{v}_i \in \varphi_i(\mathbf{v})$ for $i = 0, \ldots, m+1$.*

Proof. Φ is a non-empty compact convex subset of $\mathbb{R}^{mT+(5+2m)(T+1)}$. Each $\varphi_i : \Phi \to 2^{\Phi_i}$ is a convex (possibly empty) valued correspondence whose graph is open in $\Phi \times \Phi_i$ (Claim 10). Then the Gale and Mas-Colell (1975) fixed-point theorem applies. ■

We observe the following.

(1) By definition of φ_0 (the inequality in (11) is strict): $(\mathbf{p}, \mathbf{r}, \mathbf{w}) \notin \varphi_0(\mathbf{v})$.

(2) $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \notin P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i) \times Y_i$ implies that $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \notin \varphi_i(\mathbf{v})$ for $i = 1, \ldots, m$.

(3) By definition of φ_{m+1} (the inequality in (12) is strict): $(\mathbf{K}, \mathbf{L}) \notin \varphi_{m+1}(\mathbf{v})$.

Then, for $i = 0, \ldots, m+1$, $\mathbf{v}_i \notin \varphi_i(\mathbf{v})$.

According to Lemma 1, there exists $\bar{\mathbf{v}} \in \Phi$ such that $\varphi_i(\bar{\mathbf{v}}) = \varnothing$ for $i = 0, \ldots, m+1$, that is, there exists $\bar{\mathbf{v}} \in \Phi$ such that the following holds.

$i = 0$. For every $(\mathbf{p}, \mathbf{r}, \mathbf{w}) \in P$,

$$\begin{aligned} & \sum_{t=0}^{T} (p_t - \bar{p}_t) \left(\sum_{i=1}^{m} \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta) \bar{k}_{it} \right] - F\left(\bar{K}_t, \bar{L}_t\right) \right) \\ & + \sum_{t=0}^{T} (r_t - \bar{r}_t) \left(\bar{K}_t - \sum_{i=1}^{m} \bar{k}_{it} \right) + \sum_{t=0}^{T} (w_t - \bar{w}_t) \left(\bar{L}_t - m + \sum_{i=1}^{m} \bar{\lambda}_{it} \right) \\ \leq \; & 0 \end{aligned} \tag{13}$$

$i = 1, \ldots, m$. $\left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right) \in C_i(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}})$ and $B_i(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}) \cap \left[P_i\left(\bar{\mathbf{c}}_i, \bar{\boldsymbol{\lambda}}_i\right) \times Y_i\right] = \varnothing$ for $i = 1, \ldots, m$. Then, for $i = 1, \ldots, m$, $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in C_i(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}) = \overline{B}_i(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}})$ implies

$$\sum_{t=0}^{T} \beta_i^t u_i(c_{it}, \lambda_{it}) \leq \sum_{t=0}^{T} \beta_i^t u_i\left(\bar{c}_{it}, \bar{\lambda}_{it}\right) \tag{14}$$

$i = m+1$. For $t = 0, \ldots, T$ and for every $(\mathbf{K}, \mathbf{L}) \in Y \times Z$, we have $\sum_{t=0}^{T} \left[\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t\right] \leq \sum_{t=0}^{T} \left[\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t\right]$.

This is possible if and only if

$$\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t \leq \bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t \tag{15}$$

for any t (simply choose $(\mathbf{K}, \mathbf{L})$ such that $(K_s, L_s) = \left(\bar{K}_s, \bar{L}_s\right)$ if $s \neq t$, to prove the necessity, and sum (15) side by side to prove the sufficiency).

In particular, we have

$$\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t \geq 0 \tag{16}$$

Proposition 3 *At the prices* $(\bar{p}_t, \bar{r}_t, \bar{w}_t)$, $\left(\bar{K}_t, \bar{L}_t\right)$ *satisfies the zero-profit condition:*

$$\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) = \bar{r}_t \bar{K}_t + \bar{w}_t \bar{L}_t \tag{17}$$

Proof. From (16), we know that $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t \geq 0$. Suppose, by contradiction, that $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t > 0$. Choose a new vector of inputs $\left(\mu \bar{K}_t, \mu \bar{L}_t\right)$ with $\mu > 1$ (this is possible if bounds B_K and B_L are sufficiently large). The constant returns to scale imply

$$\begin{aligned} \bar{p}_t F\left(\mu \bar{K}_t, \mu \bar{L}_t\right) - \bar{r}_t \mu \bar{K}_t - \bar{w}_t \mu \bar{L}_t &= \mu \left[\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t\right] \\ &> \bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t \end{aligned}$$

against the fact that inequality (15) holds for every $(K_t, L_t) \in [0, B_K] \times [0, B_L]$. ■

Claim 11 *If* $\bar{p}_t > 0$, *then* $\bar{K}_t - \sum_{i=1}^{m} \bar{k}_{it} \geq 0$ *and* $\bar{L}_t - \sum_{i=1}^{m} \bar{l}_{it} \geq 0$.

Proof.

(1) We notice that, from (13), if the demand for capital is less than the supply of capital: $\bar{K}_t < \sum_{i=1}^{m} \bar{k}_{it}$, we have $\bar{r}_t = 0$. But, since $\bar{p}_t > 0$, $\bar{r}_t = 0$ implies $\bar{K}_t = B_K$ and, so, $B_K = \bar{K}_t < \sum_{i=1}^{m} \bar{k}_{it} \leq A < B_K$, a contradiction. Then $\bar{K}_t - \sum_{i=1}^{m} \bar{k}_{it} \geq 0$ for $t = 0, \ldots, T+1$.

(2) Similarly, we notice that, if the labor demand is less than the labor supply: $\bar{L}_t < \sum_{i=1}^{m} \bar{l}_{it}$, we have $\bar{w}_t = 0$. But $\bar{w}_t = 0$ implies $\bar{L}_t = B_L$ and, so, $B_L = \bar{L}_t < \sum_{i=1}^{m} \bar{l}_{it} \leq m < B_L$, a contradiction. Then $\bar{L}_t - \sum_{i=1}^{m} \bar{l}_{it} \geq 0$ for $t = 0, \ldots, T+1$. ■

Let $\bar{Z}_t \equiv \sum_{i=1}^{m} \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] - F\left(\bar{K}_t, \bar{L}_t\right)$ be the aggregate excess demand at time t. We want to prove that $\bar{Z}_t = 0$.

Assume, by contradiction, that

$$\bar{Z}_t \neq 0 \tag{18}$$

Claim 12 *If $\bar{Z}_t \neq 0$ and $p_t\bar{Z}_t \leq \bar{p}_t\bar{Z}_t$ for every p_t with $|p_t| \leq 1$, then (1) $|\bar{p}_t| = 1$ and (2) $\bar{p}_t\bar{Z}_t > 0$.*

Proof.

(1) Let us show that $-1 < \bar{p}_t < 1$ leads to a contradiction.

(1.1) If $\bar{Z}_t > 0$, we choose p_t such that $\bar{p}_t < p_t < 1$ and we find $\bar{p}_t\bar{Z}_t < p_t\bar{Z}_t$, a contradiction.

(1.2) If $\bar{Z}_t < 0$, we choose p_t such that $-1 < p_t < \bar{p}_t$ and we find $\bar{p}_t\bar{Z}_t < p_t\bar{Z}_t$, a contradiction.

(2) Clearly, if we choose $p_t = 0$, we have always $\bar{p}_t\bar{Z}_t \geq 0$. Since $\bar{p}_t = \pm 1$ and $\bar{Z}_t \neq 0$, then $\bar{p}_t\bar{Z}_t \neq 0$ and, so, $\bar{p}_t\bar{Z}_t > 0$. ■

Claim 13 *If $\bar{Z}_t \neq 0$, then $\bar{Z}_t > 0$ and, hence, $\bar{p}_t = 1$.*

Proof. First, we observe that (13) holds also with $p_t = \bar{p}_t$ for $t \neq s$ and $(r_t, w_t) = (\bar{r}_t, \bar{w}_t)$ for $t = 0, \ldots, T$, that is

$$(p_s - \bar{p}_s)\left(\sum_{i=1}^{m}\left[\bar{c}_{is} + \bar{k}_{is+1} - (1-\delta)\bar{k}_{is}\right] - F\left(\bar{K}_s, \bar{L}_s\right)\right) = (p_s - \bar{p}_s)\bar{Z}_s \leq 0$$

for every p_s with $|p_s| \leq 1$. Replacing s by t, we have $p_t\bar{Z}_t \leq \bar{p}_t\bar{Z}_t$ for every p_t with $|p_t| \leq 1$.

Claim 12 applies. Then $|\bar{p}_t| = 1$ and $\bar{p}_t\bar{Z}_t > 0$.

Suppose that the conclusion of Claim 13 is false, that is $\bar{Z}_t < 0$ and, hence, $\bar{p}_t = -1$. We obtain $\sum_{i=1}^{m}\left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] - F\left(\bar{K}_t, \bar{L}_t\right) < 0$.

But if $\bar{p}_t = -1$, we have $\bar{c}_{it} = B_c$. Indeed, if $\bar{c}_{it} < B_c$ for at least one agent, we can find $\bar{c}_{it} < c_{it} < B_c$ such that $\sum_{t=0}^{T}\beta_i^t u_i(c_{it}, \lambda_{it}) > \sum_{t=0}^{T}\beta_i^t u_i\left(\bar{c}_{it}, \bar{\lambda}_{it}\right)$ with $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in B_i(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}})$, against the definition of $\bar{\mathbf{v}}$ (see (14)). Then

$$\begin{aligned} mB_c &= \sum_{i=1}^{m}\bar{c}_{it} < F\left(\bar{K}_t, \bar{L}_t\right) + (1-\delta)\sum_{i=1}^{m}\bar{k}_{it} - \sum_{i=1}^{m}\bar{k}_{it+1} \\ &\leq F\left(\sum_{i=1}^{m}\bar{k}_{it}, \sum_{i=1}^{m}\bar{l}_{it}\right) + (1-\delta)\sum_{i=1}^{m}\bar{k}_{it} \leq F(A, m) + (1-\delta)A \leq A \end{aligned}$$

a contraction. ■

Proposition 4 *The goods market clears: $\bar{Z}_t = 0$, that is*

$$\sum_{i=1}^{m} \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\,\bar{k}_{it}\right] = F\left(\bar{K}_t, \bar{L}_t\right)$$

Proof. $\bar{p}_t = 1$ implies $\gamma\left(\bar{p}_t, \bar{r}_t, \bar{w}_t\right) = 0$. In this case, $\left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right) \in C_i\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}\right)$ implies $\bar{p}_t\left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\,\bar{k}_{it}\right] \leq \bar{r}_t\bar{k}_{it} + \bar{w}_t\left(1 - \bar{\lambda}_{it}\right)$ and, therefore, we have

$$\bar{p}_t \sum_{i=1}^{m} \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\,\bar{k}_{it}\right] \leq \bar{r}_t \sum_{i=1}^{m} \bar{k}_{it} + \bar{w}_t \sum_{i=1}^{m} \bar{l}_{it} \tag{19}$$

Assume, by contradiction, $\bar{Z}_t \neq 0$. Claim 13 implies $\bar{p}_t = 1$ and $\bar{Z}_t > 0$. This implies, in turn,

$$\bar{p}_t \sum_{i=1}^{m} \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\,\bar{k}_{it}\right] > \bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right)$$

According to (16), we have also $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) \geq \bar{r}_t \bar{K}_t + \bar{w}_t \bar{L}_t$.

Finally, we know that $\bar{K}_t \geq \sum_{i=1}^{m} \bar{k}_{it}$ and $\bar{L}_t \geq \sum_{i=1}^{m} \bar{l}_{it}$ (Claim 11).

Putting together, we have $\bar{p}_t \sum_{i=1}^{m} \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\,\bar{k}_{it}\right] > \bar{r}_t \sum_{i=1}^{m} \bar{k}_{it} + \bar{w}_t \sum_{i=1}^{m} \bar{l}_{it}$, in contradiction with (19). Thus the inequality (18) is false and $\bar{Z}_t = 0$. ■

We observe that

$$\begin{aligned}
\sum_{i=1}^{m} \bar{c}_{it} &= F\left(\bar{K}_t, \bar{L}_t\right) + \sum_{i=1}^{m} \left[(1-\delta)\,\bar{k}_{it} - \bar{k}_{it+1}\right] \\
&\leq F\left(\sum_{i=1}^{m} \bar{k}_{it}, \sum_{i=1}^{m} \bar{l}_{it}\right) + (1-\delta) \sum_{i=1}^{m} \bar{k}_{it} \\
&\leq F\left(A, m\right) + (1-\delta)\,A \leq A < B_c
\end{aligned}$$

We have now to prove that also the capital and the labor markets clear.

Proposition 5 $\bar{p}_t, \bar{r}_t, \bar{w}_t > 0$*,* $t = 0, \ldots, T$*.*

Proof. Let us show that $\bar{p}_t > 0$. Indeed, if $\bar{p}_t \leq 0$, then $\bar{c}_{it} = B_c$ for every i and $\sum_{i=1}^{m} \left(\bar{c}_{it} + \bar{k}_{it+1}\right) \geq B_c > F\left(A, m\right) + (1-\delta)\,A \geq F\left(\bar{K}_t, \bar{L}_t\right) + (1-\delta) \sum_{i=1}^{m} \bar{k}_{it}$ in contradiction with $\bar{Z}_t = 0$.

Recall that

$$\bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t \geq \bar{p}_t F(K_t, L_t) - \bar{r}_t K - \bar{w}_t L_t$$

for any pair (K_t, L_t) with $K_t, L_t \geq 0$. Assume $\bar{r}_t = 0$ and $\bar{w}_t \geq 0$. In this case, given $L_t > 0$, we have $\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t = \bar{p}_t F(K_t, L_t) - \bar{w}_t L_t \to +\infty$ if $K_t \to +\infty$, since $\bar{p}_t > 0$: a contradiction. A similar proof works when $\bar{w}_t = 0$ and $\bar{r}_t \geq 0$. ■

Proposition 6 $\bar{K}_t = \sum_{i=1}^m \bar{k}_{it}$ *and* $\bar{L}_t = \sum_{i=1}^m \bar{l}_{it}$.

Proof. Since $\bar{p}_t > 0$, we have $\bar{K}_t \geq \sum_{i=1}^m \bar{k}_{it}$ (Claim 11). If $\bar{K}_t > \sum_{i=1}^m \bar{k}_{it}$, from (13), we have $\bar{r}_t = 1 > 0$. Then

$$\begin{aligned}\bar{p}_t \sum_{i=1}^m \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] &= \bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) \geq \bar{r}_t \bar{K}_t + \bar{w}_t \bar{L}_t \\ &> \bar{r}_t \sum_{i=1}^m \bar{k}_{it} + \bar{w}_t \sum_{i=1}^m \bar{l}_{it}\end{aligned}$$

But $\left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right) \in C_i\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}\right)$ implies $\bar{p}_t \sum_{i=1}^m \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] \leq \bar{r}_t \sum_{i=1}^m \bar{k}_{it} + \bar{w}_t \sum_{i=1}^m \left(1 - \bar{\lambda}_{it}\right)$, a contradiction. Then $\bar{K}_t = \sum_{i=1}^m \bar{k}_{it}$.

We know that $\bar{L}_t \geq \sum_{i=1}^m \bar{l}_{it}$ (Claim 11). If $\bar{L}_t > \sum_{i=1}^m \bar{l}_{it}$, we have $\bar{w}_t = 1 > 0$. Then

$$\begin{aligned}\bar{p}_t \sum_{i=1}^m \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] &= \bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) \geq \bar{r}_t \bar{K}_t + \bar{w}_t \bar{L}_t \\ &> \bar{r}_t \sum_{i=1}^m \bar{k}_{it} + \bar{w}_t \sum_{i=1}^m \bar{l}_{it}\end{aligned}$$

But $\left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right) \in C_i\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}\right)$ implies $\bar{p}_t \sum_{i=1}^m \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] \leq \bar{r}_t \sum_{i=1}^m \bar{k}_{it} + \bar{w}_t \sum_{i=1}^m \left(1 - \bar{\lambda}_{it}\right)$, a contradiction. Then $\bar{L}_t = \sum_{i=1}^m \bar{l}_{it}$. ■

We observe that $\sum_{i=1}^m \bar{k}_{it} \leq A < B_k$ and $\sum_{i=1}^m \bar{l}_{it} \leq m < B_L$.

Proposition 7 *The modified budget constraint at equilibrium is a budget constraint:* $\gamma\left(\bar{p}_t, \bar{r}_t, \bar{w}_t\right) = 0$ *for* $t = 0, \ldots, T$.

Proof. $\bar{p}_t > 0$ implies that the modified budget constraint is binding:

$$\bar{p}_t\left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] = \bar{r}_t \bar{k}_{it} + \bar{w}_t \bar{l}_{it} + \gamma\left(\bar{p}_t, \bar{r}_t, \bar{w}_t\right)$$

This gives

$$\bar{p}_t \sum_{i=1}^m \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] = \bar{r}_t \sum_{i=1}^m \bar{k}_{it} + \bar{w}_t \sum_{i=1}^m \bar{l}_{it} + m\gamma\left(\bar{p}_t, \bar{r}_t, \bar{w}_t\right)$$

Proposition 4 implies $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) = \bar{r}_t \sum_{i=1}^m \bar{k}_{it} + \bar{w}_t \sum_{i=1}^m \bar{l}_{it} + m\gamma\left(\bar{p}_t, \bar{r}_t, \bar{w}_t\right)$, while Propositions 3 and 6 entail $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) = \bar{r}_t \sum_{i=1}^m \bar{k}_{it} + \bar{w}_t \sum_{i=1}^m \bar{l}_{it}$.

So, $\gamma\left(\bar{p}_t, \bar{r}_t, \bar{w}_t\right) = 0$. ■

Corollary 1 $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_h, \bar{\mathbf{k}}_h, \bar{\boldsymbol{\lambda}}_h\right)_{h=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ *is an equilibrium for the finite-horizon bounded economy* $\mathcal{E}^T$.

11 Appendix 2: infinite horizon

We want to prove Theorem 5. From now on, any variable x_t^T with subscript t and superscript T will refer to a period t in a T-truncated economy with $x_t^T = 0$ if $t > T$. As above, sequences will be denoted in bold type.

Under the Assumptions 1, 2, 3 and 5 an equilibrium

$$\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)^T$$

of a truncated economy exists. Under these assumptions, namely separability and differentiability of preferences, the following necessary conditions hold for the existence of an equilibrium in a truncated economy.

Claim 14 *Under Assumption 5, the equilibrium of a truncated economy satisfies the following conditions.*

For $t = 0, \ldots, T$*:*

(1) $\bar{p}_t^T, \bar{r}_t^T, \bar{w}_t^T > 0$ *with* $\bar{p}_t^T + \bar{r}_t^T + \bar{w}_t^T = 1$ *(normalization),*

(2) $(\partial F / \partial K_t)\left(\bar{K}_t^T, \bar{L}_t^T\right) = \bar{r}_t^T / \bar{p}_t^T$,

(3) $(\partial F / \partial L_t)\left(\bar{K}_t^T, \bar{L}_t^T\right) = \bar{w}_t^T / \bar{p}_t^T$,

(4) $\bar{K}_t^T = \sum_{i=1}^m \bar{k}_{it}^T$,

(5) $\bar{L}_t^T = \sum_{i=1}^m \bar{l}_{it}^T$,

(6) $\sum_{i=1}^m \left[\bar{c}_{it}^T + \bar{k}_{it+1}^T - (1-\delta)\bar{k}_{it}^T\right] = F\left(\bar{K}_t^T, \bar{L}_t^T\right)$ *with* $\bar{k}_{iT+1}^T = 0$.

For $i = 1, \ldots, m$, $t = 0, \ldots, T$*:*

(7) $\beta_i^t u_i'\left(\bar{c}_{it}^T\right) = \bar{\mu}_{it}^T \bar{p}_t^T \geq \bar{\mu}_{it+1}^T \bar{p}_{t+1}^T (1-\delta) + \bar{\mu}_{it+1}^T \bar{r}_{t+1}^T$, *with equality when* $\bar{k}_{it+1}^T > 0$,

(8) $v_i'\left(\bar{\lambda}_{it}^T\right) \geq u_i'\left(\bar{c}_{it}^T\right) \bar{w}_t^T / \bar{p}_t^T$, *with equality when* $\bar{\lambda}_{it}^T < 1$,

(9) $\bar{p}_t^T \left[\bar{c}_{it}^T + \bar{k}_{it+1}^T - (1-\delta)\bar{k}_{it}^T\right] = \bar{r}_t^T \bar{k}_{it}^T + \bar{w}_t^T \left(1 - \bar{\lambda}_{it}^T\right)$ *with* $\bar{k}_{it}^T \geq 0$, $\bar{k}_{iT+1}^T = 0$ *and* $0 \leq \bar{\lambda}_{it}^T \leq 1$,

where $\bar{\mu}_{it}^T$ *is the multiplier associated to the budget constraint at time* t.

Proof. See Bosi and Seegmuller (2010) among others. ■

In the following claims, we omit for simplicity any reference to Assumptions 1, 2, 3 and 5. We suppose that they are always satisfied.

Let us introduce some new variables:

$$\begin{array}{llll}
\bar{\zeta}_{it}^T \equiv \beta_i^t u_i'\left(\bar{c}_{it}^T\right) \bar{c}_{it}^T & \text{if } t \leq T, & \text{and} \quad \bar{\zeta}_{it}^T = 0 & \text{if } t > T, \\
\bar{\eta}_{it}^T \equiv \beta_i^t v_i'\left(\bar{\lambda}_{it}^T\right) \bar{\lambda}_{it}^T & \text{if } t \leq T, & \text{and} \quad \bar{\eta}_{it}^T = 0 & \text{if } t > T, \\
\bar{\theta}_{it}^T \equiv \beta_i^t v_i'\left(\bar{\lambda}_{it}^T\right) & \text{if } t \leq T, & \text{and} \quad \bar{\theta}_{it}^T = 0 & \text{if } t > T, \\
\bar{\vartheta}_{it}^T \equiv \bar{\mu}_{it}^T \bar{w}_t^T & \text{if } t \leq T, & \text{and} \quad \bar{\vartheta}_{it}^T = 0 & \text{if } t > T,
\end{array} \tag{20}$$

and $\bar{\varepsilon}_{it}^T \equiv \bar{\theta}_{it}^T - \bar{\vartheta}_{it}^T$.

We notice that points (7) and (8) of Claim 14 entail $\bar{\varepsilon}_{it}^T \geq 0$ and $\bar{\varepsilon}_{it}^T = 0$ when $\bar{\lambda}_{it}^T < 1$.

Claim 15 *For any $\varepsilon > 0$, there exists τ such that, for any $s > \tau$ and for any T, $\sum_{t=s}^{\infty} \bar{\zeta}_{it}^{T} < \varepsilon$.*

We observe that the critical τ is independent of T.

Proof. We know that, under Assumptions 1 and 2, $\bar{k}_{it}^{T} \leq A$ and $\bar{c}_{it}^{T} \leq A$. We observe that $\sum_{t=0}^{\infty} \beta_i^t u_i(A) = u_i(A) / (1-\beta_i) < \infty$. Then, there exists τ such that $\sum_{t=\tau}^{\infty} \beta_i^t u_i(A) < \varepsilon$. In addition, under Assumption 5,

$$\begin{aligned} \sum_{t=\tau}^{\infty} \beta_i^t u_i(A) &\geq \sum_{t=\tau}^{T} \beta_i^t u_i\left(\bar{c}_{it}^{T}\right) = \sum_{t=\tau}^{T} \beta_i^t \left[u_i\left(\bar{c}_{it}^{T}\right) - u_i(0)\right] \\ &\geq \sum_{t=\tau}^{T} \beta_i^t u_i'\left(\bar{c}_{it}^{T}\right) \bar{c}_{it}^{T} \end{aligned} \quad (21)$$

because of the concavity of u_i. Thus, for any $\varepsilon > 0$, there exists τ such that, for any $s > \tau$ and for any T, $\sum_{t=s}^{\infty} \bar{\zeta}_{it}^{T} < \varepsilon$. ■

Claim 16 *For any $\varepsilon > 0$, there exists τ such that, for any $s > \tau$ and for any T, $\sum_{t=s}^{\infty} \bar{\eta}_{it}^{T} < \varepsilon$.*

As above, the critical τ does not depend on T.

Proof. Since $\sum_{t=0}^{\infty} \beta_i^t v_i(1) = v_i(1) / (1-\beta_i) < \infty$, there exists τ such that $\sum_{t=\tau}^{\infty} \beta_i^t v_i(1) < \varepsilon$. In addition, under Assumption 5,

$$\begin{aligned} \sum_{t=\tau}^{\infty} \beta_i^t v_i(1) &\geq \sum_{t=\tau}^{T} \beta_i^t v_i\left(\bar{\lambda}_{it}^{T}\right) = \sum_{t=\tau}^{T} \beta_i^t \left[v_i\left(\bar{\lambda}_{it}^{T}\right) - v_i(0)\right] \\ &\geq \sum_{t=\tau}^{T} \beta_i^t v_i'\left(\bar{\lambda}_{it}^{T}\right) \bar{\lambda}_{it}^{T} \end{aligned} \quad (22)$$

because $\bar{\lambda}_{it}^{T} \leq 1$ and v_i is concave. Thus, for any $\varepsilon > 0$, there exists τ such that, for any $s > \tau$ and for any T, $\sum_{t=s}^{\infty} \bar{\eta}_{it}^{T} < \varepsilon$. ■

Notice that, as above, the critical τ does not depend on T.

Claim 17 *For any $\varepsilon > 0$, there exists τ such that, for any $s > \tau$ and for any T, $\sum_{t=s}^{\infty} \bar{\vartheta}_{it}^{T} \bar{\lambda}_{it}^{T} < \varepsilon$ and $\sum_{t=s}^{\infty} \bar{\varepsilon}_{it}^{T} < \varepsilon$. In addition, for any T, $\left(\bar{\vartheta}_{it}^{T} \bar{\lambda}_{it}^{T}\right)_{t=0}^{\infty} \in l_+^1$ and $\left(\bar{\varepsilon}_{it}^{T}\right)_{t=0}^{\infty} \in l_+^1$.*

Notice that the critical τ does not depend on T.

Proof. From (20), we observe that $\beta_i^t v_i'\left(\bar{\lambda}_{it}^{T}\right) \bar{\lambda}_{it}^{T} = \bar{\vartheta}_{it}^{T} \bar{\lambda}_{it}^{T} + \bar{\varepsilon}_{it}^{T} \bar{\lambda}_{it}^{T} = \bar{\vartheta}_{it}^{T} \bar{\lambda}_{it}^{T} + \bar{\varepsilon}_{it}^{T}$ since $\bar{\varepsilon}_{it}^{T} = 0$ when $\bar{\lambda}_{it}^{T} < 1$. For any $\varepsilon > 0$, there exists τ such that, for any $s > \tau$, $\sum_{t=s}^{\infty} \beta_i^t v_i(1) < \varepsilon$. Thus, according to (22), for any $\varepsilon > 0$, there exists τ such that, for any $s > \tau$ and for any T, $\sum_{t=s}^{T} \left(\bar{\vartheta}_{it}^{T} \bar{\lambda}_{it}^{T} + \bar{\varepsilon}_{it}^{T}\right) = \sum_{t=s}^{T} \beta_i^t v_i'\left(\bar{\lambda}_{it}^{T}\right) \bar{\lambda}_{it}^{T} < \varepsilon$. In particular, $\sum_{t=s}^{\infty} \bar{\vartheta}_{it}^{T} \bar{\lambda}_{it}^{T} < \varepsilon$ and $\sum_{t=s}^{\infty} \bar{\varepsilon}_{it}^{T} < \varepsilon$.

From (22), we have also, for any T,

$$\sum_{t=0}^{\infty}\left(\bar{\vartheta}_{it}^{T}\bar{\lambda}_{it}^{T}+\bar{\varepsilon}_{it}^{T}\right)\leq\sum_{t=0}^{\infty}\beta_{i}^{t}v_{i}\left(1\right)=v_{i}\left(1\right)/\left(1-\beta_{i}\right)$$

and, so, $\sum_{t=0}^{\infty}\bar{\vartheta}_{it}^{T}\bar{\lambda}_{it}^{T}\leq v_{i}\left(1\right)/\left(1-\beta_{i}\right)$ and $\sum_{t=0}^{\infty}\bar{\varepsilon}_{it}^{T}\leq v_{i}\left(1\right)/\left(1-\beta_{i}\right)$. Then, for any T, $\left(\bar{\vartheta}_{it}^{T}\bar{\lambda}_{it}^{T}\right)_{t=0}^{\infty}\in l_{+}^{1}$ and $\left(\bar{\varepsilon}_{it}^{T}\right)_{t=0}^{\infty}\in l_{+}^{1}$. ■

Claim 18 *For any $\varepsilon>0$ there exists τ such that for any $s>\tau$ and any $T\geq s$ we have $\sum_{t=s}^{T}\bar{\vartheta}_{it}^{T}<\varepsilon$. In addition, for any T,*

$$\sum_{t=0}^{T}\bar{\vartheta}_{it}^{T}<\frac{u_{i}\left(A\right)+v_{i}\left(1\right)}{1-\beta_{i}} \tag{23}$$

Proof. Focus now on the sequence of equilibrium budget constraints: $\bar{r}_{t}^{T}\bar{k}_{it}^{T}+\bar{w}_{t}^{T}\left(1-\bar{\lambda}_{it}^{T}\right)-\bar{p}_{t}^{T}\left[\bar{c}_{it}^{T}+\bar{k}_{it+1}^{T}-\left(1-\delta\right)\bar{k}_{it}^{T}\right]\geq 0$.

Multiplying them by the multipliers, we obtain, according to the Kuhn-Tucker method,

$$\bar{\mu}_{it}^{T}\bar{r}_{t}^{T}\bar{k}_{it}^{T}+\bar{\mu}_{it}^{T}\bar{w}_{t}^{T}\left(1-\bar{\lambda}_{it}^{T}\right)-\bar{\mu}_{it}^{T}\bar{p}_{t}^{T}\bar{c}_{it}^{T}-\bar{\mu}_{it}^{T}\bar{p}_{t}^{T}\bar{k}_{it+1}^{T}+\bar{\mu}_{it}^{T}\bar{p}_{t}^{T}\left(1-\delta\right)\bar{k}_{it}^{T}=0 \tag{24}$$

Summing them over time from $t=\tau$ to $t=T$, we get

$$\begin{aligned}
&\bar{\mu}_{i\tau}^{T}\bar{r}_{\tau}^{T}\bar{k}_{i\tau}^{T}+\bar{\mu}_{i\tau}^{T}\bar{w}_{\tau}^{T}\left(1-\bar{\lambda}_{i\tau}^{T}\right)-\bar{\mu}_{i\tau}^{T}\bar{p}_{\tau}^{T}\bar{c}_{i\tau}^{T}-\bar{\mu}_{i\tau}^{T}\bar{p}_{\tau}^{T}\bar{k}_{i\tau+1}^{T}+\bar{\mu}_{i\tau}^{T}\bar{p}_{\tau}^{T}\left(1-\delta\right)k_{i\tau}\\
&+\bar{\mu}_{i\tau+1}^{T}\bar{r}_{\tau+1}^{T}\bar{k}_{i\tau+1}^{T}+\bar{\mu}_{i\tau+1}^{T}\bar{w}_{\tau+1}^{T}\left(1-\bar{\lambda}_{i\tau+1}^{T}\right)-\bar{\mu}_{i\tau+1}^{T}\bar{p}_{\tau+1}^{T}\bar{c}_{i\tau+1}^{T}\\
&-\bar{\mu}_{i\tau+1}^{T}\bar{p}_{\tau+1}^{T}\bar{k}_{i\tau+2}^{T}+\bar{\mu}_{i\tau+1}^{T}\bar{p}_{\tau+1}^{T}\left(1-\delta\right)\bar{k}_{i\tau+1}^{T}\\
&+\ldots\\
&+\bar{\mu}_{iT}^{T}\bar{r}_{T}^{T}\bar{k}_{iT}^{T}+\bar{\mu}_{iT}^{T}\bar{w}_{T}^{T}\left(1-\bar{\lambda}_{iT}^{T}\right)-\bar{\mu}_{iT}^{T}\bar{p}_{T}^{T}\bar{c}_{iT}^{T}-\bar{\mu}_{iT}^{T}\bar{p}_{T}^{T}\bar{k}_{iT+1}^{T}\\
&+\bar{\mu}_{iT}^{T}\bar{p}_{T}^{T}\left(1-\delta\right)\bar{k}_{iT}^{T}\\
=\;&0
\end{aligned}$$

that is

$$\begin{aligned}
&\sum_{t=\tau}^{T}\bar{\vartheta}_{it}^{T}-\sum_{t=\tau}^{T}\bar{\vartheta}_{it}^{T}\bar{\lambda}_{it}^{T}\\
&-\sum_{t=\tau}^{T-1}\left[\bar{\mu}_{it}^{T}\bar{p}_{t}^{T}-\bar{\mu}_{it+1}^{T}\bar{p}_{t+1}^{T}\left(1-\delta\right)-\bar{\mu}_{it+1}^{T}\bar{r}_{t+1}^{T}\right]\bar{k}_{it+1}^{T}\\
&+\bar{\mu}_{i\tau}^{T}\bar{p}_{\tau}^{T}\left(1-\delta\right)k_{i\tau}+\bar{\mu}_{i\tau}^{T}\bar{r}_{\tau}^{T}k_{i\tau}-\bar{\mu}_{iT}^{T}\bar{p}_{T}^{T}\bar{k}_{iT+1}^{T}\\
=\;&\sum_{t=\tau}^{T}\bar{\mu}_{it}^{T}\bar{p}_{t}^{T}\bar{c}_{it}^{T}=\sum_{t=\tau}^{T}\beta_{i}^{t}u_{i}'\left(\bar{c}_{it}^{T}\right)\bar{c}_{it}^{T}=\sum_{t=\tau}^{T}\bar{\zeta}_{it}^{T}
\end{aligned}$$

We know that $\left[\bar{\mu}_{it}^{T}\bar{p}_{t}^{T}-\bar{\mu}_{it+1}^{T}\bar{p}_{t+1}^{T}\left(1-\delta\right)-\bar{\mu}_{it+1}^{T}\bar{r}_{t+1}^{T}\right]\bar{k}_{it+1}^{T}=0$ because either $\bar{\mu}_{it}^{T}\bar{p}_{t}^{T}-\bar{\mu}_{it+1}^{T}\bar{p}_{t+1}^{T}\left(1-\delta\right)-\bar{\mu}_{it+1}^{T}\bar{r}_{t+1}^{T}=0$ or $\bar{k}_{it+1}^{T}=0$ (point (7) of Claim 14). Then

$$\begin{aligned}\sum_{t=\tau}^{T}\bar{\vartheta}_{it}^{T} &= \sum_{t=\tau}^{T}\bar{\zeta}_{it}^{T}+\sum_{t=\tau}^{T}\bar{\vartheta}_{it}^{T}\bar{\lambda}_{it}^{T}\\ &\quad -\bar{\mu}_{i\tau}^{T}\bar{p}_{\tau}^{T}\left(1-\delta\right)k_{i\tau}-\bar{\mu}_{i\tau}^{T}\bar{r}_{\tau}^{T}k_{i\tau}+\bar{\mu}_{iT}^{T}\bar{p}_{T}^{T}\bar{k}_{iT+1}^{T}\end{aligned} \tag{25}$$

From the proof of Claim 15, we know that

$$\sum_{t=\tau}^{T}\bar{\zeta}_{it}^{T}\leq\sum_{t=\tau}^{T}\beta_{i}^{t}u_{i}\left(A\right)=u_{i}\left(A\right)\frac{\beta_{i}^{\tau}-\beta_{i}^{T+1}}{1-\beta_{i}}<\frac{\beta_{i}^{\tau}u_{i}\left(A\right)}{1-\beta_{i}} \tag{26}$$

Thus, for any $\varepsilon>0$, there exists τ_1 such that, for any $s>\tau_1$ and for any $T\geq s$,

$$\sum_{t=s}^{T}\bar{\zeta}_{it}^{T}<\varepsilon/2 \tag{27}$$

From the proof of Claim 17, we know also that

$$\sum_{t=\tau}^{T}\bar{\vartheta}_{it}^{T}\bar{\lambda}_{it}^{T}\leq\sum_{t=\tau}^{T}\beta_{i}^{t}v_{i}\left(1\right)=v_{i}\left(1\right)\frac{\beta_{i}^{\tau}-\beta_{i}^{T+1}}{1-\beta_{i}}<\frac{\beta_{i}^{\tau}v_{i}\left(1\right)}{1-\beta_{i}} \tag{28}$$

Thus, for any $\varepsilon>0$, there exists τ_2 such that, for any $s>\tau_2$ and for any $T\geq s$,

$$\sum_{t=s}^{T}\bar{\vartheta}_{it}^{T}\bar{\lambda}_{it}^{T}<\varepsilon/2$$

According to (25), we have that

$$\begin{aligned}\sum_{t=s}^{T}\bar{\vartheta}_{it}^{T} &\leq \sum_{t=s}^{T}\bar{\zeta}_{it}^{T}+\sum_{t=s}^{T}\bar{\vartheta}_{it}^{T}\bar{\lambda}_{it}^{T}+\bar{\mu}_{iT}^{T}\bar{p}_{T}^{T}\bar{k}_{iT+1}^{T}\\ &= \sum_{t=s}^{T}\bar{\zeta}_{it}^{T}+\sum_{t=s}^{T}\bar{\vartheta}_{it}^{T}\bar{\lambda}_{it}^{T}\end{aligned} \tag{29}$$

because in the truncated economy $\bar{k}_{iT+1}^{T}=0$.

Thus, for any $\varepsilon>0$, there exists $\tau\equiv\max\left\{\tau_1,\tau_2\right\}$ such that, for any $s>\tau$ and for any $T\geq s$,

$$\sum_{t=s}^{T}\bar{\vartheta}_{it}^{T}<\varepsilon/2+\varepsilon/2=\varepsilon$$

because in the truncated economy $\bar{k}_{iT+1}^{T}=0$.

Finally, from (26), (28) and (29), we have

$$\sum_{t=\tau}^{T} \bar{\vartheta}_{it}^{T} \leq \sum_{t=\tau}^{T} \bar{\zeta}_{it}^{T} + \sum_{t=\tau}^{T} \bar{\vartheta}_{it}^{T} \bar{\lambda}_{it}^{T} < [u_i(A) + v_i(1)] \frac{\beta_i^{\tau}}{1-\beta_i}$$

Taking $\tau = 0$, we obtain (23). ■

Claim 19 *Let* $\bar{\boldsymbol{\vartheta}}_i^T \equiv \left(\bar{\vartheta}_{it}^T\right)_{t=0}^{\infty}$. *There is a subsequence* $\left(\bar{\boldsymbol{\vartheta}}_i^{T_S}\right)_{S=0}^{\infty}$ *which converges for the* l^1*-topology to a sequence* $\bar{\boldsymbol{\vartheta}}_i \equiv \left(\bar{\vartheta}_{it}\right)_{t=0}^{\infty} \in l_+^1$. *The limit* $\bar{\boldsymbol{\vartheta}}_i$ *shares the same properties of the terms* $\bar{\boldsymbol{\vartheta}}_i^T$ *of the sequence, namely, (1) for any* $\varepsilon > 0$ *there exists* τ *(the same for all the terms) such that, for any* $s > \tau$*, we have* $\sum_{t=s}^{\infty} \bar{\vartheta}_{it} \leq \varepsilon$*, and (2)* $\sum_{t=0}^{\infty} \bar{\vartheta}_{it} \leq [u_i(A) + v_i(1)] / (1-\beta_i)$.

Proof. We apply Claim 18 and we find that, for any $\varepsilon > 0$ there exists τ such that for any $s > \tau$ and for any T, we have $\sum_{t=s}^{\infty} \bar{\vartheta}_{it}^T \leq \varepsilon$. We observe also that (23) implies $\sum_{t=0}^{\infty} \bar{\vartheta}_{it}^T \leq [u_i(A) + v_i(1)] / (1-\beta_i)$ for any T. Thus, Lemma 2 in Appendix 3 applies with a ball B of radius $\rho = [u_i(A) + v_i(1)] / (1-\beta_i)$. ■

Claim 20 *In the infinite-horizon economy, leisure demand is positive:*

$$\lim_{T\to\infty} \bar{\lambda}_{it}^T = \bar{\lambda}_{it} \in (0,1]$$

Proof. We have $\bar{\theta}_{it}^T = \bar{\vartheta}_{it}^T + \bar{\varepsilon}_{it}^T$ with $\bar{\varepsilon}_{it}^T \geq 0$ and $\bar{\varepsilon}_{it}^T = 0$ if $\bar{\lambda}_{it}^T < 1$.

From Claim 18, we know that, for any $\varepsilon > 0$, there exists τ_1 such that, for any $s > \tau_1$ and for any T, $\sum_{t=s}^{\infty} \bar{\vartheta}_{it}^T \leq \varepsilon/2$.

From Claim 17, we know that for any $\varepsilon > 0$, there exists τ_2 such that, for any $s > \tau_2$ and for any T, $\sum_{t=s}^{\infty} \bar{\varepsilon}_{it}^T < \varepsilon/2$.

Hence, for any $\varepsilon > 0$, there exists $\tau \equiv \max\{\tau_1, \tau_2\}$ such that, for any $s > \tau$ and for any T, $\sum_{t=s}^{\infty} \bar{\theta}_{it}^T = \sum_{t=s}^{\infty} \bar{\vartheta}_{it}^T + \sum_{t=s}^{\infty} \bar{\varepsilon}_{it}^T < \varepsilon$. In addition, for any T,

$$\sum_{t=0}^{\infty} \bar{\theta}_{it}^T = \sum_{t=0}^{\infty} \bar{\vartheta}_{it}^T + \sum_{t=0}^{\infty} \bar{\varepsilon}_{it}^T \leq \frac{u_i(A) + v_i(1)}{1-\beta_i} + \frac{v_i(1)}{1-\beta_i}$$

Let $\bar{\boldsymbol{\theta}}_i^T \equiv \left(\bar{\theta}_{it}^T\right)$. Then $\bar{\boldsymbol{\theta}}_i^T \to \bar{\boldsymbol{\theta}}_i \in l_+^1$ for the l^1-topology (Lemma 2 in Appendix 3 applies with $\rho = [u_i(A) + 2v_i(1)] / (1-\beta_i)$).

Therefore, for any t, $\bar{\theta}_{it}^T$ converges to $\bar{\theta}_{it} \in (0, +\infty)$. Hence, $\bar{\lambda}_{it}^T$ converges to $\bar{\lambda}_{it} > 0$ since v_i satisfies the Inada conditions (Assumption 5). Clearly, $\bar{\lambda}_{it} \leq 1$. ■

Claim 21 *In the infinite-horizon economy, the equilibrium prices are positive:* $\lim_{T\to\infty} \bar{p}_t^T = \bar{p}_t \in (0,1)$, $\lim_{T\to\infty} \bar{r}_t^T = \bar{r}_t \in (0,1)$, $\lim_{T\to\infty} \bar{w}_t = \bar{w}_t \in (0,1)$.

Proof. Focus on prices.

Suppose that $\lim_{T\to\infty} \bar{p}_t^T = 0$. We know that $\beta_i^t u_i'\left(\bar{c}_{it}^T\right) = \bar{\mu}_{it}^T \bar{p}_t^T$.

If $\bar{\mu}_{it}^T$ is bounded, we have $\lim_{T\to\infty} u_i'\left(\bar{c}_{it}^T\right) = 0$ which is impossible because $\bar{c}_{it}^T \leq A$ for every T.

Then, $\lim_{T\to\infty} \bar{\mu}_{it}^T = +\infty$. However, $\beta_i^t v_i'\left(\bar{\lambda}_{it}^T\right)/\bar{\mu}_{it}^T = \bar{w}_t^T + \bar{\varepsilon}_{it}^T/\bar{\mu}_{it}^T$ and $\lim_{T\to\infty} \bar{w}_t^T = \lim_{T\to\infty}\left(\bar{\theta}_{it}^T/\bar{\mu}_{it}^T\right) - \lim_{T\to\infty}\left(\bar{\varepsilon}_{it}^T/\bar{\mu}_{it}^T\right) = 0$ (Claim 20).

Since $\lim_{T\to\infty} \bar{p}_t^T = 0$, $\lim_{T\to\infty} \bar{w}_t^T = 0$ and $\bar{p}_t^T + \bar{w}_t^T + \bar{r}_t^T = 1$, we get $\lim_{T\to\infty} \bar{r}_t^T = 1$.

We know that $\bar{\mu}_{it-1}^T \bar{p}_{t-1}^T \geq \bar{\mu}_{it}^T \bar{p}_t^T (1-\delta) + \bar{\mu}_{it}^T \bar{r}_t^T \geq \bar{\mu}_{it}^T \bar{r}_t^T$ (point (7) of Claim 14). Then $\lim_{T\to\infty} \bar{\mu}_{it-1}^T \bar{p}_{t-1}^T \geq \lim_{T\to\infty} \bar{\mu}_{it}^T \bar{r}_t^T = +\infty$.

Similarly, $\bar{\mu}_{it-2}^T \bar{p}_{t-2}^T \geq \bar{\mu}_{it-1}^T \bar{p}_{t-1}^T (1-\delta) + \bar{\mu}_{it-1}^T \bar{r}_{t-1}^T \geq \bar{\mu}_{it-1}^T \bar{p}_{t-1}^T (1-\delta)$ and $\lim_{T\to\infty} \bar{\mu}_{it-2}^T \bar{p}_{t-2}^T \geq \lim_{T\to\infty} \bar{\mu}_{it-1}^T \bar{p}_{t-1}^T (1-\delta) = +\infty$.

Computing backward, we obtain $\lim_{T\to\infty} \bar{\mu}_{i0}^T \bar{p}_0^T = +\infty$.

If $\lim_{T\to\infty} \bar{p}_0^T > 0$, since $\bar{p}_0^T \leq 1$, then $\lim_{T\to\infty} \bar{\mu}_{i0}^T = +\infty$ and, since $\lim_{T\to\infty} \bar{\mu}_{i0}^T \bar{w}_0^T = \bar{\vartheta}_{i0} < +\infty$, this implies $\lim_{T\to\infty} \bar{w}_0^T = 0$. Thus,

$$0 = \bar{p}_0 F\left(K_0, \bar{L}_0\right) - \bar{r}_0 K_0 - \bar{w}_0 \bar{L}_0 = \bar{p}_0 F\left(K_0, \bar{L}_0\right) - \bar{r}_0 K_0$$

Choose $L_0 > \bar{L}_0$ in order to obtain a strictly higher profit and a contradiction with profit maximization.

Let $\lim_{T\to\infty} \bar{p}_0^T = 0$. We know that $u_i'(A) \leq u_i'\left(\bar{c}_{i0}^T\right) = \beta_i^0 u_i'\left(\bar{c}_{i0}^T\right) = \bar{\mu}_{i0}^T \bar{p}_0^T$.

If $\lim_{T\to\infty} \bar{\mu}_{i0}^T < +\infty$, we have $\lim_{T\to\infty} \bar{\mu}_{i0}^T \bar{p}_0^T = 0$ and $u_i'(A) \leq 0$, a contradiction.

If $\lim_{T\to\infty} \bar{\mu}_{i0}^T = +\infty$, then $\lim_{T\to\infty} \bar{\mu}_{i0}^T \bar{w}_0^T = \bar{\vartheta}_{i0} < +\infty$ gives $\lim_{T\to\infty} \bar{w}_0^T = 0$ and $\lim_{T\to\infty} \bar{r}_0^T = 1$. Focus on the first budget constraint:

$$\bar{p}_0^T \left[\bar{c}_{i0}^T + \bar{k}_{i1}^T - (1-\delta) k_{i0}\right] = \bar{r}_0^T k_{i0} + \bar{w}_0^T \left(1 - \bar{\lambda}_{i0}^T\right)$$

Assumption 3 ensures $k_{i0} > 0$. In this case, in the limit:

$$0 = \bar{p}_0 \left[\bar{c}_{i0} + \bar{k}_{i1} - (1-\delta) k_{i0}\right] = \bar{r}_0 k_{i0} + \bar{w}_0 \left(1 - \bar{\lambda}_{i0}\right) \geq k_{i0} > 0$$

a contradiction. Thus, for every t, $\bar{p}_t^T \to \bar{p}_t > 0$.

Focus now on $\bar{r}_t$ and $\bar{w}_t$. In the limit, $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t = 0$.

If $\bar{r}_t = 0$, then $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{w}_t \bar{L}_t = 0$. Fix $L_t > 0$ and choose K_t large enough such that $\bar{p}_t F(K_t, L_t) - \bar{w}_t L_t > 0$, against the equilibrium condition.

If $\bar{w}_t = 0$, then $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t = 0$. Fix $K_t > 0$ and choose L_t large enough such that $\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t > 0$, against the equilibrium condition.

Thus, $\bar{p}_t, \bar{r}_t, \bar{w}_t > 0$. ■

Claim 22 $\bar{c}_{it} = \lim_{T\to\infty} \bar{c}_{it}^T \in (0, +\infty)$.

Proof. For any t, $\sum_{i=1}^m \bar{c}_{it}^T \leq A$. This implies $\bar{c}_{it}^T \leq A$ independently on the choice of T and $\lim_{T\to\infty} \bar{c}_{it}^T \leq A < +\infty$. In addition, if $\bar{c}_{it} = \lim_{T\to\infty} \bar{c}_{it}^T = 0$, then, since $u_i'\left(\bar{c}_{it}^T\right) \bar{w}_t^T/\bar{p}_t^T \leq v_i'\left(\bar{\lambda}_{it}^T\right)$, we obtain $+\infty = \lim_{T\to\infty} u_i'\left(\bar{c}_{it}^T\right) \bar{w}_t^T/\bar{p}_t^T \leq \lim_{T\to\infty} v_i'\left(\bar{\lambda}_{it}^T\right)$, that is $\bar{\lambda}_{it} = \lim_{T\to\infty} \bar{\lambda}_{it}^T = 0$, a contradiction (see Claim 20). Then $\bar{c}_{it} > 0$. ■

Claim 23 *For any t, $\lim_{T\to\infty} \bar{K}_t^T = \bar{K}_t > 0$ and $\lim_{T\to\infty} \bar{L}_t^T = \bar{L}_t > 0$.*

Proof. We know that $\sum_{i=1}^m \bar{k}_{it+1} \geq 0$ and that $\sum_{i=1}^m \bar{c}_{it} + \sum_{i=1}^m \bar{k}_{it+1} = F\left(\bar{K}_t, \bar{L}_t\right) + (1-\delta)\bar{K}_t$. If $\bar{K}_t = 0$, then $\bar{c}_{it} = 0$ for every i, a contradiction. Now, if $\bar{L}_t = 0$, we have $\bar{r}_t \bar{K}_t = 0$ and hence $\bar{K}_t = 0$: a contradiction. ■

Claim 24 $\lim_{t\to+\infty} \bar{\mu}_{it}\bar{p}_t\bar{k}_{it+1} = 0$.

Proof. Let $\varepsilon > 0$. We know that there exists τ such that for any pair (s, s') such that $s' > s > \tau$ and for any $T > s$, we have $\sum_{t=s}^{s'} \bar{\zeta}_{it}^T < \varepsilon$ and $\sum_{t=s}^{s'} \bar{\vartheta}_{it}^T \left(1 - \bar{\lambda}_{it}^T\right) < \varepsilon$ for every i (inequality (27) and Claim 18). Taking the limit for $T \to +\infty$, we get also

$$\begin{aligned} \varepsilon &\geq \lim_{T\to+\infty} \sum_{t=s}^{s'} \bar{\zeta}_{it}^T = \sum_{t=s}^{s'} \lim_{T\to+\infty} \left[\beta_i^t u_i'\left(\bar{c}_{it}^T\right)\bar{c}_{it}^T\right] = \sum_{t=s}^{s'} \beta_i^t u_i'\left(\bar{c}_{it}\right)\bar{c}_{it} \\ &= \sum_{t=s}^{s'} \bar{\mu}_{it}\bar{p}_t\bar{c}_{it} \end{aligned}$$

(see Claim 22) and

$$\begin{aligned} \varepsilon &\geq \lim_{T\to+\infty} \sum_{t=s}^{s'} \bar{\vartheta}_{it}^T \left(1-\bar{\lambda}_{it}^T\right) = \sum_{t=s}^{s'} \lim_{T\to+\infty} \left(\bar{\mu}_{it}^T \bar{w}_t^T\right)\left(1 - \lim_{T\to+\infty} \bar{\lambda}_{it}^T\right) \\ &= \sum_{t=s}^{s'} \bar{\mu}_{it}\bar{w}_t\left(1-\bar{\lambda}_{it}\right) \end{aligned}$$

(see Claims 19 and 20). Since this holds for any $s' > s$, we get also

$$\sum_{t=s}^{\infty} \bar{\mu}_{it}\bar{p}_t\bar{c}_{it} \leq \varepsilon \text{ and } \sum_{t=s}^{\infty} \bar{\mu}_{it}\bar{w}_t\left(1-\bar{\lambda}_{it}\right) \leq \varepsilon \tag{30}$$

From the budget constraints, for any $s' \geq T$, we obtain

$$\begin{aligned} \varepsilon &> \sum_{t=s}^{s'} \bar{\mu}_{it}^T\bar{p}_t^T\bar{c}_{it}^T = \bar{\mu}_{is}^T\bar{p}_s^T(1-\delta)\bar{k}_{is}^T + \bar{\mu}_{is}^T\bar{r}_s^T\bar{k}_{is}^T + \sum_{t=s}^{s'} \bar{\vartheta}_{it}^T\left(1-\bar{\lambda}_{it}^T\right) \\ &\geq \bar{\mu}_{is}^T\bar{p}_s^T(1-\delta)\bar{k}_{is}^T + \bar{\mu}_{is}^T\bar{r}_s^T\bar{k}_{is}^T \end{aligned}$$

(see (25)). Taking the limit for $T \to +\infty$, we obtain

$$\bar{\mu}_{is}\bar{p}_s(1-\delta)\bar{k}_{is} \leq \varepsilon \text{ and } \bar{\mu}_{is}\bar{r}_s\bar{k}_{is} \leq \varepsilon$$

for every $s > \tau$. Thus, $\limsup_s \bar{\mu}_{is}\bar{p}_s(1-\delta)\bar{k}_{is} \leq \varepsilon$ and $\limsup_s \bar{\mu}_{is}\bar{r}_s\bar{k}_{is} \leq \varepsilon$. These inequalities hold for any $\varepsilon > 0$. Hence

$$\lim_{t\to+\infty} \bar{\mu}_{it}\bar{p}_t(1-\delta)\bar{k}_{it} = 0 \text{ and } \lim_{t\to+\infty} \bar{\mu}_{it}\bar{r}_t\bar{k}_{it} = 0 \tag{31}$$

Again, from the budget constraint, we have $\bar{\mu}_{it}^T \bar{p}_t^T \bar{k}_{it+1}^T = \bar{\mu}_{it}^T \bar{p}_t^T (1-\delta) \bar{k}_{it}^T + \bar{\mu}_{it}^T \bar{r}_t^T \bar{k}_{it}^T + \bar{\mu}_{it}^T \bar{w}_t^T \left(1-\bar{\lambda}_{it}^T\right) - \bar{\mu}_{it}^T \bar{p}_t^T \bar{c}_{it}^T$ (see (24)). Taking the limit for $T \to +\infty$, we obtain $\bar{\mu}_{it} \bar{p}_t \bar{k}_{it+1} = \bar{\mu}_{it} \bar{p}_t (1-\delta) \bar{k}_{it} + \bar{\mu}_{it} \bar{r}_t \bar{k}_{it} + \bar{\mu}_{it} \bar{w}_t \left(1-\bar{\lambda}_{it}\right) - \bar{\mu}_{it} \bar{p}_t \bar{c}_{it}$. We know that $\lim_{t \to +\infty} \bar{\mu}_{it} \bar{p}_t (1-\delta) \bar{k}_{it} = 0$ and $\lim_{t \to +\infty} \bar{\mu}_{it} \bar{r}_t \bar{k}_{it} = 0$ (see (31)). We know also that $\lim_{t \to +\infty} \bar{\mu}_{it} \bar{w}_t \left(1-\bar{\lambda}_{it}\right) = 0$ and $\lim_{t \to +\infty} \bar{\mu}_{it} \bar{p}_t \bar{c}_{it} = 0$ (see (30)). Therefore, $\lim_{t \to +\infty} \bar{\mu}_{it} \bar{p}_t \bar{k}_{it+1} = 0$. ■

Claim 25 $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{k}}, \bar{\mathbf{L}}\right)$ *is an equilibrium.*

Proof. Consider first the firm. For every truncated T-economy a zero profit condition holds: $\bar{p}_t^T F\left(\bar{K}_t^T, \bar{L}_t^T\right) - \bar{r}_t^T \bar{K}_t^T - \bar{w}_t^T \bar{L}_t^T = 0$. In the limit, for the infinite-horizon economy: $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t = 0$, because $\bar{p}_t^T \to \bar{p}_t \in (0,1)$, $\bar{r}_t^T \to \bar{r}_t \in (0,1)$, $\bar{w}_t \to \bar{w}_t \in (0,1)$, $\bar{K}_t^T = \sum_{i=1}^m \bar{k}_{it}^T \to \sum_{i=1}^m \bar{k}_{it} = \bar{K}_t < +\infty$, $\bar{L}_t^T = \sum_{i=1}^m \bar{l}_{it}^T \to \sum_{i=1}^m \bar{l}_{it} = \bar{L}_t < +\infty$. If $\left(\bar{K}_t, \bar{L}_t\right)$ does not maximize the profit in the infinite-horizon economy, then there exists (K_t, L_t) such that $\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t > \bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t = 0$ and, so, a critical τ, such that, for any $T > \tau$, $\bar{p}_t^T F(K_t, L_t) - \bar{r}_t^T K_t - \bar{w}_t^T L_t > \bar{p}_t^T F\left(\bar{K}_t^T, \bar{L}_t^T\right) - \bar{r}_t^T \bar{K}_t^T - \bar{w}_t^T \bar{L}_t^T = 0$ against the fact that $\left(\bar{K}_t^T, \bar{L}_t^T\right)$ maximizes the profit in the T-economy.

Focus on the consumer. Consider an alternative sequence $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)$ which satisfies the budget constraints and the Euler inequalities in the infinite-horizon economy. We have

$$\begin{aligned}
\Delta_T &\equiv \sum_{t=0}^T \beta_i^t \left[u_i(\bar{c}_{it}) + v_i\left(\bar{\lambda}_{it}\right)\right] - \sum_{t=0}^T \beta_i^t \left[u_i(c_{it}) + v_i(\lambda_{it})\right] \\
&= \sum_{t=0}^T \beta_i^t \left[u_i(\bar{c}_{it}) - u_i(c_{it})\right] + \sum_{t=0}^T \beta_i^t \left[v_i\left(\bar{\lambda}_{it}\right) - v_i(\lambda_{it})\right] \\
&\geq \sum_{t=0}^T \beta_i^t u_i'(\bar{c}_{it})(\bar{c}_{it} - c_{it}) + \sum_{t=0}^T \beta_i^t v_i'\left(\bar{\lambda}_{it}\right)\left(\bar{\lambda}_{it} - \lambda_{it}\right) \\
&\geq \sum_{t=0}^T \bar{\mu}_{it} \bar{p}_t (\bar{c}_{it} - c_{it}) + \sum_{t=0}^T \bar{\mu}_{it} \bar{w}_t \left(\bar{\lambda}_{it} - \lambda_{it}\right)
\end{aligned}$$

We observe that

$$\begin{aligned}
\bar{\mu}_{it} \bar{p}_t \bar{c}_{it} - \bar{\mu}_{it} \bar{w}_t \left(1 - \bar{\lambda}_{it}\right) &= \bar{\mu}_{it} \bar{r}_t \bar{k}_{it} + \bar{\mu}_{it} \bar{p}_t (1-\delta) \bar{k}_{it} - \bar{\mu}_{it} \bar{p}_t \bar{k}_{it+1} \\
\bar{\mu}_{it} \bar{p}_t c_{it} - \bar{\mu}_{it} \bar{w}_t (1 - \lambda_{it}) &\leq \bar{\mu}_{it} \bar{r}_t k_{it} + \bar{\mu}_{it} \bar{p}_t (1-\delta) k_{it} - \bar{\mu}_{it} \bar{p}_t k_{it+1}
\end{aligned}$$

where the first equality holds because of the Kuhn-Tucker method.

Subtracting member by member, we get

$$\begin{aligned}
&\bar{\mu}_{it} \bar{p}_t (\bar{c}_{it} - c_{it}) + \bar{\mu}_{it} \bar{w}_t \left(\bar{\lambda}_{it} - \lambda_{it}\right) \\
\geq\ & \left[\bar{\mu}_{it} \bar{r}_t \bar{k}_{it} + \bar{\mu}_{it} \bar{p}_t (1-\delta) \bar{k}_{it} - \bar{\mu}_{it} \bar{p}_t \bar{k}_{it+1}\right] \\
& - \left[\bar{\mu}_{it} \bar{r}_t k_{it} + \bar{\mu}_{it} \bar{p}_t (1-\delta) k_{it} - \bar{\mu}_{it} \bar{p}_t k_{it+1}\right]
\end{aligned}$$

Summing over t, we obtain

$$\begin{aligned} & \sum_{t=0}^{T} \bar{\mu}_{it}\bar{p}_t \left(\bar{c}_{it} - c_{it}\right) + \sum_{t=0}^{T} \bar{\mu}_{it}\bar{w}_t \left(\bar{\lambda}_{it} - \lambda_{it}\right) \\ \geq & \sum_{t=0}^{T} \left[\bar{\mu}_{it}\bar{p}_t \left(1-\delta\right) \bar{k}_{it} + \bar{\mu}_{it}\bar{r}_t \bar{k}_{it} - \bar{\mu}_{it}\bar{p}_t \bar{k}_{it+1}\right] \\ & - \sum_{t=0}^{T} \left[\bar{\mu}_{it}\bar{p}_t \left(1-\delta\right) k_{it} + \bar{\mu}_{it}\bar{r}_t k_{it} - \bar{\mu}_{it}\bar{p}_t k_{it+1}\right] \end{aligned}$$

We know also that

$$\begin{aligned} \left[\bar{\mu}_{it}\bar{p}_t - \bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta\right) - \bar{\mu}_{it+1}\bar{r}_{t+1}\right] \bar{k}_{it+1} &= 0 \\ \left[\bar{\mu}_{it}\bar{p}_t - \bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta\right) - \bar{\mu}_{it+1}\bar{r}_{t+1}\right] k_{it+1} &= 0 \end{aligned}$$

(point (7) in the Claim 14), that is

$$\begin{aligned} \bar{\mu}_{it}\bar{p}_t \left(1-\delta\right) \bar{k}_{it} + \bar{\mu}_{it}\bar{r}_t \bar{k}_{it} &= \bar{\mu}_{it-1}\bar{p}_{t-1}\bar{k}_{it} \\ \bar{\mu}_{it}\bar{p}_t \left(1-\delta\right) k_{it} + \bar{\mu}_{it}\bar{r}_t k_{it} &= \bar{\mu}_{it-1}\bar{p}_{t-1}k_{it} \end{aligned}$$

Then

$$\begin{aligned} & \sum_{t=0}^{T} \bar{\mu}_{it}\bar{p}_t \left(\bar{c}_{it} - c_{it}\right) + \sum_{t=0}^{T} \bar{\mu}_{it}\bar{w}_t \left(\bar{\lambda}_{it} - \lambda_{it}\right) \\ = & \sum_{t=0}^{T} \left(\bar{\mu}_{it-1}\bar{p}_{t-1}\bar{k}_{it} - \bar{\mu}_{it}\bar{p}_t \bar{k}_{it+1}\right) - \sum_{t=0}^{T} \left(\bar{\mu}_{it-1}\bar{p}_{t-1}k_{it} - \bar{\mu}_{it}\bar{p}_t k_{it+1}\right) \\ = & \bar{\mu}_{i,-1}\bar{p}_{-1}k_{i0} - \bar{\mu}_{iT}\bar{p}_T\bar{k}_{iT+1} - \left[\bar{\mu}_{i,-1}\bar{p}_{-1}k_{i0} - \bar{\mu}_{iT}\bar{p}_T k_{iT+1}\right] \\ = & -\bar{\mu}_{iT}\bar{p}_T\bar{k}_{iT+1} + \bar{\mu}_{iT}\bar{p}_T k_{iT+1} \\ \geq & -\bar{\mu}_{iT}\bar{p}_T\bar{k}_{iT+1} \end{aligned}$$

Thus $\lim_{T\to+\infty} \Delta_T \geq 0$ (Claim 24) and

$$\sum_{t=0}^{\infty} \beta_i^t \left[u_i\left(\bar{c}_{it}\right) + v_i\left(\bar{\lambda}_{it}\right)\right] \geq \sum_{t=0}^{\infty} \beta_i^t \left[u_i\left(c_{it}\right) + v_i\left(\lambda_{it}\right)\right]$$

Thus $\left(\bar{\mathbf{c}}_i, \bar{\boldsymbol{\lambda}}_i\right)$ maximizes the consumer's objective. ■

12 Appendix 3

Let $B\left(0,\rho\right) \equiv \left\{\mathbf{x} \in l^1 : \sum_{t=0}^{\infty} |x_t| \leq \rho\right\}$ be a ball of l^1.

Lemma 2 *Let K be a subset of $B\left(0,\rho\right)$, which satisfies the property: for any $\varepsilon > 0$, there exists τ such that for any $s > \tau$ and for any $\mathbf{x} \in K$, $\sum_{t=s}^{\infty} |x_t| \leq \varepsilon$. Then K is compact for the l^1-topology.*

Proof. Let $\left(\mathbf{y}^T\right)$ be a sequence in K. $B(0,\rho)$ is a compact set for the product topology and contains the sequence $\left(\mathbf{y}^T\right)$. Thus there exists a subsequence $\left(\mathbf{y}^{T_S}\right)$ which, for the product topology, converges to some $\mathbf{y} \in B(0,\rho)$.

For any $\varepsilon > 0$, there exists τ such that for any $s > \tau$ and for any S, $\sum_{t=s}^{\infty} \left|y_t^{T_S}\right| \leq \varepsilon$. Take now any $s' > s$. We have, for any s', for any $s > \tau$ and for any S, $\sum_{t=s}^{s'} \left|y_t^{T_S}\right| \leq \sum_{t=s}^{\infty} \left|y_t^{T_S}\right| \leq \varepsilon$. Convergence of $\left(\mathbf{y}^{T_S}\right)$ for the product topology implies, for any s' and for any $s > \tau$, $\sum_{t=s}^{s'} |y_t| \leq \varepsilon$. Thus we get, for any $s > \tau$, $\sum_{t=s}^{\infty} |y_t| \leq \varepsilon$. Then $\mathbf{y} \in K$ and K is compact for the l^1-topology. ■

13 Appendix 4

Let us prove Claim 6

Proof. For any truncated economy, the vector of equilibrium multipliers $\left(\bar{\mu}_{it}^T\right)_{t=0}^T$ exists and satisfies $\bar{\mu}_{it}^T = \beta_i^t u_i'\left(\bar{c}_{it}^T\right)/\bar{p}_t^T$ (see point (7) in Claim 14). Since $\lim_{T\to+\infty} \bar{c}_{it}^T = \bar{c}_{it} \in (0,+\infty)$ and $\lim_{T\to+\infty} \bar{p}_t^T = \bar{p}_t \in (0,+\infty)$, we obtain also

$$\bar{\mu}_{it} = \lim_{T\to+\infty} \frac{\beta_i^t u_i'\left(\bar{c}_{it}^T\right)}{\bar{p}_t^T} = \frac{\beta_i^t u_i'\left(\bar{c}_{it}\right)}{\bar{p}_t} \in (0,+\infty)$$

In addition, we obtain from point (7) in Claim 14:

$$\begin{aligned}
\bar{\mu}_{it}\bar{p}_t &= \lim_{T\to+\infty}\left(\bar{\mu}_{it}^T \bar{p}_t^T\right) \\
&\geq \lim_{T\to+\infty}\left(\bar{\mu}_{it+1}^T\left[\bar{p}_{t+1}^T(1-\delta)+\bar{r}_{t+1}^T\right]\right) = \bar{\mu}_{it+1}\left[\bar{p}_{t+1}(1-\delta)+\bar{r}_{t+1}\right]
\end{aligned}$$

because $\lim_{T\to+\infty} \bar{\mu}_{it}^T = \bar{\mu}_{it} \in (0,+\infty)$.

If $\lim_{T\to+\infty} \bar{k}_{it+1}^T = \bar{k}_{it+1} > 0$, then there exists S such that, for any $T > S$, $\bar{k}_{it+1}^T > 0$ and $\bar{\mu}_{it}^T \bar{p}_t^T = \bar{\mu}_{it+1}^T\left[\bar{p}_{t+1}^T(1-\delta)+\bar{r}_{t+1}^T\right]$. Thus, $\bar{\mu}_{it}\bar{p}_t = \bar{\mu}_{it+1}\left[\bar{p}_{t+1}(1-\delta)+\bar{r}_{t+1}\right]$ if $k_{it+1} > 0$.

Summing up, the FOC of point (7) in Claim 14 are satisfied in the limit economy:

$$\begin{aligned}
\bar{\mu}_{it}\bar{p}_t &= \beta_i^t u_i'\left(\bar{c}_{it}\right) \\
\bar{\mu}_{it}\bar{p}_t &\geq \bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta+\bar{\rho}_{t+1}\right) \\
\bar{\mu}_{it}\bar{p}_t &= \bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta+\bar{\rho}_{t+1}\right) \text{ if } \bar{k}_{it+1} > 0
\end{aligned}$$

where $\rho_t \equiv r_t/p_t$. ■

Let us prove now Proposition 2

Proof. Assume that such a sequence exists. Let $\omega_t \equiv w_t/p_t$. We have just to show that

$$\lim_{T\to+\infty}\left[\sum_{t=0}^{T} \bar{Q}_t\left(\bar{C}_t - C_t\right) + \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t\left(\sum_{i=1}^{m} \bar{\lambda}_{it} - \sum_{i=1}^{m} \lambda_{it}\right)\right] \geq 0$$

that is a contradiction.

It is enough to prove that feasibility and FOC imply

$$\left[\sum_{t=0}^{T} \bar{Q}_t \left(\bar{C}_t - C_t\right) + \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(\sum_{i=1}^{m} \bar{\lambda}_{it} - \sum_{i=1}^{m} \lambda_{it}\right)\right] \geq -\bar{Q}_T \bar{K}_{T+1} \tag{32}$$

Since capitals are uniformly bounded above, the result follows from $\lim_{T\to\infty} \bar{Q}_T = 0$.

Let us prove inequality (32). Using (8), we find

$$\begin{aligned}
\Delta_T \equiv\ & \sum_{t=0}^{T} \bar{Q}_t \left(\bar{C}_t - C_t\right) + \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(\sum_{i=1}^{m} \bar{\lambda}_{it} - \sum_{i=1}^{m} \lambda_{it}\right) \\
=\ & \sum_{t=0}^{T} \bar{Q}_t \left[F\left(\bar{K}_t, \bar{L}_t\right) + (1-\delta)\bar{K}_t - \bar{K}_{t+1}\right] \\
& - \sum_{t=0}^{T} \bar{Q}_t \left[F\left(K_t, L_t\right) + (1-\delta) K_t - K_{t+1}\right] \\
& + \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(m - \bar{L}_t\right) - \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(m - L_t\right) \\
=\ & \sum_{t=0}^{T} \bar{Q}_t \left[F\left(\bar{K}_t, \bar{L}_t\right) - F\left(K_t, L_t\right) + (1-\delta)\left(\bar{K}_t - K_t\right) - \left(\bar{K}_{t+1} - K_{t+1}\right)\right] \\
& - \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(\bar{L}_t - L_t\right) \\
\geq\ & \sum_{t=0}^{T} \bar{Q}_t \left[F_K\left(\bar{K}_t, \bar{L}_t\right)\left(\bar{K}_t - K_t\right) + F_L\left(\bar{K}_t, \bar{L}_t\right)\left(\bar{L}_t - L_t\right)\right] \\
& + (1-\delta)\sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_t - K_t\right) - \sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_{t+1} - K_{t+1}\right) \\
& - \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(\bar{L}_t - L_t\right) \\
=\ & \sum_{t=0}^{T} \bar{Q}_t \left[\bar{\rho}_t \left(\bar{K}_t - K_t\right) + \bar{\omega}_t \left(\bar{L}_t - L_t\right)\right] - \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(\bar{L}_t - L_t\right) \\
& + (1-\delta)\sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_t - K_t\right) - \sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_{t+1} - K_{t+1}\right) \\
=\ & \sum_{t=0}^{T} \bar{Q}_t \left(1 - \delta + \bar{\rho}_t\right)\left(\bar{K}_t - K_t\right) - \sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_{t+1} - K_{t+1}\right)
\end{aligned}$$

because $F_K\left(\bar{K}_t, \bar{L}_t\right) = \bar{\rho}_t$ and $F_L\left(\bar{K}_t, \bar{L}_t\right) = \bar{\omega}_t$ (see points (2) and (3) in Claim

14). Noticing that $\bar{K}_0 = K_0$ and replacing (7), we get:

$$\begin{aligned}
\Delta_T &\geq \bar{Q}_0 (1-\delta+\bar{\rho}_0)\left(\bar{K}_0 - K_0\right) + \sum_{t=1}^{T} \bar{Q}_t (1-\delta+\bar{\rho}_t)\left(\bar{K}_t - K_t\right) \\
&\quad - \sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_{t+1} - K_{t+1}\right) \\
&= \sum_{t=0}^{T-1} \bar{Q}_{t+1}\left(1-\delta+\bar{\rho}_{t+1}\right)\left(\bar{K}_{t+1} - K_{t+1}\right) \\
&\quad - \sum_{t=0}^{T-1} \bar{Q}_t \left(\bar{K}_{t+1} - K_{t+1}\right) - \bar{Q}_T \left(\bar{K}_{T+1} - K_{T+1}\right) \\
&\geq \sum_{t=0}^{T-1} \left[\bar{Q}_{t+1}\left(1-\delta+\bar{\rho}_{t+1}\right) - \bar{Q}_t\right]\left(\bar{K}_{t+1} - K_{t+1}\right) - \bar{Q}_T \bar{K}_{T+1} \\
&= -\bar{Q}_T \bar{K}_{T+1}
\end{aligned}$$

■

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