BIDDING RINGS UNDER COMPLETE INFORMATION: A BARGAINING APPROACH

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ABSTRACT. We address the issue of coalition formation in single and multi-unit Vickrey auctions with unit demand. We address this issue in a bargaining game set up under the assumption of complete information where valuation of participants is commonly known amongst themselves while the seller is unaware of these valuations. In the single goods case, we provide the necessary and sufficient conditions for formation of any bidding ring when the highest valuation agent proposes first. In the multiple goods case, we specify sufficient conditions for formation of an interesting class of coalition structures where exactly one winner colludes with all the losers and, depending on the protocol, the remaining winners either stay alone or collude in pairs.

1. INTRODUCTION

This paper analyses the formation of bidding rings in auctions in a complete information setting. The assumption of common knowledge of valuations among potential bidders is made to focus on the coalition formation problem, for which we use a non-cooperative sequential bargaining procedure. Incomplete information would introduce additional complications, which would detract from the main purpose of the paper.

We consider both the single good and multiple goods cases in which each potential buyer has unit demand. Unlike other papers (see below), we do not assume that the game ends when the first coalition forms. This is especially important in the multiple goods setting in which subsequent coalition formation will affect the payoffs to

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coalitions that have formed earlier. Thus it is important to endogenize the coalition structure in our setting.

Collusion in bidding is usually illegal and the agreements on what to bid cannot be enforced in one-shot settings without baseball-bat incentives. Thus, in our model, bidding behaviour must be a Nash equilibrium of the (simultaneous move) bidding game, which follows the sequential move coalition formation game. This leads us to consider only second-price/uniform price auctions, since collusion in first-price auctions can be sustained only through enforceable agreements on what to bid.

Coalition formation, in our model, is an outcome of a sequential bargaining game with irreversible commitments to coalition membership. (That is, coalitions once formed cannot add or subtract members.) This game determines the equilibrium coalition structure and the equilibrium payoff sharing rule for each member coalition of the coalition structure. We assume a non-strategic seller who simply sets the reservation price for the indivisible objects to be sold. We also rule out resale possibilities across bidders.

We focus on the asymptotic results as agents become sufficiently patient. We look for the subgame perfect equilibria in stationary strategies (SSPE), in the bargaining game. In the single goods case, we provide the necessary and sufficient conditions for formation of any bidding ring when the highest valuation agent proposes. In the multiple goods case, we specify sufficient conditions for formation of an interesting class of coalition structures where (a) exactly one winner (any one agent out of those who would win a good in the non-cooperative play) colludes with all the losers (the agents who would not win any of the goods in non-cooperative play) and, (b) depending on the protocol, the remaining winners either stay alone or collude in pairs. Thus, the conclusion that exactly one winner will collude with all the losers is independent of the order of players in the protocol. We present results for the single good and the multiple goods models separately, because of the qualitative difference that arises due to the externality in the latter case.

Relevant literature The sequential bargaining approach, pioneered by Rubinstein for two-player bargaining, is extended to coalition formation in characteristic function games by Chatterjee, Dutta, Ray and Sengupta [1]. We use their method of analysis in the single good bidding ring problem. Ray and Vohra [9] consider similar bargaining games but with externalities, where the entire coalition structure determines a coalition's payoff. However, our structure differs from Ray and Vohra [9] in one crucial aspect and that is the absence of binding agreements.They assume that any coalition can write binding agreements that specify (i) the strategies that each member plays at the underlying strategic game and (ii) the contingent payoffs that each member gets. This implies that the resultant strategy profile at the underlying strategic game, after the coalitions have formed, *is not* necessarily a Nash equilibrium. In our case, no such binding agreements can be enforced.

There is also the very different literature on internal mechanism design of a ring such as Graham and Marshall [3], Marshall and Marx [6], McAfee and McMillan [7], Mailath and Zemsky [5], Hendricks, Porter and Tan [4]. These papers analyze collusion in an ex-ante sense where, at the beginning of the ring formation process, the bidders are yet to know the valuations of their colluding partners. Usually, in these papers,once a coalition forms, the members play a direct mechanism within the coalition; those outside the given coalition, if any, act as individuals. The nature of the coalition formed is therefore not endogenously derived in this approach.

We note also that very few papers discuss collusion in multiple goods auctions, especially the payoff externalities across coalitions.

Coalition formation papers with incomplete information are rare. This field is relatively underdeveloped (Okada [8]).

The paper that comes closest to ours in approach, though it is still very different, is Cho, Jewell and Vohra [2]. They analyze coalitional bidding in first-price auctions of a single indivisible good with identical, budget constrained players. They, too, assume complete information among bidders and pre-auction bargaining. They find that unless the budget constraint is sufficiently acute, in the sense that budgets of all bidders put together is less than the common value for the good, the grand coalition forms. Note that they assume binding agreements on bidding behaviour, otherwise the caveat expressed earlier here about first-price auctions applies.

Section 2 states the general unified structure under which we analyze both the single and multiple good case. Sections 3.1 and 3.2 state the results for the single and the multiple goods case respectively. Section 4 states the conclusion.

2. MODEL

We consider the model of multi-unit auction with unit demand where $N = \{1, \ldots, n\}$ is the set of players and $k \in \{1, \ldots, n-1\}$ is the number of objects. Let $V = (V_1, V_2, \ldots, V_n)$ represent the valuation vector and s be the reservation price of the seller. We assume that V is common knowledge among bidders but the seller has no information about it. Let $v_l = \max\{V_l - s, 0\}, \forall\, l \in N$. Arrange the v values in a non-increasing order, and rename the players so that the first ranked player is now called 1, the second is called 2 and so on. Further, assume that the players have non-identical valuations.[1] Thus, we now have a vector $v = (v_1, \ldots, v_n)$ such that $v_1 > v_2 > \ldots > v_n > v_{n+1} = 0$. Let $K = \{1, \ldots, k\}$ denote the set of players who win a good at the non-cooperative play; henceforth, called *winners*. Similarly, let $L = \{k+1, \ldots, n\}$ denote the set of players who do not win a good at the non-cooperative play; henceforth, called *losers*. Hence, $N = K \cup L$ with $K \cap L = \emptyset$.

For each non-empty $S \subseteq N$, define the set of all possible partitions on S as $\Pi(S)$. Thus, each $\pi_S \in \Pi(S)$ is a collection of mutually exclusive and exhaustive subsets of S. Pick any $\pi_S \in \Pi(S)$ and define $L(\pi_S) := \{T \in \pi_S \mid T \cap K = \emptyset\}$ and $\bar{L}(\pi_S) :=$

[1] Allowing identical valuations would only lead to multiplicity of equilibria without adding to the qualitative analysis.

$\cup_{T \in L(\pi_S)} T$. Therefore, $\bar{L}(\pi_S)$ denotes the union of those members of π_S that do not contain any winner.

The pre-auction ring formation is captured by a Rubinstein [10] bargaining game $G \equiv (N, \bar{w}, p, \delta)$. The function $p : 2^N \mapsto N$ is the protocol function which assigns to each set of *active* players (players who are yet to form coalitions), a proposer from that set, who carries the game forward. Therefore, $p(T) \in T, \forall\ T \subseteq N$. $\delta \in (0,1)$ is the common discount factor, that is, any player receiving payoff x in period t gets a utility $\delta^{t-1}x$. A *stage* in G is given by a *substructure* (partition defined on a strict subset of N) constituting of the coalitions who have formed and left the game. The set of all possible stages is $\mathcal{P} := \cup_{S \subset N} \Pi(S)$, which is the set of all possible partitions of all possible strict subsets of N. Define $R(\pi) := N \setminus \{\cup_{T \in \pi} T\}$, $\forall\ \pi \in \mathcal{P}$. Therefore, $R(\pi)$ is the set of remaining (active) players after coalitions in substructure π have formed and left the game.

We assume that players follow stationary Markovian strategies which depend on a small set of state variables in a way that is insensitive to past history. In particular, they depend on the current set of active players, coalition (sub)structure that has already formed and, in case of response, the on-going proposal.

At any stage $\pi \in \mathcal{P}$, a player j must choose (i) a pair (T, z) with $T \subseteq R(\pi)$, $j \in T$ and $z \in \Re_+^{|T|}$; and (ii) an $a_j(\pi) \in \Re_+$. The choice of (i) signifies the proposal decision of j at stage π; where j proposes some subset T of the current set of active players $R(\pi)$ containing j and offers the members of T a payoff z.[2] The choice (ii) signifies the response decision of j at stage π; where j accepts any proposal (T', z') with $j \in T'$ only if $z'_j \geq a_j(\pi)$. At any stage, if a proposal gets rejected then all active players at that stage incur a utility loss due to delay of *one* period which is captured by the common discount factor $\delta \in (0,1)$. We make the following assumptions.

[2]In equilibrium, this payoff division z must exhaust the coalitional worth of T, no matter what the finally realized coalition structure is.

Assumption 2.1. *Any group of players agree to cooperate only if the payoff from cooperation exceeds that by staying alone.*

Assumption 2.2. *All players get* 0 *utility in case of perpetual disagreement.*

At any stage in G, we call a proposal *acceptable* if all the members of the coalition proposed, are offered a payoff no less than their respective acceptance thresholds of that stage. It, then, follows that the coalition mentioned in any acceptable proposal will form and leave the game. If an *unacceptable* proposal is made, it will be rejected by any one of the members at whom the proposal is directed. As mentioned before, this will cause a period of delay to all the players. In the next period, the *rejector will propose*.

For any coalition structure $\pi_N \in \Pi(N)$, we assume that any coalition $S \in \pi_N$ such that $S \cap K \neq \emptyset$, bids at the auction according to the following rule;

R: For any $i \in S$,

$$b_i = \begin{cases} v_i & \text{if } i \in K \\ 0 & \text{otherwise} \end{cases}$$

Note that given bidding rule **R**, for all members in $\bar{L}(\pi_N)$, the best response is to bid their true valuations.

Note that we could have endogenised the bid arrangements within a coalition, by requiring proposals in our (complete information) sequential offers bargaining game, to specify not only division of the realized coalitional worth (contingent upon final coalition structure), but also the bids that each member must submit at the auction. However, for any coalition S (containing at least one winner and one loser), bidding rule **R** generates the maximum possible worth, no matter what *substructure* $\pi \in \Pi(N \setminus S)$, the remaining $N \setminus S$ players organize themselves into and no matter how each coalition in π decides to bid at the auction. Therefore, at any stage of the bargaining game, any coalition containing winners and losers would find bidding rule **R** to be a weakly dominant strategy. Our assumption therefore, simply rules out multiple SSPE profile of strategies where the outcome coalition structure and the player specific payoffs are same, but the bid arrangements are different. This is in consonance with our primary objective of focusing on the coalition formation aspect of collusion.

In Ray and Vohra [9], the assumption of binding contracts ensures that no member of any coalition plays differently at the underlying game than the play agreed upon by the coalition (that is, cheating is ruled out). In our work, bidding rule **R** takes care of this issue, by ensuring that any deviation from the agreed upon strategy of a coalition, gives the same payoff as that resulting from any non-cooperative strategy. Therefore, bidding rule **R** implies that irrespective of the coalition structure formed, the bid profile at the underlying auction is always a Nash equilibrium. We also assume that the payoffs to the loser members of any coalition are paid by the winner members, *before* the auction begins. This eliminates the possibility of the winner members reneging on their commitments, post-auction, after the gains from cooperation have accrued to the winners

Using **R**, we define the following *partition function* which assigns a worth to each $S \in \pi_N, \forall\, \pi_N \in \Pi(N)$,

$$\bar{w}(S;\pi_N) = \sum_{j\in S\cap K} \left\{ v_j - \max_{l\in \bar{L}(\pi_N)} v_l \right\} \tag{2.1}$$

This function specifies the payoff to any member coalition S of any partition π_N of player set N.

The particular functional form of the partition function in this setting, follows from the desire of the winners to manipulate the price that they end up paying for the good. That is, winners want to collude with losers to persuade them to bid lower than their true valuations; and thereby, ensure procurement of the good at a lower price, in the auction. The extra payoff that accrues to the winner out of this enterprise, is used to compensate the losers suitably. Hence, any worthwhile collusive venture must involve at least one winner, while the losers that are not included in any such venture, cannot benefit by forming coalitions amongst themselves, and so, play non-cooperatively. So, for any $\pi_N \in \Pi(N)$, members of $\bar{L}(\pi_N)$ bid their true valuations. Therefore, the going price at the auction, when a coalition structure π_N has formed, turns out to be $\max\limits_{l \in \bar{L}(\pi_N)} v_l$. The coalitional worth of each coalition in π_N, then, is simply the sum of the payoffs of the winners in that coalition. The following example illuminates on this.

Example 2.3. *Consider $N = \{1,2,3\}$, $k = 2$. Therefore, $K = \{1,2\}$ and $L = \{3\}$. Then,*

$$\begin{array}{ll} \bar{w}(\{1\};\{1\},\{2,3\}) = v_1 & \bar{w}(\{2,3\};\{1\},\{2,3\}) = v_2 \\ \bar{w}(\{1,2\};\{1,2\},\{3\}) = v_1 + v_2 - 2v_3 & \bar{w}(\{3\};\{1,2\},\{3\}) = 0 \\ \bar{w}(\{1,3\};\{1,3\},\{2\}) = v_1 & \bar{w}(\{2\};\{1,3\},\{2\}) = v_2 \\ \bar{w}(\{1,2,3\};\{1,2,3\}) = v_1 + v_2 & \end{array}$$

while the non-cooperatively play payoffs are given as $\bar{w}(\{1\};\{1\},\{2\},\{3\}) = v_1 - v_3$, $\bar{w}(\{2\};\{1\},\{2\},\{3\}) = v_2 - v_3$ and $\bar{w}(\{3\};\{1\},\{2\},\{3\}) = 0$.

For all $i \in K$ and $S \subseteq [L \cup \{i\}]$, define $w^i(S)$ to be the maximum worth that coalition S can attain at a single good second price auction with the player set $L \cup \{i\}$. It is easy to see that

$$w^i(S) = \begin{cases} v_i - \max\limits_{l \in L \setminus S} v_l & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

Using this, we state an obvious property of the partition function given by (2.1) is the following;

Proposition 2.4. *For all $i \in K$ and non-empty $S \subseteq L$,*

$$\bar{w}(S \cup \{i\}; S \cup \{i\}, \pi_{L \setminus S}, \pi_{K \setminus \{i\}}) = w^i(S \cup \{i\}) > 0$$

for all $\pi_{L \setminus S} \in \Pi(L \setminus S)$ and all $\pi_{K \setminus \{i\}} \in \Pi(K \setminus \{i\})$.

Proof: It follows from the definition that $\forall\, \pi_{L \setminus S} \in \Pi(L \setminus S)$ and $\forall\, \pi_{K \setminus \{i\}} \in \Pi(K \setminus \{i\})$, $\bar{w}(S \cup \{i\}; S \cup \{i\}, \pi_{L \setminus S}, \pi_{K \setminus \{i\}}) = v_i - \max\limits_{l \in L \setminus S} v_l$. □

Also define $S(L) := \{S_k(m)\}_{m=k+1}^{m=n}$ where $S_k(k+1) := \{k+1\}$ and $S_k(r) := \{k+1,\dots,r\}$ for all integers $r = k+2,\dots n$. We call the partition of N having all singleton members, $\underline{\pi}$.

3. Results

3.1. Single Good. There is now single winner, and so, $K = \{1\}$ and $L = N \setminus \{1\}$. Note that the worth of partition function, now, reduces to $w^1(.)$, that is,

$$\bar{w}(S; S, \pi_{N\setminus S}) = w^1(S) = \begin{cases} v_1 - \max\limits_{l\in N\setminus S} v_l & \text{if } 1 \in S \\ 0 & \text{otherwise} \end{cases}$$

$\forall\, S \subseteq N, \forall\, \pi_{N\setminus S} \in \Pi(N \setminus S)$. This occurs because the worth of a coalition S no longer depends on the coalition structure.

Proposition 3.1. *For any $G = (N, w^1, p, \delta)$, if the SSPE outcome π^* is such that $\pi^* \neq \underline{\pi}$, then*

(i) $\exists\, S_1(m) \in S(L)$ *such that* $\{S_1(m) \cup \{1\}\} \in \pi^*$, *and*

(ii) $\forall : l \in L \setminus S_k(m), \{l\} \in \pi^*_N$

Proof: Suppose that the equilibrium coalition structure π^* is such that there exists $X \in \pi^*$ with the property that $1 \in X$ and $X \setminus \{1\} \notin S(L)$. Then there must exist an $m' > 2$ such that $m' \in X$ with $X \setminus \{1\} \subset S_1(m')$. Therefore, $w^1(X) = w^1(X \setminus \{m'\})$, that is, the marginal contribution of agent m' to the coalition X is zero. But, given Assumption 2.1, $X \in \pi^*$ implies that in equilibrium m' gets a positive payoff. This is clearly suboptimal and hence, contradiction. So any such X will never be formed in equilibrium. Thus (i) follows. Finally, (ii) follows from Assumption 2.1 that all other losers form singleton coalitions. □

It is important to observe that at any stage of the game $G = (N, w^1, p, \delta)$, *no active agent makes an unacceptable proposal*. The reason is provided in the next two paragraphs.

The $w^1(.)$ function implies that a coalition S generates positive payoff only if $S \ni 1$. Therefore at any stage where 1 is not active then, from Assumption 2.1, all the active agents stay alone. Consider any stage where 1 is active and the proposal power is with some $l \in L$. Agent l will never make an unacceptable proposal because any such proposal, given stationarity, does not change the stage of the game. It simply passes the power of proposal to some other active agent (because the rejector proposes in our bargaining game). This rejector can *either* make an acceptable proposal (which must contain 1 to have a positive worth) and leave the game; *or* propose unacceptably, in which case, the stage of the game remains unchanged even after two periods of delay. The latter possibility is undesirable to l as it causes delay without changing the stage of the game. The former possibility gives l zero payoff *if* l is not one of the members to whom an acceptable proposal is made. Even if l is a member of the said coalition, he always could have proposed the same thereby saving the cost of delay.

Now, consider the stage where 1 has the proposal power. Suppose agent 1 can get a payoff of x by making an acceptable proposal. As before, 1 observes that given stationarity, an unacceptable proposal will not change the stage of the game and will only pass the proposal power to some $l \in L$ in the present stage. By the previous argument l will never propose unacceptably. Moreover l will never leave the game alone (as it will give 0 payoff). So l must propose (and have accepted) a coalition containing 1. This can be done by offering at least δx to 1. Therefore, we see that an unacceptable proposal by 1 gives $\delta^2 x < x$. Hence making an acceptable proposal strictly dominates any unacceptable proposal.

Therefore, at any SSPE outcome, for any $G = (N, w^1, p, \delta)$, there is no delay.

Define $AV(i,l) = \frac{v_i - v_l}{l-k}$ where $i = 1, 2, \ldots, k$ and $l = k+1, k+2, \ldots, n$. Observe that for single good game $G = (N, w^1, p, \delta)$, $w^1(\{1\} \cup S_1(m)) = mAV(1, m+1)$ for all $m = 2, \ldots, n$. We refer to a coalition $\{1, 2, \ldots, r-1, r\}$ as an r-ring, for every $r \in N$. For any $T \subseteq N$, let $2^T := \{T' : T' \subseteq T\}$.

Proposition 3.2. *For any $G = (N, w^1, p, \delta)$ with $p(N) = 1$, $\exists\ \delta' \in (0,1)$ such that $\forall\ \delta \in (\delta', 1)$ the SSPE outcome is an r-ring without any delay, if and only if*

(1) $AV(1, r+1) \geq AV(1, t+1)$, $\forall\, t \in \{1, 2, \ldots, r-1\}$ *and*

(2) $AV(1, r+1) > AV(1, t+1)$, $\forall\, t \in \{r+1, r+2, \ldots, n\}$.

Proof:

Only If: Consider a stage π in the game such that $1 \in R(\pi)$. Since there can be no delay in equilibrium (since no active agent at any stage proposes unacceptably), the equilibrium acceptance threshold of any $i \in R(\pi)$ must be the one period discounted payoff that i can generate by making the equilibrium proposal, at the stage π itself. Therefore, for a given δ; from Proposition 3.1 it follows that the equilibrium acceptance thresholds $\{a_i^\delta(\pi)\}_{i \in R(\pi)}$ must satisfy the following equality[3]

$$\frac{a_i^\delta(\pi)}{\delta} = \max_{T \in [2^{R(\pi)} \cap S(L)], i \in T} \left\{ w(\{1\} \cup T) - \sum_{j \in [\{1\} \cup T] \setminus \{i\}} a_j^\delta(\pi) \right\}$$

$\forall\, i \neq 1$ and,

$$\frac{a_1^\delta(\pi)}{\delta} = \max_{T \in [2^{R(\pi)} \cap S(L)]} \left\{ w(\{1\} \cup T) - \sum_{j \in T} a_j^\delta(\pi) \right\}$$

¿From Chatterjee, Dutta, Ray and Sengupta [1], it follows that $\forall\, \pi \in \mathcal{P}$ with $1 \in R(\pi)$; the acceptance thresholds are obtained by the following recursion;

(i) $a_1^\delta(\pi) = \max_{T \in [2^{R(\pi)} \cap S(L)]} \frac{\delta w^1(\{1\} \cup T)}{1 + \delta |T|}$ and $a_i^\delta(\pi) = a_1^\delta(\pi)$ for all $i \in \bar{H}_1^\delta(\pi)$ where $\bar{H}_1^\delta(\pi) := \left[\cup_{T \in H_1^\delta(\pi)} T \right]$ with $H_1^\delta(\pi) := \operatorname{argmax}_{T \in [2^{R(\pi)} \cap S(L)]} \frac{\delta w^1(\{1\} \cup T)}{1 + \delta |T|}$.

(ii) Suppose $(\bar{H}_1^\delta, \bar{H}_2^\delta, \ldots, \bar{H}_q^\delta)$ is well defined. If $R(\pi) \setminus [\{1\} \cup \bar{H}_q^\delta(\pi)] \neq \emptyset$, then define

$$H_{q+1}^\delta(\pi) := \operatorname*{argmax}_{T \in [2^R(\pi) \cap S(L)], \bar{H}_q^\delta(\pi) \subset T} \frac{\delta \left\{ w^1(\{1\} \cup T) - \sum_{j \in \bar{H}_q^\delta(\pi)} a_j^\delta(\pi) - a_1^\delta(\pi) \right\}}{1 + \delta(|T| - |\bar{H}_q^\delta(\pi)| - 1)}$$

[3] In case the feasible set of maximizers in the following optimization problem is empty, $a_i^\delta(\pi) := 0$.

As before, $\bar{H}^{\delta}_{q+1}(\pi) := \left[\cup_{T \in H^{\delta}_{q+1}(\pi)} T\right]$. For all $i \in \bar{H}^{\delta}_{q+1}(\pi)$, $a^{\delta}_i(\pi)$ is the maximized value in the definition of $H^{\delta}_{q+1}(\pi)$.

Note that $\bar{H}^{\delta}_q(\pi) \subset H^{\delta}_q(\pi)$, $\forall\, q$ in the recursion above. This follows from the particular structure of the problem reflected in Proposition 3.1. The proposal decision at any stage π with $1 \in R(\pi)$ is as follows. Each $i \in R(\pi)$ must belong to some $\bar{H}^{\delta}_q(\pi)$ and therefore proposes any $M_i \cup \{1\}$ such that $M_i \in H^{\delta}_q(\pi)$ with $i \in M_i$.

Now, recall that any coalition not containing agent 1 has a zero worth. Therefore, at all other stages π' with $1 \notin R(\pi')$, all proposers propose singleton coalitions of themselves and $a^{\delta}_i(\pi') = 0, \forall\, i \in R(\pi')$.

Claim (a): $\exists\, \bar{\delta} \in (0,1)$ such that $\forall\, \delta \in (\bar{\delta}, 1)$, $H^{\delta}_1(\pi)$ contains *only* the largest member of

$$\underset{T \in [2^{R(\pi)} \cap S(L)]}{\operatorname{argmax}} \quad AV(1, |T| + 2)$$

for all $\pi \in \mathcal{P}$ with $1 \in R(\pi)$.

Proof: Define $H_1(\pi) := \underset{T \in [2^{R(\pi)} \cap S(L)]}{\operatorname{argmax}} \quad AV(1, |T|+2)$. Note that $\lim\limits_{\delta \to 1} \frac{\delta w^1(\{1\} \cup T)}{1+\delta|T|} = AV(1, |T| + 2)$. Therefore, it trivially follows that for values of δ sufficiently close to 1, $H^{\delta}_1(\pi) \subseteq H_1(\pi)$. Now suppose $\exists\, T, T' \in H_1(\pi)$. Therefore (A) $\frac{w^1(\{1\} \cup T)}{1+\delta|T|} - \frac{w^1(\{1\} \cup T')}{1+\delta|T'|} = 0$ with δ value fixed at 1. Given the structure of the game, it must be that $|T| \neq |T'|$; say $|T| > |T'|$. From (A), $|T| > |T'|$ implies that $w^1(\{1\} \cup T) > w^1(\{1\} \cup T')$ (because we use non-identical valuations). Also, for a "slight" fall in δ value; in the left hand side of (A), the denominator of the first term decreases by more than the second term (since $|T| > |T'|$). Hence, the 'equals to' sign in (A), changes to 'greater than' for δ values sufficiently close to 1. Therefore, proposal choice of the largest coalition in $H_1(\pi)$ dominates that of the other members of $H_1(\pi)$, for δ sufficiently close to 1. Hence, proved.

Since $p(N) = 1$, an r-ring is formed only if agent 1 proposes $\{1, 2, \ldots, r\}$ acceptably on the SSPE path. This will happen only if $S_1(r) \cup \{1\}$ is the largest coalition in $H^{\delta}_1(\emptyset)$, that is, $S_1(r) \cup \{1\}$ is the largest average worth maximizing coalition. This implies that $AV(1, r+1) \geq AV(1, t+1), \forall\, t < r$ and $AV(1, r+1) > AV(1, t+1), \forall\, t > r$. These two conditions imply results (1) and (2) respectively. □

If: Define the following strategy $\mathbf{\Sigma}$ in game G:

- At any stage π with $1 \notin R(\pi)$, all proposers choose to stay alone, and set an acceptance threshold of 0.
- Recall that for any stage π with $1 \in R(\pi)$, $H_1(\pi) := \underset{T \in [2^{R(\pi)} \cap S(L)]}{\operatorname{argmax}} AV(1, |T|+2)$. For all such π, let $\bar{H}_1(\pi)$ be the largest coalition in $H_1(\pi)$. Then, at any stage π with $1 \in R(\pi)$, all $i \in [\bar{H}_1(\pi) \cup \{1\}]$ propose $[\bar{H}_1(\pi) \cup \{1\}]$ and set their acceptance thresholds to be $\frac{\delta w^1(\bar{H}_1(\pi) \cup \{1\})}{1+\delta|\bar{H}_1(\pi)|}$. If the sequence $(\bar{H}_1, \bar{H}_2, \ldots, \bar{H}_q)$ is well defined and $R(\pi) \setminus [\bar{H}_q(\pi) \cup \{1\}] \neq \emptyset$; then
$$H_{q+1}(\pi) := \underset{T \in [2^{R(\pi)} \cap S(L)], \bar{H}_q(\pi) \subset T}{\operatorname{argmax}} \frac{w^1(\{1\} \cup T) - w^1(\{1\} \cup \bar{H}_q(\pi))}{|T| - |\bar{H}_q(\pi)|}$$
with $\bar{H}_{q+1}(\pi)$ is defined as before to be the largest coalition in $H_{q+1}(\pi)$. Then all $j \in [\bar{H}_{q+1}(\pi) \cup \{1\}]$ propose $[\bar{H}_{q+1}(\pi) \cup \{1\}]$ and set their acceptance thresholds to be
$$\frac{\delta w^1(\bar{H}_{q+1}(\pi) \cup \{1\}) - \delta w^1(\bar{H}_q(\pi) \cup \{1\})}{1 + \delta(|\bar{H}_{q+1}(\pi)| - |\bar{H}_q(\pi)| - 1)}$$

It can easily be seen that the recursion in strategy $\mathbf{\Sigma}$ is simply the limit version of the recursion given by (i) and (ii) in the proof of necessity. Then, arguing as in Claim **(a)**, for each round q of this recursion; we see that for δ values very close to 1, $\mathbf{\Sigma}$ is SSPE. So we can find a $\delta' \in (\bar{\delta}, 1)$ such that $\forall\, \delta \in (\delta', 1)$, $\mathbf{\Sigma}$ is SSPE. Then from conditions (1) and (2) in the statement of the theorem it follows that; when $p(N) = 1$, strategy $\mathbf{\Sigma}$ will lead to formation of an r-ring. Thus, the sufficiency is established. □

In Proposition 3.2 we assumed a specific protocol function where $p(N) = 1$. What happens if $p(N) \neq 1$ is explained informally using the following example.

Example 3.3. *Suppose $N = \{1, 2, 3\}$ and $K = \{1\}$ where $v \equiv (v_1 = 70, v_2 = 65, v_3 = 20)$. Note that $AV(1, 2) < AV(1, 3) > AV(1, 4)$. Invoking the strategy $\mathbf{\Sigma}$ in the sufficiency proof of the Proposition 3.2, at the stage $\emptyset$ (that is, at the beginning of the game), we see that $\bar{H}_1(\emptyset) = \bar{H}_2(\emptyset) = \{2\}$ and $\bar{H}_3(\emptyset) = \{2, 3\}$. It can be shown that $\forall\, \delta \in \left(\frac{2}{3}, 1\right)$, agents 1 and 2 propose $\{1, 2\}$, while agent 3 proposes $\{1, 2, 3\}$ at stage $\emptyset$. Therefore,*

-
- *if $p(N) \in \{1, 2\}$ then the outcome coalition structure is $\{\{1, 2\}, \{3\}\}$, that is, the 2-ring forms.*
- *if $p(N) = 3$ then the outcome coalition structure is $\{\{1, 2, 3\}\}$, that is, the 3-ring forms.*

3.2. **Multiple Goods.** Consider the subgames with the set of active agents as T such that $L \subseteq T$. For all such subgames, the substructure formed by the departed agents (who have formed coalitions and left the game) does not affect the worth of any coalitions that remaining agents may form in future. That is, at such a stage with active player set T with $L \subseteq T$; $\bar{w}(S; \pi_{N\setminus T}, S, \hat{\pi}_{T\setminus S}) = \bar{w}(S; \pi'_{N\setminus T}, S, \hat{\pi}_{T\setminus S})$, $\forall\, \pi_{N\setminus T}, \pi'_{N\setminus T} \in \Pi(N \setminus T)$, $\forall\, S \subseteq T$, $\forall\, \hat{\pi}_{T\setminus S} \in \Pi(T \setminus S)$. At these subgames, we refer to the *stage* in the game by the set of *active* agents, instead of the substructure consisting of coalitions who have (formed and) left the game.

At any such stage T (with $L \subseteq T$), define $C_i^{\delta}(T)$ to be the set of best acceptable proposals (only the coalitions) that agent i can make at that stage. Also define $T^k := \{k\} \cup L, \forall\, k \in K$.

Proposition 3.4. *For any $i, j \in K$ such that $v_i > v_j$, $\exists\, \delta' \in (0, 1)$ such that $\forall\, \delta \in (\delta', 1)$; if $S_k(m) \cup \{i\} \in C_i^{\delta}(T^i)$ then $\exists\, m' \geq m$ such that $S_k(m'^{\delta}_j(T^j)$.*

Proof: Note that at any stage T^i, *the subgame becomes equivalent to a single good auction where the only winner is agent i.* This is because the worth of any subset of T^i, irrespective of the substructure formed amongst the agents who have departed from the game, is given by the $w^i(.)$ function. Hence, we can invoke the Proposition 3.1(i) and infer that $\forall\, i \in K$, if $X \in C_i^\delta(T^i)$ and $X \neq \{i\}$ then $X \setminus \{i\} \in S(L), \forall\, \delta \in (0,1)$.

Now, from the *continuity* of the objective functions in the maximization programs of (i) and (ii) in the necessity proof of Proposition 3.2, it follows that for δ sufficiently close to 1, any agent $i \in K$ proposes acceptably the average worth maximizing coalition (containing i) at stage T^i. So, for δ sufficiently close to 1, $S_k(m) \cup \{i\} \in C_i^\delta(T^i)$ implies that $S_k(m) \cup \{i\}$ is the average worth maximizing coalition among all subsets of T^i. Therefore, $\frac{v_i - v_{m+1}}{|S_k(m)|+1} \geq \frac{v_i - v_{m-l+1}}{|S_k(m-l)|+1}$ for all $l = 0, 1, \ldots, m-k-1$. It is easy to check that this in turn implies that $\frac{v_j - v_{m+1}}{|S_k(m)|+1} \geq \frac{v_j - v_{m-l+1}}{|S_k(m-l)|+1}$ for all $l = 0, 1, \ldots, m-k-1$ when $v_j < v_i$. Now, for suitably high δ, any $j \in \{i+1, \ldots, k\}$ must also choose the average worth maximizing coalition containing j among the subsets of T^j. Hence, it follows that; for a sufficiently high δ (that is, $\exists$ some $\delta' \in (0,1)$ such that $\forall\, \delta \in (\delta', 1)$), there exists an $m' \geq m$ with $S_k(m_j'^{\delta}(T^j)$. □

Proposition 3.4 states that when δ is sufficiently high; if winners i and j separately find themselves at a stage where the remaining set of agents are T^i and T^j respectively and if i picks a set $S_k(m) \cup \{i\}$ as a best acceptable proposal then there exists an $m' \in \{m, \ldots, n\}$ such that $S_k(m') \cup \{j\}$ is a best acceptable proposal for j, whenever $v_i > v_j$.

Remark 3.5. *It also follows from Proposition 3.4 that at the stage T^k (for any $k \in K$); the game $G = (N, \bar{w}, p, \delta)$ reduces to a single good/single winner bargaining game $G^k = (T^k, w^k, p^k, \delta)$ where $p^k(.)$ is the restriction of the original protocol function $p(.)$ to the set 2^{T^k}. As mentioned earlier, in the bargaining game with single winner, at any stage, no active agent makes an unacceptable proposal. Therefore, $C_i^\delta(T^k)$ is the set of coalitions that agent i proposes in equilibrium at stage T^k, in game G; $\forall\, i \in T^k, \forall\, k \in K$.*

Define $C_i^*(T^k)$ to be the set of coalitions that any $i \in T^k$ proposes in equilibrium at any stage $T^k, k \in K$; *as δ goes to* 1 *in limit*. From the arguments in proof of Proposition 3.4, we see that at any single winner stage T^k, in limit, the winner k chooses the average worth maximizing coalition containing itself. That is, $C_k^*(T^k) = \underset{S \in S(L)}{\operatorname{argmax}} \frac{w^k(S \cup \{k\})}{1+|S|}, \forall\, k \in K$.

The following proposition states that if the winner 1 finds it optimal to collude with all the losers at stage T^1 in limit (that is, $T^1 \in C_1^*(T^1)$); then, irrespective of the value of δ, the *optimal* proposal of all winners i other than 1, at stage T^i, *can only be* T^i itself (that is, $C_i^\delta(T^i) = \{T^i\}$).

Proposition 3.6. *If* $S_k(n) \in \underset{S \in S(L)}{\operatorname{argmax}} \frac{w^1(S \cup \{1\})}{|S|+1}$ *then* $S_k(n) = \underset{S \in S(L)}{\operatorname{argmax}} \frac{w^i(S \cup \{i\})}{\delta|S|+1}$*, for all* $\delta \in (0,1)$ *and all* $i \in K$.

Proof: Since $[S_k(n) \cup \{1\}] = T^1 \in C_1^*(T^1)$ and $v_i < v_1, \forall\, i \in K \setminus \{1\}$; as in the previous proposition we can say that $\frac{w^i(S_k(n) \cup \{i\})}{|S_k(n)|+1} \geq \frac{w^i(S_k(m) \cup \{i\})}{|S_k(m)|+1}$ for all $S_k(m) \in S(L)$ and for all $i \in K$. For any $S_k(m) \in S(L) \setminus \{S_k(n)\}, \forall\, i \in K$, define the function $d_i(S_k(m), \delta) = \frac{w^i(S_k(n) \cup \{i\})}{\delta|S_k(n)|+1} - \frac{w^i(S_k(m) \cup \{i\})}{\delta|S_k(m)|+1}$. By applying proof by contradiction it is easy to prove that $d_i(S_k(m), \delta) > 0$ for all $\delta \in (0,1)$; $\forall\, m \neq n$ and $\forall\, i \in K$. Hence it follows that $S_k(n) = \underset{S \in S(L)}{\operatorname{argmax}} \frac{w^i(S \cup \{i\})}{\delta|S|+1}$, for all $i \in K$. This proves our result. □

We, now, design a recursion that will be used in the theorem to follow. For this we call the agent with the highest (the lowest) valuation in any set $T \subseteq N$ as $\bar{m}^T$ (as $\underline{m}^T$).[4] This recursion is used to optimize the proposal decision of any loser at any stage $T \cup L$ such that $T \subseteq K$. The recursion, essentially, generates the final coalition structure (for a given protocol function) subject to the choice of a set of winners (one winner from each possible stage $T' \cup L$ where $T' \subseteq T \subseteq K$). This choice is done under the assumption that at each such stage $T' \cup L$; if any winner $j \in T'$ gets to propose, he must propose $\{\bar{m}^{T'}\}$ if $j = \bar{m}^{T'}$ and $\{\bar{m}^{T'}, j\}$ otherwise.

Recursion (*): For any $T \subseteq K$, define $\mathbf{b}(T; p(.)) \equiv \{b(T'; p(.))\}_{T' \subseteq T}$ to be a sequence of members of T such that (i) $b(T'; p(.)) = \underline{m}^{T'}$ if $|T'| = 2$ and (ii) $b(T'; p(.)) \in T'$ if $|T'| \neq 2$. To simplify the notations, henceforth we ignore the argument for the protocol

[4]Given the non-identical valuations, for any set T, the agents $\bar{m}^T$ and $\underline{m}^T$ are well defined.

function when writing the $b(.)$ expression. For any such $\mathbf{b}(T;p)$ define the sequence of sets $\{B_t\}_{t=1}^{h}$ such that

- $\{B_t\}_{t=1}^{h}$ is a partition of $T \cup L$.
- $B_1 = \begin{cases} \{\bar{m}^T\} & \text{if } b(T;p) = \bar{m}^T \\ \{\bar{m}^T, b(T;p))\} & \text{otherwise} \end{cases}$
- Suppose the sequence of sets $(B_1, B_2, \dots, B_{q-1})$ is well defined. Then, define $D_1 := T \cup L$ and $D_q := [T \cup L] \setminus [\cup_{t=1}^{q-1} B_t], \forall\, q > 1$.

$$B_q = \begin{cases} D_q & \text{if } |D_q \cap K| = 1 \\ \begin{array}{ll} \{\underline{m}^{D_q}\} & \text{if } p(D_q) \neq \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}\} & \text{otherwise} \end{array} & \text{if } |D_q \cap K| = 2 \\ \begin{array}{ll} \{\bar{m}^{D_q \cap K}\} & \text{if } p(D_q) = \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}\} & \text{if } p(D_q) \in L \text{ and } b(D_q \cap K) = \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}, b(D_q \cap K; p)\} & \text{if } p(D_q) \in L \text{ and } b(D_q \cap K; p) \neq \bar{m}^{D_q \cap K} \\ \{\bar{m}^{D_q \cap K}, p(D_q)\} & \text{otherwise} \end{array} & \text{if } |D_q \cap K| > 2 \end{cases}$$

- $B_h = T^{j(\mathbf{b}(T;p))}$ for some $j(\mathbf{b}(T;p)) \in T$.

The last term of the recursion B_h should be a set of all the losers and any one winner j from T. The identity of this winner would depend on the choice of the sequence $\mathbf{b}(T;p)$. That is, for any choice of sequence $\mathbf{b}(T;p)$ we would get a $j^{\mathbf{b}(T;p)} \in T$ such that $B_h = T^{j(\mathbf{b}(T;p)}$. Define $\mathbf{b}^*(T;p)$ to be that sequence of winners that maximizes the (valuation of) agent $j^{\mathbf{b}(T;p)}$ and let $k^*(T;p) := b^*(T;p)$.[5]

Theorem 3.7. *For any $G = (N, \bar{w}, p, \delta)$ if $T^1 \in C_1^*(T^1)$ then there exists $\delta' \in (0,1)$ such that for all $\delta \in (\delta', 1)$, the SSPE strategies of G are such that $\forall\, T \subseteq K$ we have the following:*

(1) *if $|T| = 1$ then $C_t^{\delta}(T \cup L) = T \cup L$ for all $t \in T \cup L$,*

[5]It may so happen that we have multiple $k^*(T;.)$ for a given protocol function. In that case, we choose any one.

(2) *if* $|T| = 2$ *then* $C^{\delta}_t(T \cup L) = \{t\}$ *if* $t \in T$ *and any loser* $t \in L$ *proposes unacceptably to* $\underline{m}^T$ *where* $\underline{m}^T = \operatorname*{argmin}_{j \in T} v_j$, *and*

(3) *if* $|T| > 2$ *then*

$$C^{\delta}_t(T \cup L) = \begin{cases} \{\overline{m}^T, t\} & \text{if } t \in T \setminus \{\overline{m}^T\} \\ \{t\} & \text{if } t = \overline{m}^T \\ \end{cases}$$

where $\overline{m}^T = \operatorname*{argmax}_{j \in T} v_j$ *and any* $t \in L$ *proposes unacceptably to* $k^*(T; p)$ *where* $k^*(T; p)$ *follows from Recursion (*).*

Proof: Pick any $i \in K$ and consider the stage T^i. At this stage the only winner i and all the losers are active. From Propositions 3.2 and 3.6 it follows that if $T^1 \in C^*_1(T^1)$ then $\exists\ \delta(i) \in (0, 1)$ such that $\forall\, \delta \in (\delta(i), 1)$; $C^{\delta}_l(T^i) = \{T^i\}$, $\forall\, l \in T^i$. Define $\delta(1) := \max\{\delta(i)\}_{i \in K}$. Therefore, $\forall\, \delta \in (\delta(1), 1)$, $C^{\delta}_l(T^i) = \{T^i\}$, $\forall\, l \in T^i$, $\forall\, i \in K$; and thus result (1) follows.

Consider any stage $T' = \{i, j\} \cup L$, for any $i, j \in K$. Pick any $\delta \in (\delta(1), 1)$. Then $C^{\delta}_t(T^j) = \{T^j\}$, $\forall\, t \in T^j$and $C^{\delta}_t(T^i) = \{T^i\}$, $\forall\, t \in T^i$. W.l.o.g. assume $v_i > v_j$. If i has the proposal power then the first possibility is that he chooses to stay alone, so that in the next stage with T^j agents, the coalition T^j forms (since $C^{\delta}_l(T^j) = \{T^j\}$ for all $l \in T^j$) and i gets a payoff of v_i. The remaining possibilities do not give agent i any more than $\frac{v_i + v_j}{1+\delta}$.[6] For all $\delta \in \left(\frac{v_j}{v_i}, 1\right)$, $v_i > \frac{v_i + v_j}{1+\delta}$ and so agent i will find it optimal to stay alone. Hence $\forall\, \delta \in \left(\max\left\{\delta(1), \frac{v_j}{v_i}\right\}, 1\right)$, agent i stays alone (that is, $C^{\delta}_i(T') = \{\{i\}\}$). Pick any $\delta \in \left(\max\left\{\delta(1), \frac{v_j}{v_i}\right\}, 1\right)$. As before, if j has the proposal power and he chooses to stay alone then he gets v_j. Otherwise, knowing that winner i can reject any proposal and get a payoff of δv_i, the best agent j can achieve, by proposing some coalition that includes

[6]Note that this payoff is the outcome of two member bargaining over $v_i + v_j$. Such a payoff will never materialize if both winners form a coalition and exit the game (because the losers have not colluded with any winner and so will bid their true valuations leading to a price equal to the third highest valuation).

i, is no more than $\frac{v_j+(1-\delta)v_i}{1+\delta}$. Also any non-singleton coalition excluding i gives j less than $\frac{v_j+(1-\delta)v_i}{1+\delta}$. There is also the possibility that agent j proposes $\{i,j\}$ acceptably to get $(1-\delta)v_i + v_j - 2v_{k+1}$. Note that if $v_j \leq 4v_{k+1}$ then $\exists\ \bar{\bar{\delta}} \in (0,1)$ such that $\forall\ \delta \in (\bar{\bar{\delta}}, 1)$, $\frac{v_j+(1-\delta)v_i}{1+\delta} > (1-\delta)v_i + v_j - 2v_{k+1}$. If $v_j > 4v_{k+1}$ then $\exists\ \underline{\underline{\delta}} \in (0,1)$ such that $\forall\ \delta \in (\underline{\underline{\delta}}, 1)$, $\frac{v_j+(1-\delta)v_i}{1+\delta} < (1-\delta)v_i + v_j - 2v_{k+1}$. Let $\tilde{\bar{\delta}} := \max\left\{\delta(1), \bar{\bar{\delta}}, \underline{\underline{\delta}}, \frac{v_i-2v_{k+1}}{v_i}, \frac{v_j}{v_i}, \frac{v_i}{v_i+v_j}\right\}$. Therefore $\forall\ \delta \in \left(\tilde{\bar{\delta}}, 1\right)$, $C_t(T') = \{\{t\}\}, \forall\ t = i,j$.

We now consider the possible proposals of any loser for $\delta \in \left(\tilde{\bar{\delta}}, 1\right)$. If any loser $l \in L$ has the proposal power, then he has two choices, (i) to make an acceptable proposal and (ii) to make an unacceptable proposal. If he chooses the former, then $C_l^{\delta}(T') \subset \{\{i,j\} \cup S_k(t)\}_{t=l}^{n}$ because, given $S_k(t)$, it is better to take both winners instead of one. For each $t \in \{l, l+1, \ldots, n\}$, the loser can attain a payoff of $\frac{(1-\delta)(v_i+v_j)-2v_{t+1}}{1+\delta(t-k-1)}$. If $\delta \in \left(1-\frac{2v_n}{v_i+v_j}, 1\right)$ then the maximum attainable payoff is $\frac{(1-\delta)(v_i+v_j)}{1+\delta(n-k-1)}$, resulting from a proposal $T' = \{i,j\} \cup L$. If agent l makes an unacceptable proposal, it may either be directed at a winner or a loser. If it is directed at a winner, the winner (say i) would get the proposer power in the next period and given our restriction on δ, would stay alone and exit the game. This would drive the game to the stage T^j where, as mentioned above, the coalition T^j would form giving l a payoff $\frac{\delta v_j}{1+\delta(n-k)}$ in *the next period*. Observe that given $v_i > v_j$, the loser will never unacceptably propose to i, because he could do better by unacceptably proposing to j and getting a payoff $\frac{\delta v_i}{1+\delta(n-k)}$ in the next period. If the unacceptable proposal is directed at a loser l', the stage of the game would not change, there would be a period of delay, and in the next period the proposal power would be with loser l' who faces the same options as l with one period delay. Thus unacceptable proposal directed to a loser is suboptimal. Thus given $\delta \in \left(1-\frac{2v_n}{v_i+v_j}, 1\right)$, loser l has two options. Either propose T' acceptably and get a payoff $\frac{(1-\delta)(v_i+v_j)}{1+\delta(n-k-1)}$ or propose unacceptably to j and get one-period discounted payoff $\frac{\delta^2 v_i}{1+\delta(n-k)}$. Define $F(\delta) := \frac{\delta^2 v_i}{1+\delta(n-k)} - \frac{(1-\delta)(v_i+v_j)}{1+\delta(n-k-1)}$. Note that $F(\delta)$ is strictly increasing and continuous in δ and $\lim_{\delta\to 1} F(\delta) = \frac{v_i}{n-k+1} > 0$. Therefore $\exists\ \bar{\delta} \in (0,1)$

such that $\forall\, \delta \in (\bar{\delta}, 1)$; $F(\delta) > 0$, and so, given the restriction on δ, making unacceptable proposal strictly dominates making acceptable proposal for the loser l. Define $\delta(i,j) := \max\left\{\tilde{\tilde{\delta}}, 1 - \frac{2v_n}{v_i+v_j}, \bar{\delta}\right\}$. So $\forall\, \delta \in (\delta(i,j), 1)$, $C_t^{\delta}(T') = \{\{t\}\}$, $\forall\, t = i, j$ and any loser proposing at stage T' unacceptably proposes to j (the lower valuation winner). Hence, for all $\delta \in (\delta(2), 1)$ result(2) follows where $\delta(2) := \max\{\delta(i,j)\}_{i,j \in N, i \neq j}$.

Suppose that at the stage $T'' \cup L$ with $T'' \subset K$ and $2 \leq |T''| \leq m-1$ result (3) holds $\forall\, \delta \in (\delta^{m-1}, 1)$. Then consider the stage $T \cup L$ where $|T| = m$. Define the winners $\{j_t\}_{t=1}^{m}$ in T, where $j_1 = \bar{m}^T$ and $j_t = \bar{m}^{T \setminus \{j_1, j_2, \ldots, j_{t-1}\}}$. Fix a $\delta \in (\max\{\delta(2), \delta^{m-1}\}, 1)$. The following STEPS 1 and 2 describe the proposal choice of j_1 and the winners other than j_1, respectively; *when they propose acceptably* at stage $T \cup L$. STEP 3 establishes that no winner in T *proposes unacceptably* at stage $T \cup L$. Finally, STEP 4 describes the proposal choice of the losers at stage $T \cup L$.

STEP 1: Pick the agent $j_1 \in T$ ($j_1 = \bar{m}^T$). Strict inequality guarantees that j_1 is well defined. Now, by staying alone j_1 can get at least $\delta^{m-2} v_{j_1}$. This is because, from our hypothesis (and the specified range of δ) it follows that at all the later stages (consequent to j_1 staying alone) other than the single winner stage; *only* the winners make acceptable proposals, and all these acceptable proposals are either directed at themselves (that is, they stay alone) or at exactly one active winner (that is, forming a two agent coalition). This implies that after j_1 has stayed alone, the game must arrive at a single winner stage. From Proposition 3.2, 3.4 and 3.6; it follows that given $T^1 \in C_1^*(T^1)$, all the active agents in this single winner stage collude amongst themselves (irrespective of the identity of that single winner) and the game ends. Therefore, the final coalition structure yields j_1 a payoff of v_{j_1}[7]. Given our hypothesis, delay can occur along this path if (and only if) at some intermediate stage, an active loser gets to propose. There can be at most $m-2$ such stages; and so staying alone yields j_1 at least $\delta^{m-2} v_{j_1}$.

[7]This follows from our worth of partition function; where any singleton (winner) member of a partition gets his valuation as payoff, if that partition contains another member set where all the losers collude with one or more winners.

The maximum that j_1 can get by colluding with any other active agent is given by $\max\left\{\frac{v_{j_1}+v_{j_2}}{1+\delta},\ldots,\frac{\sum_{t=1}^{m-1} v_{j_t}}{1+(m-2)\delta}\right\}$[8]. For any $t'=2,\ldots,m-1$, the difference $\left[\delta^{m-2}v_{j_1}-\frac{\sum_{t=1}^{t'} v_{j_t}}{1+(t'-1)\delta}\right]$ is continuous and strictly increasing in δ with the $\delta\to 1$ limit being positive. Therefore, for δ sufficiently close to 1, this difference is positive. Thus, $\exists\ \delta_1\in(\max\{\delta(2),\delta^{m-1}\},1)$ such that $\delta^{m-2}v_{j_1}>\max\left\{\frac{\sum_{t=1}^{t'} v_{j_t}}{1+(t'-1)\delta}\right\}_{t'=2}^{m-1}\ \forall\,\delta\in(\delta_1,1)$. Therefore $C^{\delta}_{j_1}(T\cup L)\equiv C^{\delta}_{\bar{m}^T}(T\cup L)=\{\{j_1\}\},\forall\,\delta\in(\delta_1,1)$.

STEP 2: Fix a $\delta\in(\delta_1,1)$ and consider the agent j_2. For such a δ, our hypothesis implies that if j_2 stays alone, the maximum payoff (attained if no delay occurs in the intermediate stages) he can get is v_{j_2} and the minimum payoff (attained if there is delay at each of the intermediate stages) that he can get is $\delta^{m-3}v_{j_2}$. If j_2 acceptably proposes $\{j_1,j_2\}$, he gets *at least*, $\delta^{m-3}\left[v_{j_2}+(1-\delta)v_{j_1}\right]$. Any other collusive venture gives j_2 a maximum possible payoff of $\max\left\{\frac{(1-\delta)v_{j_1}+\sum_{t=2}^{t'} v_{j_t}}{1+(t'-2)\delta}\right\}_{t'=3}^{m-1}$. Like in the previous case, there exists a $\bar{\delta}_2\in(\delta_1,1)$ such that $\forall\,\delta\in(\bar{\delta}_2,1)$, $\delta^{m-3}\left[v_{j_2}+(1-\delta)v_{j_1}\right]>\max\left\{v_{j_2},\max\left\{\frac{(1-\delta)v_{j_1}+\sum_{t=2}^{t'} v_{j_t}}{1+(t'-2)\delta}\right\}_{t'=3}^{m-1}\right\}$.[9] That is, $C^{\delta}_{j_2}(T\cup L)=\{\{j_1,j_2\}\}$, and $a_{j_2}(T\cup L)\geq\delta^{m-2}[v_{j_2}+(1-\delta)v_{j_1}]$.[10]

[8] Agent j_1 attains the payoff of $\frac{v_{j_1}+\ldots+v_{j_t}}{1+(t-1)\delta}$, for any $t<m$; when j_1 acceptably proposes $\{j_1,j_2,\ldots,j_t\}$ at this stage and the remaining winners (or winner) colludes with all the losers in the next stage.

[9] For any $t'=3,\ldots,m-1$, the difference $\delta^{m-3}v_{j_2}-\frac{(1-\delta)v_{j_1}+\sum_{t=2}^{t'} v_{j_t}}{1+(t'-2)\delta}$ is continuous and strictly increasing in δ with a positive value in the limit (tends to 1). Therefore for δ *sufficiently close* to 1, the difference is always positive. Hence for δ sufficiently close to 1, the difference $\left[\delta^{m-3}v_{j_2}-\max\left\{\frac{(1-\delta)v_{j_1}\sum_{t=2}^{t'} v_{j_t}}{1+(t'-2)\delta}\right\}_{t'=3}^{m-1}\right]$ is positive and so staying alone strictly dominates formation of any coalition other than $\{j_1,j_2\}$. However, the difference $\delta^{m-3}[v_{j_2}+(1-\delta)v_{j_1}]-v_{j_2}=(1-\delta)\left[\delta^{m-3}v_{j_1}-\left(1+\delta+\delta^2+\ldots+\delta^{m-4}\right)v_{j_2}\right]$ is positive *iff* $H(\delta)>\frac{v_{j_2}}{v_{j_1}}$ where $H(\delta):=\frac{\delta^{m-3}}{1+\delta+\delta^2+\ldots+\delta^{m-4}}$. Since $\frac{v_{j_2}}{v_{j_1}}\in(0,1)$ and $H(\delta)$ is a strictly increasing function of δ, once again the for *sufficiently high* δ, the difference $\delta^{m-3}[v_{j_2}+(1-\delta)v_{j_1}]-v_{j_2}$ is positive. Thus a $\bar{\delta}_2$ can indeed be found.

[10] Recall that j_2 can always reject a proposal, incur a period of delay, and then acceptably propose $\{j_1,j_2\}$.

Note that we can also find a $\delta_2 \in (\bar{\delta}_2, 1)$ such that $\forall\, \delta \in (\delta_2, 1)$, $\delta^{m-2}[v_{j_2} + (1-\delta)v_{j_1}] > v_{j_2}$. This means that for this range of discount factor; any acceptable proposal directed at j_2 must give him at least a payoff greater than v_{j_2} (which is the maximum possible marginal contribution that j_2 can make to any coalition containing it). This is always suboptimal and therefore, for this range of δ, no acceptable proposal directed at j_2 is ever made in equilibrium. In this manner we can generate a sequence $\{\delta_t\}_{t=3}^{m}$ such that $\delta_3 \in (\delta_2, 1)$ and $\delta_{t+1} \in (\delta_t, 1)$, $\forall\, t$ with the property that (i) $C_t^{\delta}(T \cup L) = \{j_1, j_t\}$, $\forall\, \delta \in (\delta_t, 1)$, $\forall\, t \geq 3$; and (ii) *no acceptable proposal* containing any of the members in $\{j_2, \ldots, j_t\}$ will be made in the equilibrium if the discount factor exceeds $\bar{\delta}_2$.[11]

STEP 3: Fix a $\delta \in (\delta_m, 1)$. Consider any $j \in T \setminus \{j_1\}$ and suppose that j makes an unacceptable proposal. This is optimal only if, this leads to transfer of proposal power to some other active player $j' \in T$ who makes an acceptable proposal $S^{j'}$ excluding j.

If $j' \in L$, then from (ii) in STEP 2, $S^{j'} \in \{\{j_1\} \cup S_k(t)\}_{t=l}^{n}$. For any $t \in \{l, \ldots, n\}$, the acceptable proposal $[\{j_1\} \cup S_k(t)]$ gives l a maximum possible payoff $\frac{(1-\delta)v_{j_1} - v_{t+1}}{1+(t-k-1)\delta}$. Observe that if $\delta > \frac{v_{j_1} - v_n}{v_{j_1}}$, then only acceptable proposal giving l a positive (maximum possible) payoff $\frac{(1-\delta)v_{j_1}}{1+(n-k-1)\delta}$ is $[\{j_1\} \cup S_k(n)]$. Then $\forall\, \delta \in \left(\max\left\{\delta_m, \frac{v_{j_1} - v_n}{v_{j_1}}\right\}, 1\right)$, $C_l^{\delta}(T \cup L) = [\{j_1\} \cup S_k(n)]$. But then, the game goes to stage $T \setminus \{j_1\}$, where the maximum possible payoff that j can get is $\delta[v_j + (1-\delta)v_{\bar{m}^{T\setminus\{j_1\}}}]$[12] It can be easily seen that $\exists\ \delta'_m \in \left(\max\left\{\delta_m, \frac{v_{j_1} - v_n}{v_{j_1}}\right\}, 1\right)$ such that $\forall\, \delta \in (\delta'_m, 1)$, $\delta^{m-2}[v_{j_2} + (1-\delta)v_{j_1}] > \delta[v_j + (1-\delta)v_{\bar{m}^{T\setminus\{j_1\}}}]$ (since $v_{j_1} > \bar{m}^{T\setminus\{j_1\}}$) and so making an acceptable proposal dominates making an unacceptable proposal.

Fix any $\delta \in (\delta'_m, 1)$. If $j' \in T \setminus \{j\}$, then either $j' = j_1$ or $j' \neq j_1$. If $j' = j_1$, then from STEP 1, $C_{j_1}(T \cup L) = \{j_1\}$, and so the game goes to the stage $[T \setminus \{j_1\}] \cup L$ with $m-1$ winners. Then from the induction hypothesis, we get that $C_j([T \setminus \{j_1\}] \cup L) =$

[11]This is because the expression $\delta^{m-2}[x + (1-\delta)v_{j_1}] - x$ is decreasing in x.

[12]As in STEP 2, we can show that it is suboptimal for j to acceptably propose to any other winner in $T \setminus \left\{j_1, \bar{m}^{T\setminus\{j_1\}}\right\}$, at stage $T \setminus \{j_1\}$.

$\left\{\bar{m}^{T\setminus\{j_1\}}, j\right\}$. Given the range of δ, from STEP 2, $C_j(T \cup L) = \{j_1, j\}$; and so, payoff to j from proposing $\{j_1, j\}$ acceptably at stage $T \cup L$ exceeds that from proposing $\left\{\bar{m}^{T\setminus\{j_1\}}, j\right\}$ acceptably, at stage $T \cup L$ (which, in turn, is weakly greater than doing the same at stage $[T \setminus \{j_1\}] \cup L$). Therefore, proposing acceptably dominates doing otherwise. When $j' \neq j_1$, from STEP 2, agent j' acceptably proposes $\{j_1, j'\}$ (since $C_{j'}(T \cup L) = \{j_1, j'\}$); and so, the game proceeds to the next stage $[T \setminus \{j_1, j'\}] \cup L$ with $m-2$ winners. Then, from induction hypothesis, $C_j([T \setminus \{j_1, j'\}] \cup L) = \left\{\bar{m}^{T\setminus\{j_1,j'\}]\cup L}, j\right\}$. Given the range of δ, from STEP 2, $C_j(T \cup L) = \{j_1, j\}$; and so, arguing as before, proposing acceptably dominates doing otherwise.

Finally, consider the possibility that $j = j_1$. Then, $j' \neq j_1$. Therefore, $\forall\, \delta \in (\delta'_m, 1)$, as mentioned before, $j' \in L \Rightarrow C_l^\delta(T \cup L) = [\{j_1\} \cup S_k(n)]$ and $j' \in T \setminus \{j_1\} \Rightarrow C_{j'}(T \cup l) = \{j_1, j'\}$. But for both these cases, j_1 could have proposed the same coalition acceptably, in the first place; thereby saving a period of delay (and getting the (higher) proposer's share out of the worth of $[\{j_1\} \cup S_k(n)]$, in case of $j' \in L$). Hence, proposing unacceptably turns out to be sub-optimal for j_1 at stage $T \cup L$. Therefore, $\forall\, \delta \in (\delta'_m, 1)$, no winner in T makes an unacceptable proposal at stage $T \cup L$.

STEP 4: If any loser l proposes acceptably at stage $T \cup L$, then, from STEP 3, $\forall\, \delta \in (\delta'_m, 1)$, $C_l^\delta(T \cup L) = [\{j_1\} \cup S_k(n)]$ and l gets a maximum possible payoff of $\frac{(1-\delta)v_{j_1}}{1+(n-k-1)\delta}$ is $[\{j_1\} \cup S_k(n)]$. On the other hand, like in the two winner stage $\{i, j\} \cup L$, given the specified range of δ and our hypothesis, an unacceptable proposal by l to some winner, yields at least $\frac{\delta v_{j_1}}{1+(n-k)\delta}$ in the final single winner stage, at most $m-1$ periods later. That is, the least l gets by making an unacceptable proposal when $\delta \in (\delta'_m, 1)$ is $\frac{\delta^m v_{j_1}}{1+(n-k)\delta}$. The difference $\frac{\delta^m v_{j_1}}{1+(n-k)\delta} - \frac{(1-\delta)v_{j_1}}{1+(n-k-1)\delta}$ is continuous and strictly increasing in δ and this difference is positive in the limit. Therefore, $\exists\ \underline{\delta} \in (\delta'_m, 1)$ such that for all $\delta \in (\underline{\delta}, 1)$ the difference is positive, that is, making unacceptable proposal is the optimal action. The particular identity of the winner in T to whom any l must unacceptably propose is given Recursion (*).

Define $\delta^m := \max\left\{\delta'_m, \frac{v_{j_1}-v_n}{v_{j_1}}, \underline{\delta}\right\}$. Then, $\forall\, \delta \in (\delta^m, 1)$; at the stage $T \cup L$ such that $|T| = m$, all losers make an unacceptable proposal at some active winner and $C^{\delta}_t(T \cup L) = \{j_1, t\} = \{\bar{m}^T, t\}$, $\forall\, t \neq j_1 = \bar{m}^T$ with $C^{\delta}_{\bar{m}^T}(T \cup L) = \{\bar{m}^T\}$. We can continue such a recursion to get a sequence of $\{\delta^m\}_{m=3}^n$ such that result (3) follows by simply choosing $\delta'^m\}_{m=3}^n$. □

An obvious consequence of Theorem 3.7 is the resulting coalition structure contingent on the protocol function. This is summarized in the next corollary using the δ' obtained in Theorem 3.7.

Corollary 3.8. *For any $G = (N, \bar{w}, p, \delta)$ if $T^1 \in C_1^*(T^1)$ and $\delta \in (\delta', 1)$, then for any given $p(.)$, the SSPE coalition structure is a protocol contingent partition $(E_1, \ldots, E_s)$ of the agent set N such that*

$$E_1 = \begin{cases} \{1\} & \textit{if } \{p(N) = 1\} \textit{ or } \{p(N) \in L \textit{ and } k^*(K;p) = 1\} \\ \{p(N), 1\} & \textit{if } p(N) \in K \setminus \{1\} \\ \{k^*(K;p), 1\} & \textit{if } p(N) \in L \textit{ and } k^*(K;p) \neq 1 \end{cases}$$

Suppose the sequence is $\{E_1 \cup \ldots \cup E_q\}$ well defined and $R_q := N \setminus \{E_1 \cup \ldots \cup E_q\} \neq \emptyset$. Then

$$E_{q+1} = \begin{cases} T^i & \textit{if } R_q \cap K = \{i\} \\ \{p(R_q)\} & \textit{if } \{p(R_q) = \overline{m}^{R_q}\} \textit{ or if } \{p(R_q) \in K \setminus \{\overline{m}^{R_q}\} \textit{ and } |R_q \cap K| = 2\} \\ \{p(R_q), \overline{m}^{R_q}\} & \textit{if } p(R_q) \in K \setminus \{\overline{m}^{R_q}\} \textit{ and } |R_q \cap K| > 2 \\ \left\{\underline{m}^{[R_q \cap K]}\right\} & \textit{if } p(R_q) \in L \textit{ and } |R_q \cap K| = 2 \\ \{k^*(R_q \cap K; p), \overline{m}^{R_q}\} & \textit{if } p(R_q) \in L \textit{ and } |R_q \cap K| > 2 \end{cases}$$

Example 3.9. *Suppose $N = \{1, 2, 3, 4, 5\}$, $K = \{1, 2, 3\}$. Let the protocol function $p(S) := \max_{j \in S}\{\succ\}$, $\forall\, S \subseteq N$ for some linear order "$\succ$" defined on the agent set N. For δ sufficiently close to 1, if*

(1)
(2) $1 \succ 2 \succ 3 \succ 4 \succ 5$ *then final coalition structure is* $\{\{1\}, \{2\}, \{3, 4, 5\}\}$.
(3) $3 \succ 2 \succ 5 \succ 4 \succ 1$ *then final coalition structure is* $\{\{1, 3\}, \{2, 4, 5\}\}$.

(4) $4 \succ 1 \succ 5 \succ 2 \succ 3$ *then final coalition structure can either be* $\{\{1,3\},\{2,4,5\}\}$ *or it can be* $\{\{1\},\{3\},\{2,4,5\}\}$.

Note that two possibilities arise for the coalition structure in the third case. That is because a loser (agent 4) gets to propose at a stage with more than 2 *winners. Recall that the* ***Recursion*** *(*) did not guarantee a unique* $k^*(.)$*; which is why* $k^*(N;p) \in \{1,3\}$*, thereby leading to two possible coalition structures.*

An interesting coalition structure is the one where the lowest valuation winner k colludes with all the losers in L while all other winners stay alone. This is interesting because the coalition $T^k = \{k\} \cup L$ ensures that all the losers bid zero at the auction, thereby reducing the $(k+1)$th price to zero. Thus the other winners $\{1,\ldots,k-1\}$ get their own valuations as the equilibrium payoff in the limit as δ tends to 1. Agent k, however, gets only $\frac{v_k}{n-k+1}$ in the limit. In other words, winner k generates the gains from cooperation while the other winners *free ride*. The following proposition provides the restriction on the protocol function that characterizes formation of this coalition structure in equilibrium.

Proposition 3.10. *For any* $G = (N,p,\bar{w},\delta)$*, if* $T^1 \in C_1^*(T^1)$ *then* $\forall\, \delta \in (\delta',1)$[13]*; the SSPE outcome is* $\{\{1\},\ldots,\{k-1\},\{k,k+1,\ldots,n\}\}$ *if and only if the* $p(.)$ *satisfies the property*

$$p(N) = 1, p\left(N \setminus \{1,\ldots,i\}\right) = i+1, \forall\, i \in K \setminus \{k-1,k\} \tag{3.1}$$

Proof: The sufficiency of condition 3.1 follows from Corollary 3.8. To establish the necessity, consider the member $T^k = \{k,k+1,\ldots,n\}$. For T^k to have formed; on the equilibrium path, at some stage $\hat{T}$ (such that $T^k \subseteq \hat{T}$), some member $i \in T^k$ must have acceptably proposed T^k. Now if $|\hat{T} \cap K| \geq 2$ then, given the specified range of δ, irrespective of whether $i = k$ or $i \in L$, we get a contradiction to the equilibrium strategies defined in Theorem 3.7. Hence $\hat{T} = T^k$.

[13]The δ' is taken from Theorem 3.7.

Now consider the singleton coalition $\{k-1\}$. Since T^k must have formed at the stage T^k itself, $\{k-1\}$ must have formed at a stage $\bar{T}$ such that $\{\{k-1\} \cup T^k\} \subseteq \bar{T}$. Given the range of δ, the only possibility where agent $k-1$ would choose to stay alone without contradicting our findings in Theorem 3.7; is when $\bar{T} = \{\{k-1\} \cup T^k\}$. Now if $p(\bar{T}) \in L$, then it must unacceptably propose to the lower value winner k, who would then stay alone. If $p(\bar{T}) = k$ then it is optimal for k to stay alone so that T^{k-1} forms in the next stage. Therefore in either case we have a contradiction. Therefore, $p(\{k-1\} \cup T^k) = k-1 \Rightarrow p(N \setminus \{1, \ldots, k-2\}) = k-2+1 = k-1$. Continuing in this manner, for the rest of the singleton coalitions, $\{k-2\}, \{k-3\}, \ldots, \{1\}$; the result follows.[14] □

In fact, the strategies in Theorem 3.7 generate a class of coalition structures where any one winner colludes with all the losers on the equilibrium path as δ approaches 1 *irrespective of the protocol function*. This is presented formally in the following corollary.

Corollary 3.11. *For any $G = (N, \bar{w}, p, \delta)$, if $T^1 \in C_1^*(T^1)$ then $\forall\, \delta \in (\delta', 1)$, the SSPE outcome belongs to the class of coalition structures $\bar{\mathcal{P}} \subset \Pi(N)$ such that $\forall\, \pi \in \bar{\mathcal{P}}$,*

(1)
(2) $\exists\, j(\pi) \in K$ *such that* $T^{j(\pi)} \in \pi$.[15]
(3) *if* $S \in \pi \setminus \left\{T^{j(\pi)}\right\}$ *then* $|S| \in \{1, 2\}$.
(4) $|\{j \in K : \{j\} \in \pi, v_j < v_{j(\pi)}\}| \in \{0, 1\}$.

Proof: (1) and (2) follow from the Theorem 3.7. To prove (3), suppose the contrary holds. That is, there exists a $\pi \in \mathcal{P}$ and a pair of winners $j, j' \in K$ such that $v_j < v_{j'} < v_{j(\pi)}$ and $\{j\}, \{j'\} \in \pi$.[16] Now, from Theorem 3.7 it follows that coalition $T^{j(\pi)}$ forms at stage $T^{j(\pi)}$, that is, the single winner stage (after which the game ends). This means

[14]Note that $p(T^k) = p(N \setminus \{1, \ldots, k-1\})$ is free from any restriction because any agent in T^k proposes T^k optimally.

[15]If $|N| > 2$ then $j(\pi) \in K \setminus \{1\}, \forall\, \pi \in \mathcal{P}$.

[16]This means $|\{j \in K : \{j\} \in \pi, v_j < v_{j(\pi)}\}| = 2$.

either $\{j\}$ or $\{j'\}$ must have formed at some stage T' such that $\left\{\{j\} \cup \{j'^{j(\pi)}\right\} \subseteq T'$. In either case, this is in contradiction to the equilibrium proposal decisions in Theorem 3.7 for the specified range of δ. Hence the result (3) follows. $\square$

Example 3.12. *Take the simplest multiple goods case where there are two goods, that is,* $N = \{1,2,3,4,5,6\}$ *and* $K = \{1,2\}$. *Fix the* δ *value sufficiently high so that the comparison amongst the average worths gives us the ranking between different coalitions according to their profitability as collusive ventures. Assume that* (**a**) $\frac{v_i - v_5}{3} > \max\left\{v_i - v_3, \frac{v_i - v_4}{2}, \frac{v_i - v_6}{4}, \frac{v_i}{5}\right\}$, $\forall\, i = 1,2$; *that is,* $C_i^\delta(\{i,3,4,5,6\}) = \{i\} \cup \{3,4\}$, $\forall\, i = 1,2$.

Suppose $p(N) = 1$ *and* $p(N \setminus \{1\}) = 2$. *Therefore, if agent 1 stays alone at the stage* N, *then at the next stage 2 acceptably proposes* $\{2,3,4\}$ *leading to the coalition structure* $\{\{1\},\{2,3,4\},\{5\},\{6\}\}$ *which gives agent 1 a payoff of* $v_1 - v_5$. *However, if 1 forms* $\{1,6\}$ *at stage* N, *then the payoff is*

$$\begin{cases} \frac{v_1 - v_3}{2} & \textit{if 2 stays alone at the next stage } \{2,3,4,5\} \\ \frac{v_1 - v_4}{2} & \textit{if 2 forms } \{2,3\} \textit{ at the next stage } \{2,3,4,5\} \\ \frac{v_1 - v_5}{2} & \textit{if 2 forms } \{2,3,4\} \textit{ at the next stage } \{2,3,4,5\} \\ \frac{v_1}{2} & \textit{if 2 forms } \{2,3,4,5\} \textit{ at the next stage } \{2,3,4,5\} \end{cases}$$

Therefore, for 1 to make the optimal proposal choice at stage N *(that is, to evaluate the proposal* $\{1,6\}$ *at stage* N*), it needs to know the proposal choice of agent 2 at stage* $\{2,3,4,5\}$. *Note that our assumption (**a**) puts no restriction on the ranking of average worths of subsets of* $\{2,3,4,5\}$, *that agent 2 can propose acceptably (keeping in mind that agent 6 has already colluded with agent 1 and so will bid zero at the auction) at stage* $\{2,3,4,5\}$. *That is, (**a**) does not impart any ranking of the numbers* $v_2 - v_3, \frac{v_2 - v_4}{2}, \frac{v_2 - v_5}{3}$ *(payoffs from forming* $\{2\}$, $\{2,3\}$ *and* $\{2,3,4\}$ *respectively) with respect to* $\frac{v_2}{4}$ *(payoff from forming* $\{2,3,4,5\}$*). Hence the problem becomes fairly intractable, even with two goods case, once we allow* $C_i^\delta(T^i)$ *to be strict subset of* T^i *for all (or some)* $i \in K$.

Also in such a case the final coalition structure may or may not have one winner colluding with all the losers, depending upon the protocol function. That is, if we use the protocol function

$p(N) = 1, p(N \setminus \{1\}) = 6$; *then it is optimal for agent 1 to stay alone at stage N since, in the next stage, the loser 6 proposes (who has no choice but to acceptably propose)* $\{2, 3, 4, 5, 6\}$ *leading to formation of the coalition structure* $\{\{1\}, \{2, 3, 4, 5, 6\}\}$ *giving 1 a payoff of* v_1 *(which is the best that agent 1 can get).*

4. Conclusion

In this chapter, we analyze coalition formation at Vickrey auction with single as well as multiple identical indivisible identical goods; with unit demand and complete information. The assumption of complete information is restrictive but it turns out that this case is already quite rich. We provide, for sufficiently patient bidders, the necessary and sufficient conditions for formation of bidding ring at the single good auction, when the highest valuation agent is the first proposer. In the multiple goods case, we specify the sufficient conditions for formation of the class of coalition structures, where exactly one winner colludes with all the losers *irrespective of the protocol function*. Our work, therefore, turns out to be the complete information benchmark with regard to collusion at such auctions. Of course, further research needs to be done to extend this line of coalition formation to the incomplete information case.

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